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Eigenfunction Expansions for Symmetric Systems of First Order in the Half-Space $\mathbf{R}_+^n$

By

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§0. Introduction

The eigenfunction expansion theory for partial differential operators has been investigated by many authors. T. Carleman [1], A. Ya. Povzner [10] and T. Ikebe [2] treated the Schrödinger operator $-\Delta+q(x)$ in the whole 3-dimensional Euclidean space $\mathbf{R}^3$. Especially Ikebe gave an explicit eigenfunction expansion in terms of distorted plane waves. For $-\Delta$ in an exterior domain of $\mathbf{R}^3$, the first result from Ikebe's point of view was obtained by Y. Shizuta [13] (see also Ikebe [3]) and the result is generalized by N. A. Shenk II [12] to the higher dimensional case. K. Mochizuki [8] derived the eigenfunction expansions in terms of distorted plane waves for symmetric systems in an exterior domain of $\mathbf{R}^n$ and J. R. Schulenberger and C. H. Wilcox [11] in the whole space $\mathbf{R}^n$. An other approach to spectral representations for the operators associated with the wave equation and symmetric hyperbolic systems in an exterior domain in $\mathbf{R}^n$ is developed by P. D. Lax and R. S. Phillips [6]. As for the eigenfunction expansions for more general partial differential operators there are important works by F. E. Browder, L. Gårding, F. I. Mautner and others.

In this paper we shall derive eigenfunction expansions associated with the stationary problems in the half-space $\mathbf{R}_+^n$ for symmetric hyperbolic systems with constant coefficients. We note that this problem cannot be treated as a perturbation of the whole space problem. In fact, our theory is a generalization of the sine and cosine transformations in

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the L^2 space on the positive half-line which are eigenfunction expansions for $-\frac{d^2}{dx^2}$ in $(0, \infty)$ with Dirichlet or Neumann conditions at $x=0$.

Let $\mathbf{R}^n$ denote the n-dimensional Euclidean space. Denote by x the generic point of $\mathbf{R}^n$ and write $x'=(x_1,\dots, x_{n-1})$. We shall also denote by $\mathbf{R}^n_+$ the half-space $\{x=(x', x_n)\in \mathbf{R}^n;\ x_n>0\}$ and by t the time variable. Let L be a first order symmetric hyperbolic operator with constant coefficients:

$$L=I\frac{\partial}{\partial t}-\sum_{j=1}^{n}A_j\frac{\partial}{\partial x_j}, \tag{0.1}$$

where I is the identity matrix of order N and the A_j are $N\times N$ constant Hermitian matrices.

We consider the mixed initial and boundary value problem in $\mathbf{R}^n_+$ for the operator L:

$$L[u(t, x)]=f(t, x), \qquad t>0, \quad x\in \mathbf{R}^n_+, \tag{0.2}$$

$$u(0, x)=u_0(x), \qquad x\in \mathbf{R}^n_+, \tag{0.3}$$

$$Bu(t, x)|_{x_n=0}=0, \qquad t>0, \tag{0.4}$$

where $u(t, x), f(t, x)$ and $u_0(x)$ are vector-valued functions whose values lie in the N-dimensional complex space $\mathbf{C}^N$ and B is an $l\times N$ constant matrix with rank l, which stipulates l linear homogeneous relations between the components of u on the boundary $x_n=0$.

Replacing $u(t, x)$ and $f(t, x)$ in (0.2) by $e^{ikt}v(x)$ and $-ie^{ikt}g(x)$, respectively, we obtain the corresponding stationary problem:

$$(A-kI)v(x)=g(x), \qquad x\in \mathbf{R}^n_+, \tag{0.5}$$

$$Bv(x)|_{x_n=0}=0, \tag{0.6}$$

where

$$A=\frac{1}{i}\sum_{j=1}^{n}A_j\frac{\partial}{\partial x_j}. \tag{0.7}$$

Our aim is to expand an arbitrary function in $L^2(\mathbf{R}^n_+)$ by means of the generalized or improper eigenfunctions for the self-adjoint operator associated with this problem under some suitable conditions of L (or A) and B. In order to state our assumptions on the operators A and B, let us now recall some basic notations and terminology about hyperbolic mixed problems. For more details in this subject, see, for example, Courant and Hilbert book and Lax and Phillips [6].

Let $p(\lambda, \eta)$ be the characteristic polynomial associated with the operator L:

$$p(\lambda, \eta)=\det(\lambda I - A(\eta)), \tag{0.8}$$

where η denotes a generic point of the real dual space Ξ^n of $\mathbf{R}^n$ by the duality $x\cdot\eta = x_1\eta_1 + \cdots + x_n\eta_n$ and

$$A(\eta)=\sum_{j=1}^{n} \eta_j A_j . \tag{0.9}$$

The polynomial $p(\lambda, \eta)$ has a factorization

$$p(\lambda, \eta)=Q_1(\lambda, \eta)^{m_1}\dots Q_q(\lambda, \eta)^{m_q}, \tag{0.10}$$

where the factors $Q_j(\lambda, \eta)$ are distinct homogeneous polynomials in (λ, η), irreducible over the complex number field $\mathbf{C}$. Since the coefficient of λ^N in $p(\lambda, \eta)$ is 1, the factors are unique, apart from their order, by requiring the coefficient of the highest power of λ in each $Q_j(\lambda, \eta)$ be 1. Put

$$Q(\lambda, \eta)=Q_1(\lambda, \eta)\dots Q_q(\lambda, \eta). \tag{0.11}$$

Definition 0.1. The operator L is said to be uniformly propagative if the roots $\lambda_j(\eta)$, $1 \leqq j \leqq \mu$, of the equation $Q(\lambda, \eta)=0$ in λ satisfy the following conditions where μ is the order of $Q(\lambda, \eta)$: (1) The roots $\lambda_j(\eta)$ are all distinct for every η with $|\eta|=1$. Thus we may assume that the $\lambda_j(\eta)$ are enumerated in the following way.

$$\lambda_1(\eta) > \lambda_2(\eta) > \cdots > \lambda_\mu(\eta). \tag{0.12}$$

(2) A root function $\lambda_j(\eta)$ vanishes for some real $\eta \neq 0$ if and only if it vanishes identically.

Wilcox [15] gave an elegant characterization of this class of symmetric hyperbolic operators. For a uniformly propagative operator, the eigenvalues of $A(\eta)$ for $\eta\neq 0$ have constant multiplicity.

$$p(\lambda, \eta)=(\lambda-\lambda_1(\eta))^{\nu_1}\dots(\lambda-\lambda_\mu(\eta))^{\nu_\mu},\ \nu_1+\cdots+\nu_\mu=N. \tag{0.13}$$

From the enumeration (0.12) it follows that the roots $\lambda_j(\eta)$ are C^∞ functions of η in $\Xi^n\backslash\{0\}$ and positively homogeneous of degree 1. Further we have

$$\lambda_j(\eta)=-\lambda_{\mu-j+1}(-\eta) \quad\text{and}\quad \nu_j=\nu_{\mu-j+1}, \qquad j=1,\dots,\mu. \tag{0.14}$$

We consider only the case $n\geqq 2$. Hence by the condition (2) we have for $\eta\neq 0$

$$\lambda_1(\eta)>\cdots>\lambda_\rho(\eta)>0>\lambda_{\rho+1}(\eta)>\cdots>\lambda_{2\rho}(\eta), \tag{0.15}$$

if $\mu=2\rho$ is even, and

$$\lambda_1(\eta)>\cdots>\lambda_\rho(\eta)>\lambda_{\rho+1}(\eta)\equiv 0>\lambda_{\rho+2}(\eta)>\cdots>\lambda_{2\rho+1}(\eta), \tag{0.16}$$

if $\mu=2\rho+1$ is odd.

The $l\times N$ matrix B in (0.6) defines a linear operator of $\mathbf{C}^N$ into $\mathbf{C}^l$ (under the respective canonical bases).

Definition 0.2. The linear operator defined above is called boundary operator and denoted by the same letter B. The kernel $\{\zeta\in\mathbf{C}^N;\ B\zeta=0\}$ is called boundary space and denoted by $\mathscr{B}$ or $\ker B$. A boundary operator B (or space $\mathscr{B}$) is said to be conservative or energy preserving if the quadratic form $A_n\zeta\cdot\bar{\zeta}$ associated with A_n is zero on $\mathscr{B}$, that is,

$$A_n\zeta\cdot\bar{\zeta}=0 \qquad \text{for all} \quad \zeta\in\mathscr{B}, \tag{0.17}$$

where $\zeta\cdot\bar{\zeta}'$ denotes the scalar product $\zeta_1\bar{\zeta}'_1+\cdots+\zeta_N\bar{\zeta}'_N$ for $\zeta,\ \zeta'\in\mathbf{C}^N$.

Under this condition we can easily see that every solution $u(t,\cdot)\in L^2(\mathbf{R}^n_+)$ of the mixed problem (0.2)–(0.4) with $f=0$ satisfies the following energy equality which expresses the conservation of energy:

$$\|u(t,\cdot)\|_{L^2(\mathbf{R}^n_+)}=\|u_0(\cdot)\|_{L^2(\mathbf{R}^n_+)}. \tag{0.18}$$

This equality immediately assures uniqueness for the solution and continuous dependence on the initial data. But the above condition does not guarantee existence of the solution. Indeed, we cannot expect a solution to exist if too many boundary conditions are imposed, that is, if fewer boundary conditions would guarantee the uniqueness of the solution. To guarantee general existence of the solution, we require the following concept.

Definition 0.3. A boundary space $\mathscr{B}$ is called maximally conservative if $\mathscr{B}$ cannot be enlarged to a larger linear space over which the quadratic form $A_n\zeta\cdot\bar{\zeta}$ is still everywhere zero. Since the boundary space $\mathscr{B}$ is larger when we impose fewer conditions, the boundary operator B then is called minimally conservative.

The following lemma due to Lax and Phillips [6] give a complete description of all maximally conservative subspaces of $\mathbf{C}^N$ with respect to the quadratic form $A_n\zeta\cdot\bar{\zeta}$ of signature zero.

Lemma 0.4 ([6], p. 199). *Let S be a symmetric $N\times N$ matrix of signature zero, and denote by $\mathscr{P}$ and $\mathscr{N}$ the $N/2$ dimensional subspaces spanned by the eigenvectors corresponding to the positive and negative eigenvalues, respectively. Then S is positive and negative definite over $\mathscr{P}$ and $\mathscr{N}$, respectively. Let $e_1^+,\dots, e_{N/2}^+$ be any orthonormal basis in $\mathscr{P}$ with respect to S and $e_1^-,\dots, e_{N/2}^-$ be any orthonormal basis in $\mathscr{N}$ with respect to $-S$. Then*

$$S\zeta\cdot\bar{\zeta}=0 \tag{0.17)'}$$

for all ζ in the subspace $\mathscr{B}$ spanned by $e_1^+ + e_1^-,\dots, e_{N/2}^+ + e_{N/2}^+$ and $\mathscr{B}$ is maximal with respect to this property. Conversely, every $\mathscr{B}$ which has property (0.17)′ *maximally can be constructed in this way.*

From now on we shall assume that the hyperplane $x_n=0$ is not characteristic for L, i.e., the matrix A_n is non-singular. We define

$$M(\xi;\lambda)=A_n^{-1}(\lambda I-\sum_{j=1}^{n-1}\xi_j A_j), \tag{0.19}$$

where λ is a complex parameter and ξ is the generic point of Ξ^{n-1}.

From the hyperbolicity of L it follows that the eigenvalues of $M(\xi;\lambda)$ are never real when $\operatorname{Im}\lambda\neq 0$ and $\xi\in\Xi^{n-1}$. Since the eigenvalues of $M(\xi;\lambda)$ are (multivalued) continuous functions of (ξ,λ) this implies that the number of roots with positive imaginary part, counted according to multiplicity, is constant when $\operatorname{Im}\lambda>0$ (resp. $\operatorname{Im}\lambda<0$) and $\xi\in\Xi^{n-1}$. If the operator A is elliptic in the sense that $A(\eta)$ is non-singular for all non-zero $\eta\in\Xi^n$, the matrix $M(\xi;0)$ has the same property for non-zero $\xi\in\Xi^{n-1}$. We denote by $E^+(\xi;\lambda)$ (resp. $E^-(\xi;\lambda)$) the subspace of $\mathbf{C}^N$ spanned by all the ordinary and generalized eigenvectors of $M(\xi;\lambda)$ corresponding to eigenvalues with positive (resp. negative) imaginary part and call it the positive (resp. negative) eigenspace of $M(\xi;\lambda)$. Then we have

$$E^+(\xi;\lambda)\oplus E^-(\xi;\lambda)=\mathbf{C}^N \tag{0.20}$$

when $\operatorname{Im}\lambda\neq 0$ and $\xi\in\Xi^{n-1}$. If the operator A is elliptic we have further

$$\dim E^+(\xi;\lambda)=\dim E^-(\xi;\lambda)=\frac{N}{2}\equiv m\ (n\geqq 2) \tag{0.21}$$

for $\operatorname{Im}\lambda\neq 0$ and $\xi\in\Xi^{n-1}$.

Definition 0.5. A boundary operator B is called coercive for an operator A if there exists a positive constant C such that

$$\sum_{j=1}^{n}\left\|\frac{\partial v}{\partial x_j}\right\|\leqq C(\|Av\|+\|v\|) \tag{0.22}$$

for all functions $v\in C_0^\infty(\overline{\mathbf{R}_+^n})$[1] which satisfy the boundary condition $Bv|_{x_n=0}=0$. Here $\|v\|$ denotes the L^2 norm of v over $\mathbf{R}_+^n$.

The following lemma is also due to Lax and Phillips [6].

Lemma 0.6 ([6], p. 202). *The boundary operator B is coercive for $A=\frac{1}{i}\sum_{j=1}^{n}A_j\frac{\partial}{\partial x_j}$ if and only if the following condition are satisfied:*

(i) $l=N/2$.

(ii) $\mathscr{B}\cap E^+(\xi;0)=\{0\}$ *for any non-zero* $\xi\in\Xi^{n-1}$.

1) $C_0^\infty(\overline{\mathbf{R}_+^n})=\{v\in C^\infty(\overline{\mathbf{R}_+^n});\ \textit{Supp } v \textit{ is bounded}\}$.

Here l is the number of column vectors of B.

Now we state precisely the assumptions that we impose on L and B:

(L.1) The operator L is uniformly propagative.

(L.2) The operator A is elliptic.

(L.3) The multiplicity of the real roots of $Q(\lambda, \eta)|_{\eta=(\xi,\tau)}=0$ with respect to τ is not greater than two for every $\xi \in \Xi^{n-1}$ and real $\lambda \neq 0$. Moreover the equation has at most only one couple of real double roots for every $(\xi, \lambda) \neq (0, 0)$.

(B.1) The boundary operator B is minimally conservative.

Remark 1. The condition (L.2) implies that the matrices A_j are non-singular and that μ and N are even. Hence we put $\mu=2\rho$ and $N=2m$.

Remark 2. The condition (B.1) implies that $l=\frac{N}{2}\equiv m$.

The differential operator A defines a non-bounded linear operator $\mathscr{A}$ in $L^2(\mathbf{R}^n_+)$ with domain

$$D(\mathscr{A})=\{v(x);\ v \in C_0^\infty(\overline{\mathbf{R}^n_+}),\ Bv(x)|_{x_n=0}=0\}\,.$$

$\mathscr{A}$ is closable and we denote by $\mathbf{A}$ its closure. Then $\mathbf{A}$ is a self-adjoint operator in $L^2(\mathbf{R}^n_+)$. Moreover if we assume that $\mathscr{B} \cap E^+(\xi; 0)=\{0\}$ for any non-zero $\xi \in \Xi^{n-1}$, the domain $D(\mathbf{A})$ of the operator $\mathbf{A}$ is the set $\{v(x) \in H^1(\mathbf{R}^n_+);\ Bv(x)|_{x_n=0}=0\}$, where $H^1(\mathbf{R}^n_+)$ denotes the space of vector-valued functions whose derivatives of order $\leqq 1$ belong to $L^2(\mathbf{R}^n_+)$ and $Bv(x)|_{x_n=0}$ is interpreted as the trace of $Bv(x)$ on the hyperplane $x_n=0$. These were proved by Lax and Phillips [5].

Let $G(x, y; \lambda)$ be the Green function for $(\mathbf{A}-\lambda\mathbf{I})$, $\operatorname{Im}\lambda \neq 0$, which will be constructed in §3 according to M. Matsumura [7]. $G(x, y; \lambda)$ is defined in $\mathbf{R}^n_+ \times \mathbf{R}^n_+ \times \{\mathbf{C}\backslash\mathbf{R}\}$ and C^∞ function of (x, y) outside the diagonal in $\mathbf{R}^n_+ \times \mathbf{R}^n_+$. We extend $G(x, y; \lambda)$ over $\mathbf{R}^n$ with respect to y by defining $G(x, y; \lambda)=0$ for $x \in \mathbf{R}^n_+$, $y \notin \mathbf{R}^n_+$ and $\operatorname{Im}\lambda \neq 0$. Then for $x \in \mathbf{R}^n_+$ and $\operatorname{Im}\lambda \neq 0$, $G(x, y, \lambda)$ is a temperate distribution of y. Now we put, for $x \in \mathbf{R}^n_+$, $\eta \in \Xi^n$ and $\operatorname{Im}\lambda \neq 0$, (0.23) $\Psi_j(x, \eta; \lambda)=\bar{\mathscr{F}}_y[G(x, y; \lambda)](\eta)(\lambda_j(\eta)-\lambda)P_j(\eta), \quad 1\leqq j\leqq 2\rho,$

where $\bar{\mathscr{F}}_y[f(y)](\eta)$ denotes the conjugate Fourier transform of a temperate distribution $f(y)$ and $P_j(\eta)$ the orthogonal projection of $\mathbf{C}^N$ onto the eigenspace corresponding to the eigenvalue $\lambda_j(\eta)$ of the matrix $A(\eta)$. We can show that for all $x\in\mathbf{R}^n_+$ and almost all $\eta\in\Xi^n$ $\Psi_j^{\pm}(x, \eta)\equiv\Psi_j(x, \eta; \lambda_j(\eta)\pm i0)$ $(=\lim_{\varepsilon\to 0+}\Psi_j(x, \eta; \lambda_j(\eta)\pm i\varepsilon))$ exists and satisfies

$$A_x\Psi_j^{\pm}(x, \eta)=\lambda_j(\eta)\Psi_j^{\pm}(x, \eta), \tag{0.24}$$

$$B\Psi_j^{\pm}(x, \eta)|_{x_n=0}=0. \tag{0.25}$$

Thus the $\Psi_j^{\pm}(x, \eta)$ are (improper) eigenfunctions for the system $\{A, B\}$ and therefore for the operator $\mathbf{A}$ in $L^2(\mathbf{R}^n_+)$.

Under our assumptions we need generally new eigenfunctions corresponding to the real zeros of the Lopatinski determinant defined in §2 in order to derive the eigenfunction expansions for our problem. If we assume in addition to the conditions (L.1)–(L.3) and (B.1) that (L.1)′ the operator L is strictly hyperbolic, and that (B.2) $E^+(\xi; k)\cap\mathscr{B}=\{0\}$ for every $\xi\in\Xi^{n-1}$ and every real k with $|\xi|+|k|\neq 0$, then our expansion theorem is stated by the following form.

Theorem 0.7. *Under the conditions* (L.1)′, (L.2), (L.3), (B.1) *and* (B.2), *we have the following*:

(i) *For all* $f\in L^2(\mathbf{R}^n_+)$

$$f(x)=\sum_{j=1}^{2\rho}\int_{\Xi^n}\Psi_j^{\pm}(x, \eta)\hat{f}_j^{\pm}(\eta)d\eta, \tag{0.26}$$

$$\hat{f}_j^{\pm}(\eta)=\int_{\mathbf{R}^n_+}\Psi_j^{\pm}(x, \eta)^*f(x)dx. \tag{0.27}$$

Here the above integrals are taken in the sense of the limit in the mean.

(ii) $f\in D(\mathbf{A})$ *if and only if* $\hat{f}_j^{\pm}(\eta)$, $\lambda_j(\eta)\hat{f}_j^{\pm}(\eta)\in P_j(\eta)L^2(\Xi^n)\equiv\{f\in L^2(\Xi^n); P_j(\eta)f(\eta)=f(\eta)\}$, $1\leqq j\leqq 2\rho$. *Then we have*

$$(\mathbf{A}f)(x)=\sum_{j=1}^{2\rho}\int_{\Xi^n}\lambda_j(\eta)\Psi_j^{\pm}(x, \eta)\hat{f}_j^{\pm}(\eta)d\eta, \tag{0.28}$$

(0.29) $$(\mathbf{A}f)\hat{}_j^{\pm}(\eta)=\lambda_j(\eta)\hat{f}_j^{\pm}(\eta).$$

From our proof of the above expansion theorem we can see that $\sigma(\mathbf{A})=\sigma_{ac}(\mathbf{A})=\mathbf{R}^1$, where $\sigma(\mathbf{A})$ and $\sigma_{ac}(\mathbf{A})$ denote the spectrum and the absolutely continuous spectrum of $\mathbf{A}$, respectively. Moreover we can obtain explicit representations of the eigenfunctions $\Psi_j^{\pm}(x, \eta)$. Let $\Phi_j^{\pm}: L^2(\mathbf{R}_+^n)\to L^2(\Xi^n)$ be the mappings defined by

(0.30) $$\Phi_j^{\pm}f=\hat{f}_j^{\pm} \quad \text{for all} \quad f\in L^2(\mathbf{R}_+^n), \quad 1\leqq j\leqq 2\rho.$$

Put

(0.31) $$\Phi^{\pm}\equiv\sum_{j=1}^{2\rho}\Phi_j^{\pm}.$$

Then we can prove that the $\Phi_j^{\pm}$ and $\Phi^{\pm}$ are (partial) isometries and give explicitly the ranges of $\Phi_j^{\pm}$ and $\Phi^{\pm}$.

Under the conditions (L.1)–(L.3) and (B.1) we shall prove the expansion theorem in §6. Further we shall also show that the condition (L.2) can be removed in the last section.

The plan of the remainder of this paper is as follows: In §1, we study some behaviors of the eigenvalues with respect to the parameter (ξ, λ) and construct continuous bases of the positive and negative eigenspaces $E^{\pm}(\xi;\lambda)$ of $M(\xi;\lambda)$. In §2 some behaviors of the Lopatinski determinant in the neighborhood of the zeros are studied. In §3 the Green function $G(x, y;\lambda)$ of the operator $\mathbf{A}-\lambda\mathbf{I}$ is constructed and a representation of its partial Fourier image is given. In §4 improper eigenfunctions for $\mathbf{A}$ are defined. A construction of the spectral family of $\mathbf{A}$ by means of the improper eigenfunctions is given in §5 and is applied in §6 to prove the expansion theorem. Some examples are given in §7.

§1. Eigenvalues of $M(\xi;\lambda)$ and Continuous Bases of $E^{\pm}(\xi;\lambda)$

We shall construct the Green function $G(x, y;\lambda)$ for the operator $\mathbf{A}-\lambda\mathbf{I}$ $(\mathrm{Im}\,\lambda\neq 0)$ in the form

$$(1.1)\qquad G(x, y; \lambda)=E(x-y; \lambda)-E_c(x, y; \lambda),$$

where $E(x; \lambda)$ is the fundamental solution of $A=\frac{1}{i}\sum_{j=1}^{n} A_j\frac{\partial}{\partial x_j}$ in the free space $\mathbf{R}^n$, defined by

$$(1.2)\qquad E(x; \lambda)=(2\pi)^{-\frac{n}{2}}\bar{\mathscr{F}}[(A(\eta)-\lambda I)^{-1}].$$

On the other hand $E_c(x, y; \lambda)$, called compensating kernel, will be obtained as a solution of the following boundary value problem:

$$(1.3)\qquad (A_x-\lambda I)E_c(x, y; \lambda)=0, \qquad x, y\in\mathbf{R}^n_+, \quad \operatorname{Im}\lambda\neq 0,$$

$$(1.4)\qquad BE_c(x, y; \lambda)|_{x_n=0}=BE(x-y; \lambda)|_{x_n=0}.$$

Taking formally partial Fourier transforms with respect to $x'=(x_1,\dots, x_{n-1})$ in (1.3) and (1.4), we obtain the first order system of ordinary differential equations depending on parameters (ξ, λ):

$$(1.5)\qquad \left(\frac{1}{i}\frac{d}{dx_n}-M(\xi; \lambda)\right)\tilde{E}_c(\xi, x_n, y; \lambda)=0, \; x_n>0, \; \xi\in\Xi^{n-1},$$

$$(1.6)\qquad B\tilde{E}_c(\xi, 0, y; \lambda)=\mathscr{F}_{x'}[BE(x-y; \lambda)|_{x_n=0}],$$

where $\tilde{E}_c(\xi, x_n, y; \lambda)=\mathscr{F}_{x'}[E_c(x, y; \lambda)]$. In order to construct the solutions and to investigate their properties, we first study some behaviors of the eigenvalues of $M(\xi; \lambda)$ and construct continuous bases of the positive and negative eigenspaces $E^{\pm}(\xi; \lambda)$ of $M(\xi; \lambda)$.

Eigenvalues of the matrix $M(\xi; \lambda)$

Let k^0 be a non-zero real and $\xi^0\in\Xi^{n-1}$. Assume that the matrix $M(\xi^0; k^0)$ has a real eigenvalue σ^0. From the relation

$$(1.7)\qquad p(\lambda, \xi, \sigma)=\det(\lambda I-A(\xi, \sigma))=\det(-A_n)\det(\sigma I-M(\xi; \lambda))$$

and the definition (0.11) of $Q(\lambda, \xi, \sigma)$ we have $Q(k^0, \xi^0, \sigma^0)=0$. Then there exists in the enumeration (0.12) a unique number r, $1\leqq r\leqq\mu$ $(=2\rho)$, such that $k^0=\lambda_r(\xi^0, \sigma^0)$. We know that the $\lambda_j(\xi, \sigma)$ are real-valued analytic functions of real variables $\eta=(\xi, \sigma)$ in $\Xi^n\backslash\{0\}$. We consider $\lambda=\lambda_r(\xi, \sigma)$ in a neighborhood of the point (ξ^0, σ^0) and extend the real variable σ to the complex variable τ. By the assumption (L.3) the real root σ^0 of $Q(k^0, \xi^0, \sigma)=0$ is either simple or double. This implies

$$(1.8)\quad \text{(i) } \frac{\partial\lambda_r}{\partial\tau}(\xi^0, \sigma^0)\neq 0 \text{ or (ii) } \frac{\partial\lambda_r}{\partial\tau}(\xi^0, \sigma^0)=0 \text{ and } \frac{\partial^2\lambda_r}{\partial\tau^2}(\xi^0, \sigma^0)\neq 0.$$

In the first case where σ^0 is simple, there exists by the implicit function theorem an analytic function $\tau=\tau(\xi; \lambda)$ defined in a neighborhood of (ξ^0, σ^0) such that $\lambda\equiv\lambda_r(\xi, \tau(\xi; \lambda))$, $\sigma^0=\tau(\xi^0; k^0)$ and the value $\tau(\xi; k)$ is real for real k. $\tau(\xi; \lambda)$ is an eigenvalue of the matrix $M(\xi; \lambda)$ with multiplicity ν_r and a simple root of $Q(\lambda, \xi, \tau)=0$. Moreover $\tau(\xi; \lambda)$ is not real for non-real λ. Thus $\mathrm{Im}\,\tau(\xi; \lambda)$ is always either positive or negative for $\mathrm{Im}\,\lambda>0$ and the same for $\mathrm{Im}\,\lambda<0$. Write $\tau=\tau^+(\xi; \lambda)$ or $\tau=\tau^-(\xi; \lambda)$ according as $\mathrm{Im}\,\tau(\xi; \lambda)>0$ or $\mathrm{Im}\,\tau(\xi; \lambda)<0$. Let W be a sufficiently small neighborhood of $\xi^0\in\Xi^{n-1}$ and put

$$(1.9)\qquad \Lambda_\delta^\pm(k^0)=\{\lambda\in\mathbf{C};\ |\mathrm{Re}\,\lambda-k^0|<\delta,\ 0<\pm\mathrm{Im}\,\lambda<\delta\}.$$

Then $\tau=\tau^+(\xi; \lambda)$ or $\tau^-(\xi; \lambda)$ is analytic in $W\times\Lambda_\delta^+(k^0)$ $(W\times\Lambda_\delta^-(k^0))$ and continuous in $W\times\overline{\Lambda_\delta^+(k^0)}$ $(W\times\overline{\Lambda_\delta^-(k^0)})$,[2] where δ is chosen sufficiently small. Moreover, making use of the Taylor expansion in τ of $\lambda_r(\xi, \tau)$ about the point σ:

$$(1.10)\qquad \lambda_r(\xi, \tau)=\sum_{j=0}^{\infty}\frac{1}{j!}\frac{\partial^j\lambda_r}{\partial\tau^j}(\xi, \sigma)(\tau-\sigma)^j,$$

we can show that

$$(1.11)\qquad \lim_{\substack{\lambda\to k\\ \mathrm{Im}\,\lambda\neq 0}}\frac{\mathrm{Im}\,\lambda}{\mathrm{Im}\,\tau(\xi; \lambda)}=\frac{\partial\lambda_r}{\partial\tau}(\xi, \tau(\xi; k))\neq 0,$$

where k is real.

Next let us consider the case (ii) of (1.8) where σ^0 is a double root. From the implicit function theorem there exists a unique real-valued analytic function $\sigma=\sigma(\xi)$ in a neighborhood W of ξ^0 which satisfies

$$(1.12)\qquad \frac{\partial\lambda_r}{\partial\tau}(\xi, \sigma(\xi))=0 \text{ and } \sigma^0=\sigma(\xi^0).$$

2) $\overline{\Lambda_\delta^\pm(k^0)}=\{\lambda\in\mathbf{C};\ |\mathrm{Re}\,\lambda-k^0|\leqq\delta,\ 0\leqq\pm\mathrm{Im}\,\lambda\leqq\delta\}$.

Regarding $\lambda_r(\xi, \tau)$ as a function of τ with parameters ξ we expand it in a Taylor series centered at the point $\sigma=\sigma(\xi)$

$$(1.13)\quad \lambda=\lambda_r(\xi, \tau)=\lambda_r(\xi, \sigma(\xi))+\frac{1}{2!}\frac{\partial^2\lambda_r}{\partial\tau^2}(\xi, \sigma(\xi))(\tau-\sigma(\xi))^2+\cdots.$$

From this there exists by an inverse function theorem in analytic function theory an algebraic function $\tau(\xi; \lambda)$ in λ defined in a neighborhood of (ξ^0, k^0) which satisfies

$$(1.14)\qquad \lambda\equiv\lambda_r(\xi, \tau(\xi; \lambda)),\ \sigma^0=\tau(\xi^0; k^0) \quad \text{and} \quad k^0=\lambda_r(\xi^0, \sigma^0).$$

$\tau=\tau(\xi; \lambda)$ is represented by a development in a Puiseux series of the form

$$(1.15)\quad \tau-\sigma(\xi)=\sum_{j=1}^{\infty}\alpha_j(\xi)t^j,\ t=(\lambda-\lambda_r(\xi, \sigma(\xi)))^{\frac{1}{2}},\ \alpha_1(\xi)=\sqrt{\frac{2}{\frac{\partial^2\lambda_r}{\partial\tau^2}(\xi, \sigma(\xi))}}.$$

Define $\tau^+(\xi; \lambda)$ and $\tau^-(\xi; \lambda)$ by the branches of $\tau(\xi; \lambda)$ such that $\operatorname{Im}\tau^+(\xi; \lambda)>0$ and $\operatorname{Im}\tau^-(\xi; \lambda)<0$ for $\operatorname{Im}\lambda\neq 0$, respectively. The validity of these definitions follows from the fact that $\operatorname{Im}\tau(\xi; \lambda)$ has the same sign as $\operatorname{Im}(\alpha_1(\xi)t)$. Then $\tau^+(\xi; \lambda)$ and $\tau^-(\xi; \lambda)$ are analytic in $W\times\Lambda_\delta^+(k^0)(W\times\Lambda_\delta^-(k^0))$ and continuous in $W\times\overline{\Lambda_\delta^+(k^0)}(W\times\overline{\Lambda_\delta^-(k^0)})$. Moreover $\tau^+(\xi; \lambda)$ and $\tau^-(\xi; \lambda)$ are eigenvalues of the matrix $M(\xi; \lambda)$ in $W\times\overline{\Lambda_\delta^+(k^0)}$ $(W\times\overline{\Lambda_\delta^-(k^0)})$ and their multiplicities are equal to ν_r, respectively, unless $\tau^+(\xi; \lambda)=\tau^-(\xi; \lambda)$. In this case, from the development (1.13), we also obtain

$$(1.16)\qquad \lim_{\substack{\lambda\to k\\ \operatorname{Im}\lambda\neq 0}}\frac{\operatorname{Im}\lambda}{\operatorname{Im}\tau^\pm(\xi; \lambda)}=0,$$

where $k=\lambda_r(\xi, \sigma(\xi))$.

Lemma 1.1. *Let us consider the case (ii) of* (1.8). *Let* $\lambda=k\pm i\varepsilon$. *In a sufficiently small neighborhood of* (ξ^0, k^0)

$$(1.17)\qquad \frac{1}{C}|t|\leqq|\tau^\pm(\xi; \lambda)-\sigma(\xi)|\leqq C|t|,$$

$$\frac{1}{C}|t|\leqq|\tau^+(\xi; \lambda)-\tau^-(\xi; \lambda)|\leqq C|t|.$$

hold.[3] *Moreover the following inequalities hold*:

(i) *If* $\tau^{\pm}(\xi; k)$ *are real, then*

$$(\mathrm{Re}\,\tau^{+}(\xi;\lambda)-\sigma(\xi))(\mathrm{Re}\,\tau^{-}(\xi;\lambda)-\sigma(\xi))\leqq 0, \tag{1.18}$$

$$\frac{1}{C}\varepsilon\cdot|t|^{-1}\leqq|\mathrm{Im}\,\tau^{\pm}(\xi;\lambda)|\leqq C\varepsilon\cdot|t|^{-1}. \tag{1.19}$$

(ii) *If* $\tau^{\pm}(\xi; k)$ *are not real, then*

$$|\mathrm{Im}(\tau^{\pm}(\xi;\lambda)-\sigma(\xi))|\geqq\frac{1}{2}|\tau^{\pm}(\xi;\lambda)-\sigma(\xi)|.$$

Proof. (1.17) follows from (1.15). (1.19) follows from (1.13) and the fact that $\frac{1}{C}|t|\leqq|\mathrm{Re}\,\tau^{\pm}(\xi;\lambda)-\sigma(\xi)|\leqq C|t|$ if $\tau^{\pm}(\xi; k)$ are real. (1.18) follows from the fact that $sgn(\mathrm{Re}\,\tau(\xi;\lambda)-\sigma(\xi))=sgn\,\mathrm{Re}(\alpha_1(\xi)t)$ if $\tau^{\pm}(\xi; k)$ are real. The assertion (ii) is obvious. Q.E.D.

Since the matrices A_j are Hermitian, the eigenvalues of the matrix $M(\xi;\lambda)$ coincide with the complex conjugate of those of the matrix $M(\xi;\bar{\lambda})$. Let $\tau_1^0,\dots,\tau_{2\rho}^0$ be the roots of $Q(k^0,\xi^0,\tau)=0$ (counted according to multiplicity). Then the number of the non-real roots of $Q(k^0,\xi^0,\tau)=0$ is even. Thus that of its real roots is also even. Let $\tau=\tau_j(\xi;\lambda)$ be the functions defined in a neighborhood of (ξ^0, k^0) which correspond to the real roots τ_j^0, respectively. Then the condition (L.2) implies that for $\mathrm{Im}\,\lambda>0$ $(\mathrm{Im}\,\lambda<0)$ the number of $\tau_j(\xi;\lambda)$ with positive imaginary part is equal to that of $\tau_j(\xi;\lambda)$ with negative imaginary part. We can rewrite the above $\tau_j(\xi;\lambda)$ in $W\times\overline{\Lambda_\delta^+(k^0)}\,(W\times\overline{\Lambda_\delta^-(k^0)})$ as

$$\tau_1^+(\xi;\lambda),\dots,\tau_p^+(\xi;\lambda),\quad \tau_1^-(\xi;\lambda),\dots,\tau_p^-(\xi;\lambda), \tag{1.20}$$

where $2p$ is equal to the number of the real roots τ_j^0 and $\tau_j^{\pm}(\xi;\lambda)$ are taken to be $\pm\mathrm{Im}\,\tau_j(\xi;\lambda)>0$ for $\mathrm{Im}\,\lambda>0$ $(\mathrm{Im}\,\lambda<0)$. The $\tau_j^{\pm}(\xi;\lambda)$, $1\leqq j\leqq p$, are analytic in $W\times\Lambda_\delta^+(k^0)\,(W\times\Lambda_\delta^-(k^0))$ and continuous in $W\times\overline{\Lambda_\delta^+(k^0)}\,(W\times\overline{\Lambda_\delta^-(k^0)})$. Moreover there exist continuous functions $\tau_j^{\pm}(\xi;\lambda)$, $p+1\leqq j\leqq\rho$, defined in $W\times\overline{\Lambda_\delta^+(k^0)}\,(W\times\overline{\Lambda_\delta^-(k^0)})$ such that the $\tau_j^{\pm}(\xi^0;k^0)$

3) Here and in sequal C denotes a positive constant.

are the non-real roots of $Q(k^0, \xi^0, \tau)=0$, the $\tau_j^{\pm}(\xi; \lambda)$ are roots of $Q(\lambda, \xi, \tau)=0$ and $\pm\operatorname{Im}\tau_j^{\pm}(\xi; \lambda)>0$. Thus we obtain the 2ρ roots of $Q(\lambda, \xi, \tau)=0$ in $W\times\overline{\Lambda_\delta^+(k^0)}(W\times\overline{\Lambda_\delta^-(k^0)})$:

$$(1.21)\qquad \begin{aligned}&\tau_1^+(\xi; \lambda),\dots, \tau_p^+(\xi; \lambda),\quad \tau_{p+1}^+(\xi; \lambda),\dots, \tau_\rho^+(\xi; \lambda),\\ &\tau_1^-(\xi; \lambda),\dots, \tau_p^-(\xi; \lambda),\quad \tau_{p+1}^-(\xi; \lambda),\dots, \tau_\rho^-(\xi; \lambda),\end{aligned}$$

where $\tau_1^{\pm}(\xi^0; k^0),\dots, \tau_p^{\pm}(\xi^0; k^0)$ are real and $\tau_{p+1}^{\pm}(\xi^0; k^0),\dots, \tau_\rho^{\pm}(\xi^0; k^0)$ are not real. Let $\tilde{\nu}_j$ be the multiplicities of the eigenvalues $\tau_j^{\pm}(\xi; \lambda)$ $(1\leqq j\leqq p, \operatorname{Im}\lambda\neq 0)$ of $M(\xi; \lambda)$. By the condition (L.3) $Q(k^0, \xi^0, \tau)=0$ has at most only one couple of real double roots. Thus when $Q(k^0, \xi^0, \tau)=0$ has real double roots we may assume without loss of generality that

$$(1.21)'\qquad \tau_1^+(\xi^0; k^0)=\tau_1^-(\xi^0; k^0)$$

is the real double root.

Construction of continuous bases of the positive and negative eigenspaces $E^{\pm}(\xi; \lambda)$

Slightly modifying a construction in [7], we construct a system of vectors which satisfy the following properties:

(i) $h_{jk}^+(\xi; \lambda)$ $(1\leqq j\leqq p, 1\leqq k\leqq\tilde{\nu}_j)$ and $h_j^+(\xi; \lambda)$ $(p_0+1\leqq j\leqq m)$ are defined and continuous in $W\times\overline{\Lambda_\delta^+(k^0)}$ and are linearly independent, where $p_0=\sum_{k=1}^{p}\tilde{\nu}_k$.

(ii) $h_{jk}^+(\xi; \lambda)$ $(1\leqq j\leqq p, 1\leqq k\leqq\tilde{\nu}_j)$ are eigenvectors corresponding to eigenvalues $\tau_j^+(\xi; \lambda)$ of the matrix $M(\xi; \lambda)$.

(iii) $\{h_{jk}^+(\xi; \lambda), h_l^+(\xi; \lambda)\}_{1\leqq j\leqq p; 1\leqq k\leqq\tilde{\nu}_j; p_0+1\leqq l\leqq m}$ is a basis of the positive eigenspace $E^+(\xi; \lambda)$ in $W\times\Lambda_\delta^+(k^0)$.

First let us define $\{h_j^+(\xi; \lambda)\}_{p_0+1\leqq j\leqq m}$. We choose a basis $\{h_j^0\}_{p_0+1\leqq j\leqq m}$ of the subspace $E^+(\xi^0; k^0)$ of $\mathbf{C}^{2m}$ generated by all root vectors corresponding to eigenvalues $\tau_{p+1}^+(\xi^0; k^0),\dots, \tau_\rho^+(\xi^0; k^0)$ of $M(\xi^0; k^0)$ and put

$$(1{,}22)\qquad h_j^+(\xi; \lambda)=\frac{1}{2\pi i}\int_{\gamma_+}(\tau I-M(\xi; \lambda))^{-1}h_j^0 d\tau,\ p_0+1\leqq j\leqq m,$$

where γ_+ is a simple closed curve enclosing only the eigenvalues $\tau^+_{p+1}(\xi; \lambda),\dots, \tau^+_\rho(\xi; \lambda)$ and away from the real axis. Next let us define $\{h^+_{jk}(\xi; \lambda)\}_{1\leqq j\leqq p; 1\leqq k\leqq \tilde{\nu}_j}$. There exists a number $\pi(l)$, $1\leqq \pi(l)\leqq 2\rho$, for each l, $1\leqq l\leqq p$, such that

$$k^0=\lambda_{\pi(l)}(\xi^0, \tau^+_l(\xi^0; k^0)). \tag{1.23}$$

The rank of a projection

$$P_{\pi(l)}(\xi, \tau^+_l(\xi; \lambda))=\frac{1}{2\pi i}\int_{|\lambda_1-\lambda|=\delta}(\lambda_1 I-A(\xi, \tau^+_l(\xi; \lambda)))^{-1}d\lambda_1 \tag{1.24}$$

is equal to the multiplicity $\nu_{\pi(l)}$ of the eigenvalue $\lambda_{\pi(l)}(\eta)$ of $A(\eta)$, which is also equal to $\tilde{\nu}_l$. Thus there exists a set of $\tilde{\nu}_l$ column vectors $h^+_{lk}(\xi; \lambda)$, $1\leqq k\leqq \tilde{\nu}_l$, of the matrix $P_{\pi(l)}(\xi, \tau^+_l(\xi; \lambda))$ which are linearly independent in $W\times\overline{\Lambda^+_\delta(k^0)}$. From the equation

$$\begin{aligned}(\tau^+_l(\xi; \lambda)I-M(\xi; \lambda))P_{\pi(l)}(\xi, \tau^+_l(\xi; \lambda))\\ =-A_n^{-1}(\lambda I-A(\xi, \tau^+_l(\xi; \lambda)))P_{\pi(l)}(\xi, \tau^+_l(\xi; \lambda))=0\end{aligned} \tag{1.25}$$

it follows that $h^+_{lk}(\xi; \lambda)$, $1\leqq k\leqq \tilde{\nu}_l$, are eigenvectors corresponding to $\tau^+_l(\xi; \lambda)$. The equation (1.25) is easily proved by the following

Lemma 1.2. *Let $T(\kappa)$ be a matrix-valued analytic function in neighborhood of $\kappa=0$, $\mu_j(\kappa)$ $(1\leqq j\leqq s)$ its eigenvalues and Γ a small circle enclosing only $\mu_1(0)$ with multiplicity ν for $\kappa=0$. Put*

$$P(\kappa)=\frac{1}{2\pi i}\int_\Gamma(\zeta-T(\kappa))^{-1}d\zeta,$$

$$D(\kappa)=\frac{1}{2\pi i}\int_\Gamma(\zeta-\mu_1(\kappa))(\zeta-T(\kappa))^{-1}d\zeta.$$

Then $P(\kappa)$ and $D(\kappa)$ are analytic in κ. Further if $T(\kappa)=T(\kappa)^*$ for real κ, then $D(\kappa)\equiv 0$. Thus $(\zeta-T(\kappa))^{-1}$ has a simple pole in ζ at $\zeta=\mu_1(\kappa)$ and $P(\kappa)$ is a projection onto the proper eigenspace.

Remark. In [4] more precise results are proved (see, Theorem II–1.10, e.t.c., in [4]).

We have constructed a basis of $E^+(\xi; \lambda)$ satisfying (i), (ii) and (iii).

Moreover our construction shows that these vectors are analytic in $W\times\Lambda_\delta^+(k^0)$. Similarly we can construct $\{h_{jk}^+, h_l^+\}$ defined in $W\times\overline{\Lambda_\delta^-(k^0)}$ satisfying (i), (ii) and (iii) and $\{h_{jk}^-, h_l^-\}$ for the negative eigenspace $E^-(\xi;\lambda)$. When $Q(k^0, \xi^0, \tau)=0$ has a real double root $\tau_1^+(\xi^0; k^0)=\tau_1^-(\xi^0; k^0)$, we may assume that

$$h_{1\mu}^+(\xi^0; k^0)=h_{1\mu}^-(\xi^0; k^0), \qquad 1\leqq\mu\leqq\tilde{\nu}_1\,. \tag{1.26}$$

For simplicity we sometimes rewrite $\{h_{jk}^+\}_{1\leqq j\leqq p;1\leqq k\leqq\tilde{\nu}_j}$ in the form $\{h_j^+\}_{1\leqq j\leqq p_0}$.

§2. Behavior of Lopatinski Determinant in the Neighborhood of the Zeros

Consider the system of ordinary differential equations depending on the parameters $(\xi, \lambda)\in\Xi^{n-1}\times(\mathbf{C}\backslash\mathbf{R})$:

$$\left(\frac{1}{i}\frac{d}{dx_n}-M(\xi;\lambda)\right)U(x_n, \xi;\lambda)=0,\ \ x_n>0, \tag{2.1}$$

with the boundary condition

$$BU(0, \xi;\lambda)=g\,. \tag{2.2}$$

The question first arise is: what are the condition on B in order that for any $g\in\mathbf{C}^m$ the boundary value problem (2.1), (2.2) has one and only one solution $U(x_n, \xi;\lambda)$ which is temperate in x_n (or belonging to $L^2(\mathbf{R}_+^1)$). Noting that $M(\xi;\lambda)$ has no real eigenvalues for non-real λ, a solution $U(x_n, \xi;\lambda)$ of (2.1) is temperate in x_n if and only if $U(0, \xi;\lambda)\in E^+(\xi;\lambda)$. Thus our problem turns to find the conditions on B so that for any $g\in\mathbf{C}^m$ the linear equation $BU(0, \xi;\lambda)=g$ has a unique solution in $E^+(\xi;\lambda)$.

Here we define the Lopatinski determinant $\Delta(\xi;\lambda)$ by using the basis $\{h_j^+(\xi;\lambda)\}_{1\leqq j\leqq m}$ constructed in the previous section:

$$\Delta(\xi;\lambda)=\det(Bh_1^+(\xi;\lambda),\dots, Bh_m^+(\xi;\lambda))\,. \tag{2.3}$$

We defined this locally, but we can define it globally in $\Xi^{n-1}\times\overline{\mathbf{C}^+}\backslash$

$\{(0, 0)\}$ $(\Xi^{n-1}\times\overline{\mathbf{C}^-}\setminus\{(0, 0)\})$ making use of a suitable C^∞ partition of unity.

Lemma 2.1. *Let λ be non-real and $\xi\in\Xi^{n-1}$. Then the following propositions are equivalent:*

(1) *For any $g\in\mathbf{C}^m$ there exists a unique solution $U(x_n, \xi; \lambda)$ of* (2.1) *and* (2.2) *which is temperate in x_n.*

(2) *The linear operator B is a one-to-one mapping of $E^+(\xi; \lambda)$ onto $\mathbf{C}^m$.*

(3) $l=m$ *and* $\mathscr{B}\cap E^+(\xi; \lambda)=\{0\}$, *i.e.*, $\mathscr{B}\oplus E^+(\xi; \lambda)=\mathbf{C}^{2m}$, *where* $\mathscr{B}=\ker B$.

(4) $\Delta(\xi; \lambda)\neq 0$.

Lemma 2.2. *If the boundary matrix B is conservative for $L=I\frac{\partial}{\partial t}-iA$ $\left(\text{or for } A=\frac{1}{i}\sum A_j\frac{\partial}{\partial x_j}\right)$, i.e., $A_n\zeta\cdot\bar{\zeta}=0$ for every $\zeta\in\mathscr{B}=\ker B$, then $\mathscr{B}\cap E^+(\xi; \lambda)=\{0\}$ holds for every non-real λ and $\xi\in\Xi^{n-1}$ and, therefore, $\Delta(\xi; \lambda)\neq 0$.*

Lemma 2.3. *Let $|\xi|+|k|\neq 0$, where k is real. Then we have*

(1) $A_nh\cdot\bar{g}=0$ *for any* $h, g\in E^+(\xi; k)$.

(2) $A_nh\cdot\bar{g}=0$ *for any* $h\in E^+(\xi; k)$ *and any* $g\in E^0(\xi; k)$.

(3) (i) $A_nh_j^+(\xi; k\pm i0)\cdot\overline{h_l^+(\xi; k\pm i0)}=0$ *for* $1\leqq j, l\leqq m, j\neq l$.

(ii) $A_nh_j^+(\xi; k\pm i0)\cdot\overline{h_j^+(\xi; k\pm i0)}=0$ *for the eigenvectors* $h_j^+(\xi; k\pm i0)$, $1\leqq j\leqq\tilde{\nu}_1$, *corresponding to a real double root, i.e., for* $h_{1\mu}^+(\xi; k\pm i0)$, $1\leqq\mu\leqq\tilde{\nu}_1$.

(iii) $A_nh_{j\mu}^+(\xi; k+i0)\cdot\overline{h_{j\mu}^+(\xi; k+i0)}>0$,
$A_nh_{j\mu}^+(\xi; k-i0)\cdot\overline{h_{j\mu}^+(\xi; k-i0)}<0$,

for the eigenvectors $h_{j\mu}^+(\xi; k\pm i0)$, $1\leqq\mu\leqq\tilde{\nu}_j$, *corresponding to each real simple root* $\tau_j^+(\xi; k\pm i0)$, $2\leqq j\leqq p$.

Here $E^0(\xi; k)$ denotes the linear subspace spanned by the eigenvectors corresponding to all the real eigenvalues of $M(\xi; k)$.

Lemmas 2.1, 2.2 and 2.3 are proved in [7] when L is strictly hyperbolic. In our case we can also prove the above lemmas in the same way.

Now put $e_h={}^t(0,\dots, 0, \overset{h}{\overset{\cup}{1}}, 0,\dots, 0)\in\mathbf{C}^m$, $1\leqq h\leqq m$. Let us consider

$$(2.4) \qquad (Bh_1^+(\xi;\, k\pm i\varepsilon),\dots,\, Bh_m^+(\xi;\, k\pm i\varepsilon))\begin{pmatrix} C_{1h} \\ \vdots \\ C_{mh} \end{pmatrix} = e_h\,,$$

where $0<\varepsilon\leqq\varepsilon^0$. Then we have

$$(2.5) \qquad C_{lh}(\xi;\, k\pm i\varepsilon) = \frac{1}{\Delta(\xi;\, k\pm i\varepsilon)} \det(Bh_1^+(\xi;\, k\pm i\varepsilon),\dots \underset{\overset{\cup}{l}}{\dots}, e_h,\dots,\, Bh_m^+(\xi;\, k\pm i\varepsilon))\,.$$

In order to estimate $C_{lh}(\xi;\, k\pm i\varepsilon)$ we prepare the following three lemmas.

Lemma 2.4 (Schur). *Let $A=(a_{ij})$ be an $N\times N$ positive definite Hermitian matrix and $B=(b_{ij})$ an $N\times N$ non-negative definite Hermitian matrix. Then*

$$(2.6) \qquad \sum_{i,j=1}^{N} a_{ij} b_{ij} \zeta_i \bar{\zeta}_j \geqq h \sum_{i=1}^{N} b_{ii} |\zeta_i|^2$$

holds for any $\zeta\in\mathbf{C}^N$, where h is the smallest eigenvalue of A.

Proof. Using the existence of a unitary matrix T such that $A = T\begin{pmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_N \end{pmatrix}T^*$, one can easily prove this lemma (see, [9]). Q.E.D.

Lemma 2.5. *Let λ be non-real and $\xi\in\Xi^{n-1}$. If the boundary matrix B is conservative for A, then the following inequality holds for the temperate solution $U(x_n,\, \xi;\, \lambda)$ of the equations* (2.1) *and* (2.2):

$$(2.7) \qquad (\operatorname{Im}\lambda)^2 \int_0^\infty |U(x_n,\, \xi;\, \lambda)|^2 dx_n \leqq C|g|^2\,,$$

where C is a constant independent of $(\xi,\, \lambda)$ which varies in a bounded set of $\Xi^{n-1}\times\mathbf{C}^+(\Xi^{n-1}\times\mathbf{C}^-)$.

This lemma is also proved in [7].

Put

$$(2.8) \qquad Q(\xi;\, \lambda) = \frac{1}{2\pi i}\int_{\gamma_+} (\tau I - M(\xi;\, \lambda))^{-1} d\tau,$$

$$(2.9) \qquad S(x_n,\, \xi;\, \lambda) = \frac{1}{2\pi i}\int_{\gamma_+} e^{i\tau x_n} (\tau I - M(\xi;\, \lambda))^{-1} d\tau\,,$$

where γ_+ is the simple closed curve defined in § 1. Then we have

$$(2.10)\qquad Q(\xi;\lambda)S(x_n,\xi;\lambda)=S(x_n,\xi;\lambda)=Q(\xi;\lambda)e^{iM(\xi;\lambda)x_n}$$
$$=e^{iM(\xi;\lambda)x_n}Q(\xi;\lambda)=S(x_n,\xi;\lambda)Q(\xi;\lambda),$$

and, therefore,

$$(2.11)\qquad R(Q(\xi;\lambda))=R(S(x_n,\xi;\lambda))$$ [4]

is generated by $\{h_j^+(\xi;\lambda)\}_{p_0+1\leqq j\leqq m}$. Thus we put

$$(2.12)\quad S(x_n,\xi;\lambda)h_j^+(\xi;\lambda)=\sum_{k=p_0+1}^{m}f_{jk}(x_n,\xi;\lambda)h_k^+(\xi;\lambda)\quad\text{for}\quad p_0+1\leqq j\leqq m.$$

Lemma 2.6. *Let $x_n>0$ and (ξ,λ) be in a neighborhood of (ξ^0,k^0).*

(i) *The $f_{jk}(x_n,\xi;\lambda)$ are continuous in (x_n,ξ,λ) and the inequalities*

$$(2.13)\qquad |f_{jk}(x_n,\xi;\lambda)|\leqq Ce^{-dx_n},\quad p_0+1\leqq j,\ k\leqq m,$$

hold, where $2d=\mathrm{dis}(\gamma_+,\mathbf{R}^1)$.

(ii) *Put $F=(f_{jk})$. Then F is a $(m-p_0)\times(m-p_0)$ matrix and non-singular. In particular $F(0,\xi;\lambda)=I_{m-p_0}\equiv\begin{pmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{pmatrix}$.*

Proof. (i) The rank of the projection $(I-Q(\xi;\lambda))$ is equal to $(m+p_0)$. Hence it has $(m+p_0)$ linearly independent column vectors at $(\xi,\lambda)=(\xi^0,k^0)$. By continuity they are also linearly independent in a neighborhood of (ξ^0,k^0). Let $v_j(\xi;\lambda)$, $1\leqq j\leqq m+p_0$, be these column vectors. Therefore we get

$$(2.14)\qquad f_{jk}(x_n,\xi;\lambda)=\frac{\det(v_1,\dots,v_{m+p_0},h_{p_0+1}^+,\dots,\overset{m+k}{\overset{\cup}{Sh_j^+}},\dots,h_m^+)}{\det(v_1,\dots,v_{m+p_0},h_{p_0+1}^+,\dots,h_m^+)}.$$

Since the denominator of the right-hand side of (2.14) does not vanish, the functions $f_{jk}(x_n,\xi;\lambda)$ are continuous in (x_n,ξ,λ). The inequalities (2.13) immediately follow from the estimates of $Sh_j^+(\xi;\lambda)$.

(ii) Suppose that $\det F(x_n,\xi;\lambda)=0$ for some $(x_n,\xi;\lambda)$, i.e., there exists $(d_{p_0+1},\dots,d_m)\neq(0,\dots,0)$ such that $\sum_{j=p_0+1}^{m}d_jf_{jk}(x_n,\xi;\lambda)=0$ for $p_0+1\leqq k\leqq m$. Then,

4) $R(T)$ denotes the range of T.

$$\sum_{j=p_0+1}^{m} d_j S(x_n, \xi; \lambda) h_j^+(\xi; \lambda) = \sum_{j,k=p_0+1}^{m} d_j f_{jk} h_k^+$$

$$= \sum_{k=p_0+1}^{m} \left(\sum_{j=p_0+1}^{m} d_j f_{jk} \right) h_k^+ = 0 .$$

This contradicts the linear independence of

$$S(x_n, \xi; \lambda) h_j^+(\xi; \lambda) = e^{iM(\xi;\lambda)x_n} h_j^+(\xi; \lambda), \qquad p_0 + 1 \leqq j \leqq m .$$

Thus the assertion (ii) follows. Q.E.D.

Now we can estimate $C_{lh}(\xi; k \pm i\varepsilon)$.

Lemma 2.7. *Let* $\varepsilon > 0$ *and* $(\xi, k \pm i\varepsilon)$ *be a point in a neighborhood of* (ξ^0, k^0). *Then the following inequalities hold*:

$$(2.15) \qquad |C_{lh}(\xi; k \pm i\varepsilon)| \leqq C[\operatorname{Im} \tau_{p(l)}^+(\xi; k \pm i\varepsilon)]^{\frac{1}{2}}/\varepsilon \quad \textit{for} \quad 1 \leqq l \leqq p_0 ,$$

where $\tau_{p(l)}^+(\xi; k \pm i\varepsilon)$ *is the eigenvalue of* $M(\xi; k \pm i\varepsilon)$ *corresponding to each vector* $h_l^+(\xi; k \pm i\varepsilon)$.

$$(2.16) \qquad |C_{lh}(\xi; k \pm i\varepsilon)| \leqq C/\varepsilon \qquad \textit{for} \quad p_0 + 1 \leqq l \leqq m .$$

Proof. Let $U_h(x_n, \xi; \lambda)$ be the temperate solution of (2.1) with the initial value $U_h(0, \xi; \lambda) = \sum_{l=1}^{m} C_{lh}(\xi; \lambda) h_l^+(\xi; \lambda)$. Then it is represented as

$$U_h(x_n, \xi; \lambda) = \sum_{l=1}^{p_0} C_{lh}(\xi; \lambda) e^{i\tau_{p(l)}^+(\xi;\lambda)x_n} h_l^+(\xi; \lambda)$$

$$+ \sum_{k=p_0+1}^{m} \left\{ \sum_{l=p_0+1}^{m} C_{lh}(\xi; \lambda) f_{lk}(x_n, \xi; \lambda) h_k^+(\xi; \lambda) \right\} .$$

By Lemma 2.4 we obtain the inequality

$$\int_0^\infty |U_h(x_n, \xi; \lambda)|^2 dx_n \geqq \gamma(\pm\varepsilon) \left[\sum_{l=1}^{p_0} \frac{|C_{lh}(\xi; \lambda)|^2}{2 \operatorname{Im} \tau_{p(l)}^+(\xi; \lambda)} \right.$$

$$\left. + \sum_{k=p_0+1}^{m} \int_0^\infty \left| \sum_{l=p_0+1}^{m} C_{lh}(\xi; \lambda) f_{lk}(x_n, \xi; \lambda) \right|^2 dx_n \right] ,$$

where $\gamma(\pm\varepsilon)$ is the smallest eigenvalue of the Gram matrix $(h_j^+ \cdot \overline{h_k^+})$ and $\operatorname{Im} \lambda = \pm\varepsilon$. Note that

$$\sum_{k=p_0+1}^{m} \Big| \sum_{l=p_0+1}^{m} C_{lh}(\xi;\lambda) f_{lk}(x_n,\xi;\lambda)\Big|^2 = \sum_{l,k} C_{lh}(FF^*)_{lk}\bar{C}_{kh},$$

and that $\int_0^\infty FF^* dx_n$ exists and is positive definite. Thus we have

$$\sum_{k=p_0+1}^{m} \int_0^\infty \Big| \sum_{l=p_0+1}^{m} C_{lh}(\xi;\lambda) f_{lk}(x_n,\xi;\lambda)\Big|^2 dx_n \geqq \gamma' \sum_{l=p_0+1}^{m} |C_{lh}(\xi;\lambda)|^2,$$

where γ' is a positive constant. Then from Lemma 2.5 the assertion of the above lemma follows. Q.E.D.

Lemma 2.8. *Assume that the operator L satisfies the conditions* (L.1)–(L.3) *and the matrix B satisfies the conditions* (B.1) *and* (B.2). *Moreover let $|\xi|+|k|\neq 0$. If the Lopatinski determinant $\Delta(\xi; k\pm i0)$ vanishes, then $Q(k,\xi,\tau)=0$ have a real double root in τ.*

Lemma 2.9. *Assume that the operator L satisfies the conditions* (L.1)′, (L.2) *and* (L.3) *and that the matrix B satisfies the conditions* (B.1) *and* (B.2). *Moreover let $\xi^0 \in \Xi^{n-1}$ and $k^0 \in \mathbf{R}$ with $|\xi^0|+|k^0|\neq 0$, and suppose that $Q(k^0,\xi^0,\tau)=0$ has a real double root in τ. Then we have the development*

$$\Delta(\xi;\lambda) = \sum_{j=0}^{\infty} \beta_j(\xi)(\lambda-\lambda_r(\xi,\sigma(\xi)))^{\frac{j}{2}} \tag{2.17}$$

in a neighborhood of (ξ^0, k^0) and

$$|\beta_0(\xi^0)|+|\beta_1(\xi^0)|\neq 0, \tag{2.18}$$

where $k^0=\lambda_r(\xi^0, \tau_1^\dagger(\xi^0; k^0))$, i.e., $r=\pi(1)$.

These two lemmas are also proved in [7].

Remark. These lemmas are used only to prove Theorem 0.7 where eigenfunctions corresponding to boundary waves do not appear in the expansion formula.

For our purpose we need to study some properties of zeros of the Lopatinski determinant $\Delta(\xi;\lambda)$.

Lemma 2.10. *Put*

$$p(\xi;\lambda)=\lambda^l+a_1(\xi)\lambda^{l-1}+\cdots+a_l(\xi),$$

where the $a_j(\xi)$ are analytic in a complex domain V. Then there exists an analytic function $D(\xi)$ ($\not\equiv 0$) in V such that the roots of $p(\xi;\lambda)=0$ with respect to λ have constant multiplicities for ξ in $\tilde{V}\equiv\{\xi\in V;\ D(\xi)\neq 0\}$ and are analytic functions of ξ in $\tilde{V}$.

Proof. We apply the Euclidean algorithm to $p(\xi;\lambda)$ and $\frac{\partial p}{\partial\lambda}(\xi;\lambda)$ as polynomials of λ. Put

$$p(\xi;\lambda)=q_1(\xi;\lambda)\frac{\partial p}{\partial\lambda}(\xi;\lambda)+r_1(\xi;\lambda)\equiv q_1(\xi;\lambda)r_0(\xi;\lambda)+r_1(\xi;\lambda),$$

where $q_1(\xi;\lambda)$, $r_1(\xi;\lambda)$ are polynomials in λ and the order l_1 of $r_1(\xi;\lambda)$ is less than $l-1$. Write

$$r_1(\xi;\lambda)=a_{10}(\xi)\lambda^{l_1}+a_{11}(\xi)\lambda^{l_1-1}+\cdots+a_{1l_1}(\xi),\quad a_{10}(\xi)\not\equiv 0.$$

Moreover, put

$$r_{j-1}(\xi;\lambda)=q_{j+1}(\xi;\lambda)r_j(\xi;\lambda)+r_{j+1}(\xi;\lambda),\qquad j=1,2,\dots,$$

where the $q_j(\xi;\lambda)$ and $r_j(\xi;\lambda)$ are polynomials in λ and the order l_{j+1} of $r_{j+1}(\xi;\lambda)$ is less than the order l_j of $r_j(\xi;\lambda)$. Write

$$r_j(\xi;\lambda)=a_{j0}(\xi)\lambda^{l_j}+a_{j1}(\xi)\lambda^{l_j-1}+\cdots+a_{jl_j}(\xi),\ a_{j0}(\xi)\not\equiv 0,\qquad \text{if}\quad r_j(\xi;\lambda)\not\equiv 0.$$

Then there exists a number α, which is less than $l-1$, such that $r_{\alpha+1}(\xi;\lambda)\equiv 0$ and $r_j(\xi;\lambda)\not\equiv 0$ for $0\leqq j\leqq\alpha$. We may write $a_{j0}(\xi)$, $1\leqq j\leqq\alpha$, as

$$a_{j0}(\xi)=\frac{b_j(\xi)}{c_j(\xi)},$$

where $b_j(\xi)$ and $c_j(\xi)$ are analytic in V. Put

$$D(\xi)=b_1(\xi)\dots b_\alpha(\xi).$$

Then for each fixed $\xi\in\tilde{V}\equiv\{\xi\in V;\ D(\xi)\neq 0\}$ the greatest common divisor of $p(\xi;\lambda)$ and $\frac{\partial p}{\partial\lambda}(\xi;\lambda)$ is $r_\alpha(\xi;\lambda)$. Thus

$$p(\xi;\lambda)=r_\alpha(\xi;\lambda)m(\xi;\lambda),\ \frac{\partial p}{\partial\lambda}(\xi;\lambda)=r_\alpha(\xi;\lambda)n(\xi;\lambda),\qquad \text{for } \xi\in\tilde{V}.$$

It is easy to see that $p(\xi;\lambda)=0$ has just $l-l_\alpha$ distinct roots for $\xi\in\tilde{V}$ and that the totality of the distinct roots of $p(\xi;\lambda)=0$ is the totality of the roots of $m(\xi;\lambda)=0$ for $\xi\in\tilde{V}$. This implies that $m(\xi;\lambda)=0$ has only simple roots for each $\xi\in\tilde{V}$ and that the roots of $m(\xi;\lambda)=0$ (or $p(\xi;\lambda)=0$) are (multi-valued) analytic functions in $\tilde{V}$. Let $\lambda_1(\xi),\dots,\lambda_\beta(\xi)$ be the distinct roots of $p(\xi;\lambda)=0$, where $\beta=l-l_\alpha$. Here we consider the $\lambda_k(\xi)$ locally with respect to ξ. Then we have

$$\frac{\partial p}{\partial\lambda}(\xi;\lambda)/p(\xi;\lambda)=\frac{n(\xi;\lambda)}{m(\xi;\lambda)}=\sum_{j=1}^{\beta}\frac{A_j(\xi)}{(\lambda-\lambda_j(\xi))}$$

where the $A_j(\xi)$ are analytic. Further the $A_j(\xi)$ are equal to the multiplicities of the roots $\lambda_j(\xi)$. It is clear that $\tilde{V}$ is connected. Therefore the $A_j(\xi)$ are constant for $\xi\in\tilde{V}$. Q.E.D.

Lemma 2.11. (i) *For each fixed $\xi\in\Xi^{n-1}$ the zeros of $\Delta(\xi;\lambda)$ in $\overline{\mathbf{C}^+}$ ($\overline{\mathbf{C}^-}$) are all real and the number of the zeros is finite.* (ii) $\Delta(\xi; k+i0)=0$ *if and only if* $\Delta(\xi;k-i0)=0$. (iii) *There exist real-valued continuous functions $k_1(\xi),\dots,k_s(\xi)$, which are defined on open sets $D_1,\dots,D_s$ ($D_1\supset\cdots\supset D_s$), respectively, and a closed null set $N(\subset\Xi^{n-1})$ such that the totality of non-vanishing zeros of the Lopatinski determinant $\Delta(\xi;\lambda)$ is the set $\{k_j(\xi)\}_{j\in\{\nu;\xi\in D_\nu\}}$ for any $\xi\notin N$, $k_i(\xi)\neq k_j(\xi)$ for $\xi\in D_i\cap D_j$ and $i\neq j$, and the $k_j(\xi)$ are positively homogeneous of degree* 1.

Remark. The Lopatinski determinant $\Delta(\xi;\lambda)$ is defined only on $\Xi^{n-1}\times\overline{\mathbf{C}^+}(\Xi^{n-1}\times\overline{\mathbf{C}^-})$. We shall often regard $\Delta(\xi;\lambda)$ as a function to be continued analytically across the real axis into $\Xi^{n-1}\times\mathbf{C}^-(\Xi^{n-1}\times\mathbf{C}^+)$. However, $\Delta(\xi;\lambda)$ does not coincide with the Lopatinski determinant in $\Xi^{n-1}\times\mathbf{C}^-(\Xi^{n-1}\times\mathbf{C}^+)$. Thus Lemma 2.11 does not give any information about the zeros of $\Delta(\xi;\lambda)$ in $\mathbf{C}^-(\mathbf{C}^+)$.

Proof. $\Delta(\xi;k\pm i0)=0$ if and only if there exists $(C_1^\pm,\dots,C_m^\pm)\neq(0,\dots,0)$ satisfying $\zeta\equiv\sum_{l=1}^{m}C_l^\pm h_l^\pm(\xi;k\pm i0)\in\mathscr{B}$. From Lemma 2.3 it follows that $C_l^\pm=0$ for $\tilde{\nu}_1+1\leqq l\leqq p_0$. On the other hand $Q(k,\xi,\tau)=0$ has only real simple roots with respect to τ when $|k|$ is sufficiently large. Then if $\zeta^\pm\equiv\sum_{l=1}^{m}C_l^\pm h_l^\pm(\xi;k\pm i0)$ belongs to $\mathscr{B}$, all the $C_l^\pm$ must vanish. Therefore $\Delta(\xi; k\pm i0)\neq 0$, if $|k|$ is sufficiently large. Thus from analyticity of $\Delta(\xi; \lambda)$ with respect to λ (or $t=(\lambda-\lambda_r(\xi, \sigma(\xi)))^{\frac{1}{2}}$) we see that the number of the zeros of the Lopatinski determinant $\Delta(\xi; \lambda)$ is finite. This proves the assertion (i). The assertion (ii) is easily verified. The assertion (iii) can be proved by Lemma 2.10 and Weierstrass' preparation theorem. In fact, let $\kappa_1,\dots, \kappa_\alpha$ be the (real) distinct roots of $\Delta(\xi^0; \lambda)=0$ for fixed ξ^0 in Ξ^{n-1}, $|\xi^0|=1$. Moreover it suffices to consider the case where the Lopatinski determinant $\Delta(\xi; \lambda)$ is defined in $\Xi^{n-1}\times\overline{\mathbf{C}^+}$. If $Q(\kappa_j, \xi^0, \tau)=0$ has no real double roots, then $\Delta(\xi; \lambda)$ is analytic in $W\times\Lambda_\delta^+(\kappa_j)$ and can be continued analytically across the real axis in a complex neighborhood of (ξ^0, κ_j). Thus applying Weierstrass' preparation theorem to $\Delta(\xi; \lambda)$, we have

$$\Delta(\xi; \lambda)=(\lambda^{l_j}+a_{1j}(\xi)\lambda^{l_j-1}+\cdots+a_{l_j j}(\xi))q_j(\xi; \lambda)$$

in a small neighborhood of (ξ^0, κ_j), where the $a_{ij}(\xi)$ are analytic in ξ and $a_{ij}(\xi^0)=(-1)^i\binom{l_j}{i}\kappa_j^i$, and $q_j(\xi; \lambda)$ is analytic in $(\xi; \lambda)$ and $q_j(\xi; \lambda)\neq 0$ in the above neighborhood. It follows from Lemma 2.10 that there exist functions $\tilde{k}_1(\xi),\dots, \tilde{k}_\beta(\xi)$ defined in a real neighborhood W_0 of ξ^0 and an open set $\tilde{W}_0(\subset W_0)$ such that the $\tilde{k}_\nu(\xi)$ are the zeros of $\Delta(\xi; \lambda)$ in $\Lambda_\delta(\kappa_j)$[5] for each ξ in W_0 and analytic in $\tilde{W}_0$, $\tilde{k}_\nu(\xi)\neq\tilde{k}_\mu(\xi)$ for each ξ in $\tilde{W}_0$ and $W_0\backslash\tilde{W}_0$ is a closed null subset of W_0. Since

$$\operatorname{Im}\tilde{k}_\nu(\xi)=\frac{1}{2i}\{\tilde{k}_\nu(\xi)-\overline{\tilde{k}_\nu(\bar{\xi})}\} \qquad \text{for } \xi \text{ in } \tilde{W}_0,$$

$\operatorname{Im}\tilde{k}_\nu(\xi)$ is analytic in $\tilde{W}_0$. Thus if $\operatorname{Im}\tilde{k}_\nu(\xi)$ does not vanish identically in a component $\hat{W}_0$ of $\tilde{W}_0$, $\operatorname{Im}\tilde{k}_\nu(\xi)$ is non-zero for almost every ξ in $\hat{W}_0$. Define $\tilde{\tilde{W}}_0$ by removing the set $\{\xi\in\hat{W}_0; \operatorname{Im}\tilde{k}_\nu(\xi)=0\ (\tilde{k}_\nu(\xi)=0)\}$ from $\hat{W}_0$ if $\operatorname{Im}\tilde{k}_\nu(\xi)\not\equiv 0\ (\tilde{k}_\nu(\xi)\not\equiv 0)$. Let $\hat{\hat{W}}_0$ be a component of $\tilde{\tilde{W}}_0$. Then we have, by modifying the enumeration of the $\tilde{k}_\nu(\xi)$,

$$\tilde{k}_1(\xi)<\tilde{k}_2(\xi)<\cdots<\tilde{k}_q(\xi),$$

$$\operatorname{Im}\tilde{k}_{q+1}(\xi)<0,\dots, \operatorname{Im}\tilde{k}_\beta(\xi)<0, \qquad \text{for} \quad \xi\in\hat{\hat{W}}_0.$$

Here we note that $\hat{W}_0\backslash\tilde{\tilde{W}}_0$ is a closed null set of $\hat{W}_0$ and that $\hat{\hat{W}}_0$ is

5) $\Lambda_\delta(\kappa)=\{\lambda\in\mathbf{C};\ |\operatorname{Re}(\lambda-\kappa)|<\delta,\ |\operatorname{Im}(\lambda-\kappa)|<\delta\}$

an open connected set of Ξ^{n-1}. Next let us consider the case where $Q(\kappa_j, \xi^0, \tau)=0$ has real double roots. Moreover it suffices to consider the case where the Lopatinski determinant $\Delta(\xi; t^2+\lambda_r(\xi, \sigma(\xi)))$ is defined in $W\times\{t\in\mathbf{C}; \mathrm{Re}\, t\geqq 0$ and $\mathrm{Im}\, t\geqq 0\}$, where $t=(\lambda-\lambda_r(\xi, \sigma(\xi)))^{\frac{1}{2}}$ and $\kappa_j=\lambda_r(\xi^0, \sigma(\xi^0))$. Then $\Delta(\xi; t^2+\lambda_r(\xi, \sigma(\xi)))$ can be continued analytically to a complex neighborhood of $(\xi^0, 0)$. Thus applying Weierstrass' preparation theorem to $\Delta(\xi; t^2+\lambda_r(\xi, \sigma(\xi)))$ we also have

$$(2.19)\qquad \Delta(\xi; t^2+\lambda_r(\xi, \sigma(\xi)))=(t^{l_j}+a_{1j}(\xi)t^{l_j-1}+\cdots+a_{l_jj}(\xi))q_j(\xi; t)$$

in a small neighborhood of $(\xi^0, 0)$, where the $a_{ij}(\xi)$ are analytic in ξ and $a_{ij}(\xi^0)=0$, and $q_j(\xi; t)$ is analytic in (ξ, t) and $q_j(\xi; t)\neq 0$ in the above neighborhood. It follows from Lemma 2.10 that there exist functions $t_1(\xi),\dots, t_\beta(\xi)$ defined in a real neighborhood W_0 of ξ^0 and an open set $\tilde{W}_0(\subset W_0)$ such that the $t_\nu(\xi)$ are the zeros of $\Delta(\xi; t^2+\lambda_r(\xi, \sigma(\xi)))$ in $\Lambda_{\sqrt{\delta}}(0)$ for each ξ in W_0 and analytic in $\tilde{W}_0$, $t_\nu(\xi)\neq t_\mu(\xi)$ for each ξ in $\tilde{W}_0$ and $W_0\backslash\tilde{W}_0$ is a closed null subset of W_0. Since

$$\mathrm{Im}\, t_\nu(\xi)=\frac{1}{2i}\{t_\nu(\xi)-\overline{t_\nu(\bar{\xi})}\},\quad \mathrm{Re}\, t_\nu(\xi)=\frac{1}{2}\{t_\nu(\xi)+\overline{t_\nu(\bar{\xi})}\}\ \text{ for } \xi\in\tilde{W}_0,$$

$\mathrm{Im}\, t_\nu(\xi)$ and $\mathrm{Re}\, t_\nu(\xi)$ are analytic in $\tilde{W}_0$. Thus if $\mathrm{Im}\, t_\nu(\xi)\not\equiv 0$ $(\mathrm{Re}\, t_\nu(\xi)\not\equiv 0)$ in a component $\hat{W}_0$ of $\tilde{W}_0$, $\mathrm{Im}\, t_\nu(\xi)$ $(\mathrm{Re}\, t_\nu(\xi))$ does not vanish for almost every ξ in $\hat{W}_0$. Define $\tilde{\tilde{W}}_0$ by removing the set $\{\xi\in\hat{W}_0; \mathrm{Im}\, t_\nu(\xi)=0$ $(\mathrm{Re}\, t_\nu(\xi)=0)\}$ from $\hat{W}_0$ if $\mathrm{Im}\, t_\nu(\xi)\not\equiv 0$ $(\mathrm{Re}\, t_\nu(\xi)\not\equiv 0)$. Let $\hat{\hat{W}}_0$ be a component of $\tilde{\tilde{W}}_0$ and put

$$(2.20)\qquad \Omega=\{t\in\mathbf{C}; \mathrm{Im}\, t<0 \text{ or } \mathrm{Re}\, t<0\},\qquad \omega=\partial\Omega.$$ [6]

Then we have, by modifying the enumeration of the $t_\nu(\xi)$,

$$t_1(\xi),\dots, t_q(\xi)\in\omega,\quad t_{q+1}(\xi),\dots, t_\beta(\xi)\in\Omega,$$
$$\tilde{k}_1(\xi)<\cdots<\tilde{k}_q(\xi),$$

where $\tilde{k}_\nu(\xi)=t_\nu(\xi)^2+\lambda_r(\xi, \sigma(\xi))$, $1\leqq\nu\leqq q$. Thus we can define a system

6) ∂S denotes the boundary of S.

$\{\tilde{k}_1(\xi),\dots, \tilde{k}_{s(\xi)}(\xi)\}$ of continuous functions in $\Xi^{n-1}\backslash N$ such that $\{\tilde{k}_1(\xi),\dots, \tilde{k}_{s(\xi)}(\xi)\}$ is the totality of non-vanishing real zeros of the Lopatinski determinant $\Delta(\xi;\lambda)$ for $\xi\notin N$ and N is a closed null set of Ξ^{n-1}. Here the number $s(\xi)$ ($\geqq 0$) depends on ξ. However $s(\cdot)$ is constant in a small neighborhood of each $\xi\in\Xi^{n-1}\backslash N$, therefore in each component of $\Xi^{n-1}\backslash N$, and $\operatorname{Sup}_{\xi\notin N} s(\xi)$ is finite. Hence we put $s=\operatorname{Sup}_{\xi\notin N} s(\xi)$ and $D_\nu=\{\xi\in\Xi^{n-1}\backslash N;\ s(\xi)\geqq\nu\}$, $k_\nu(\xi)=\tilde{k}_\nu(\xi)$ in D_ν, $1\leqq\nu\leqq s$. This proves the assertion (iii). Q.E.D.

Remark. From the proof of Lemma 2.11 it follows that for any $\xi^0\notin N$ there exist a small neighborhood $W(\subset\Xi^{n-1}\backslash N)$ and $\delta\,(>0)$ such that $\Delta(\xi;\lambda)=0$ has no roots in

$$\bigcup_{j\in\{\nu;\,\xi\in D_\nu\}}\{\lambda\in\mathbf{C};\ 0<|k_j(\xi)-\lambda|<\delta\}$$

for every ξ in W. Moreover we note that $N\subset\Xi^{n-1}\backslash\bigcup_{\nu=1}^{s}D_\nu$ and $N=\bigcup_{\nu=1}^{s}\partial D_\nu$. Also we note that the $k_j(\xi)$ do not vanish for $\xi\in D_j$ although $\Delta(\xi;0)$ may identically vanish.

Define

$$\tilde{N}=\{(\xi,k)\in\Xi^{n-1}\times\mathbf{R};\ Q(k,\xi,\tau)=0\ \textit{has real double roots with respect to}\ \tau\},\tag{2.21}$$

$$\tilde{N}_j=\{\eta\in\Xi^n;\ (\xi,\lambda_j(\eta))\in\tilde{N}\},\qquad 1\leqq j\leqq 2\rho,\tag{2.22}$$

$$\Delta_\nu=\{\xi\in D_\nu;\ (\xi,k_\nu(\xi))\in\tilde{N}\},\qquad 1\leqq\nu\leqq s.\tag{2.23}$$

$\tilde{N}$ is a null set of $\Xi^{n-1}\times\mathbf{R}$ and the $\tilde{N}_j$ are null sets of Ξ^n. Further we have the following

Lemma 2.12. (i) *The $k_\nu(\xi)$ are analytic in D_ν.*
(ii) *The $\partial\Delta_\nu$ are null sets of Ξ^{n-1}.*

Proof. The assertion (i) follows from the proof of Lemma 2.11. We observe that $\xi\in\Delta_i$ if and only if $k_i(\xi)=\lambda_r(\xi,\sigma(\xi))$, where $\sigma(\xi)$ is defined in §1. Thus the assertion (ii) easily follows from analyticity of $\lambda_r(\xi,\sigma(\xi))$. Q.E.D.

Put

$$\hat{N}=N\cup\bigcup_{\nu=1}^{s}\partial\Lambda_\nu \qquad \text{(a null set of } \Xi^{n-1}). \tag{2.24}$$

Next let us consider the $C_{lh}(\xi;\lambda)$ defined by (2.5). The following lemma immediately follows from Lemma 2.7.

Lemma 2.13. *Let $\xi^0\notin\hat{N}\cup\bigcup_{\mu=1}^{s}\Lambda_\mu$ and $k^0=k_\nu(\xi^0)$ for a fixed ν. Then we have the following*:

(i) *When $\tau^+_{p(l)}(\xi^0;k^0)$ is a real simple root of $Q(k^0,\xi^0,\tau)=0$, that is, $1\leqq l\leqq p_0$,*

$$|C_{lh}(\xi;\lambda)|\leqq C \ \textit{for}\ (\xi,\lambda)\ \textit{in}\ W\times\Lambda^+_\delta(k^0)(W\times\Lambda^-_\delta(k^0)).$$

(ii) *When $\tau^+_{p(l)}(\xi^0;k^0)$ is non-real, that is, $p_0+1\leqq l\leqq m$,*

$$|C_{lh}(\xi;\lambda)|\leqq\frac{C}{|\lambda-k_\nu(\xi)|}\ \textit{for}\ (\xi,\lambda)\ \textit{in}\ W\times\Lambda^+_\delta(k^0)(W\times\Lambda^-_\delta(k^0)).$$

Here and in sequel W denotes a small neighborhood of ξ^0 in Ξ^{n-1} and $\delta\,(>0)$ is chosen sufficiently small.

Proof. From Lemma 2.7 we have

$$|(\lambda-k_\nu(\xi))C_{lh}(\xi;\lambda)|_{\lambda=k_\nu(\xi)\pm i\varepsilon}\leqq C[\operatorname{Im}\tau^+_{p(l)}(\xi;k_\nu(\xi)\pm i0)]^{\frac{1}{2}}.$$

This implies the above lemma. Q.E.D.

Lemma 2.14. *Let $\xi^0\in\Lambda_\nu\backslash\partial\Lambda_\nu$ and $k^0=k_\nu(\xi^0)$ for a fixed ν. Then we have the following*:

(i) *When $\tau^+_{p(l)}(\xi^0;k^0)$ is a real double root of $Q(k^0,\xi^0,\tau)=0$, that is, $1\leqq l\leqq\tilde{\nu}_1$,*

$$|C_{lh}(\xi;\lambda)|\leqq\frac{C}{|\lambda-k_\nu(\xi)|^{\frac{1}{2}}}\ \textit{for}\ (\xi,\lambda)\ \textit{in}\ W\times\Lambda^+_\delta(k^0)(W\times\Lambda^-_\delta(k^0)).$$

(ii) *When $\tau^+_{p(l)}(\xi^0;k^0)$ is a real simple root of $Q(k^0,\xi^0,\tau)=0$, that is, $\tilde{\nu}_1+1\leqq l\leqq p_0$,*

$$|C_{lh}(\xi;\lambda)|\leqq C\ \textit{for}\ (\xi,\lambda)\ \textit{in}\ W\times\Lambda^+_\delta(k^0)(W\times\Lambda^-_\delta(k^0)).$$

(iii) *When* $\tau^+_{p(l)}(\xi^0; k^0)$ *is non-real, that is,* $p_0+1\leqq l\leqq m$,

$$|C_{lh}(\xi;\lambda)|\leqq\frac{C}{|\lambda-k_\nu(\xi)|}\ \textit{for}\ (\xi,\lambda)\ \textit{in}\ \ W\times\Lambda^+_\delta(k^0)(W\times\Lambda^-_\delta(k^0)).$$

Here $k_\nu(\xi)\equiv\lambda_r(\xi,\sigma(\xi))$ *in* W.

Proof. The assertions (i) and (iii) immediately follow from Lemma 2.7. The assertion (ii) is proved by showing that $t\equiv(\lambda-\lambda_r(\xi,\sigma(\xi))^{\frac{1}{2}}=0$ is not a pole of $C_{lh}(\xi; t^2+\lambda_r(\xi,\sigma(\xi)))$ for each $\xi\in W$, $\tilde{\nu}_1+1\leqq l\leqq p_0$, which will be shown in the proof of Lemma 2.16 for more complicate case. Q.E.D.

Next we investigate the behavior of $C_{lh}(\xi;\lambda)$ in a neighborhood of $\hat{N}\times\mathbf{R}$. Here we assume for simplicity that the Lopatinski determinant $\Delta(\xi;\lambda)$ is defined in $\Xi^{n-1}\times\overline{\mathbf{C}^+}$. First we consider the case where $\xi^0\in\hat{N}$, $\Delta(\xi^0;k^0)=0$ and $(\xi^0,k^0)\notin\tilde{N}$. Then it suffices to consider $C_{lh}(\xi;\lambda)$ for ξ in each component $\hat{\hat{W}}_0$ constructed in the proof of Lemma 2.11. Let $\tilde{k}_1(\xi),\dots,\tilde{k}_\beta(\xi)$ be the zeros of $\Delta(\xi;\lambda)$, continued analytically, in $\Lambda_\delta(k^0)$ for each $\xi\in\hat{\hat{W}}_0$. Then we can assume without loss of generality that

$$k^0=\tilde{k}_1(\xi^0)=\cdots=\tilde{k}_\beta(\xi^0),\quad \tilde{k}_1(\xi)<\cdots<\tilde{k}_q(\xi),$$
$$\mathrm{Im}\,\tilde{k}_{q+1}(\xi)<0,\dots,\mathrm{Im}\,\tilde{k}_\beta(\xi)<0,\qquad \text{for}\ \ \xi\ \text{in}\ \hat{\hat{W}}_0.$$

We note that $\{\tilde{k}_1(\xi),\dots,\tilde{k}_q(\xi)\}$ is contained in $\{k_j(\xi)\}_{j\in\{\nu;\xi\in D_\nu\}}$ for $\xi\in\hat{\hat{W}}_0$.

Lemma 2.15. *Assume that* $\xi^0\in\hat{N}$, $\Delta(\xi^0;k^0)=0$ *and* $(\xi^0,k^0)\notin\tilde{N}$. *Then we have the following*:

(i) *When* $\tau^+_{p(l)}(\xi^0;k^0)$ *is a real simple root of* $Q(k^0,\xi^0,\tau)=0$, *that is,* $1\leqq l\leqq p_0$,

$$|C_{lh}(\xi;\lambda)|\leqq\sum_{j=q+1}^{\beta}\frac{C|\mathrm{Im}\tilde{k}_j(\xi)|^{\frac{1}{2}}}{|\lambda-\tilde{k}_j(\xi)|}+C,\ \textit{for}\ (\xi,\lambda)\ \textit{in}\ \hat{\hat{W}}_0\times\Lambda^+_\delta(k^0).$$

(ii) *When* $\tau^+_{p(l)}(\xi^0;k^0)$ *is non-real, that is,* $p_0+1\leqq l\leqq m$,

$$|C_{lh}(\xi;\lambda)|\leqq\sum_{j=1}^{\beta}\frac{C}{|\lambda-\tilde{k}_j(\xi)|}+C,\ \textit{for}\ (\xi,\lambda)\ \textit{in}\ \ \hat{\hat{W}}_0\times\Lambda^+_\delta(k^0).$$

Proof. By Weierstrass' preparation theorem, more precisely Späth's theorem, $C_{lh}(\xi;\lambda)$ is written in the form

$$C_{lh}(\xi;\lambda)=q(\xi;\lambda)\frac{p_2(\xi;\lambda)}{p_1(\xi;\lambda)}+C(\xi;\lambda),$$

where $q(\xi;\lambda)$, $C(\xi;\lambda)$, $p_1(\xi;\lambda)$ and $p_2(\xi;\lambda)$ are analytic in $W\times\Lambda_\delta(k^0)$, $q(\xi;\lambda)$ is bounded away from zero, $p_1(\xi;\lambda)$ and $p_2(\xi;\lambda)$ are polynomials with respect to λ, that is, pseudo-polynomials, and $\deg p_1(\xi;\lambda)>\deg p_2(\xi;\lambda)$. Thus, decomposing $\frac{p_1(\xi;\lambda)}{p_1(\xi;\lambda)}$ into a sum of partial fractions, we have

$$C_{lh}(\xi;\lambda)=q(\xi;\lambda)\left[\sum_{j=1}^{q}\frac{g_j(\xi)}{(\lambda-\tilde{k}_j(\xi))}+\sum_{j=q+1}^{\beta}\sum_{\nu=1}^{r_j}\frac{d_{j\nu}(\xi)}{(\lambda-\tilde{k}_j(\xi))^\nu}\right]+C(\xi;\lambda)$$

$$\text{for}\quad(\xi,\lambda)\text{ in }\hat{\hat{W}}_0\times\Lambda_\delta(k^0).$$

In fact, it follows from Lemma 2.7 that $\lambda=\tilde{k}_j(\xi)$, $1\leqq j\leqq q$, are at most simple poles of $C_{lh}(\xi;\lambda)$. Moreover Lemma 2.7 implies that

$$|g_j(\xi)|=|(\lambda-\tilde{k}_j(\xi))C_{lh}(\xi;\lambda)|_{\lambda=\tilde{k}_j(\xi)}\leqq C[\operatorname{Im}\tau^+_{p(l)}(\xi;\tilde{k}_j(\xi)+i0)]^{\frac{1}{2}}\text{ in }\hat{\hat{W}}_0,$$

$$1\leqq j\leqq q.$$

Therefore,

$$\text{(2.25)}\qquad |F_{lh}(\xi;\lambda)|\equiv\left|\sum_{j=q+1}^{\beta}\sum_{\nu=1}^{r_j}\frac{d_{j\nu}(\xi)}{(\lambda-\tilde{k}_j(\xi))^\nu}\right|$$

$$=\left|\frac{C_{lh}(\xi;\lambda)}{q(\xi;\lambda)}-\sum_{j=1}^{q}\frac{g_j(\xi)}{(\lambda-\tilde{k}_j(\xi))}-\frac{C(\xi;\lambda)}{q(\xi;\lambda)}\right|$$

$$\leqq\frac{C[\operatorname{Im}\tau^+_{p(l)}(\xi;\lambda)]^{\frac{1}{2}}}{\varepsilon}+C\text{ for }(\xi,\lambda)\text{ in }\hat{\hat{W}}_0\times\Lambda^+_\delta(k^0),$$

where $\operatorname{Im}\lambda=\varepsilon$. We rewrite $F_{lh}(\xi;\lambda)$ in the form

$$\text{(2.26)}\qquad F_{lh}(\xi;\lambda)=\prod_{j=q+1}^{\beta-1}(\lambda-\tilde{k}_j(\xi))^{-r_j}\cdot\sum_{\nu=-\gamma_1}^{r_\beta}\frac{d_\nu(\xi)}{(\lambda-\tilde{k}_\beta(\xi))^\nu},$$

where $\gamma_1=r_{q+1}+\cdots+r_{\beta-1}-1$. Put

$$\tilde{k}(a)=\operatorname{Re}\tilde{k}_\beta(\xi)-ia\operatorname{Im}\tilde{k}_\beta(\xi),\qquad a=1,\dots,\gamma_2,$$

where $\gamma_2 = r_{q+1} + \cdots + r_\beta$. Then, from (2.25) and (2.26) it follows that

$$\left| \sum_{\nu=-\gamma_1}^{r_\beta} \frac{d_\nu(\xi)}{(1+a)^\nu(-i\mathrm{Im}\tilde{k}_\beta(\xi))^\nu} \right| \leqq \prod_{j=q+1}^{\beta-1} |\tilde{k}(a) - \tilde{k}_j(\xi)|^{r_j}$$

$$\times \left\{ \frac{C[\mathrm{Im}\,\tau^+_{p(l)}(\xi;\, \tilde{k}(a))]^{\frac{1}{2}}}{a|\mathrm{Im}\tilde{k}_\beta(\xi)|} + C \right\}.$$

Since the matrix $\left(\frac{1}{(1+a)^\nu}\right)_{\substack{1\leqq a\leqq \gamma_2 \\ -\gamma_1 \leqq \nu \leqq r_\beta}}$ is non-singular, we have

$$|d_\nu(\xi)| \leqq \sum_{a=1}^{\gamma_2} C|\mathrm{Im}\,\tilde{k}_\beta(\xi)|^\nu \prod_{j=q+1}^{\beta-1} |\tilde{k}(a) - \tilde{k}_j(\xi)|^{r_j} \tag{2.27}$$

$$\times \left\{ \frac{C[\mathrm{Im}\,\tau^+_{p(l)}(\xi;\, \tilde{k}(a))]^{\frac{1}{2}}}{a|\mathrm{Im}\tilde{k}_\beta(\xi)|} + C \right\}, \quad -\gamma_1 \leqq \nu \leqq r_\beta.$$

Let us divided $\{q+1,\ldots, \beta-1\}$ into two parts for fixed $\xi \in \hat{\hat{W}}_0$ as follows: For u_j, $1 \leqq j \leqq b$, $\tilde{k}_{u_j}(\xi)$ satisfy the relations

$$\frac{1}{\alpha}|\mathrm{Im}\,\tilde{k}_\beta(\xi)| \leqq |\mathrm{Im}\,\tilde{k}_{u_j}(\xi)| \leqq \alpha|\mathrm{Im}\,\tilde{k}_\beta(\xi)|,$$

$$\alpha|\tilde{k}_\beta(\xi) - \tilde{k}_{u_j}(\xi)| \leqq |\mathrm{Im}\,(\tilde{k}_\beta(\xi) + \tilde{k}_{u_j}(\xi))|,$$

and for u_j, $b+1 \leqq j \leqq \beta-1-q$, $\tilde{k}_{u_j}(\xi)$ do not satisfy at least one of these relations, where α is a fixed large positive integer. We also rewrite $F_{lh}(\xi;\lambda)$ in the following form:

$$F_{lh}(\xi;\lambda) = \prod_{j=1}^{b}(\lambda - \tilde{k}_{u_j}(\xi))^{-r_{u_j}} \cdot \sum_{\nu=-\gamma_3}^{r_\beta} \frac{\tilde{d}_\nu(\xi)}{(\lambda - \tilde{k}_\beta(\xi))^\nu} + \sum_{j=b+1}^{\beta-1-q} \sum_{\nu=1}^{r_{u_j}} \frac{d_{u_j\nu}(\xi)}{(\lambda - \tilde{k}_{u_j}(\xi))^\nu},$$

where $\gamma_3 = r_{u_1} + \cdots + r_{u_b} - 1$. Then, from (2.26) and (2.27),

$$|\tilde{d}_{r_\beta}(\xi)| = |\prod_{j=1}^{b}(\lambda - \tilde{k}_{u_j}(\xi))^{r_{u_j}} \cdot (\lambda - \tilde{k}_\beta(\xi))^{r_\beta} F_{lh}(\xi;\lambda)|_{\lambda=\tilde{k}_\beta(\xi)} \tag{2.28}$$

$$= |d_{r_\beta}(\xi) \prod_{j=b+1}^{\beta-1-q}(\lambda - \tilde{k}_{u_j}(\xi))^{-r_{u_j}}|_{\lambda=\tilde{k}_\beta(\xi)}$$

$$\leqq \sum_{a=1}^{\gamma_2} C|\mathrm{Im}\,\tilde{k}_\beta(\xi)|^{r_\beta-1} \prod_{j=1}^{b} |\tilde{k}(a) - \tilde{k}_{u_j}(\xi)|^{r_{u_j}} \times$$

$$\times \{C[\mathrm{Im}\,\tau^+_{p(l)}(\xi;\, \tilde{k}(a))]^{\frac{1}{2}} + C|\mathrm{Im}\,\tilde{k}_\beta(\xi)|\}.$$

Here we have used the inequalities

$$\frac{|\tilde{k}(a)-\tilde{k}_{u_j}(\xi)|}{|\tilde{k}_\beta(\xi)-\tilde{k}_{u_j}(\xi)|}\leqq C,\ \ b+1\leqq j\leqq \beta-1-q\,.$$

Noting that for $\operatorname{Im}\lambda\geqq 0$ and $1\leqq j\leqq b$

$$\frac{|\tilde{k}(a)-\tilde{k}_{u_j}(\xi)|}{|\lambda-\tilde{k}_{u_j}(\xi)|}\leqq C,$$

it follows from (2.28) that

$$\Big|\prod_{j=1}^{b}(\lambda-\tilde{k}_{u_j}(\xi))^{-r_{u_j}}\cdot\frac{\tilde{d}_{r_\beta}(\xi)}{(\lambda-\tilde{k}_\beta(\xi))^{r_\beta}}\Big|$$

$$\leqq\sum_{a=1}^{\gamma_2}\frac{C|\operatorname{Im}\tilde{k}_\beta(\xi)|^{r_\beta-1}}{|\lambda-\tilde{k}_\beta(\xi)|^{r_\beta}}\{C[\operatorname{Im}\tau^+_{p(l)}(\xi;\,\tilde{k}(a))]^{\frac{1}{2}}+C|\operatorname{Im}\tilde{k}_\beta(\xi)|\}$$

$$\leqq\frac{C[\operatorname{Im}\tau^+_{p(l)}(\xi;\,\overline{\tilde{k}_\beta(\xi)})]^{\frac{1}{2}}}{|\lambda-\tilde{k}_\beta(\xi)|}+C,\ \text{for }\operatorname{Im}\lambda>0\,.$$

Observe that the above constants C's are independent of ξ in $\hat{\hat{W}}_0$. Put

$$F^{(1)}_{lh}(\xi;\,\lambda)=F_{lh}(\xi;\,\lambda)-\prod_{j=1}^{b}(\lambda-\tilde{k}_{u_j}(\xi))^{-r_{u_j}}\cdot\frac{\tilde{d}_{r_\beta}(\xi)}{(\lambda-\tilde{k}_\beta(\xi))^{r_\beta}}\,.$$

Then $F^{(1)}_{lh}(\xi;\,\lambda)$ is also estimated as follows:

$$|F^{(1)}_{lh}(\xi;\,\lambda)|\leqq\frac{C[\operatorname{Im}\tau^+_{p(l)}(\xi;\,\lambda)]^{\frac{1}{2}}}{\varepsilon}+C,\ \ \varepsilon=\operatorname{Im}\lambda>0\,.$$

In fact, for $1\leqq l\leqq p_0$

$$\frac{[\operatorname{Im}\tau^+_{p(l)}(\xi;\,\overline{\tilde{k}_\beta(\xi)})]^{\frac{1}{2}}}{|\lambda-\tilde{k}_\beta(\xi)|}\leqq\frac{C}{|\lambda-\tilde{k}_\beta(\xi)|^{\frac{1}{2}}}\leqq\frac{C\varepsilon^{\frac{1}{2}}}{\varepsilon}$$

$$\leqq\frac{C[\operatorname{Im}\tau^+_{p(l)}(\xi;\,\lambda)]^{\frac{1}{2}}}{\varepsilon}\,.$$

Thus we can estimate $\prod_{j=1}^{b}(\lambda-\tilde{k}_{u_j}(\xi))^{-r_{u_j}}\cdot\frac{\tilde{d}_{r_\beta-1}(\xi)}{(\lambda-\tilde{k}_\beta(\xi))^{r_\beta-1}}$ by applying the above argument and, inductively, $\prod_{j=1}^{b}(\lambda-k_{u_j}(\xi))^{-r_{u_j}}\cdot\frac{\tilde{d}_\nu(\xi)}{(\lambda-\tilde{k}_\beta(\xi))^\nu}$, $\nu=r_\beta-2,\dots,1$. Put

$$F_{lh}^{(r_\beta)}(\xi;\lambda)=F_{lh}(\xi;\lambda)-\prod_{j=1}^{b}(\lambda-\tilde{k}_{u_j}(\xi))^{-r_{u_j}}\times\sum_{\nu=1}^{r_\beta}\frac{\tilde{d}_\nu(\xi)}{(\lambda-\tilde{k}_\beta(\xi))^\nu}.$$

Then we have

$$|F_{lh}^{(r_\beta)}(\xi;\lambda)|\leqq\frac{C[\operatorname{Im}\tau_{p(l)}^{+}(\xi;\lambda)]^{\frac{1}{2}}}{\varepsilon}+C,\ \varepsilon=\operatorname{Im}\lambda>0,$$

and $\lambda=\tilde{k}_\beta(\xi)$ is no pole of $F_{lh}^{(r_\beta)}(\xi;\lambda)$. Apply the same argument for $F_{lh}^{(r_\beta)}(\xi;\lambda)$ replacing $\tilde{k}_\beta(\xi)$ by $\tilde{k}_{\beta-1}(\xi)$. Repeating the above argument, we conclude

$$|F_{lh}(\xi;\lambda)|\leqq\sum_{j=q+1}^{\beta}\frac{C[\operatorname{Im}\tau_{p(l)}^{+}(\xi;\overline{\tilde{k}_j(\xi)})]^{\frac{1}{2}}}{|\lambda-k_j(\xi)|},\ \operatorname{Im}\lambda>0,$$

and, therefore,

$$|C_{lh}(\xi;\lambda)|\leqq\sum_{j=1}^{q}\frac{C[\operatorname{Im}\tau_{p(l)}^{+}(\xi;\tilde{k}_j(\xi)+i0)]^{\frac{1}{2}}}{|\lambda-\tilde{k}_j(\xi)|}$$

$$+\sum_{j=q+1}^{\beta}\frac{C[\operatorname{Im}\tau_{p(l)}^{+}(\xi;\overline{\tilde{k}_j(\xi)})]^{\frac{1}{2}}}{|\lambda-\tilde{k}_j(\xi)|}+C,\ \text{for } (\xi;\lambda)\ \textit{in}\ \hat{\hat{W}}_0\times\Lambda_\delta^{+}(k^0).$$

Lemma 2.15 easily follows from this. Q.E.D.

Next let us consider $C_{lh}(\xi;\lambda)$ in the case where $\xi^0\in\hat{N}$, $\Delta(\xi^0;k^0)=0$ and $(\xi^0,k^0)\in\tilde{N}$. Then it suffices to estimate $C_{lh}(\xi;\lambda)$ for ξ in each component $\hat{\hat{W}}_0$ constructed in the proof of Lemma 2.11. Thus let $t_1(\xi),\dots,t_\beta(\xi)$ be the zeros of $\Delta(\xi;t^2+\lambda_r(\xi,\sigma(\xi)))$, continued analytically, in $\Lambda_{\sqrt{\delta}}(0)$ for each $\xi\in\hat{\hat{W}}_0$. Then we can assume without loss of generality that

$$t_1(\xi^0)=\cdots=t_\beta(\xi^0)=0,\quad t_1(\xi),\dots,t_q(\xi)\in\omega,$$

$$t_{q+1}(\xi),\dots,t_\beta(\xi)\in\Omega\qquad\text{for }\ \xi\ \text{in}\ \hat{\hat{W}}_0.$$

Moreover, since $t_1(\xi),\dots,t_\beta(\xi)$ are distinct in $\hat{\hat{W}}_0$ we may assume that

$t_1(\xi)\equiv 0$ and $t_j(\xi)\neq 0$, $2\leqq j\leqq q$, in $\hat{\hat{W}}_0$. We observe that $\{\tilde{k}_1(\xi),\dots, \tilde{k}_q(\xi)\}$ is contained in $\{k_j(\xi)\}_{j\in\{\nu;\xi\in D_\nu\}}$, where $\tilde{k}_j(\xi)=t_j(\xi)^2+\lambda_r(\xi, \sigma(\xi))$.

Lemma 2.16. *Assume that* $\xi^0\in\hat{N}$, $\Delta(\xi^0; k^0)=0$ *and* $(\xi^0, k^0)\in\tilde{N}$. *Then* $C_{lh}(\xi;\lambda)$ *is decomposed into a sum of* $J_j(\xi;\lambda)$, $1\leqq j\leqq\beta$, *which satisfy the following estimates in* $\hat{\hat{W}}_0\times\Lambda_\delta^+(k^0)$:

(1) *In the case where* $\tau_{p(l)}^+(\xi^0; k^0)$ *is a real double root of* $Q(k^0, \xi^0, \tau)=0$, *that is,* $1\leqq l\leqq\tilde{\nu}_1$,

(i) *when* $\tau_1^+(\xi; k+i0)$ *is real,* $\lambda=k+i\varepsilon$,

$$|J_j(\xi;\lambda)|\leqq|t|^{-\frac{1}{2}}\Big[\frac{C|t_j(\xi)|^{\frac{1}{2}}}{\{(k-\lambda_r(\xi, \sigma(\xi)))^2+|t_j(\xi)|^4\}^{3/8}}+\beta_j\frac{C|\mathrm{Im}\tilde{k}_j(\xi)|^{\frac{1}{2}}}{|\lambda-\tilde{k}_j(\xi)|}$$

$$+\frac{C}{|t|^{1-\theta}|t_j(\xi)|^{\theta}}+\frac{C}{|t|^{\frac{1}{2}}}\Big],\quad 2\leqq j\leqq\beta,$$

$$|J_1(\xi;\lambda)|\leqq\frac{C}{|t|^{\frac{1}{2}}\,|t|^{\frac{1}{2}}},$$

(ii) *when* $\tau_1^+(\xi; k+i0)$ *is non-real,*

$$|J_j(\xi;\lambda)|\leqq[\mathrm{Im}\tau_1^+(\xi;\lambda)]^{\frac{1}{2}}\Big\{\frac{C}{|\lambda-\lambda_r(\xi, \sigma(\xi))|}+\beta_j\frac{C}{|\lambda-\tilde{k}_j(\xi)|}\Big\}.$$

(2) *In the case where* $\tau_{p(l)}^+(\xi^0; k^0)$ *is a real simple root of* $Q(k^0, \xi^0, \tau)=0$, *that is,* $\tilde{\nu}_1+1\leqq l\leqq p_0$,

$$|J_j(\xi;\lambda)|\leqq\frac{C|t_j(\xi)|^{\frac{1}{2}}}{\{(k-\lambda_r(\xi, \sigma(\xi)))^2+|t_j(\xi)|^4\}^{3/8}}+\beta_j\frac{C|\mathrm{Im}\tilde{k}_j(\xi)|^{\frac{1}{4}}}{|\lambda-\tilde{k}_j(\xi)|^{3/4}}$$

$$+\frac{C}{|t|^{1-\theta}|t_j(\xi)|^{\theta}},\quad 2\leqq j\leqq\beta,$$

$$|J_1(\xi;\lambda)|\leqq C.$$

(3) *In the case where* $\tau_{p(l)}^+(\xi^0; k^0)$ *is non-real, that is,* $p_0+1\leqq l\leqq m$,

$$|J_j(\xi;\lambda)|\leqq\frac{C}{|\lambda-\lambda_r(\xi, \sigma(\xi))|}+\beta_j\frac{C}{|\lambda-\tilde{k}_j(\xi)|}.$$

Here θ *is an arbitrary positive constant less than* $\frac{1}{2}$ *and* $\beta_j=1$ (if $\operatorname{Im}\tilde{k}_j(\xi)\leqq 0$), $=0$ (if $\operatorname{Im}\tilde{k}_j(\xi)>0$).

Proof. By Späth's theorem $C_{lh}(\xi;\, t^2+\lambda_r(\xi,\, \sigma(\xi)))$ is written in the form

$$C_{lh}(\xi\,;\, t^2+\lambda_r(\xi,\, \sigma(\xi)))=q(\xi\,;\, t)\frac{p_2(\xi\,;\, t)}{p_1(\xi\,;\, t)}+C(\xi\,;\, t) \text{ in } W\times\Lambda_{\sqrt{\delta}}(0)\,,$$

where $q(\xi;\, t)$, $C(\xi;\, t)$, $p_1(\xi;\, t)$ and $p_2(\xi;\, t)$ are analytic in $W\times\Lambda_{\sqrt{\delta}}(0)$, $q(\xi;\, t)$ is bounded away from zero, $p_1(\xi;\, t)$ and $p_2(\xi;\, t)$ are polynomials with respect to t and $\deg p_1(\xi;\, t)>\deg p_2(\xi;\, t)$. Put

$$F_{lh}(\xi\,;\, t)=\frac{C_{lh}(\xi\,;\, t)}{q(\xi\,;\, t)}-\frac{C(\xi\,;\, t)}{q(\xi\,;\, t)}-\sum_{j=1}^{q}\frac{g_j(\xi)}{t^2-t_j(\xi)^2}\,,$$

where

$$g_j(\xi)=(t^2-t_j(\xi)^2)C_{lh}(\xi;\, t)/q(\xi;\, t)|_{t=t_j(\xi)}\,.$$

Then we have by Lemma 2.7

$$|g_j(\xi)|\leqq C[\operatorname{Im}\tau^+_{p(l)}(\xi;\, t_j(\xi)^2+\lambda_r(\xi,\, \sigma(\xi))+i0)]^{\frac{1}{2}}$$

and

$$|F_{lh}(\xi\,;t)|\leqq\frac{C[\operatorname{Im}\tau^+_{p(l)}(\xi\,;\, t^2+\lambda_r(\xi,\, \sigma(\xi)))]^{\frac{1}{2}}}{\varepsilon}+C,\quad t\notin\Omega\cup\omega\,, \tag{2.29}$$

where $\varepsilon=\operatorname{Im}t^2>0$. In fact, the following estimates hold for $t\notin\Omega\cup\omega$ and $1\leqq l\leqq\tilde{\nu}_1$, $1\leqq j\leqq q$:

(1) Let $\tau^+_1(\xi;\, \tilde{k}_j(\xi)+i0)$ be non-real.

(i) If $\operatorname{Im}\tau^+_1(\xi;\, k+i0)\neq 0$, $\lambda\equiv k+i\varepsilon\equiv t^2+\lambda_r(\xi,\, \sigma(\xi))$,

$$\frac{|g_j(\xi)|}{|t^2-t_j(\xi)^2|}\leqq\frac{C[\operatorname{Im}\tau^+_1(\xi\,;\lambda)]^{\frac{1}{2}}}{|\lambda-\tilde{k}_j(\xi)|}\quad(2|t|\geqq|t_j(\xi)|)$$

$$\leqq\frac{C[\operatorname{Im}\tau^+_1(\xi\,;\lambda)]^{\frac{1}{2}}}{|\lambda-\lambda_r(\xi,\, \sigma(\xi))|}\quad(2|t|\leqq|t_j(\xi)|)\,.$$

(ii) If $\operatorname{Im}\tau_1^\dagger(\xi;\, k+i0)=0$,

$$\frac{|g_j(\xi)|}{|t^2-t_j(\xi)^2|}\leqq\frac{C|t_j(\xi)|^{\frac{1}{2}}}{(|t|^4+|t_j(\xi)|^4)^{\frac{1}{2}}}$$

$$\leqq\frac{C|t_j(\xi)|^{\frac{1}{2}}}{|t|^{\frac{1}{2}}\{(k-\lambda_r(\xi,\,\sigma(\xi)))^2+\varepsilon^2+|t_j(\xi)|^4\}^{3/8}}$$

$$\left(\leqq\frac{C[\operatorname{Im}\tau_1^\dagger(\xi;\,\lambda)]^{\frac{1}{2}}}{\varepsilon}\right).$$

(2) When $\operatorname{Im}\tau_1^\dagger(\xi;\, \tilde{k}_j(\xi)+i0)=0$,

$$\frac{|g_j(\xi)|}{|t^2-t_j(\xi)^2|}=0\,.$$

Here we have used Lemma 1.1 and the relation $\frac{1}{C}|t|\leqq\operatorname{Im}\tau_1^\dagger(\xi;\,\lambda)\leqq C|t|$ when $\operatorname{Im}\tau_1^\dagger(\xi;\, k+i0)\neq 0$. Moreover we have used the inequality $|t\pm t_j(\xi)|\geqq C\{|t|+|t_j(\xi)|\}$ in the case (1) (ii) which follows from the fact that $|\arg t-\arg(\pm t_j(\xi))|\geqq\frac{\pi}{4}$. (2.29) follows from the above estimates and Lemma 2.7. Removing a closed null set from $\hat{\hat{W}}_0$ if necessary, we may assume that $-t_j(\xi)\neq t_i(\xi)$ in $\hat{\hat{W}}_0$ if $1\leqq j\leqq q$ and $q+1\leqq i\leqq\beta$. In the case where $-t_j(\xi)\equiv t_i(\xi)$ in $\hat{\hat{W}}_0$ for some j and i we can easily modify our proof. Since $F_{lh}(\xi;\, t)=\frac{p_2(\xi;\,t)}{p_1(\xi;\,t)}-\sum_{j=1}^{q}\frac{g_j(\xi)}{t^2-t_j(\xi)^2}$ has no poles on $\omega\backslash\{0\}$, $F_{lh}(\xi;\, t)$ can be written in the following form:

$$F_{lh}(\xi;t)=\sum_{j=1}^{q}\frac{d_{j1}(\xi)}{t+t_j(\xi)}+\sum_{j=q+1}^{\beta}\sum_{\nu=1}^{r_j}\frac{d_{j\nu}(\xi)}{(t-t_j(\xi))^\nu}\,.$$

To unify the treatment we rewrite $-t_j(\xi)$ instead of $t_j(\xi)$ in the remainder of this proof, $1\leqq j\leqq q$. Hence, $t_1(\xi)=0$ and $t_j(\xi)\in\Omega$, $2\leqq j\leqq\beta$. Moreover put $r_j=1$, $1\leqq j\leqq q$. For $t\in\Omega\cup\omega$ define $\mathrm{d}(t)=\mathrm{dis}(t,\,\omega)$ and $\mathrm{r}(t)\in\omega$ in such a way that $|t-\mathrm{r}(t)|=\mathrm{d}(t)$. Then let us divide $\{1,\ldots,\beta-1\}$ into two parts for fixed $\xi\in\hat{\hat{W}}_0$ as follows: for u_j, $1\leqq j\leqq b$, $t_{u_j}(\xi)$ satisfy the relations

$$\frac{1}{\alpha}\mathrm{d}(t_{u_j}(\xi))\leqq\mathrm{d}(t_\beta(\xi))\leqq\alpha\mathrm{d}(t_{u_j}(\xi))\,, \tag{2.30}$$

$$(2.31)\qquad \alpha|t_\beta(\xi)-t_{u_j}(\xi)|\leqq \mathrm{d}(t_\beta(\xi))+\mathrm{d}(t_{u_j}(\xi)),$$

and for u_j, $b+1\leqq j\leqq \beta-1$, do not satisfy at least one of these relations, where α is a fixed large positive integer. We rewrite $F_{lh}(\xi;t)$ in the form

$$(2.32)\qquad F_{lh}(\xi;t)=\prod_{j=1}^{\beta-1}(t-t_j(\xi))^{-r_j}\cdot\sum_{\nu=-\gamma_1}^{r_\beta}\frac{d_\nu(\xi)}{(t-t_\beta(\xi))^\nu},$$

where $\gamma_1=r_1+\cdots+r_{\beta-1}-1$. Put

$$t(a)=\mathrm{r}(t_\beta(\xi))+aw\mathrm{d}(t_\beta(\xi)),\qquad a=1,\dots,\gamma_2,$$

where $w=e^{\frac{\pi}{4}i}$ and $\gamma_2=r_1+\cdots+r_\beta$. Substituting $t=t(a)$ in (2.32) and using (2.29), we obtain

$$\left|\sum_{\nu=-\gamma_1}^{r_\beta}\frac{d_\nu(\xi)}{y_a^\nu(t(1)-t_\beta(\xi))^\nu}\right|\leqq\prod_{j=1}^{\beta-1}|t(a)-t_j(\xi)|^{r_j}$$

$$\times\left\{\frac{C[\mathrm{Im}\,\tau_{p(l)}^+(\xi;t(a)^2+\lambda_r(\xi,\sigma(\xi)))]^{\frac{1}{2}}}{\mathrm{Im}\,t(a)^2}+C\right\},$$

where $y_a=\dfrac{t(a)-t_\beta(\xi)}{t(1)-t_\beta(\xi)}$. From $\left|\dfrac{1}{y_a}-\dfrac{1}{y_{a'}}\right|\geqq C>0$, $a\neq a'$, the matrix $\left(\dfrac{1}{y_a^\nu}\right)_{\substack{1\leqq a\leqq\gamma_2\\ -\gamma_1\leqq\nu\leqq r_\beta}}$ is non-singular. Therefore we have

$$|d_\nu(\xi)|\leqq\sum_{a=1}^{\gamma_2}C\prod_{j=1}^{\beta-1}|t(a)-t_j(\xi)|^{r_j}\cdot\mathrm{d}(t_\beta(\xi))^\nu$$

$$\times\left\{\frac{C[\mathrm{Im}\,\tau_{p(l)}^+(\xi;t(a)^2+\lambda_r(\xi,\sigma(\xi)))]^{\frac{1}{2}}}{\mathrm{Im}\,t(a)^2}+C\right\}.$$

We also rewrite $F_{lh}(\xi;t)$ in the following form:

$$F_{lh}(\xi;t)=\prod_{j=1}^{b}(t-t_{u_j}(\xi))^{-r_{u_j}}\cdot\sum_{\nu=-\gamma_3}^{r_\beta}\frac{\tilde{d}_\nu(\xi)}{(t-t_\beta(\xi))^\nu}$$

$$+\sum_{j=b+1}^{\beta-1}\sum_{\nu=1}^{r_{u_j}}\frac{d_{u_j\nu}(\xi)}{(t-t_{u_j}(\xi))^\nu},$$

where $\gamma_3=r_{u_1}+\cdots+r_{u_b}-1$. By the same argument as in the proof of Lemma 2.15 we obtain

$$(2.33)\qquad |G(\xi;t)| \equiv \left| \prod_{j=1}^{b} (t-t_{u_j}(\xi))^{-r_{u_j}} \cdot \frac{\tilde{d}_{r_\beta}(\xi)}{(t-t_\beta(\xi))^{r_\beta}} \right|$$

$$\leqq \sum_{a=1}^{\nu_2} \frac{Cd(t_\beta(\xi))^{r_\beta-1}}{|t-t_\beta(\xi)|^{r_\beta}} \left\{ \frac{C[\operatorname{Im} \tau^+_{p(l)}(\xi;t(a)^2+\lambda_r(\xi,\sigma(\xi)))]^{\frac{1}{2}}}{|t_\beta(\xi)|} + Cd(t_\beta(\xi)) \right\}$$

$$\leqq \frac{Cd(t_\beta(\xi))^{r_\beta-1}[\operatorname{Im} \tau^+_{p(l)}(\xi;t(1)^2+\lambda_r(\xi,\sigma(\xi)))]^{\frac{1}{2}}}{|t-t_\beta(\xi)|^{r_\beta}|t_\beta(\xi)|}.$$

Here we have also used the relation

$$\operatorname{Im} t(a)^2 = a^2 d(t_\beta(\xi))^2 + \sqrt{2}\, a d(t_\beta(\xi)) |r(t_\beta(\xi))|.$$

First let us consider $|G(\xi;t)|$ for $(\xi,\lambda) \in \hat{W}_0 \times \Lambda^+_\delta(k^0)$ $(t \notin \Omega \cup \omega)$ in the case where $r_\beta \geqq 2$.

(1) Let $1 \leqq l \leqq \nu_1$.

(i) If $\operatorname{Im} \tau^+_1(\xi;k+i0) \neq 0$,

$$|G(\xi;t)| \leqq \frac{C[\operatorname{Im} \tau^+_1(\xi;\lambda)]^{\frac{1}{2}}}{|\lambda-\lambda_r(\xi,\sigma(\xi))|} \quad (r(t_\beta(\xi))=0 \text{ or } 2|t| \leqq |t_\beta(\xi)|)$$

$$\leqq \frac{C[\operatorname{Im} \tau^+_1(\xi;\lambda)]^{\frac{1}{2}}}{|\lambda-\tilde{k}_\beta(\xi)|} \quad (|t_\beta(\xi)| \leqq 2|t| \text{ and } r(t_\beta(\xi)) \neq 0).$$

(ii) If $\operatorname{Im} \tau^+_1(\xi;k+i0)=0$ and $\operatorname{Im} \tau^+_1(\xi;r(t_\beta(\xi))^2+\lambda_r(\xi,\sigma(\xi))+i0) \neq 0$,

$$|G(\xi;t)| \leqq \frac{C|t_\beta(\xi)|^{\frac{1}{2}}}{|t|^{\frac{1}{2}}\{(k-\lambda_r(\xi,\sigma(\xi)))^2+\varepsilon^2+|t_\beta(\xi)|^4\}^{3/8}} \left(\leqq \frac{C[\operatorname{Im} \tau^+_1(\xi;\lambda)]^{\frac{1}{2}}}{\varepsilon} \right).$$

(iii) If $\operatorname{Im} \tau^+_1(\xi;k+i0)=0$ and $\operatorname{Im} \tau^+_1(\xi;r(t_\beta(\xi))^2+\lambda_r(\xi,\sigma(\xi))+i0)=0$,

$$|G(\xi;t)| \leqq \frac{C|t_\beta(\xi)|^{\frac{1}{2}}}{|t|^{\frac{1}{2}}\{(k-\lambda_r(\xi,\sigma(\xi)))^2+\varepsilon^2+|t_\beta(\xi)|^4\}^{3/8}}$$

$$\left(\leqq \frac{C[\operatorname{Im}\tau_1^\dagger(\xi;\lambda)]^{\frac{1}{2}}}{\varepsilon}\right)$$

$$(2|t| \leqq |t_\beta(\xi)| \text{ or } 2|t_\beta(\xi)| \leqq |t|)$$

$$\leqq \frac{C|\operatorname{Im}\tilde{k}_\beta(\xi)|^{\frac{1}{2}}}{|t|^{\frac{1}{2}}|\lambda-\tilde{k}_\beta(\xi)|} \quad \left(\leqq \frac{C[\operatorname{Im}\tau_1^\dagger(\xi;\lambda)]^{\frac{1}{2}}}{\varepsilon}\right)$$

$$\left(\frac{1}{2}|t| < |t_\beta(\xi)| < 2|t| \text{ and } \mathrm{d}(t_\beta(\xi)) \leqq |\mathrm{r}(t_\beta(\xi))|\right)$$

$$\leqq \frac{C|t_\beta(\xi)|^{\frac{1}{2}}}{|t|^{\frac{1}{2}}\{(k-\lambda_r(\xi,\sigma(\xi)))^2+\varepsilon^2+|t_\beta(\xi)|^4\}^{3/8}}$$

$$\left(\frac{1}{2}|t| < |t_\beta(\xi)| < 2|t| \text{ and } \mathrm{d}(t_\beta(\xi)) > |\mathrm{r}(t_\beta(\xi))|\right).$$

(2) When $\tilde{\nu}_1+1 \leqq l \leqq p_0$,

$$|G(\xi;t)| \leqq \frac{C|\operatorname{Im}\tilde{k}_\beta(\xi)|^{\frac{1}{2}}}{|\lambda-\tilde{k}_\beta(\xi)|} \qquad (|\mathrm{r}(t_\beta(\xi))| \geqq \mathrm{d}(t_\beta(\xi)))$$

$$\leqq \frac{C|t_\beta(\xi)|^{\frac{1}{2}}}{\{(k-\lambda_r(\xi,\sigma(\xi)))^2+\varepsilon^2+|t_\beta(\xi)|^4\}^{3/8}}$$

$$(|\mathrm{r}(t_\beta(\xi))| < \mathrm{d}(t_\beta(\xi))).$$

(3) When $p_0+1 \leqq l \leqq m$,

$$|G(\xi;t)| \leqq \frac{C}{|\lambda-\lambda_r(\xi,\sigma(\xi))|} \quad (|t| \geqq 2|t_\beta(\xi)| \text{ or } \mathrm{r}(t_\beta(\xi))=0)$$

$$\leqq \frac{C}{|\lambda-\tilde{k}_\beta(\xi)|} \qquad (|t| \leqq 2|t_\beta(\xi)| \text{ and } \mathrm{r}(t_\beta(\xi)) \neq 0).$$

Here we have used Lemma 1.1 and the facts that if $|\mathrm{r}(t_\beta(\xi))| < \mathrm{d}(t_\beta(\xi))$ $|t-t_\beta(\xi)| \geqq C\{|t|+|t_\beta(\xi)|\}$ for $t \notin \Omega \cup \omega$, in the case (1) (ii) $|t-t_\beta(\xi)| \geqq C\{|t|+|t_\beta(\xi)|\}$ and that $|\operatorname{Im}\tilde{k}_\beta(\xi)| = |2\mathrm{r}(t_\beta(\xi))\mathrm{d}(t_\beta(\xi))|$ if $\mathrm{r}(t_\beta(\xi)) \neq 0$. Put

$$F_{lk}^{(1)}(\xi;t)=F_{lh}(\xi;t)-\prod_{j=1}^{b}(t-t_{u_j}(\xi))^{-r_{u_j}}\cdot\frac{\tilde{d}_{r_\beta}}{(t-t_\beta(\xi))^{r_\beta}},\quad r_\beta\geqq 2.$$

Then it follows from the above estimates and (2.29) that

$$|F_{lh}^{(1)}(\xi;t)|\leqq\frac{C[\mathrm{Im}\ \tau_{p(l)}^{+}(\xi;\lambda)]^{\frac{1}{2}}}{\varepsilon}+C,\ t\notin\Omega\cup\omega,$$

where $\lambda\equiv k+i\varepsilon\equiv t^2+\lambda_r(\xi,\sigma(\xi))$. Thus we can inductively estimate $\prod_{j=1}^{b}(t-t_{u_j}(\xi))^{-r_{u_j}}\cdot\frac{\tilde{d}_\nu(\xi)}{(t-t_\beta(\xi))^\nu}$ by applying the above argument, $\nu=r_\beta-1,\dots,2$. Put

$$F_{lh}^{(r_\beta-1)}(\xi;t)=F_{lh}(\xi;t)-\prod_{j=1}^{b}(t-t_{u_j}(\xi))^{-r_{u_j}}\cdot\sum_{\nu=2}^{r_\beta}\frac{\tilde{d}_\nu(\xi)}{(t-t_\beta(\xi))^\nu}.$$

Then,

$$|F_{lh}^{(r_\beta-1)}(\xi;t)|\leqq\frac{C[\mathrm{Im}\ \tau_{p(l)}^{+}(\xi;\lambda)]^{\frac{1}{2}}}{\varepsilon}+C,\ t\notin\Omega\cup\omega.$$

Apply the same argument for $F_{lh}^{(r_\beta-1)}(\xi;t)$ replacing $t_\beta(\xi)$ by $t_{\beta-1}(\xi)$. Repeating the above argument, we conclude

$$\tilde{F}_{lh}(\xi;t)=\sum_{j=1}^{\beta}\frac{d_j(\xi)}{t-t_j(\xi)},\tag{2.34}$$

$$|\tilde{F}_{lh}(\xi;t)|\leqq\frac{C[\mathrm{Im}\ \tau_{p(l)}^{+}(\xi;\lambda)]^{\frac{1}{2}}}{\varepsilon}+C,\ t\notin\Omega\cup\omega.\tag{2.35}$$

Next we consider the case where there exists a number u, $1\leqq u\leqq\beta-1$, satisfying the relations (2.30) and (2.31). Write $\tilde{F}_{lh}(\xi;t)$ in the form

$$\tilde{F}_{lh}(\xi;t)=\prod_{j=1}^{b}(t-t_{u_j}(\xi))^{-1}\sum_{\nu=-b+1}^{1}\frac{\tilde{d}_\nu(\xi)}{(t-t_\beta(\xi))^\nu}+\sum_{j=b+1}^{\beta-1}\frac{d_{u_j}(\xi)}{t-t_{u_j}(\xi)}.$$

Then we have

$$|G(\xi;t)|\equiv\left|\prod_{j=1}^{b}(t-t_{u_j}(\xi))^{-1}\cdot\frac{\tilde{d}_1(\xi)}{(t-t_\beta(\xi))}\right|$$

$$\leqq\frac{C[\mathrm{Im}\ \tau_{p(l)}^{+}(\xi;t(1)^2+\lambda_r(\xi,\sigma(\xi)))]^{\frac{1}{2}}}{|t-t_\beta(\xi)|\,|t_\beta(\xi)|},$$

$$|G(\xi;t)|\leqq\frac{C[\mathrm{Im}\,\tau^+_{p(l)}(\xi;t(1)^2+\lambda_r(\xi,\sigma(\xi)))]^{\frac{1}{2}}}{|t-t_\beta(\xi)|\,|t-t_{u_1}(\xi)|},\ t\notin\Omega\cup\omega\,.$$

$|G(\xi;t)|$ is estimated for (ξ,λ) in $\hat{\hat{W}}_0\times\Lambda^+_\delta(k^0)$ $(t\notin\Omega\cup\omega)$ as follows:

(1) Let $1\leqq l\leqq\tilde{\nu}_1$.

(i) If $\mathrm{Im}\,\tau^+_1(\xi;k+i0)=0$ and $\mathrm{Im}\,\tau^+_1(\xi;\mathrm{r}(t_\beta(\xi))^2+\lambda_r(\xi,\sigma(\xi))+i0)=0$,

$$|G(\xi;t)|\leqq\frac{C|t_\beta(\xi)|^{\frac{1}{2}}}{|t|^{\frac{1}{2}}\{(k-\lambda_r(\xi,\sigma(\xi)))^2+\varepsilon^2+|t_\beta(\xi)|^4\}^{3/8}}$$

$$\left(\leqq\frac{C[\mathrm{Im}\,\tau^+_1(\xi;\lambda)]^{\frac{1}{2}}}{\varepsilon}\right)$$

$$(2|t|\leqq|t_\beta(\xi)|\ \text{or}\ |t|\geqq4|t_\beta(\xi)|)$$

$$\leqq\frac{C|\mathrm{Im}\,\tilde{k}_\beta(\xi)|^{\frac{1}{2}}}{|t|^{\frac{1}{2}}|\lambda-\tilde{k}_\beta(\xi)|}$$

$$\left(\frac{1}{4}|t|\leqq|t_\beta(\xi)|\leqq2|t|\ \text{and}\ \mathrm{d}(t_\beta(\xi))\leqq|\mathrm{r}(t_\beta(\xi))|\right)$$

$$\leqq\frac{C|t_\beta(\xi)|^{\frac{1}{2}}}{|t|^{\frac{1}{2}}\{(k-\lambda_r(\xi,\sigma(\xi)))^2+\varepsilon^2+|t_\beta(\xi)|^4\}^{3/8}}$$

$$\left(\frac{1}{4}|t|\leqq|t_\beta(\xi)|\leqq2|t|\ \text{and}\ \mathrm{d}(t_\beta(\xi))>|\mathrm{r}(t_\beta(\xi))|\right).$$

(ii) If $\mathrm{Im}\,\tau^+_1(\xi;k+i0)=0$ and $\mathrm{Im}\,\tau^+_1(\xi;\mathrm{r}(t_\beta(\xi))^2+\lambda_r(\xi,\sigma(\xi))+i0)\neq0$,

$$|G(\xi;t)|\leqq\frac{C|t_\beta(\xi)|^{\frac{1}{2}}}{|t|^{\frac{1}{2}}\{(k-\lambda_r(\xi,\sigma(\xi)))^2+\varepsilon^2+|t_\beta(\xi)|^4\}^{3/8}}\,.$$

(iii) If $\mathrm{Im}\,\tau^+_1(\xi;k+i0)\neq0$,

$$|G(\xi;t)|\leqq\frac{C[\mathrm{Im}\,\tau^+_1(\xi;\lambda)]^{\frac{1}{2}}}{|\lambda-\lambda_r(\xi,\sigma(\xi))|}\quad(|t|\geqq4|t_\beta(\xi)|\ \text{or}\ 2|t|\leqq|t_\beta(\xi)|)$$

$$\leqq\frac{C[\mathrm{Im}\,\tau^+_1(\xi;\lambda)]^{\frac{1}{2}}}{|\lambda-\tilde{k}_\beta(\xi)|}\quad\left(\frac{1}{4}|t|\leqq|t_\beta(\xi)|\leqq2|t|\ \text{and}\ \mathrm{r}(t_\beta(\xi))\neq0\right)$$

$$\leqq \frac{C[\operatorname{Im} \tau_1^\dagger(\xi;\lambda)]^{\frac{1}{2}}}{|\lambda-\lambda_r(\xi,\sigma(\xi))|} \quad \left(\frac{1}{4}|t| \leqq |t_\beta(\xi)| \leqq 2|t| \text{ and } \mathrm{r}(t_\beta(\xi))=0\right).$$

(2) When $\tilde{\nu}_1+1 \leqq l \leqq p_0$,

$$|G(\xi;t)| \leqq \frac{C|t_\beta(\xi)|^{\frac{1}{2}}}{\{(k-\lambda_r(\xi,\sigma(\xi)))^2+\varepsilon^2+|t_\beta(\xi)|^4\}^{3/8}}$$

$$(4|t_\beta(\xi)| \leqq |t| \text{ or } |t_\beta(\xi)| \geqq 4|t|)$$

$$\leqq \frac{C|\operatorname{Im} \tilde{k}_\beta(\xi)|^{\frac{1}{4}}}{|\lambda-\tilde{k}_\beta(\xi)|^{3/4}}$$

$$\left(\frac{1}{4}|t| \leqq |t_\beta(\xi)| \leqq 4|t| \text{ and } |\mathrm{r}(t_\beta(\xi))| \geqq \mathrm{d}(t_\beta(\xi))\right)$$

$$\leqq \frac{C|t_\beta(\xi)|^{\frac{1}{2}}}{\{(k-\lambda_r(\xi,\sigma(\xi)))^2+\varepsilon^2+|t_\beta(\xi)|^4\}^{3/8}}$$

$$\left(\frac{1}{4}|t| \leqq |t_\beta(\xi)| \leqq 4|t| \text{ and } |\mathrm{r}(t_\beta(\xi))| < \mathrm{d}(t_\beta(\xi))\right).$$

(3) When $p_0+1 \leqq l \leqq m$,

$$|G(\xi;t)| \leqq \frac{C}{|\lambda-\lambda_r(\xi,\sigma(\xi))|} \quad (4|t| \leqq |t_\beta(\xi)| \text{ or } 4|t_\beta(\xi)| \leqq |t|)$$

$$\leqq \frac{C}{|\lambda-\tilde{k}_\beta(\xi)|} \quad \left(\frac{1}{4}|t| \leqq |t_\beta(\xi)| \leqq 4|t| \text{ and } \mathrm{r}(t_\beta(\xi)) \neq 0\right)$$

$$\leqq \frac{C}{|\lambda-\lambda_r(\xi,\sigma(\xi))|} \quad \left(\frac{1}{4}|t| \leqq |t_\beta(\xi)| \leqq 4|t| \text{ and } \mathrm{r}(t_\beta(\xi))=0\right).$$

Here we have used the fact that if $4|t_\beta(\xi)| \leqq |t|$ or $4|t| \leqq |t_\beta(\xi)|$,

$$|t-t_\beta(\xi)| \geqq \frac{1}{4}\{|t|+|t_\beta(\xi)|\} \text{ and } |t-t_{u_1}(\xi)| \geqq C\{|t|+|t_\beta(\xi)|\}$$

hold. Put

$$\tilde{F}_{lh}^{(1)}(\xi;t)=\tilde{F}_{lh}(\xi;t)-\prod_{j=1}^{b}(t-t_{u_j}(\xi))^{-1}\cdot\frac{\tilde{d}_1(\xi)}{(t-t_\beta(\xi))}.$$

Then it follows from the above estimates that

$$|\tilde{F}_{lh}^{(1)}(\xi;t)|\leqq\frac{C[\operatorname{Im}\tau_{p(l)}^{+}(\xi;\lambda)]^{\frac{1}{2}}}{\varepsilon}+C,\ t\notin\Omega\cup\omega\,.$$

Thus we assume for simplicity that for $1\leqq i\neq j\leqq\beta$ both the relations

$$\frac{1}{\alpha}\mathrm{d}(t_i(\xi))\leqq\mathrm{d}(t_j(\xi))\leqq\alpha\,\mathrm{d}(t_i(\xi)),\ \alpha|t_i(\xi)-t_j(\xi)|\leqq\mathrm{d}(t_i(\xi))+\mathrm{d}(t_j(\xi)),$$

do not hold. Therefore, by (2.33) we obtain for $\tilde{\nu}_1+1\leqq l\leqq p_0$ and $2\leqq j\leqq\beta$

$$\frac{|d_j(\xi)|}{|t-t_j(\xi)|}\leqq\frac{C}{|t|^{1-\theta}|t_j(\xi)|^{\theta}}\quad(2|t|\leqq|t_j(\xi)|\ \text{or}\ 2|t_j(\xi)|\leqq|t|)$$

$$\leqq\frac{C|\operatorname{Im}\tilde{k}_j(\xi)|^{\frac{1}{2}}}{|\lambda-\tilde{k}_j(\xi)|}$$

$$\left(\frac{1}{2}|t|\leqq|t_j(\xi)|\leqq2|t|\ \text{and}\ |\mathrm{r}(t_j(\xi))|\geqq\mathrm{d}(t_j(\xi))\right)$$

$$\leqq\frac{C}{|t|^{1-\theta}|t_j(\xi)|^{\theta}}\quad(|\mathrm{r}(t_j(\xi))|\leqq\mathrm{d}(t_j(\xi)))\,,$$

where $d_j(\xi)$ is defined by (2.34). $\tilde{F}_{lh}(\xi;t)$ can also be rewritten in the form

$$\tilde{F}_{lh}(\xi;t)=\frac{1}{t}\left(\sum_{j=2}^{\beta}\frac{\tilde{d}_j(\xi)}{t-t_j(\xi)}+\tilde{d}_0(\xi)\right)\ \text{for}\ l=1,\dots,\nu_1,p_0+1,\dots,m\,.$$

Then, for $l=1,\dots,\nu_1,\ p_0+1,\dots,m$

$$|\tilde{d}_j(\xi)|=|t_j(\xi)d_j(\xi)|\leqq C[\operatorname{Im}\tau_{p(l)}^{+}(\xi;\ t_j(1)^2+\lambda_r(\xi,\sigma(\xi)))]^{\frac{1}{2}},$$

$$2\leqq j\leqq\beta,$$

$$|\tilde{d}_0(\xi)|\leqq C,$$

where $t_j(1)=\mathrm{r}(t_j(\xi))+e^{\frac{\pi}{4}i}\,\mathrm{d}(t_j(\xi))$. Let us estimate $\frac{\tilde{d}_j(\xi)}{t(t-t_j(\xi))}$, $2\leqq j\leqq\beta$, for $(\xi,\lambda)\in\hat{W}_0\times\Lambda_\delta^{+}(k^0)$ $(t\notin\Omega\cup\omega)$ and $l=1,\dots,\nu_1,\ p_0+1,\dots,m$.

(1) Let $1\leqq l\leqq\tilde{\nu}_1$.

(i) If $\operatorname{Im}\tau_1^+(\xi;\, k+i0)=0$ and $\operatorname{Im}\tau_1^+(\xi;\, \mathrm{r}(t_j(\xi))^2+\lambda_r(\xi,\, \sigma(\xi))+i0)=0$,

$$\left|\frac{\tilde{d}_j(\xi)}{t(t-t_j(\xi))}\right| \leqq \frac{C|\operatorname{Im}\tilde{k}_j(\xi)|^{\frac{1}{2}}}{|t|^{\frac{1}{2}}|\lambda-\tilde{k}_j(\xi)|}$$

$$\left(\frac{1}{2}|t| \leqq |t_j(\xi)| \leqq 2|t| \text{ and } |\mathrm{r}(t_j(\xi))| \geqq \mathrm{d}(t_j(\xi))\right)$$

$$\leqq \frac{C}{|t|^{\frac{1}{2}}|t|^{1-\theta}|t_j(\xi)|^{\theta}} \quad \text{(otherwise)}.$$

(ii) If $\operatorname{Im}\tau_1^+(\xi;\, k+i0)=0$ and $\operatorname{Im}\tau_1^+(\xi;\, \mathrm{r}(t_j(\xi))^2+\lambda_r(\xi,\, \sigma(\xi))+i0)\neq 0$,

$$\left|\frac{\tilde{d}_j(\xi)}{t(t-t_j(\xi))}\right| \leqq \frac{C}{|t|^{\frac{1}{2}}|t|^{1-\theta}|t_j(\xi)|^{\theta}}.$$

(iii) If $\operatorname{Im}\tau_1^+(\xi;\, k+i0)\neq 0$,

$$\left|\frac{\tilde{d}_j(\xi)}{t(t-t_j(\xi))}\right| \leqq \frac{C[\operatorname{Im}\tau_1^+(\xi;\, \lambda)]^{\frac{1}{2}}}{|\lambda-\tilde{k}_j(\xi)|}$$

$$\left(\frac{1}{2}|t| \leqq |t_j(\xi)| \leqq 2|t| \text{ and } \mathrm{r}(t_j(\xi))\neq 0\right)$$

$$\leqq \frac{[C\operatorname{Im}\tau_1^+(\xi;\, \lambda)]^{\frac{1}{2}}}{|\lambda-\lambda_r(\xi;\, \sigma(\xi))|} \quad \text{(otherwise)}.$$

(2) When $p_0+1\leqq l\leqq m$,

$$\left|\frac{\tilde{d}_j(\xi)}{t(t-t_j(\xi))}\right| \leqq \frac{C}{|\lambda-\tilde{k}_j(\xi)|}$$

$$\left(\frac{1}{2}|t| \leqq |t_j(\xi)| \leqq 2|t| \text{ and } \mathrm{r}(t_j(\xi))\neq 0\right)$$

$$\leqq \frac{C}{|\lambda-\lambda_r(\xi,\, \sigma(\xi))|} \quad \text{(otherwise)}.$$

Finally we prove that for $\tilde{\nu}_1+1\leqq l\leqq p_0$ $t=t_1(\xi)=0$ is not a pole of $C_{lh}(\xi;\, t)$. Note that

$$e_h=\sum_{j=1}^{m} C_{jh}(\xi;\, \lambda)Bh_j^+(\xi;\, \lambda), \qquad \operatorname{Im}\lambda>0\ (t\notin\Omega\cup\omega). \tag{2.4}$$

Put

$J=\{j;\ 1\leqq j\leqq m$ and $C_{jh}(\xi;\ t^2+\lambda_r(\xi,\ \sigma(\xi)))$ has a simple pole at $t=0\}$,

$J'=\{j;\ 1\leqq j\leqq m$ and $C_{jh}(\xi;\ t^2+\lambda_r(\xi,\ \sigma(\xi)))$ has a pole of order 2 at $t=0\}$.

Then it is clear that for $j\in\{1,\dots,m\}\backslash\{J\cup J'\}$ $t=0$ is not a pole of $C_{jh}(\xi;\ t^2+\lambda_r(\xi,\ \sigma(\xi)))$ and that $J'\subset\{p_0+1,\dots,m\}$. Put

$$tC_{jh}(\xi;\ t^2+\lambda_r(\xi,\ \sigma(\xi)))=\tilde{C}_{jh}(\xi;\ t),\qquad j\in J,$$

$$t^2C_{jh}(\xi;\ t^2+\lambda_r(\xi,\ \sigma(\xi)))=\tilde{C}^0_{jh}(\xi)+t\tilde{C}^1_{jh}(\xi;\ t),\qquad j\in J',$$

where $\tilde{C}_{jh}(\xi;\ t)$ and $\tilde{C}^1_{jh}(\xi;\ t)$ are analytic at $t=0$ and $\tilde{C}_{jh}(\xi;\ 0)$, $\tilde{C}^0_{jh}(\xi)\neq 0$. Multiplying (2.4) by t^2 and making t tend to zero ($t\notin\Omega\cup\omega$), we obtain

$$0=\sum_{j\in J'}\tilde{C}^0_{jh}(\xi)Bh^+_j(\xi;\ \lambda_r(\xi,\ \sigma(\xi))+i0).$$

For $j\in J'\subset\{p_0+1,\dots,m\}$ $h^+_j(\xi;\ \lambda)$ is analytic in λ. Therefore we can put

$$\sum_{j\in J'}\tilde{C}^0_{jh}(\xi)Bh^+_j(\xi;\ \lambda)=(\lambda-\lambda_r(\xi,\ \sigma(\xi)))v(\xi;\ \lambda)=t^2v(\xi;\ \lambda),$$

where $v(\xi;\ \lambda)$ is analytic in λ. Thus multiplying (2.4) by t and making t tend to zero, we have

$$0=\sum_{j\in J}\tilde{C}_{jh}(\xi;\ 0)Bh^+_j(\xi;\ \lambda_r(\xi,\ \sigma(\xi))+i0)$$
$$+\sum_{j\in J'}\tilde{C}^1_{jh}(\xi;\ 0)Bh^+_j(\xi;\ \lambda_r(\xi,\ \sigma(\xi))+i0).$$

It follows from Lemma 2.3 that $\tilde{C}_{jh}(\xi;\ 0)=0$ for $j\in J\cap\{\tilde{\nu}_1+1,\dots,p_0\}$. This implies that $J\cap\{\tilde{\nu}_1+1,\dots,p_0\}=\phi$, that is, $C_{jh}(\xi;\ t)$, $\tilde{\nu}_1+1\leqq j\leqq p_0$, are analytic at $t=0$. Thus for $\tilde{\nu}_1+1\leqq l\leqq p_0$ $\tilde{F}_{lh}(\xi;\ t)=\sum_{j=2}^{\beta}\frac{d_j(\xi)}{t-t_j(\xi)}$.

Q.E.D.

§3. Green Function $G(x, y; \lambda)$ of the Operator $A-\lambda I$

By the hyperbolicity of $L=I\frac{\partial}{\partial t}-iA$, the matrix $A(\eta)-\lambda I$ is non-singular for every non-real λ. Therefore $(A(\eta)-\lambda I)^{-1}$ has the conjugate Fourier transform with respect to η

$$(3.1)\qquad E(x; \lambda)=(2\pi)^{-\frac{n}{2}}\bar{\mathscr{F}}_{\eta}[(A(\eta)-\lambda I)^{-1}](x), \qquad \operatorname{Im}\lambda\neq 0,$$

in the distribution sense. $E(x; \lambda)$ is a fundamental solution in $\mathbf{R}^n$ of the differential operator $A-\lambda I$ with non-real λ, i.e., $E(x; \lambda)$ satisfies the equation

$$(3.2)\qquad (A-\lambda I)E(x; \lambda)=\delta(x)I.$$

It is well known that $E(x; \lambda)$ is analytic in $(\mathbf{R}^n\backslash\{0\})\times(\mathbf{C}\backslash\mathbf{R})$. From (3.1) and the relation $(A(\eta)-\lambda I)^{-1}=(\tau I-M(\xi; \lambda))^{-1}A_n^{-1}$, $\eta=(\xi, \tau)$, we have

$$(3.3)\qquad \mathscr{F}_{x'}[E(x-y; \lambda)|_{x_n=0}]$$
$$=(2\pi)^{-\frac{n+1}{2}}e^{-iy'\cdot\xi}\int_{-\infty}^{\infty}e^{-iy_n\tau}(\tau I-M(\xi; \lambda))^{-1}A_n^{-1}d\tau.$$

Consider the first order system of ordinary differential equations depending on parameters (ξ, λ)

$$(3.4)\quad \left(\frac{1}{i}\frac{d}{dx_n}-M(\xi; \lambda)\right)\tilde{E}_c(\xi, x_n, y; \lambda)=0,\ x_n>0,\ y\in\mathbf{R}_+^n,\ \xi\in\Xi^{n-1},$$

and the condition

$$(3.5)\qquad B\tilde{E}_c(\xi, 0, y; \lambda)=\mathscr{F}_{x'}[BE(x-y; \lambda)|_{x_n=0}].$$

Under the assumptions that L is hyperbolic and B is minimally conservative, there exists, by Lemmas 2.1 and 2.2, a unique solution $\tilde{E}_c(\xi, x_n, y; \lambda)$ of (3.4) satisfying (3.5) which belongs to $L^2(0, \infty)$ in x_n and $\tilde{E}_c(\xi, x_n, y; \lambda)$ has the conjugate Fourier transform $E_c(x', x_n, y; \lambda)$ with respect to ξ. Define for non-real λ

$$(3.6)\qquad G(x, y; \lambda)=E(x-y; \lambda)-E_c(x, y; \lambda).$$

Then $G(x, y; \lambda)$ satisfies the equation

$$(3.7) \qquad (A_x - \lambda I)G(x, y; \lambda) = \delta(x-y)I, \qquad x, y \in \mathbf{R}_+^n, \quad \operatorname{Im}\lambda \neq 0$$

and

$$(3.8) \qquad BG(x, y; \lambda)|_{x_n=0} = 0.$$

Moreover for every $g \in C_0^\infty(\mathbf{R}_+^n)$ $v(x; \lambda) = (\mathbf{A} - \lambda \mathbf{I})^{-1} g(x)$ is given by

$$(3.9) \qquad v(x; \lambda) = \int_{\mathbf{R}_+^n} G(x, y; \lambda) g(y) dy, \qquad x \in \mathbf{R}_+^n, \quad \operatorname{Im}\lambda \neq 0.$$

We call $G(x, y; \lambda)$ Green function of the operator $\mathbf{A} - \lambda \mathbf{I}$ (or the system $\{A - \lambda I, B\}$). From the self-adjointness of $\mathbf{A}$ it follows that

$$(3.10) \qquad G(x, y; \lambda)^* = G(y, x; \bar{\lambda}),$$

where S^* denotes the Hermitian adjoint of a matrix S. More precisely we have

$$(3.11) \qquad E(x-y; \lambda)^* = E(y-x; \bar{\lambda}),$$

$$(3.12) \qquad E_c(x, y; \lambda)^* = E_c(y, x; \bar{\lambda}).$$

Let us find a local but more explicit representation of $\mathscr{F}_{x'}[G(x, y; \lambda)]$. Let ξ^0 be a point of Ξ^{n-1}, k^0 a non-zero real, W a small neighborhood of ξ^0 and $\Lambda_\delta^\pm(k^0)$ the regions defined in §1. Then it suffices to consider the case when the roots of $Q(\lambda, \xi, \tau) = 0$ in τ for $(\xi, \lambda) \in W \times \Lambda_\delta^\pm(k^0)$ are in the situation (1.21) and (1.21)'. In fact, in other cases we obtain corresponding representations by obvious modifications. First we consider $\mathscr{F}_{x'}[E(x-y; \lambda)|_{x_n=0}]$. From (3.3) we have

$$(3.13) \qquad \mathscr{F}_{x'}[E(x-y; \lambda)|_{x_n=0}]$$
$$= -(2\pi)^{-\frac{n+1}{2}} e^{-iy'\cdot\xi} \sum_{j=1}^{p} \int_{|\tau - \tau_j^-(\xi; \lambda)| = \delta_j} e^{-iy_n\tau} (\tau I - M(\xi; \lambda))^{-1} A_n^{-1} d\tau$$
$$- (2\pi)^{-\frac{n+1}{2}} e^{-iy'\cdot\xi} \int_{\gamma_-} e^{-iy_n\tau} (\tau I - M(\xi; \lambda))^{-1} A_n^{-1} d\tau,$$

where γ_- is a simple closed curve in the lower half-plane enclosing only the eigenvalues $\tau_{p+1}^-(\xi; \lambda), \ldots, \tau_\rho^-(\xi; \lambda)$, $(\xi, \lambda) \in W \times \overline{\Lambda_\delta^+(k^0)}\,(W \times \overline{\Lambda_\delta^-(k^0)})$.

Now,

$$\frac{1}{2\pi i}\int_{|\tau-\tau_j^-(\xi;\lambda)|=\delta_j} e^{-iy_n\tau}(\tau I-M(\xi;\lambda))^{-1}A_n^{-1}d\tau$$

$$=-\frac{1}{2\pi i}\int_{|\tau-\tau_j^-(\xi;\lambda)|=\delta_j} e^{-iy_n\tau}\sum_{\mu=1}^{2\rho}\frac{1}{\lambda-\lambda_\mu(\xi,\tau)}P_\mu(\xi,\tau)d\tau$$

$$=\frac{e^{-i\tau_j^-(\xi;\lambda)y_n}}{\frac{\partial\lambda_{\pi(j)}}{\partial\tau}(\xi,\tau_j^-(\xi;\lambda))}P_{\pi(j)}(\xi,\tau_j^-(\xi;\lambda)).$$

Put

$$(3.14)\quad (q_{j1}(\xi;\lambda),\cdots,q_{j2m}(\xi;\lambda))=\frac{1}{\frac{\partial\lambda_{\pi(j)}}{\partial\tau}(\xi,\tau_j^-(\xi;\lambda))}P_{\pi(j)}(\xi,\tau_j^-(\xi;\lambda)).$$

Then the column vectors $q_{j1}(\xi;\lambda),\dots,q_{j2m}(\xi;\lambda)$ are eigenvectors corresponding to the eigenvalue $\tau_j^-(\xi;\lambda)$ of $M(\xi;\lambda)$. Note that the $h_{jk}^-(\xi;\lambda)$, $1\leqq k\leqq\tilde{\nu}_j$, defined in §1 are $\tilde{\nu}_j$ linearly independent column vectors of the matrix $P_{\pi(j)}(\xi,\tau_j^-(\xi;\lambda))$. We also put

$$(3.15)\quad (q_1(y_n,\xi;\lambda),\dots,q_{2m}(y_n,\xi;\lambda))$$
$$=\frac{1}{2\pi i}\int_{\gamma_-}e^{-iy_n\tau}(\tau I-M(\xi;\lambda))^{-1}A_n^{-1}d\tau.$$

The column vectors $q_1(y_n,\xi;\lambda),\dots,q_{2m}(y_n,\xi;\lambda)$ belong to the subspace generated by the root vectors corresponding to the eigenvalues $\tau_{p+1}^-(\xi;\lambda),\dots,\tau_\rho^-(\xi;\lambda)$ and therefore they are represented as linear combinations of $h_{p_0+1}^-(\xi;\lambda),\dots,h_m^-(\xi;\lambda)$. From (3.13), (3.14) and (3.15) we have

$$(3.16)\quad \mathscr{F}_{x'}[BE(x-y;\lambda)|_{x_n=0}]=-i(2\pi)^{-\frac{n-1}{2}}e^{-iy'\cdot\xi}\sum_{j=1}^{p}e^{-i\tau_j^-(\xi;\lambda)y_n}$$
$$\times(Bq_{j1}(\xi;\lambda),\dots,Bq_{j2m}(\xi;\lambda))$$
$$-i(2\pi)^{-\frac{n-1}{2}}e^{-iy'\cdot\xi}(Bq_1(y_n,\xi;\lambda),\dots,Bq_{2m}(y_n,\xi;\lambda)).$$

Next we find a local representation $\tilde{E}_c(\xi,x_n,y;\lambda)=\mathscr{F}_{x'}[E_c(x,y;\lambda)]$. Denote by $U_j(\xi,x_n,y;\lambda)$ the j-th column vector of $\tilde{E}_c(\xi,x_n,y;\lambda)$. By Lemmas 2.1 and 2.2, $U_j(\xi,+0,y;\lambda)\in E^+(\xi;\lambda)$. Hence it can be written in the form

$$(3.17)\qquad U_j(\xi, +0, y; \lambda)=\sum_{l=1}^{m} C_j^l h_l^+(\xi; \lambda).$$

Since $U_j(\xi, x_n, y; \lambda)$ is given by the formula

$$(3.18)\quad U_j(\xi, x_n, y; \lambda)=\frac{1}{2\pi i}\int_{\Gamma^+} e^{i\tau x_n}(\tau I-M(\xi; \lambda))^{-1}U_j(\xi, 0, y; \lambda)d\tau ,$$

where $\Gamma^+=\Gamma^+(\xi; \lambda)$ is a positively oriented simple closed curve in the upper half-plane, enclosing the eigenvalues $\tau_1^+(\xi; \lambda),\dots, \tau_p^+(\xi; \lambda)$, we have

$$(3.19)\qquad U_j(\xi, x_n, y; \lambda)=\sum_{l=1}^{p_0} C_j^l e^{i\tau^+_{p(l)}(\xi;\lambda)x_n} h_l^+(\xi; \lambda)$$

$$+\frac{1}{2\pi i}\sum_{l=p_0+1}^{m} C_j^l\Big(\int_{\gamma_+} e^{ix_n\tau}(\tau I-M(\xi; \lambda))^{-1}d\tau\Big)h_l^+(\xi; \lambda).$$

Let us determine the coefficients C_j^l. From (3.5), (3.16) and (3.17) we have

$$(3.20)\quad C_j^l(y, \xi; \lambda)=-i(2\pi)^{-\frac{n-1}{2}}e^{-iy'\cdot\xi}\sum_{\nu=1}^{p}\frac{e^{-i\tau_\nu^-(\xi;\lambda)y_n}}{\Delta(\xi; \lambda)}\det(Bh_1^+(\xi; \lambda),$$

$$\dots, \overset{l}{\underset{\cup}{B}}q_{\nu j}(\xi; \lambda),\dots, Bh_m^+(\xi; \lambda))$$

$$-i(2\pi)^{-\frac{n-1}{2}}\frac{e^{-iy'\cdot\xi}}{\Delta(\xi; \lambda)}\det(Bh_1^+(\xi; \lambda),\dots, \overset{l}{\underset{\cup}{B}}q_j(y_n, \xi; \lambda), \dots, Bh_m^+(\xi; \lambda)).$$

We extend $G(x, y; \lambda)$ over $\mathbf{R}^n$ with respect to y by defining $G(x, y; \lambda)=0$ for $x\in\mathbf{R}^n_+$ and $y_n\notin\mathbf{R}^n_+$. Then we have

Lemma 3.1. *Let λ be non-real. Then*

$$(3.21)\qquad \mathscr{F}_y[G(x, y; \lambda)](\eta)(A(\eta)-\lambda I)$$

$$=(2\pi)^{-\frac{n}{2}}e^{ix\cdot\eta}I-\frac{1}{i}(2\pi)^{-\frac{1}{2}}\mathscr{F}_{y'}[G(x, y', +0; \lambda)](\xi)A_n.$$

Proof. Define

$$G_1(x, y; \lambda) = \begin{cases} G(x, y; \lambda), & x \in \mathbf{R}_+^n \text{ and } y \in \mathbf{R}_+^n, \\ 0 \quad , & x \notin \mathbf{R}_+^n \text{ and } y \in \mathbf{R}_+^n. \end{cases}$$

Then we have

$$\text{(3.22)} \qquad G(x, y; \lambda)^* = G_1(y, x; \bar{\lambda}), \qquad x \in \mathbf{R}_+^n, \quad y \in \mathbf{R}^n,$$

and

$$\text{(3.23)} \qquad (A_y - \bar{\lambda} I) G_1(y, x; \bar{\lambda}) = \delta(y-x) I$$
$$+ \frac{1}{i} A_n G(y', +0, x; \bar{\lambda}) \delta(y_n) I, \; x \in \mathbf{R}_+^n, \; y \in \mathbf{R}^n,$$

in the distribution sense. Since every term of (3.23) is a temperate distribution in y, we take the Fourier transforms of both sides of (3.23) with respect to y.

$$\text{(3.24)} \qquad (A(\eta) - \bar{\lambda} I) \mathscr{F}_y [G_1(y, x; \bar{\lambda})](\eta)$$
$$= (2\pi)^{-\frac{n}{2}} e^{-ix \cdot \eta} I + \frac{1}{i} (2\pi)^{-\frac{1}{2}} A_n \mathscr{F}_{y'} [G(y', +0, x; \bar{\lambda})](\xi).$$

From (3.22) and (3.24), (3.21) follows. Q.E.D.

Now let us give a representation of $\bar{\mathscr{F}}_{y'}[G(x, y', +0; \lambda)](\xi)$. From the formula

$$\text{(3.25)} \quad E(x-y; \lambda) = (2\pi)^{-\frac{n}{2}} \mathscr{F}_\eta [e^{ix \cdot \eta} (A(\eta) - \lambda I)^{-1}](y), \qquad \operatorname{Im} \lambda \neq 0,$$

we have

$$\text{(3.26)} \qquad \bar{\mathscr{F}}_{y'} [E(x-y; \lambda)|_{y_n = 0}]$$
$$= (2\pi)^{-\frac{n+1}{2}} e^{ix' \cdot \xi} \int_{-\infty}^{\infty} e^{ix_n \tau} (\tau I - M(\xi; \lambda))^{-1} A_n^{-1} d\tau.$$

Therefore if $(\xi, \lambda) \in W \times \Lambda_\delta^\pm(k^0)$,

$$(3.27)\qquad \bar{\mathscr{F}}_{y'}[E(x-y;\lambda)|_{y_n=0}]$$

$$=(2\pi)^{-\frac{n+1}{2}}e^{ix'\cdot\xi}\sum_{j=1}^{p}\int_{|\tau-\tau_j^+(\xi;\lambda)|=\delta_j}e^{ix_n\tau}(\tau I-M(\xi;\lambda))^{-1}A_n^{-1}d\tau$$

$$+(2\pi)^{-\frac{n+1}{2}}e^{ix'\cdot\xi}\int_{\gamma_+}e^{ix_n\tau}(\tau I-M(\xi;\lambda))^{-1}A_n^{-1}d\tau,$$

where γ_+ is the simple closed curve in the upper half-plane defined in §1 and $\delta_j\leqq\frac{1}{2}\operatorname{Im}\tau_j^+(\xi;\lambda)$. Next consider $\bar{\mathscr{F}}_{y'}[E_c(x,y',+0;\lambda)]$. From (3.12) we have

$$(3.28)\qquad \bar{\mathscr{F}}_{y'}[E_c(x,y',+0;\lambda)]=\{\mathscr{F}_{y'}[E_c(y',+0,x;\bar{\lambda})]\}^*.$$

From (3.17) and (3.20), we obtain the following representation of $\bar{\mathscr{F}}_{y'}[E_c(x,y',+0;\lambda)]$:

$$(3.29)\qquad \bar{\mathscr{F}}_{y'}[E_c(x,y',+0;\lambda)]=(U_1(\xi,+0,x;\bar{\lambda}),\dots,U_{2m}(\xi,+0,x;\bar{\lambda}))^*,$$

$$(3.30)\qquad U_j(\xi,+0,x;\bar{\lambda})=\sum_{l=1}^{m}C_j^l h_l^+(\xi;\bar{\lambda}),$$

$$(3.31)\qquad C_j^l(x,\xi;\bar{\lambda})=-i(2\pi)^{-\frac{n-1}{2}}e^{-ix'\cdot\xi}\sum_{\nu=1}^{p}\frac{e^{-i\tau_\nu^-(\xi;\bar{\lambda})x_n}}{\Delta(\xi;\bar{\lambda})}$$

$$\times\det(Bh_1^+(\xi;\bar{\lambda}),\dots,\overset{l}{\underset{\cup}{B}}q_{\nu j}(\xi;\bar{\lambda}),\dots,Bh_m^+(\xi;\bar{\lambda}))$$

$$-i(2\pi)^{-\frac{n-1}{2}}\frac{e^{-ix'\cdot\xi}}{\Delta(\xi;\bar{\lambda})}\det(Bh_1^+(\xi;\bar{\lambda}),\dots,\overset{l}{\underset{\cup}{B}}q_j(x_n,\xi;\bar{\lambda}),\dots,Bh_m^+(\xi;\bar{\lambda})).$$

§4. Eigenfunctions for the Operator A

We defined $\Psi_j(x,\eta;\lambda)$, $1\leqq j\leqq 2\rho$, by (0.23):

$$\Psi_j(x,\eta;\lambda)=\bar{\mathscr{F}}_y[G(x,y;\lambda)](\eta)(\lambda_j(\eta)-\lambda)P_j(\eta).$$

The projection $P_j(\eta)$ are represented as

$$(4.1)\qquad P_j(\eta)=\begin{cases}\dfrac{1}{2\pi i}\displaystyle\int_{|\lambda-\lambda_j(\eta)|=\delta}(\lambda I-A(\eta))^{-1}d\lambda, & \eta\neq 0,\\ 0\quad, & \eta=0,\end{cases}$$

where δ is chosen sufficiently small such that the set $\{\lambda; |\lambda-\lambda_j(\eta)|\leqq\delta\}$ contains no roots of $Q(\lambda, \eta)=0$ except $\lambda_j(\eta)$.

Lemma 4.1. *Let λ be non-real. Then*

$$(A_x-\lambda I)\Psi_j(x, \eta; \lambda)=(2\pi)^{-\frac{n}{2}}e^{ix\cdot\eta}(\lambda_j(\eta)-\lambda)P_j(\eta), \tag{4.2}$$

$$B\Psi_j(x, \eta; \lambda)|_{x_n=+0}=0 \tag{4.3}$$

hold for $x\in\mathbf{R}^n_+$, $\eta\in\Xi^n$ and $1\leqq j\leqq 2\rho$.

Proof. Let $\phi=(\phi_\iota\delta_{ik})\in C_0^\infty(\mathbf{R}^n_+)$ and $\psi\in C_0^\infty(\Xi^n)$. Then

$$\begin{aligned}
&\langle(A_x-\lambda I)\bar{\mathscr{F}}_y[G(x, y; \lambda)], \phi(x)\psi(\eta)\rangle_{x,\eta}\\
&=\langle G(x, y; \lambda), {}^t(-A_x-\lambda I)\phi(x)\bar{\mathscr{F}}[\psi](y)\rangle_{x,y}{}^{7)}\\
&=\langle{}^t\langle G(x, y; \lambda), {}^t(-A_x-\lambda I)\phi(x)\rangle_x, \bar{\mathscr{F}}[\psi](y)\rangle_y\\
&=\langle{}^t\langle\delta(x-y)I, \phi(x)\rangle_x, \bar{\mathscr{F}}[\psi](y)\rangle_y\\
&=\int_{\mathbf{R}^n}\tilde{\phi}(y)\bar{\mathscr{F}}[\psi](y)dy=\int_{\mathbf{R}^n_+}\phi(x)\bar{\mathscr{F}}[\psi](x)dx\\
&=\int_{\mathbf{R}^n_+}dx\int_{\Xi^n}d\eta(2\pi)^{-\frac{n}{2}}e^{ix\cdot\eta}\phi(x)\psi(\eta)\\
&=\langle(2\pi)^{-\frac{n}{2}}e^{ix\cdot\eta}I, \phi(x)\psi(\eta)\rangle_{x,\eta},
\end{aligned}$$

where $\tilde{\phi}(x)=\phi(x)$ for $x\in\mathbf{R}^n_+$, and $=0$ for $x\notin\mathbf{R}^n_+$. This implies (4.2). The equation (4.3) is obvious. Q.E.D.

From Lemma 3.1 it follows that

$$\Psi_j(x, \eta; \lambda)=(2\pi)^{-\frac{n}{2}}e^{ix\cdot\eta}P_j(\eta)-\frac{1}{i}(2\pi)^{-\frac{1}{2}}\bar{\mathscr{F}}_{y'}[G(x, y', +0; \lambda)](\xi)A_nP_j(\eta), \quad \mathrm{Im}\,\lambda\neq 0, \quad 1\leqq j\leqq 2\rho. \tag{4.4}$$

Put

7) ${}^tS$ denotes the transposed matrix of S.

(4.5) $$N_{j\nu}=\{\eta\in\Xi^n;\ \xi\in N \ or\ \xi\in D_\nu \ and\ k_\nu(\xi)=\lambda_j(\eta)\},\quad 1\leqq j\leqq 2\rho,\ 1\leqq\nu\leqq s,$$

(4.6) $$N_j=\bigcup_{\nu=1}^{s} N_{j\nu}\cup\tilde{N}_j,\qquad 1\leqq j\leqq 2\rho.$$

The $N_{j\nu}$ and N_j are null sets of Ξ^n. By the local representation of $\mathscr{F}_{y'}[G(x, y', +0; \lambda)](\xi)$ in §3 and Lemma 4.1 the limits $\Psi_j^{\pm}(x, \eta)\equiv\Psi_j(x, \eta; \lambda_j(\eta)\pm i0)$ exist and satisfy (0.24) and (0.25) for $x\in\mathbf{R}_+^n$ and $\eta\notin N_j$:

$$A_x\Psi_j^{\pm}(x, \eta)=\lambda_j(\eta)\Psi_j^{\pm}(x, \eta),\quad B\Psi_j^{\pm}(x, \eta)|_{x_n=0}=0,\qquad 1\leqq j\leqq 2\rho.$$

Next define for $x\in\mathbf{R}_+^n$, $\xi\in D_\nu$ and non-real λ

(4.7) $$\Psi_{j+2\nu\rho}(x, \eta; \lambda)=\frac{\lambda-k_\nu(\xi)}{\lambda-\lambda_j(\eta)}\Psi_j(x, \eta; \lambda),\ 1\leqq j\leqq 2\rho,\ 1\leqq\nu\leqq s.$$

We denote by $\hat{N}_{j\nu}$ and $D_{j\nu}$ the sets

(4.8) $$\{\eta\in\Xi^n;\ \xi\in\hat{N} \text{ or } \xi\in D_\nu \text{ and } k_\nu(\xi)=\lambda_j(\eta)\},$$

(4.9) $$\{\eta\in\Xi^n;\ \xi\in D_\nu,\ \eta\notin\hat{N}_{j\nu}\},$$

respectively. Then we define new eigenfunctions corresponding to boundary waves by

(4.10) $$\Psi_{j+2\nu\rho}^{\pm}(x, \eta)=\Psi_{j+2\nu\rho}(x, \eta; k_\nu(\xi)\pm i0),\qquad \eta\in D_{j\nu}.$$

The validity of the above definitions follows from the estimates for the $\Psi_{j+2\nu\rho}(x, \eta; \lambda)$ which can be derived from Lemmas 2.13 and 2.14. We also note that the $\hat{N}_{j\nu}$ are null sets of Ξ^n. Moreover from Lemma 4.1 it follows that

(4.11) $$A_x\Psi_{j+2\nu\rho}^{\pm}(x, \eta)=k_\nu(\xi)\Psi_{j+2\nu\rho}^{\pm}(x, \eta),$$

(4.12) $$B\Psi_{j+2\nu\rho}^{\pm}(x, \eta)|_{x_n=0}=0,\qquad \text{for}\quad x\in\mathbf{R}_+^n,\quad \eta\in D_{j\nu}.$$

§5. Construction of the Spectral Family

The self-adjoint operator $\mathbf{A}$ admits a uniquely determined spectral resolution:

$$\mathbf{A}=\int_{-\infty}^{\infty}\lambda dE(\lambda), \tag{5.1}$$

where $\{E(\lambda)\}_{-\infty<\lambda<\infty}$ denotes the right-continuous spectral family of $\mathbf{A}$. Put

$$R(\lambda)=(\mathbf{A}-\lambda)^{-1}, \qquad \mathrm{Im}\,\lambda\neq 0. \tag{5.2}$$

Then we have

$$\begin{aligned}&\frac{E(b)+E(b-0)}{2}-\frac{E(a)+E(a-0)}{2} \\ &=s-\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_a^b[R(k+i\varepsilon)-R(k-i\varepsilon)]dk,\ b>a\end{aligned} \tag{5.3}$$

(see, e.g., [14]). From (5.3) we obtain the following

Lemma 5.1. *Let $f\in C_0^\infty(\mathbf{R}_+^n)$ and $b>a$. Then we have*

$$\begin{aligned}&\left(\left\{\frac{E(b)+E(b-0)}{2}-\frac{E(a)+E(a-0)}{2}\right\}f,f\right) \\ &=\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\sum_{j=1}^{2\rho}\int_a^b dk\int_{\Xi^n}d\eta\frac{\varepsilon}{(\lambda_j(\eta)-k)^2+\varepsilon^2}|f_j(\eta;\,k\pm i\varepsilon)|^2 \\ &=\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\sum_{j=1}^{2\rho}\int_{\Xi^n}d\eta\int_a^b dk\frac{\varepsilon}{(\lambda_j(\eta)-k)^2+\varepsilon^2}|f_j(\eta;\,k\pm i\varepsilon)|^2,\end{aligned} \tag{5.4}$$

where $(\cdot,\cdot)$ denotes the inner product of $L^2(\mathbf{R}_+^n)$ and

$$\hat{f}_j(\eta;\,\lambda)=\int_{\mathbf{R}_+^n}\Psi_j(x,\,\eta;\,\lambda)^*f(x)dx, \qquad \mathrm{Im}\,\lambda\neq 0, \quad 1\leqq j\leqq 2\rho. \tag{5.5}$$

Proof. Let $h(x)\in C_0^\infty(\mathbf{R}^n)$ and $\mathrm{Im}\,\lambda\neq 0$, and let $\tilde{h}(x)$ denote the restriction to $\mathbf{R}_+^n$ of $h(x)$. Then for $x\in\mathbf{R}_+^n$

$$\begin{aligned}&[R(\lambda)\tilde{h}](x)=\int_{\mathbf{R}_+^n}G(x,\,y;\,\lambda)\tilde{h}(y)dy=\langle{}^tG(x,\,y;\,\lambda),\,h(y)_y\rangle \\ &=\langle\mathscr{F}_y{}^tG(x,\,y;\,\lambda),\,\bar{\mathscr{F}}_y h(y)\rangle_y=\sum_{j=1}^{2\rho}\langle\frac{1}{\lambda_j(\eta)-\lambda}{}^t\Psi_j(x,\,\eta;\,\lambda),\,\hat{h}(\eta)\rangle_\eta\,.\end{aligned}$$

Since $[R(\lambda)\tilde{h}](x)\in L^2(\mathbf{R}_+^n)$ and $f\in C_0^\infty(\mathbf{R}_+^n)$, we have

$$(5.6)\qquad (R(\lambda)\tilde{h}, f)=(\tilde{h}(x), [R(\bar{\lambda})f](x))=(\hat{h}(\eta), \mathscr{F}[R(\bar{\lambda})f](\eta))_0,$$

$$(R(\lambda)\tilde{h}, f)=\sum_{j=1}^{2\rho}\int_{\mathbf{R}_+^n}dx\left(\int_{\Xi^n}\frac{1}{\lambda_j(\eta)-\lambda}\Psi_j(x, \eta; \lambda)\hat{h}(\eta)d\eta\right)\cdot\overline{f(x)},$$

where $(\cdot, \cdot)_0$ denotes the inner product of $L^2(\Xi^n)$. When λ are fixed, $\mathrm{Im}\,\lambda\neq 0$, there exist a positive constant C and a non-negative integer α such that for $x\in\mathbf{R}_+^n$ and $\eta\in\Xi^n$

$$(5.7)\qquad |\Psi_j(x, \eta; \lambda)|\leqq C(1+|\eta|)^\alpha.$$

This follows from the representations of the $\Psi_j(x, \eta; \lambda)$ which have been obtained in §§3 and 4. So we can apply Fubini's theorem to (5.6) and obtain

$$(R(\lambda)\tilde{h}, f)=\sum_{j=1}^{2\rho}\int_{\Xi^n}d\eta\frac{1}{\lambda_j(\eta)-\lambda}\hat{h}(\eta)\cdot\overline{\int_{\mathbf{R}_+^n}\Psi_j(x, \eta;\lambda)^*f(x)dx}$$

$$=\sum_{j=1}^{2\rho}\int_{\Xi^n}\frac{1}{\lambda_j(\eta)-\lambda}\hat{h}(\eta)\cdot\overline{\hat{f}_j(\eta; \lambda)}d\eta.$$

Thus we have

$$\mathscr{F}R(\bar{\lambda})f=\sum_{j=1}^{2\rho}\frac{1}{\lambda_j(\eta)-\lambda}\hat{f}_j(\eta; \lambda)\in L^2(\Xi^n).$$

Here $\mathscr{F}R(\bar{\lambda})f$ is the Fourier transform of $[R(\bar{\lambda})f](x)$ extended as $[R(\bar{\lambda})f](x)=0$ for $x\notin\mathbf{R}_+^n$. From (0.23) and (5.5) it follows that

$$P_j(\eta)\mathscr{F}[R(\bar{\lambda})f](\eta)=\frac{1}{\lambda_j(\eta)-\lambda}\hat{f}_j(\eta;\lambda)\in L^2(\Xi^n).$$

Using $P_j(\eta)P_k(\eta)=\delta_{jk}P_j(\eta)$, the resolvent equation and the above results we have

$$(\{R(k+i\varepsilon)-R(k-i\varepsilon)\}f, f)=\sum_{j=1}^{2\rho}\int_{\Xi^n}\frac{2i\varepsilon}{(\lambda_j(\eta)-k)^2+\varepsilon^2}|\hat{f}_j(\eta; k\pm i\varepsilon)|^2d\eta.$$

Thus (5.4) follows from (5.3). Q.E.D.

In order to represent the spectral family $\{E(\lambda)\}$ by means of the eigenfunctions $\Psi_j^{\pm}(x, \eta)$ and $\Psi_{j+2\nu\rho}^{\pm}(x, \eta)$ we investigate some properties of the $\Psi_j(x, \eta; \lambda)$. Let us recall that

$$\text{(4.4)} \qquad \Psi_j(x, \eta; \lambda) = (2\pi)^{-\frac{n}{2}} e^{ix\cdot\eta} P_j(\eta) - \frac{1}{i}(2\pi)^{-\frac{1}{2}} \bar{\mathscr{F}}_{y'}[G\,(x, y', +0; \lambda)](\xi) A_n P_j(\eta),$$

$$\operatorname{Im}\lambda \neq 0,\ 1 \leqq j \leqq 2\rho,$$

$$\text{(1.1)} \qquad G(x, y; \lambda) = E(x-y; \lambda) - E_c(x, y; \lambda).$$

Our aim is to analyse the behavior around the singular points of the second term on the right hand side of (4.4). To this end we consider the term in the region $W\times\Lambda_\delta^+(k^0)(W\times\Lambda_\delta^-(k^0))$. Then it suffices to deal with the case when the roots of $Q(\lambda, \xi, \tau)=0$ in τ for $(\xi, \lambda)\in W\times\Lambda_\delta^+(k^0)$ $(W\times\Lambda_\delta^-(k^0))$ are in the situation (1.21). From now on this will not be stated explicitly every time. First we consider $\bar{\mathscr{F}}_{y'}[E(x-y; \lambda)|_{y_n=0}](\xi)$ $A_n P_j(\eta)$. Put

$$\text{(5.8)} \qquad I_j^\mu(\eta;\lambda) = \frac{e^{-i\tau_\mu^+(\xi;\lambda)x_n}}{2\pi i}\int_{|\tau-\tau_\mu^+(\xi;\lambda)|=\delta_\mu} e^{ix_n\tau}(\tau I - M(\xi;\lambda))^{-1} d\tau P_j(\eta),$$

$$1 \leqq \mu \leqq p,$$

$$\text{(5.9)} \qquad I_j^0(x_n, \eta; \lambda) = \frac{1}{2\pi i}\int_{\gamma_+} e^{ix_n\tau}(\tau I - M(\xi;\lambda))^{-1} d\tau P_j(\eta).$$

Then

$$\text{(5.10)} \qquad \bar{\mathscr{F}}_{y'}[E(x-y; \lambda)|_{y_n=0}](\xi) A_n P_j(\eta)$$
$$= i(2\pi)^{-\frac{n-1}{2}} e^{ix'\cdot\xi}\Big\{\sum_{\mu=1}^{p} e^{i\tau_\mu^+(\xi;\lambda)x_n} I_j^\mu(\eta;\lambda) + I_j^0(x_n, \eta; \lambda)\Big\},$$

$$1 \leqq j \leqq 2\rho.$$

Lemma 5.2. *Let* $1 \leqq j \leqq 2\rho$ *and* $(\xi, \lambda)\in W\times\Lambda_\delta^+(k^0)(W\times\Lambda_\delta^-(k^0))$. *Then we have*

$$\text{(1)} \qquad I_j^0(x_n, \eta; \lambda) = (\lambda_j(\eta) - \lambda)\tilde{I}_j^0(x_n, \eta; \lambda),$$

where $\tilde{I}_j^0(x_n, \eta; \lambda)$ *is a matrix-valued continuous function of* (x_n, η, λ)

and the limit $\tilde{I}_j^0(x_n, \eta; k \pm i0)$ *exists and is continuous in* (x_n, η, k) *where* $x_n \in \mathbf{R}_+^n$, $\eta \in W \times \Xi$ *and* $k^0 - \delta < k < k^0 + \delta$.

(2) *Let* $2 \leqq \mu \leqq p$.

(i) *If* $j \neq \pi(\mu)$, *then*

$$I_j^\mu(\eta; \lambda) = (\lambda_j(\eta) - \lambda)\tilde{I}_j^\mu(\eta; \lambda).$$

(ii)
$$I_{\pi(\mu)}^\mu(\eta; \lambda) = \frac{\lambda_{\pi(\mu)}(\eta) - \lambda}{\sigma - \tau_\mu^+(\xi; \lambda)} \tilde{I}_{\pi(\mu)}^\mu(\eta; \lambda).$$

(3) (i) *If* $j \neq r$, *where* $r = \pi(1)$, *then*

$$I_j^1(\eta; \lambda) = \frac{\lambda_j(\eta) - \lambda}{\tau_1^+(\xi; \lambda) - \tau_1^-(\xi; \lambda)} \tilde{I}_j^1(\eta; \lambda)$$

(ii)
$$I_r^1(\eta; \lambda) = \frac{\lambda_r(\eta) - \lambda}{(\sigma - \tau_1^+(\xi; \lambda))(\tau_1^+(\xi; \lambda) - \tau_1^-(\xi; \lambda))} \tilde{I}_r^1(\eta; \lambda).$$

Here the $\tilde{I}_j^\mu(\eta; \lambda)$ *are continuous functions of* $(\eta; \lambda)$ *and the limits* $\tilde{I}_j^\mu(\eta; k \pm i0)$ *exist and are continuous in* $(\eta; k)$.

Proof. We have

$$(5.11) \qquad (\tau_1 I - M(\xi; \lambda_1))^{-1} - (\tau_2 I - M(\xi; \lambda_2))^{-1}$$
$$= (\tau_2 - \tau_1)(\tau_1 I - M(\xi; \lambda_1))^{-1}(\tau_2 I - M(\xi; \lambda_2))^{-1}$$
$$+ (\lambda_1 - \lambda_2)(\tau_1 I - M(\xi; \lambda_1))^{-1} A_n^{-1} (\tau_2 I - M(\xi; \lambda_2))^{-1}.$$

Hence,

$$\frac{1}{2\pi i}\int_{|\tau - \tau_\mu^+(\xi; \lambda)| = \delta_\mu} e^{ix_n\tau}(\tau I - M(\xi; \lambda))^{-1} d\tau P_j(\eta)$$
$$= \frac{1}{4\pi^2}\int_{|\tau_1 - \tau_\mu^+(\xi; \lambda)| = \delta_\mu} d\tau_1 \int_{|\lambda_2 - \lambda_j(\eta)| = \delta}$$
$$d\lambda_2 e^{ix_n\tau_1}(\tau_1 I - M(\xi; \lambda))^{-1}(\sigma I - M(\xi; \lambda_2))^{-1} A_n^{-1}$$
$$= \frac{1}{4\pi^2}\int_{|\tau_1 - \tau_\mu^+(\xi; \lambda)| = \delta_\mu} d\tau_1 \int_{|\lambda_2 - \lambda_j(\eta)| = \delta}$$

$$d\lambda_2\, e^{ix_n\tau_1}\frac{1}{\sigma-\tau_1}[(\tau_1 I-M(\xi;\lambda))^{-1}-(\sigma I-M(\xi;\lambda_2))^{-1}$$
$$+(\lambda_2-\lambda)(\tau_1 I-M(\xi;\lambda))^{-1}A_n^{-1}(\sigma I-M(\xi;\lambda_2))^{-1}]A_n^{-1}$$
$$=e^{i\tau_\mu^\pm(\xi;\lambda)x_n}\frac{\lambda_j(\eta)-\lambda}{\sigma-\tau_\mu^+(\xi;\lambda)}\Big/\frac{\partial\lambda_{\pi(\mu)}}{\partial\tau}(\xi,\tau_\mu^+(\xi;\lambda))$$
$$\times P_{\pi(\mu)}(\xi,\tau_\mu^+(\xi;\lambda))P_j(\eta)$$

(see, Lemmas 1.2 and 3.1). If $\pi(\mu)\neq j$, we have

$$P_{\pi(\mu)}(\xi,\tau_\mu^+(\xi;\lambda))P_j(\eta)=(\sigma-\tau_\mu^+(\xi;\lambda))R_{\pi(\mu)j}(\xi;\lambda),$$

where $R_{\pi(\mu)j}$ is continuous in (ξ,λ). The above calculations imply the assertions (2) and (3). The assertion (1) can be proved in a similar way. Q.E.D.

Next we consider $\mathscr{F}_{y'}[E_c(x,y',+0;\lambda)]A_nP_j(\eta)$, $1\leqq j\leqq 2\rho$, in $W\times\Lambda_\delta^+(k^0)$ $(W\times\Lambda_\delta^-(k^0))$.

Lemma 5.3. *q_{jl} and q_l $(1\leqq j\leqq p,\ 1\leqq l\leqq 2m)$ defined by* (3.14) *and* (3.15) *are evaluated as follows:*

(i) $$|q_l(x_n,\xi;\lambda)|\leqq Ce^{-dx},\qquad 1\leqq l\leqq 2m,$$

where $2d=\mathrm{dis}(\gamma_-,\mathbf{R}^1)$.

(ii) $$|q_{jl}(\xi;\lambda)|\leqq C,\qquad 2\leqq j\leqq p,\quad 1\leqq l\leqq 2m$$
$$|q_{1l}(\xi;\lambda)|\leqq C|\tau_1^+(\xi;\lambda)-\tau_1^-(\xi;\lambda)|^{-1},\qquad 1\leqq l\leqq 2m.$$

These assertions follow directly from the definition of the q_{jl} and q_l.

Suppose that $\Delta(\xi^0;k^0\pm i0)=0$. From (3.29) and (3.30) we see that the ν-th row vector of $\mathscr{F}_{y'}[E_c(x,y',+0;\lambda)]A_nP_j(\eta)$ are equal to $\sum_{l=1}^{m}\overline{C_\nu^l(x,\xi;\bar\lambda)}[P_j(\eta)A_nh_l^+(\xi;\bar\lambda)]^*$, $1\leqq\nu\leqq 2m$, where the coefficients $C_\nu^l(x,\xi;\bar\lambda)$ are quantities defined by (3.31). Put

$$(5.12)\qquad J_{1\nu}^l(x_n,\xi;\lambda)=\frac{e^{i\tau_1^+(\xi;\lambda)x_n}}{\overline{\Delta(\xi;\bar\lambda)}}\overline{\det(Bh_1^+(\xi;\bar\lambda),\cdots,\overset{l}{\underset{\smile}{Bq_{1\nu}(\xi;\bar\lambda)}},\ldots}$$
$$\overline{\ldots,Bh_m^+(\xi;\bar\lambda))},$$

(5.13) $$J_{2\nu}^{l}(x_n, \xi; \lambda)=\sum_{\mu=2}^{p}\frac{e^{i\tau_{\mu}^{+}(\xi;\lambda)x_n}}{\Delta(\xi);\bar{\lambda})}\overline{\det(Bh_1^{+}(\xi;\bar{\lambda}),\dots\overset{l}{\underset{\smile}{}}\dots, Bq_{\mu\nu}(\xi;\bar{\lambda}),\dots, Bh_m^{+}(\xi;\bar{\lambda}))}+\frac{1}{\Delta(\xi;\bar{\lambda})}\overline{\det(Bh_1^{+}(\xi;\bar{\lambda}),\dots\overset{l}{\underset{\smile}{}}\dots, Bq_{\nu}(x_n, \xi;\bar{\lambda}),\dots, Bh_m^{+}(\xi;\bar{\lambda}))}\,.$$

Then

(5.14) $$\overline{C_{\nu}^{l}(x, \xi; \bar{\lambda})}=i(2\pi)^{-\frac{n-1}{2}}e^{ix'\cdot\xi}\{J_{1\nu}^{l}(x_n, \xi; \lambda)+J_{2\nu}^{l}(x_n, \xi; \lambda)\}$$
$$=i(2\pi)^{-\frac{n-1}{2}}e^{ix'\cdot\xi}J_{\nu}^{l}(x_n, \xi; \lambda)\,.$$

Here we have used $\tau_{\mu}^{-}(\xi;\bar{\lambda})=\overline{\tau_{\mu}^{+}(\xi;\lambda)}$ which can be assumed in the enumeration (1.21).

Lemma 5.4. *Let* $(\xi^0, k^0)\in\tilde{N}$ *and* $(\xi, \lambda)\in W\times\Lambda_{\delta}^{+}(k^0)$ $(W\times\Lambda_{\delta}^{-}(k^0))$. *Then*

(i) *For* $1\leqq l\leqq\tilde{\nu}_1$

$$|J_{1\nu}^{l}(x_n, \xi; \lambda)|\leqq C\sum_{h=1}^{m}|C_{lh}(\xi;\bar{\lambda})|+C|\bar{\lambda}-\lambda_r(\xi, \sigma(\xi))|^{-\frac{1}{2}}\,.$$

(ii) *For* $\tilde{\nu}_1+1\leqq l\leqq m$

$$|J_{1\nu}^{l}(x_n, \xi; \lambda)|\leqq C\sum_{h=1}^{m}|C_{lh}(\xi;\bar{\lambda})|\,.$$

(iii) $|J_{2\nu}^{l}(x_n, \xi; \lambda)|\leqq C\sum_{h=1}^{m}|C_{lh}(\xi;\bar{\lambda})|\,.$

Proof. For $1\leqq j\leqq\tilde{\nu}_1$ we have

$$|h_j^{+}(\xi;\bar{\lambda})-h_j^{-}(\xi;\bar{\lambda})|\leqq C|\lambda-\lambda_r(\xi, \sigma(\xi))|^{\frac{1}{2}}\,.$$

From (3.14), Lemma 5.3 and the above inequalities Lemma 5.4 easily follows. Q.E.D.

Lemma 5.5. *Let* $\xi^0\notin\hat{N}\cup\bigcup_{\mu=1}^{s}\Delta_{\mu}$, $k^0=k_j(\xi^0)$ *and* $(\xi, \lambda)\in W\times\Lambda_{\delta}^{+}(k^0)$ $(W\times\Lambda_{\delta}^{-}(k^0))$.

(i) *For* $1 \leqq l \leqq p_0$ *the* $J_\nu^l(x_n, \xi; \lambda)$ *are continuous functions of* (x_n, ξ, λ) *and the limits* $J_\nu^l(x_n, \xi; k \pm i0)$ *exist and are continuous in* (x_n, ξ, k), *where* $k^0 - \delta < k < k^0 + \delta$.

(ii) *For* $p_0 + 1 \leqq l \leqq m$

$$J_\nu^l(x_n, \xi; \lambda) = \frac{1}{\lambda - k_j(\xi)} \tilde{J}_\nu^l(x_n, \xi; \lambda),$$

where the $\tilde{J}_\nu^l(x_n, \xi; \lambda)$ *have the same properties as* $J_\nu^\mu(x_n, \xi; \lambda)$, $1 \leqq \mu \leqq p_0$.

Proof. The above assertions follow from Lemmas 2.13 and 5.4.

Q.E.D.

Lemma 5.6. *Let* $\xi^0 \in \Delta_j \backslash \partial \Delta_j$ *and* $k^0 = k_j(\xi^0)$ *for a fixed* j *and* $(\xi, \lambda) \in W \times \Lambda_\delta^+(k^0)(W \times \Lambda_\delta^-(k^0))$. *Then* $\lambda_r(\xi, \sigma(\xi)) = k_j(\xi)$ *in* W.

(i) *For* $\tilde{\nu}_1 + 1 \leqq l \leqq p_0$ *the* $J_\nu^l(x_n, \xi; \lambda)$ *are continuous function of* (x_n, ξ, λ) *and the limits* $J_\nu^l(x_n, \xi; k \pm i0)$ *exist and continuous in* (x_n, ξ, k), *where* $k^0 - \delta < k < k^0 + \delta$.

(ii) *For* $1 \leqq l \leqq \tilde{\nu}_1$

$$J_\nu^l(x_n, \xi; \lambda) = \frac{1}{(\lambda - \lambda_r(\xi, \sigma(\xi)))^{\frac{1}{2}}} \tilde{J}_\nu^l(x_n, \xi; \lambda).$$

(iii) *For* $p_0 + 1 \leqq l \leqq m$

$$J_\nu^l(x_n, \xi; \lambda) = \frac{1}{\lambda - k_j(\xi)} \tilde{J}_\nu^l(x_n, \xi; \lambda).$$

Here the $\tilde{J}_\nu^l(x_n, \xi; \lambda)$ *have the same properties as* $J_\nu^\mu(x_n, \xi; \lambda)$, $\tilde{\nu}_1 + 1 \leqq \mu \leqq p_0$.

Proof. The above assertions follows from Lemmas 2.14 and 5.4.

Q.E.D.

Lemma 5.7. *Let* $1 \leqq j \leqq 2\rho$.

(1) (i) *If* $j \neq \pi(l)$,

$$P_j(\eta) A_n h_{l\mu}^+(\xi; \bar{\lambda}) = (\lambda_j(\eta) - \bar{\lambda}) \gamma_{jl\mu}(\eta; \lambda), \qquad 1 \leqq l \leqq p, \quad 1 \leqq \mu \leqq \tilde{\nu}_l.$$

(ii)
$$P_{\pi(l)}(\eta) A_n h_{l\mu}^+(\xi; \bar{\lambda}) = \frac{\lambda_{\pi(l)}(\eta) - \bar{\lambda}}{\sigma - \tau_l^+(\xi; \bar{\lambda})} \gamma_{\pi(l) l\mu}(\eta; \lambda), \quad 1 \leqq l \leqq p, \quad 1 \leqq \mu \leqq \tilde{\nu}_l.$$

Here the $\gamma_{jl\mu}(\eta;\lambda)$ *are continuous in* $(\eta,\lambda)\in W\times\Xi\times\Lambda_\delta^+(k^0)$ $(W\times\Xi\times\Lambda_\delta^-(k^0))$ *and the limits* $\gamma_{jl\mu}(\eta;k\pm i0)$ *exist and are continuous in* (η,k), *where* $k^0-\delta<k<k^0+\delta$.

(2) *For* $p_0+1\leqq l\leqq m$

$$P_j(\eta)A_nh_l^+(\xi;\bar{\lambda})=(\lambda_j(\eta)-\bar{\lambda})\gamma_{jl}(\eta;\lambda),$$

where $\gamma_{jl}(\eta;\lambda)$ *have the same properties as* $\gamma_{jl\mu}(\eta;\lambda)$ *in* (1).

Proof.

$$P_j(\eta)A_nP_{\pi(l)}(\xi,\tau_l^+(\xi;\lambda))=\frac{1}{4\pi^2}\int_{|\lambda_1-\lambda_j(\eta)|=\delta_1}$$

$$(\sigma I-M(\xi;\lambda))^{-1}d\lambda_1\int_{|\lambda_2-\lambda|=\delta}(\tau_l^+(\xi;\lambda)I-M(\xi;\lambda_2))^{-1}A_n^{-1}d\lambda_2$$

$$=\frac{\lambda_j(\eta)-\lambda}{\tau_l^+(\xi;\lambda)-\sigma}P_j(\eta)P_{\pi(l)}(\xi,\tau_l^+(\xi;\lambda)).$$

Here we have used (5.11) and the fact that $(\lambda I-A(\xi,\sigma))^{-1}$ has a simple pole at $\lambda=\lambda_j(\eta)$. Therefore we immediately obtain the assertion (1). The assertion (2) is proved in the same way. Q.E.D.

Lemma 5.8. *Assume that* $g(\xi)$ *is analytic in* W *and does not vanish identically. Then there exists a positive constant* θ_0 *such that for* $0<\theta<\theta_0$

$$\int_W\frac{d\xi}{|g(\xi)|^\theta}<+\infty.$$

Proof. From $f(\xi)\not\equiv 0$, linearly transforming ξ, if necessary, we can assume that $g(\xi_1,0,\dots,0)\not\equiv 0$. Thus by Weierstrass' preparation theorem we have

$$g(\xi)=q(\xi)\prod_{j=1}^{\gamma}(\xi_1-g_j(\xi_2,\dots,\xi_{n-1}))^{\nu_j},$$

where $q(\xi)$ is bounded away from zero. This completes the proof. Q.E.D.

Lemma 5.9. *Let* $f\in C_0^\infty(\mathbf{R}_+^n)$ *and* $0<a<b<\infty$ $(-\infty<a<b<0)$. *Then*

$$(5.15)\qquad \left(\left\{\frac{E(b)+E(b-0)}{2}-\frac{E(a)+E(a-0)}{2}\right\}f, f\right)$$

$$=\sum_{j=1}^{2\rho}\int_{a\leqq\lambda_j(\eta)\leqq b}|\hat{f}_j^{\pm}(\eta)|^2 d\eta$$

$$+\sum_{\nu=1}^{s}\sum_{j=1}^{2\rho}\int_{\{a\leqq k_\nu(\xi)\leqq b\}\cap D_\nu\times\Xi}|\hat{f}_{j+2\nu\rho}^{\pm}(\eta)|^2 d\eta$$

holds, where

$$(5.16)\quad \hat{f}_j^{\pm}(\eta)=\int_{\mathbf{R}_+^n}\Psi_j^{\pm}(x, \eta)^* f(x)dx \quad \textit{for} \quad \eta\notin N_j \quad (\textit{almost every } \eta\in\Xi^n)$$

and

$$(5.17)\qquad \hat{f}_{j+2\nu\rho}^{\pm}(\eta)=\int_{\mathbf{R}_+^n}\Psi_{j+2\nu\rho}^{\pm}(x, \eta)^* f(x)dx \qquad \textit{for} \quad \eta\in D_{j\nu}$$

$$(\textit{almost every } \eta\in D_\nu).$$

Proof. Now let us consider the case where $0<a<b<\infty$.

$$\left(\left\{\frac{E(b)+E(b-0)}{2}-\frac{E(a)+E(a-0)}{2}\right\}f, f\right)$$

$$=\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\sum_{j=1}^{2\rho}\int_{\Xi^n}d\eta\int_a^b dk\frac{\varepsilon}{(\lambda_j(\eta)-k)^2+\varepsilon^2}|\hat{f}_j(\eta; k\pm i\varepsilon)|^2$$

$$=\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\sum_{j=1}^{2\rho}\left[\int_{\{|\eta|<R\}\setminus\Delta\times\Xi}d\eta\int_a^b dk+\int_{\{|\eta|<R\}\cap\Delta\times\Xi}d\eta\int_a^b dk\right.$$

$$\left.+\int_{|\eta|<R}d\eta\int_a^b dk\right]\equiv\lim_{\varepsilon\downarrow 0}\sum_{j=1}^{2\rho}[I_{Rj}^1(\varepsilon)+I_{Rj}^2(\varepsilon)+I_{Rj}^3(\varepsilon)],$$

where Δ is a neighborhood of $\hat{N}$ and R is chosen sufficiently large. First we consider $\lim_{\varepsilon\downarrow 0}I_{Rj}^1(\varepsilon)$. Divide the domain of integration $D_R=\{(\eta, k); |\eta|<R, \xi\notin\Delta$ and $k\in[a, b]\}$ into a neighborhood of $\{(\eta, k)\in D_R; (\xi, k)\in\tilde{N}\}$ and the remainder. It is easy to prove that we can interchange the order of $\lim_{\varepsilon\downarrow 0}$ and $\int d\eta$ in the latter domain. Let us show that we can interchange the order in the former domain. It suffices to prove this in a neighborhood of a point (η^0, k^0) such that $(\xi^0, k^0)\in\tilde{N}$ and $(\eta^0, k^0)\in D_R$, where $\eta^0=(\xi^0, \sigma^0)$. Let (η, k) belong to such a neighborhood and $\lambda=k+i\varepsilon$ $(\lambda=k-i\varepsilon)$, $0<\varepsilon\leqq\varepsilon_0$, where ε_0 is chosen sufficiently small. Hence we may assume that $(\xi, \lambda) \in W \times \Lambda_\delta^+(k^0)$ $(W \times \Lambda_\delta^-(k^0))$. Then, by Lemmas 1.1 and 5.2–5.7, we have

$$\frac{1}{|\lambda_j(\eta)-\lambda|}|\hat{f}_j(\eta;\lambda)| \leqq |\hat{f}_{j1}(\eta;\lambda)| + \frac{1}{|\lambda_j(\eta)-\lambda|}|\hat{f}_{j2}(\eta;\lambda)|$$

$$+\frac{1}{|\lambda_r(\xi,\sigma(\xi))-\lambda|}|\hat{f}_{j3}(\eta;\lambda)| \quad \text{for} \quad i \neq r = \pi(1),$$

and

$$\frac{1}{|\lambda_r(\eta)-\lambda|}|\hat{f}_r(\eta;\lambda)| \leqq |\hat{f}_{r1}(\eta;\lambda)| + \frac{1}{|\lambda_r(\eta)-\lambda|}|\hat{f}_{r2}(\eta;\lambda)|$$

$$+\frac{1}{|\lambda_r(\xi,\sigma(\xi))-\lambda|}|\hat{f}_{r3}(\eta;\lambda)| + \frac{1}{|\sigma-\tau_1^+(\xi;\lambda)|\,|\tau_1^+(\xi;\lambda)-\tau_1^-(\xi;\lambda)|}$$

$$\times |\hat{f}_{r4}(\eta;\lambda)| + \frac{1}{|\sigma-\tau_1^-(\xi;\lambda)|\,|\tau_1^+(\xi;\lambda)-\tau_1^-(\xi;\lambda)|}|\hat{f}_{r5}(\eta;\lambda)|,$$

where the $\hat{f}_{j\mu}(\eta;\lambda)$ are continuous in (η,λ) and the limits $\hat{f}_{j\mu}(\eta;k+i0)$ $(f_{j\mu}(\eta;k-i0))$ exist and are continuous in (η, k). Hence, in order to show that the order of $\lim_{\varepsilon\downarrow 0}$ and $\int d\eta$ can be interchanged it suffices to investigate the term

$$\frac{1}{|\sigma-\tau_1^\pm(\xi;\lambda)|\,|\tau_1^+(\xi;\lambda)-\tau_1^-(\xi;\lambda)|}|\hat{f}_{r\mu}(\eta;\lambda)|, \quad \mu=4, 5.$$

Since

$$\frac{1}{(\sigma-\tau_1^+(\xi;\lambda))\,(\tau_1^+(\xi;\lambda)-\tau_1^-(\xi;\lambda))}$$

$$=\frac{1}{(\sigma-\tau_1^+(\xi;\lambda))\,(\sigma-\tau_1^-(\xi;\lambda))}+\frac{1}{(\sigma-\tau_1^-(\xi;\lambda))\,(\tau_1^+(\xi;\lambda)-\tau_1^-(\xi;\lambda))}$$

holds, by Lemma 1.1 we obtain

$$\frac{1}{|\sigma-\tau_1^\pm(\xi;\lambda)|\,|\tau_1^+(\xi;\lambda)-\tau_1^-(\xi;\lambda)|} \leqq \frac{C}{|\lambda-\lambda_r(\xi,\sigma(\xi))|}+\frac{C}{|\lambda-\lambda_r(\eta)|}.$$

Thus

$$\int_{k^0-\delta}^{k^0+\delta}\frac{\varepsilon}{(\lambda_j(\eta)-k)^2+\varepsilon^2}|\hat{f}_j(\eta;k\pm i\varepsilon)|^2 dk \leqq C,$$

where C is independent of ε and η ($0<\varepsilon\leqq\varepsilon_0$ and $|\eta|<R$). Therefore it follows from the Lebesgue bounded convergence theorem that

$$\lim_{\varepsilon\downarrow 0} I^1_{Rj}(\varepsilon)=\int_{\{|\eta|\leqq R\}\setminus\Delta\times\Xi} d\eta\Big[\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_a^b\frac{\varepsilon}{(\lambda_j(\eta)-k)^2+\varepsilon^2}|\hat{f}_j(\eta;\, k\pm i\varepsilon)|^2dk\Big]$$

$$=\int_{[\{|\eta|\leqq R\}\setminus\Delta\times\Xi]\cap\{a\leqq\lambda_j(\eta)\leqq b\}}|\hat{f}^{\pm}_j(\eta)|^2d\eta$$

$$+\sum_{\nu=1}^{s}\int_{[\{|\eta|\leqq R\}\setminus\Delta\times\Xi]\cap\{a\leqq k_\nu(\xi)\leqq b\}\cap D_\nu\times\Xi}|\hat{f}^{\pm}_{j+2\nu p}(\eta)|^2d\eta\,.$$

Next we shall consider the $I^2_{Rj}(\varepsilon)$. It suffices to estimate

$$I^2_{Rj}(\varepsilon)=\frac{1}{\pi}\int d\eta\int_a^b dk\frac{\varepsilon}{(\lambda_j(\eta)-k)^2+\varepsilon^2}|\hat{f}_j(\eta;\, k-i\varepsilon)|^2\,.$$

Assume that $\xi^0\in\hat{N}$, $\Delta(\xi^0;\, k^0)=0$ and $(\xi^0, k^0)\in\tilde{N}$ and that $\hat{\hat{W}}_0$ is a component constructed in the proof of Lemma 2.11. Let $t_1(\xi),\dots, t_\beta(\xi)$ be the zeros of $\Delta(\xi;\, t^2+\lambda_r(\xi,\sigma(\xi)))$ in $\Lambda_{\sqrt{\delta}}(0)$ constructed in the proof of Lemma 2.11. Then we can assume without loss of generality that $t_1(\xi),\dots, t_q(\xi)\in\omega$ and $t_{q+1}(\xi),\dots, t_\beta(\xi)\in\Omega$ in $\hat{\hat{W}}_0$, and that $t_1(\xi)\equiv 0$ and $t_j(\xi)\neq 0$, $2\leqq j\leqq q$, in $\hat{\hat{W}}_0$. Put $\frac{1}{\lambda_j(\eta)-\lambda}\hat{f}_j(\eta;\,\bar{\lambda})=\sum_{\nu=0}^{\beta}\hat{f}_{j\nu}(\eta;\,\bar{\lambda})$, $\lambda=k+i\varepsilon$. Then from Lemmas 2.16, 5.2, 5.4 and 5.7, we have the following estimates:

(1) For $j\neq r$ $(=\pi(1))$ and $(\xi,\lambda)\in\hat{\hat{W}}_0\times\Lambda^+_\delta(k^0)$,

$$|\hat{f}_{j0}(\eta;\,\bar{\lambda})|\leqq\frac{C}{|\lambda-\lambda_j(\eta)|}+\frac{C}{|t|}\,,$$

$$|\hat{f}_{j\nu}(\eta;\,\bar{\lambda})|\leqq\frac{C}{|\lambda-\lambda_r(\xi,\sigma(\xi))|}+\beta_\nu\frac{C}{|\lambda-\tilde{k}_\nu(\xi)|}$$

$$+\Big(\frac{1}{\varepsilon^{\frac{1}{4}}}+\frac{1}{|\sigma-\tau^+_{l_j}(\xi;\,\lambda)|}\Big)\Big(\beta_\nu\frac{C|\mathrm{Im}\,\tilde{k}_\nu(\xi)|^{\frac{1}{4}}}{|\lambda-\tilde{k}_\nu(\xi)|^{3/4}}$$

$$+\frac{C|t_\nu(\xi)|^{\frac{1}{2}}}{\{(k-\lambda_r(\xi,\sigma(\xi)))^2+|t_\nu(\xi)|^4\}^{3/8}}+\frac{C}{|t|^{1-\theta}|t_\nu(\xi)|^\theta}\Big)(2\leqq\nu\leqq\beta)$$

$$|\hat{f}_{j1}(\eta;\,\bar{\lambda})|\leqq\frac{C}{|\lambda-\lambda_r(\xi,\sigma(\xi))|}+\frac{C}{|\lambda-\lambda_j(\eta)|}\,.$$

(2) For $(\xi, \lambda) \in \hat{\hat{W}}_0 \times \Lambda_\delta^+(k^0)$,

$$|\hat{f}_{r0}(\eta; \bar{\lambda})| \leqq \frac{C}{|\sigma - \tau_1^-(\xi; \lambda)|\,|\tau_1^+(\xi; \lambda) - \tau_1^-(\xi; \lambda)|}.$$

(i) When $\tau_1^+(\xi; k+i0)$ is real, for $(\xi, \lambda) \in \hat{\hat{W}}_0 \times \Lambda_\delta^+(k^0)$

$$|\hat{f}_{r\nu}(\eta; \bar{\lambda})| \leqq \frac{|t|^{-\frac{1}{2}}}{|\sigma - \tau_1^+(\xi; \lambda)|}\Bigg[\frac{C|t_\nu(\xi)|^{\frac{1}{2}}}{\{(k - \lambda_r(\xi, \sigma(\xi)))^2 + |t_\nu(\xi)|^4\}^{3/8}}$$

$$+ \beta_\nu \frac{C|\mathrm{Im}\, \tilde{k}_\nu(\xi)|^{\frac{1}{4}}}{|\lambda - \tilde{k}_\nu(\xi)|^{3/4}} + \frac{C}{|t|^{1-\theta}|t_\nu(\xi)|^\theta} + \frac{C}{|t|^{\frac{1}{2}}}\Bigg]$$

$$+ \beta_\nu \frac{C}{|\lambda - \tilde{k}_\nu(\xi)|} + \frac{C}{|\lambda - \lambda_r(\xi, \sigma(\xi))|} \qquad (2 \leqq \nu \leqq \beta)$$

$$|\hat{f}_{r1}(\eta; \bar{\lambda})| \leqq \frac{C}{|\lambda - \lambda_r(\xi, \sigma(\xi))|} + \frac{C}{|\lambda - \lambda_r(\eta)|}$$

$$+ \frac{|t|^{-\frac{1}{2}}}{|\sigma - \tau_1^+(\xi; \lambda)|\,|t|^{\frac{1}{2}}}.$$

(ii) When $\tau_1^+(\xi; k+i0)$ is non-real, for $(\xi, \lambda) \in \hat{\hat{W}}_0 \times \Lambda_\delta^+(k^0)$

$$|\hat{f}_{r\nu}(\eta; \bar{\lambda})| \leqq \frac{[\mathrm{Im}\, \tau_1^+(\xi; \lambda)]^{\frac{1}{2}}}{|\sigma - \tau_1^+(\xi; \lambda)|}\Bigg[\frac{C}{|\lambda - \lambda_r(\xi, \sigma(\xi))|}$$

$$+ \beta_\nu \frac{C}{|\lambda - \tilde{k}_\nu(\xi)|}\Bigg] + \beta_\nu \frac{C|\mathrm{Im}\, \tilde{k}_\nu(\xi)|^{\frac{1}{4}}}{|\lambda - \tilde{k}_\nu(\xi)|^{3/4}}$$

$$+ \frac{C|t_\nu(\xi)|^{\frac{1}{2}}}{\{(k - \lambda_r(\xi, \sigma(\xi)))^2 + |t_\nu(\xi)|^4\}^{3/8}} + \frac{C}{|t|^{1-\theta}|t_\nu(\xi)|^\theta}$$

$$+ \beta_\nu \frac{C}{|\lambda - \tilde{k}_\nu(\xi)|} + \frac{C}{|\lambda - \lambda_r(\xi, \sigma(\xi))|} \qquad (2 \leqq \nu \leqq \beta)$$

$$|\hat{f}_{r1}(\eta; \bar{\lambda})| \leqq \frac{C[\mathrm{Im}\, \tau_1^+(\xi; \lambda)]^{\frac{1}{2}}}{|\sigma - \tau_1^+(\xi; \lambda)|\,|\lambda - \lambda_r(\xi, \sigma(\xi))|}$$

$$+ \frac{C}{|\lambda - \lambda_r(\eta)|} + \frac{C}{|\lambda - \lambda_r(\xi, \sigma(\xi))|}$$

Here l_j is a number satisfying $j=\pi(l_j)$ and $\beta_\nu=1$ (if $\operatorname{Im}\tilde{k}_\nu(\xi)\leqq 0$), $=0$ (if $\operatorname{Im}\tilde{k}_\nu(\xi)>0$).

From the following estimates we can deduce that $\lim_{\varepsilon\downarrow 0} I_{Rj}^2(\varepsilon)\to 0$ as $m(\Delta)\to 0$.[8)]

$$\int d\xi\int dk\int_{-\infty}^{\infty} d\sigma\frac{\varepsilon}{|\sigma-\tau_{l_j}^+(\xi;\lambda)|^2}\frac{|\operatorname{Im}\tilde{k}_\nu(\xi)|^{\frac{1}{2}}}{|\lambda-\tilde{k}_\nu(\xi)|^{3/2}}$$

$$\leqq C\int d\xi\int dk\int_{-\infty}^{\infty} d\sigma\frac{\operatorname{Im}\tau_{l_j}^+(\xi;\lambda)}{|\sigma-\tau_{l_j}^+(\xi;\lambda)|^2}\frac{|\operatorname{Im}\tilde{k}_\nu(\xi)|^{\frac{1}{2}}}{|\lambda-\tilde{k}_\nu(\xi)|^{3/2}}$$

$$\leqq C\int d\xi\int dk\frac{|\operatorname{Im}\tilde{k}_\nu(\xi)|^{\frac{1}{2}}}{|\lambda-\tilde{k}_\nu(\xi)|^{3/2}}\leqq C\int d\xi\qquad(\operatorname{Im}\tilde{k}_\nu(\xi)\leqq 0)\,,$$

$$\int d\xi\int dk\int_{-\infty}^{\infty} d\sigma\frac{\varepsilon}{|\sigma-\tau_{l_j}^+(\xi;\lambda)|^2}\frac{|t_\nu(\xi)|}{\{(k-\lambda_r(\xi,\sigma(\xi)))^2+|t_\nu(\xi)|^4\}^{3/4}}$$

$$\leqq C\int d\xi,$$

$$\int d\xi\int dk\int_{-\infty}^{\infty} d\sigma\frac{\varepsilon}{|\sigma-\tau_{l_j}^+(\xi;\lambda)|^2}\frac{1}{|t|^{2-2\theta}|t_\nu(\xi)|^{2\theta}}$$

$$\leqq C\int d\xi\int dk\frac{1}{|\lambda-\lambda_r(\xi,\sigma(\xi))|^{1-\theta}}\frac{1}{|t_\nu(\xi)|^{2\theta}}$$

$$\leqq C\int\frac{d\xi}{|t_\nu(\xi)|^{2\theta}}\qquad(2\leqq\nu\leqq\beta)\,.$$

In the case where $\tau_1^+(\xi;k+i0)$ is real

$$\int d\xi\int dk\int_{-\infty}^{\infty} d\sigma\frac{\varepsilon|t|^{-1}}{|\sigma-\tau_1^+(\xi;\lambda)|^2}\frac{|t_\nu(\xi)|}{\{(k-\lambda_r(\xi,\sigma(\xi)))^2+|t_\nu(\xi)|^4\}^{3/4}}$$

$$\leqq C\int d\xi\int dk\int_{-\infty}^{\infty} d\sigma\frac{\operatorname{Im}\tau_1^+(\xi;\lambda)}{|\sigma-\tau_1^+(\xi;\lambda)|^2}\frac{|t_\nu(\xi)|}{\{(k-\lambda_r(\xi,\sigma(\xi)))^2+|t_\nu(\xi)|^4\}^{3/4}}$$

$$\leqq C\int d\xi,$$

$$\int d\xi\int dk\int_{-\infty}^{\infty} d\sigma\frac{\varepsilon|t|^{-1}}{|\sigma-\tau_1^+(\xi;\lambda)|^2}\frac{|\operatorname{Im}\tilde{k}_\nu(\xi)|^{\frac{1}{2}}}{|\lambda-\tilde{k}_\nu(\xi)|^{3/2}}$$

$$\leqq C\int d\xi\qquad(\operatorname{Im}\tilde{k}_\nu(\xi)\leqq 0)\,,$$

8) $m(\cdot)$ denotes the Lebesgue measure in Ξ^{n-1}.

$$\int d\xi \int dk \int_{-\infty}^{\infty} d\sigma \frac{\varepsilon |t|^{-1}}{|\sigma-\tau_1^+(\xi;\lambda)|^2} \frac{1}{|t|^{2-2\theta}|t_\nu(\xi)|^{2\theta}}$$

$$\leqq C \int \frac{d\xi}{|t_\nu(\xi)|^{2\theta}} \qquad (2 \leqq \nu \leqq \beta).$$

Here we have used Lemma 1.1. In the case where $\tau_1^+(\xi; k+i0)$ is non-real,

$$\int d\xi \int dk \int_{-\infty}^{\infty} d\sigma \frac{\operatorname{Im} \tau_1^+(\xi;\lambda)}{|\sigma-\tau_1^+(\xi;\lambda)|^2} \frac{\varepsilon}{|\lambda-\lambda_r(\xi, \sigma(\xi))|^2}$$

$$\leqq C \int d\xi,$$

$$\int d\xi \int dk \int_{-\infty}^{\infty} d\sigma \frac{\operatorname{Im} \tau_1^+(\xi;\lambda)}{|\sigma-\tau_1^+(\xi;\lambda)|^2} \frac{\varepsilon}{|\lambda-\tilde{k}_\nu(\xi)|^2}$$

$$\leqq C \int d\xi \qquad (\operatorname{Im} \tilde{k}_\nu(\xi) \leqq 0).$$

When $\xi^0 \in \hat{N}$, $\Delta(\xi^0; k^0)=0$ and $(\xi^0, k^0) \notin \tilde{N}$, by Lemma 2.15 the estimates for $\dfrac{1}{\lambda_j(\eta)-\bar{\lambda}} \hat{f}_j(\eta; \bar{\lambda})$ are easily obtained. Since $\hat{N}$ is a null set of Ξ^{n-1}, we can take Δ to be sufficiently small, that is, $m(\Delta) \to 0$. Thus we have

$$\lim_{\varepsilon \downarrow 0} I_{Rj}^2(\varepsilon) \longrightarrow 0 \quad \text{as} \quad m(\Delta) \longrightarrow 0.$$

In fact, from Lemma 5.8 and the fact that $\prod_{\nu=2}^{\beta} t_\nu(\xi)^{r_\nu}$ is an analytic function and equal to one of the coefficients $a_{ij}(\xi)$ in (2.19), it follows that there exists $\theta\ (>0)$ such that

$$\int_{\hat{W}_0} \frac{d\xi}{|t_\nu(\xi)|^{2\theta}} < +\infty, \qquad 2 \leqq \nu \leqq \beta.$$

Finally let us prove that $\lim_{\varepsilon \downarrow 0} I_{Rj}^3(\varepsilon) \to 0$ as $R \to \infty$. By applying the same argument as for $I_{Rj}^2(\varepsilon)$ it is easy to see that

$$\int_{|\xi|^2+\sigma^2 \geqq R^2} d\xi \int_a^b \frac{\varepsilon}{|\lambda_j(\eta)-\lambda|^2} |\hat{f}_j(\eta; \bar{\lambda})|^2 dk \leqq \frac{C}{\sigma^2+R^2}.$$

In fact, $|\lambda_j(\eta)-\lambda|^2 \geqq C(\sigma^2+|\xi|^2+R^2)$ holds for $|\eta|>R$ and $k \in [a, b]$

and $\mathscr{F}_{x'}[f(x)](\xi)$ is rapidly decreasing with respect to ξ. For $|\xi|\leqq R$ we apply the same argument as for $I_{Rj}^2(\varepsilon)$, noting that $|\sigma-\tau_l^+(\xi;\lambda)|^2 \geqq C(\sigma^2+|\xi|^2+R^2)$ for $|\eta|>R$ and $k\in[a, b]$. Moreover for $|\xi|>R$ we make use of Lemma 2.15 and the facts that (ξ, k) does not belong to $\tilde{N}$ for $k\in[a, b]$ and that

$$\varepsilon\int_a^b |\hat{f}_j(\eta;\bar{\lambda})|^2 dk \leqq |\tilde{f}_j(\xi)|^2,$$

where $\tilde{f}_j(\xi)$ is rapidly decreasing. Thus we have

$$\lim_{\varepsilon\downarrow 0} I_{Rj}^3(\varepsilon)\longrightarrow 0 \qquad \text{as} \quad R\longrightarrow\infty.$$

Q.E.D.

We can easily extend the equation (5.15) for all $f\in L^2(\mathbf{R}_+^n)$ and obtain $E(a)=E(a-0)$, $a\neq 0$, making $b\downarrow a$ in (5.15). Hence $\sigma_p(\mathbf{A})\subset\{0\}$, where $\sigma_p(\mathbf{A})$ denotes the point spectrum of $\mathbf{A}$.

Theorem 5.10. *Assume that the conditions* (L.1)–(L.3) *and* (B.1) *are satisfied and that f and $g\in L^2(\mathbf{R}_+^n)$.*

(i) *We have for $0<a<b<\infty$ $(-\infty<a<b<0)$*

$$(5.18)\qquad (\{E(b)-E(a)\}f, g)=\sum_{j=1}^{2p}\int_{a\leqq\lambda_j(\eta)\leqq b}\hat{f}_j^{\pm}(\eta)\cdot\overline{\hat{g}_j^{\pm}(\eta)}d\eta,$$

$$+\sum_{\nu=1}^{s}\sum_{j=1}^{2p}\int_{\{a\leqq k_\nu(\xi)\leqq b\}\cap D_\nu\times\Xi}\hat{f}_{j+2\nu p}^{\pm}(\eta)\cdot\overline{\hat{g}_{j+2\nu p}^{\pm}(\eta)}d\eta,$$

where

$$(5.19)\qquad \hat{f}_j^{\pm}(\eta)=\underset{r\to\infty}{\text{l.i.m.}}\int_{\mathbf{R}_+^n\cap\{|x|<r\}}\Psi_j^{\pm}(x,\eta)^*f(x)dx,$$

$$(5.20)\qquad \hat{f}_{j+2\nu p}^{\pm}(\eta)=\underset{r\to\infty}{\text{l.i.m.}}\int_{\mathbf{R}_+^n\cap\{|x|<r\}}\Psi_{j+2\nu p}^{\pm}(x,\eta)^*f(x)dx.$$

(ii) $\sigma_p(\mathbf{A})\subset\{0\}$. *Moreover if $E^+(\xi;0)\cap\mathscr{B}=\{0\}$ for almost all $\xi\in\Xi^{n-1}$, then $\sigma_p(\mathbf{A})=\phi$.*

(iii) *Let P^0 be the orthogonal projection onto the subspace of dis-*

continuity with respect to **A**. *Then* $P\equiv(\mathbf{I}-P^0)$ *is the orthogonal projection onto the subspace of absolute continuity with respect to* **A** *and* $R(P^0)=N(\mathbf{A})$ *and* $R(P)=R(\mathbf{A})^a$. *Furthermore we have the Parseval formula*

$$(5.21)\qquad (Pf, g)=\sum_{j=1}^{2\rho}\int_{\Xi^n}\hat{f}_j^{\pm}(\eta)\cdot\overline{\hat{g}_j^{\pm}(\eta)}d\eta$$

$$+\sum_{\nu=1}^{s}\sum_{j=1}^{2\rho}\int_{D_\nu\times\Xi}\hat{f}_{j+2\nu\rho}^{\pm}(\eta)\cdot\overline{\hat{g}_{j+2\nu\rho}^{\pm}(\eta)}d\eta$$

Proof. It suffices to prove that $0\notin\sigma_p(\mathbf{A})$, if $E^+(\xi; 0)\cap\mathscr{B}=\{0\}$ for almost all $\xi\in\Xi^{n-1}$. Thus let us show that there does not exist any non-trivial solution $v\in L^1(\mathbf{R}_+^n)$ satisfying the following system of equations:

$$(5.22)\qquad \begin{cases} Av=0, & x_n>0,\\ Bv|_{x_n=0}=0. \end{cases}$$

By taking the Fourier transforms with respect to x' in (5.22), we have

$$\begin{cases} \dfrac{1}{i}\dfrac{d}{dx_n}\tilde{v}(\xi, x_n)+\sum_{j=1}^{n-1}\xi_jA_n^{-1}A_j\tilde{v}(\xi; x_n)=0, & x_n>0,\\ B\tilde{v}(\xi, 0)=0, \end{cases}$$

where $\tilde{v}(\xi, x_n)=\mathscr{F}_{x'}[v(x', x_n)]\in L^2_{x_n}(\mathbf{R}_+^1)$. Thus $\tilde{v}(\xi, 0)\in E^+(\xi; 0)$, that is,

$$\tilde{v}(\xi, 0)=\sum_{l=1}^{m}C_l(\xi)h_l^+(\xi; 0).$$

From $E^+(\xi; 0)\cap\mathscr{B}=\{0\}$ it follows that $C_l(\xi)=0$ for almost every $\xi\in\Xi^{n-1}$. Therefore $\tilde{v}(\xi, x_n)=0$ for almost every $\xi\in\Xi^{n-1}$, that is, $v(x)=0$.

Q.E.D.

§6. Eigenfunction Expansions

First we restate the properties and local representations of the eigenfunctions $\Psi_j^{\pm}(x, \eta)$ and $\Psi_{j+2\nu\rho}^{\pm}(x, \eta)$.

Theorem 6.1. *Assume that the conditions* (L.1)-(L.3) *and* (B.1) *are satisfied.*

(i) $\Psi_j^{\pm}(x, \eta)$, $1 \leqq j \leqq 2\rho$, *are defined for* $x \in \mathbf{R}_+^n$ *and* $\eta \notin N_j$. *Moreover, the* $\Psi_j^{\pm}(x, \eta)$ *are infinitely differentiable in* $(x, \eta) \in \mathbf{R}_+^n \times (\Xi^n \backslash N_j)$ *and belong to* $L^2_{\eta, \mathrm{loc}}(\Xi^n)$ *and*

$$A_x \Psi_j^{\pm}(x, \eta) = \lambda_j(\eta) \Psi_j^{\pm}(x, \eta), \quad B\Psi_j^{\pm}(x', +0, \eta) = 0$$

hold for all $(x, \eta) \in \mathbf{R}_+^n \times (\Xi^n \backslash N_j)$.

(ii) $\Psi_{j+2\nu\rho}^{\pm}(x, \eta)$ $(1 \leqq j \leqq 2\rho,\ 1 \leqq \nu \leqq s)$ *are defined for* $(x, \eta) \in \mathbf{R}_+^n \times D_{j\nu}$ *and infinitely differentiable in* $(x, \eta) \in \mathbf{R}_+^n \times D_{j\nu}$ *and belong to* $L^2_{\eta, \mathrm{loc}}(D_\nu \times \Xi)$.

Moreover,

$$A_x \Psi_{j+2\nu\rho}^{\pm}(x, \eta) = k_\nu(\xi) \Psi_{j+2\nu\rho}^{\pm}(x, \eta), \quad B\Psi_{j+2\nu\rho}^{\pm}(x', +0, \eta) = 0$$

hold for $(x, \eta) \in \mathbf{R}_+^n \times D_{j\nu}$.

(iii) *The* $\Psi_j^{\pm}(x, \eta)$ *are represented in a neighborhood of* $\eta^0 \in \Xi^n$, $\eta^0 \notin N_j$, *as*

(6.1) $\Psi_j^{\pm}(x, \eta)$

$$= \begin{cases} 0 \\ \quad (\textit{if there exists } l_j,\ 1 \leqq l_j \leqq \rho, \textit{ such that } \tau_{l_j}^{+}(\xi; \lambda_j(\eta) \pm i0) = \sigma), \\ (2\pi)^{-\frac{n}{2}} \Big[e^{ix\cdot\eta} P_j(\eta) + \sum\limits_{p(l)=l_j} \Big\{ \sum\limits_{\mu=1}^{p} e^{ix'\cdot\xi} e^{i\tau_\mu^{\pm}(\xi; \lambda_j(\eta) \pm i0)x_n} \\ \quad \times \dfrac{\frac{\partial \lambda_j}{\partial \tau}(\eta)}{\Delta(\xi; \lambda_j(\eta) \mp i0)} \overline{\det(Bh_1^{+}(\xi; \lambda_j(\eta) \mp i0), \ldots, \underset{\overset{\cup}{l}}{Bq_{\mu\nu}(\xi; \lambda_j(\eta) \mp i0)}, \ldots,} \\ \quad \overline{Bh_m^{+}(\xi; \lambda_j(\eta) \mp i0))} h_l^{+}(\xi; \lambda_j(\eta) \mp i0)^* + e^{ix'\cdot\xi} \dfrac{\frac{\partial \lambda_j}{\partial \tau}(\eta)}{\Delta(\xi); \lambda_j(\eta) \mp i0)} \\ \quad \overline{\det(Bh_1^{+}, \ldots, \underset{\overset{\cup}{l}}{Bq_\nu(x_n, \xi; \lambda_j(\eta) \mp i0)}, \ldots, Bh_m^{+})} h_l^{+}(\xi; \lambda_j(\eta) \mp i0)^* \Big\}_{\nu=1,\ldots,2m\downarrow} \Big] \\ \quad (\textit{if there exists } l_j,\ 1 \leqq l_j \leqq \rho, \textit{ such that } \tau_{l_j}^{+}(\xi; \lambda_j(\eta) \mp i0) = \sigma, \\ \quad \textit{that is, if } \tau_l^{+}(\xi; \lambda_j(\eta) \pm i0) \neq \sigma \textit{ for all } l,\ 1 \leqq l \leqq \rho). \end{cases}$$

Here $p(\cdot)$ *is a mapping of* $\{1, 2,\dots, p_0\}$ *onto* $\{1,\dots, p\}$ *such that* $\tau^{+}_{p(l)}(\xi; \lambda_j(\eta)\mp i0)$ *is the eigenvalue of* $M(\xi; \lambda_j(\eta)\mp i0)$ *corresponding to the eigenvector* $h^{+}_{l}(\xi; \lambda_j(\eta)\mp i0)$.

Proof. The assertion (iii) follows from (5.10) and Lemmas 5.2 and 5.7 after some calculations. From Lemmas 2.15, 2.16 and 5.2–5.8 it follows that $\Psi^{\pm}_{j}(x, \eta)\in L^2_{\eta,\mathrm{loc}}(\Xi^n)$ and $\Psi^{\pm}_{j+2\nu\rho}(x, \eta)\in L^2_{\eta,\mathrm{loc}}(D_\nu\times\Xi)$. In fact, for any a and b, $0<a<b$, we have

$$\int_{\{a\leqq\lambda_j(\eta)\leqq b\}\cap\{|\eta|\leqq R\}}|\Psi^{\pm}_{j}(x, \eta)|^2 d\eta+\sum_{\nu=1}^{s}\int_{D_\nu\times\Xi\cap\{a\leqq k_\nu(\xi)\leqq b\}\cap\{|\eta|\leqq R\}}$$

$$|\Psi^{\pm}_{j+2\nu\rho}(x, \eta)|^2 d\eta=\lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{|\eta|\leqq R}d\eta\int_a^b\frac{\varepsilon}{(\lambda_j(\eta)-k)^2+\varepsilon^2}$$

$$\times|\Psi_j(x, \eta; k\pm i\varepsilon)|^2 dk\leqq C,$$

where C depends only on R. Q.E.D.

Theorem 6.2. *Assume that the conditions* (L.1)–(L.3) *and* (B.1) *are satisfied and that* $f\in L^2(\mathbf{R}^n_+)$.

(i) *The expansion formula*

$$Pf(x)=\sum_{j=1}^{2\rho}\int_{\Xi^n}\Psi^{\pm}_{j}(x, \eta)\hat{f}^{\pm}_{j}(\eta)d\eta \tag{6.2}$$

$$+\sum_{\nu=1}^{s}\sum_{j=1}^{2\rho}\int_{D_\nu\times\Xi}\Psi^{\pm}_{j+2\nu\rho}(x, \eta)\hat{f}^{\pm}_{j+2\nu\rho}(\eta)d\eta$$

holds, where

$$\hat{f}^{\pm}_{j}(\eta)=\int_{\mathbf{R}^n_+}\Psi^{\pm}_{j}(x, \eta)^* f(x)dx, \qquad 1\leqq j\leqq 2\rho, \tag{6.3}$$

$$\hat{f}^{\pm}_{j+2\nu\rho}(\eta)=\int_{\mathbf{R}^n_+}\Psi^{\pm}_{j+2\nu\rho}(x, \eta)^* f(x)dx, \qquad 1\leqq j\leqq 2\rho,\quad 1\leqq\nu\leqq s. \tag{6.4}$$

Here the above integrals are taken in the sense of the limit in the mean.

(ii) $f\in D(\mathbf{A})$ *if and only if* $\lambda_j(\eta)\hat{f}^{\pm}_{j}(\eta)\in P_j(\eta)L^2(\Xi^n)$, $k_\nu(\xi)\hat{f}^{\pm}_{j+2\nu\rho}(\eta)\in P_j(\eta)L^2(D_\nu\times\Xi)$, $1\leqq j\leqq 2\rho$, $1\leqq\nu\leqq s$. *Then*

$$(6.5)\qquad (\mathbf{A}f)(x)=\sum_{j=1}^{2\rho}\int_{\Xi^n}\lambda_j(\eta)\Psi_j^{\pm}(x,\eta)\hat{f}_j^{\pm}(\eta)d\eta$$

$$+\sum_{\nu=1}^{s}\sum_{j=1}^{2\rho}\int_{D_\nu\times\Xi}k_\nu(\xi)\Psi_{j+2\nu\rho}^{\pm}(x,\eta)\hat{f}_{j+2\nu\rho}^{\pm}(\eta)d\eta\,,$$

$$(6.6)\qquad (\mathbf{A}f)\hat{}_j^{\pm}(\eta)=\lambda_j(\eta)\hat{f}_j^{\pm}(\eta),\qquad 1\leqq j\leqq 2\rho\,,$$

$$(6.7)\qquad (\mathbf{A}f)\hat{}_{j+2\nu\rho}^{\pm}(\eta)=k_\nu(\xi)\hat{f}_{j+2\nu\rho}^{\pm}(\eta),\qquad 1\leqq j\leqq 2\rho,\quad 1\leqq\nu\leqq s\,.$$

Proof. Let $\Phi_j^{\pm};\ L^2(\mathbf{R}_+^n)\to L^2(\Xi^n)\times L^2(D_1\times\Xi)\times\cdots\times L^2(D_s\times\Xi)$ be mappings defined by

$$\Phi_j^{\pm}f=(\hat{f}_j^{\pm}(\eta),\hat{f}_{j+2\rho}^{\pm}(\eta),\dots,\hat{f}_{j+2s\rho}^{\pm}(\eta)),\quad 1\leqq j\leqq 2\rho\,.$$

Put $\Phi^{\pm}=\sum_{j=1}^{2\rho}\Phi_j^{\pm}$. Then we have the following

Lemma 6.3.

$$(6.8)\qquad P_j(\eta)\Phi_j^{\pm}=\Phi_j^{\pm},\qquad 1\leqq j\leqq 2\rho\,,$$

$$(6.9)\qquad \Phi_l^{\pm *}\Phi_j^{\pm}=0\qquad \textit{if}\ \ j\neq l\,.$$

Moreover $\Phi^{\pm}$ *are isometries, that is,*

$$(6.10)\qquad \Phi^{\pm *}\Phi^{\pm}=\mathbf{I}_{L^2(\mathbf{R}_+^n)},\qquad \Phi^{\pm}\Phi^{\pm *}=P^{\pm}\,,$$

where $P^{\pm}$ *are orthogonal projections in* $L^2(\Xi^n)\times L^2(D_1\times\Xi)\times\cdots\times L^2(D_s\times\Xi)$ *whose ranges are equal to* $R(\Phi^{\pm})$.

We can easily verify that

$$(6.11)\qquad (\Phi_j^{\pm *}f)(x)=\int_{\Xi^n}\Psi_j^{\pm}(x,\eta)f_0(\eta)d\eta$$

$$+\sum_{\nu=1}^{s}\int_{D_\nu\times\Xi}\Psi_{j+2\nu\rho}^{\pm}(x,\eta)f_\nu(\eta)d\eta\,,$$

where $f=(f_0,f_1,\dots,f_s)\in C_0^\infty(\Xi^n)\times C_0^\infty(D_1\times\Xi)\times\cdots\times C_0^\infty(D_s\times\Xi)$. By the boundedness of $\Phi_j^{\pm *}$ (6.11) holds for all $f\in L^2(\Xi^n)\times L^2(D_1\times\Xi)\times\cdots\times L^2(D_s\times\Xi)$, where the integrals are taken in the sense of the limit in the mean. Therefore (6.2) follows from (6.9) and (6.10).

Next we prove the diagonal representation of **A**. From Theorem 5.10 we have

$$(E(\mu)Pf, g)=\sum_{j=1}^{2\rho}\int_{\lambda_j(\eta)\leqq\mu}\hat{f}_j^{\pm}(\eta)\cdot\overline{\hat{g}_j^{\pm}(\eta)d\eta}$$

$$+\sum_{\nu=1}^{s}\sum_{j=1}^{2\rho}\int_{D_\nu\times\Xi\cap\{k_\nu(\xi)\leqq\mu\}}\hat{f}_{j+2\nu\rho}^{\pm}(\eta)\cdot\overline{\hat{g}_{j+2\nu\rho}^{\pm}(\eta)}d\eta,\quad f,\ g\in L^2(\mathbf{R}_+^n).$$

It is well known that $f\in D(\mathbf{A})$ if and only if

$$\int_{-\infty}^{\infty}\lambda^2 d(E(\lambda)f, f)<\infty$$

(see, e.g., [16]). Thus it is easy to see that $f\in D(\mathbf{A})$ if and only if $\hat{f}_j^{\pm}(\eta)$, $\lambda_j(\eta)\hat{f}_j^{\pm}(\eta)\in P_j(\eta)L^2(\Xi'')$, $\hat{f}_{j+2\nu\rho}^{\pm}(\eta)$, $k_\nu(\xi)\hat{f}_{j+2\nu\rho}^{\pm}(\eta)\in P_j(\eta)L^2(D_\nu\times\Xi)$, $1\leqq j\leqq 2\rho$, $1\leqq\nu\leqq s$. Let $\alpha_r(x)$ be a C^∞ scalar function such that

$$\alpha_r(x)=\begin{cases}1, & |x|<r,\\ 0, & |x|>r+1.\end{cases}$$

Let $f\in D(\mathbf{A})$. Then $\alpha_r(x)f(x)\in D(\mathbf{A})$, and

$$\begin{aligned}(\mathbf{A}f)_j^{\wedge\pm}(\eta)&=\underset{r\to\infty}{\text{l.i.m.}}\int_{\mathbf{R}_+^n}\Psi_j^{\pm}(x,\eta)^*\alpha_r(x)(\mathbf{A}f)(x)dx\\ &=\underset{r\to\infty}{\text{l.i.m.}}\int_{\mathbf{R}_+^n}[A_x(\alpha_r(x)\Psi_j^{\pm}(x,\eta))]^*f(x)dx\\ &=\underset{r\to\infty}{\text{l.i.m.}}\int_{\mathbf{R}_+^n}(A_x\Psi_j(x,\eta))^*\alpha_r(x)f(x)dx\\ &\quad+\underset{r\to\infty}{\text{l.i.m.}}\int_{\{x\in\mathbf{R}_+^n;r\leqq|x|\leqq r+1\}}[(A_x\alpha_r(x))\Psi_j^{\pm}(x,\eta)]^*f(x)dx\\ &=\underset{r\to\infty}{\text{l.i.m.}}\int_{\mathbf{R}_+^n}\lambda_j(\eta)\Psi_j^{\pm}(x,\eta)^*\alpha_r(x)f(x)dx=\lambda_j(\eta)\hat{f}_j^{\pm}(\eta).\end{aligned}$$

Similarly we can show that $(\mathbf{A}f)_{j+2\nu\rho}^{\wedge\pm}(\eta)=k_\nu(\xi)\hat{f}_{j+2\nu\rho}^{\pm}(\eta)$. This proves (6.5), (6.6) and (6.7). Q.E.D.

Proof. of Theorem 0.7.

Assume that the operator L satisfies the conditions (L.1)′, (L.2) and (L.3) and that the matrix B satisfies the conditions (B.1) and (B.2). Then from Lemmas 2.8 and 2.9 it follows that

$$\Psi^{\pm}_{j+2\nu\rho}(x, \eta)\equiv 0, \qquad 1\leqq j\leqq 2\rho, \quad 1\leqq \nu\leqq s.$$

This proves Theorem 0.7. Q.E.D.

The following theorem gives the ranges $R(\Phi^{\pm})$ in an explicit form under the conditions (L.1)′, (L.2), (L.3), (B.1) and (B.2), where $\Phi^{\pm}$ are defined by (0.30) and (0.31).

Theorem 6.4. *Assume that the conditions* (L.1)′, (L.2), (L.3), (B.1) *and* (B.2) *are satisfied.*

(i) *We have*

$$R(\Phi^{\pm})=P_1(\eta)L^2(F^{\pm}_1)\oplus\cdots\oplus P_{2\rho}(\eta)L^2(F^{\pm}_{2\rho}), \tag{6.12}$$

where the $F^{\pm}_j$ *denote the sets* $\{\eta\in\Xi^n;\ \tau^{+}_l(\xi;\ \lambda_j(\eta)\pm i0)\neq\sigma$ *for all* l, $1\leqq l\leqq\rho\}^a$ [9] *and* $P_j(\eta)L^2(F^{\pm}_j)\equiv\{f\in L^2(\Xi^n);\ P_jf(\eta)=f(\eta),\ \mathrm{Supp}f\subset F^{\pm}_j\}$. *Moreover* $F^{\pm}_j=\mathrm{Supp}_\eta[\Psi^{\pm}_j(x, \eta)]$, $F^{+}_j\cap F^{-}_j=N_j$ *and* $F^{+}_j\cup F^{-}_j=\Xi^n$ *hold*, $1\leqq j\leqq 2\rho$.

(ii) *The functions* $\Phi^{\pm}_j$ *are partial isometries and*

$$R(\Phi^{\pm}_j)=P_j(\eta)L^2(F^{\pm}_j), \quad \Phi^{\pm}_j\Phi^{\pm *}_k=0, \qquad j\neq k, \tag{6.13}$$

hold.

Proof. It suffices to prove that $g\in N(\Phi^{\pm *})\cap\bigoplus_{j=1}^{2\rho}P_j(\eta)L_2(F^{\pm}_j)$[10] implies $g\equiv 0$. Let $g(\eta)\equiv g_1(\eta)\oplus\cdots\oplus g_{2\rho}(\eta)\in N(\Phi^{\pm *})\cap\bigoplus_{j=1}^{2\rho}P_j(\eta)L^2(F^{\pm}_j)$. Then

$$0=\Phi^{\pm *}g=\underset{N\to\infty}{\mathrm{l.i.m.}}\int_{\Xi^n}\sum_{j=1}^{2\rho}\Psi^{\pm}_j(x, \eta)g_N(\eta)d\eta,$$

where $g_N(\eta)=g(\eta)$ for $|\eta|<N$, $=0$ for $|\eta|>N$. Hence, for non-real λ

$$\mathscr{F}_{x'}(\mathbf{A}-\lambda)^{-1}\Phi^{\pm *}g_N\longrightarrow 0 \qquad \text{in } L^2(\Xi^{n-1}\times\mathbf{R}^1_+) \text{ as } N\longrightarrow\infty. \tag{6.14}$$

9) S^a denotes the closure of S.

10) $N(\Phi^{\pm *})$ denote the null space of $\Phi^{\pm *}$.

Let $f\in L^2(\Xi^{n-1}\times\mathbf{R}^1_+)$ such that $\bar{\mathscr{F}}_\xi f\in C_0^\infty(\mathbf{R}^n_+)$. By (6.14) we have

$$(f,\ \mathscr{F}_{x'}(\mathbf{A}-\lambda)^{-1}\Phi^{\pm *}g_N)=(\Phi^\pm(\mathbf{A}-\bar{\lambda})^{-1}\bar{\mathscr{F}}_\xi f,\ g_N)_0$$

$$=(\sum_{j=1}^{2\rho}(\lambda_j(\eta)-\bar{\lambda})^{-1}\Phi_j^\pm\bar{\mathscr{F}}_\xi f,\ g_N)_0\longrightarrow 0\qquad\text{as}\quad N\longrightarrow\infty .$$

Thus

$$0=\sum_{j=1}^{2\rho}\int_{\Xi^n}\frac{1}{\lambda_j(\eta)-\lambda}\Phi_j^\pm\bar{\mathscr{F}}_\xi f\cdot\bar{g}\,d\eta$$

and, therefore, we obtain

$$0=\sum_{j=1}^{2\rho}\int_{\Xi^n}d\eta\frac{\varepsilon}{\pi}\int_a^b dk\frac{\varepsilon}{(\lambda_j(\eta)-k)^2+\varepsilon^2}\Phi_j^\pm\bar{\mathscr{F}}_\xi f\cdot\bar{g}$$

$$=\sum_{j=1}^{2\rho}\int_{a\leqq\lambda_j(\eta)\leqq b}\Phi_j^\pm\bar{\mathscr{F}}_\xi f\cdot\bar{g}\,d\eta=\sum_{j=1}^{2\rho}(\Phi_j^\pm\bar{\mathscr{F}}_\xi f,\ g_j(\Delta))_0$$

$$=\sum_{j=1}^{2\rho}(f,\ \mathscr{F}_{x'}\Phi_j^{\pm *}g_j(\Delta)),$$

where $g_j(\Delta)=g(\eta)$ for $a\leqq\lambda_j(\eta)\leqq b$, $=0$ otherwise. So we have

(6.15) $$0=\sum_{j=1}^{2\rho}\mathscr{F}_{x'}\Phi_j^{\pm *}g_j(\Delta)$$

$$=\sum_{j=1}^{2\rho}\mathscr{F}_{x'}\left[\int_{\Xi^n}(2\pi)^{-\frac{n-1}{2}}e^{ix'\cdot\xi}\tilde{\Psi}_j^\pm(x_n,\ \eta)g_j(\Delta)d\eta\right]$$

$$=\sum_{j=1}^{2\rho}\int_{-\infty}^{\infty}\tilde{\Psi}_j^\pm(x_n,\ \eta)g_j(\Delta)d\sigma$$

$$=\sum_{j=1}^{2\rho}\int_{\tilde{F}_j^\pm(\xi)\cap\{a\leqq\lambda_j(\eta)\leqq b\}}\tilde{\Psi}_j^\pm(x_n,\ \eta)g(\eta)d\sigma,$$

where the $\tilde{F}_j^\pm(\xi)$ denote the sets $\{\sigma\in\Xi;\ \tau_l^\pm(\xi;\ \lambda_j(\eta)\pm i0)\neq\sigma$ for all l, $1\leqq l\leqq\rho\}$. It follows from (6.15) that

(6.16) $$\sum_{j=1}^{2\rho}\sum_{\sigma\in\tilde{F}^\pm(\xi),\lambda_j(\eta)=v}\tilde{\Psi}_j^\pm(x_n,\ \eta)g(\eta)=0$$

for almost every $(x_n,\ \xi,\ v)\in\mathbf{R}^1_+\times\Xi^{n-1}\times\mathbf{R}^1$. Let $(\xi,\ v)$ be fixed such that (6.16) holds for almost every $x_n\in\mathbf{R}^1_+$. The number of σ, which satisfy $\sigma \in \tilde{F}_j^{\pm}(\xi)$ and $\lambda_j(\eta)=v$, is at most finite. Denote by $\sigma_{j1},\dots,\sigma_{jk_j}$ these values of σ. Then,

$$\sum_{j=1}^{2\rho}\sum_{\nu=1}^{k_j}\tilde{\Psi}_j^{\pm}(x_n,\xi,\sigma_{j\nu})g(\xi,\sigma_{j\nu})=0.$$

On the other hand we can write $\tilde{\Psi}_j^{\pm}(x_n,\xi,\sigma)$ in the form

$$\tilde{\Psi}_j^{\pm}(x_n,\xi,\sigma)=[(2\pi)^{-\frac{1}{2}}e^{ix_n\sigma}+(2\pi)^{-\frac{1}{2}}\sum_{\nu=1}^{p}C_\nu(\eta)e^{i\tau_\nu^{+}(\xi;v\pm i0)x_n}$$

$$+(2\pi)^{-\frac{1}{2}}\sum_{\nu=p+1}^{\rho}P_\nu(x_n,\eta)e^{i\tau_\nu^{+}(\xi;v\pm i0)x_n}]P_j(\eta),$$

where the $P_\nu(x_n,\eta)$ are matrices whose entries are polynomials with respect to x_n. Therefore it follows that

$$0=\sum_{j=1}^{2\rho}\sum_{\nu=1}^{k_j}(2\pi)^{-\frac{1}{2}}e^{ix_n\sigma_{j\nu}}g_j(\xi,\sigma_{j\nu}) \tag{6.17}$$

$$+\sum_{\mu=1}^{p}g_\mu(\xi;\sigma_{11},\dots,\sigma_{2\rho k_{2\rho}})e^{i\tau_\mu^{+}(\xi;v\pm i0)x_n}$$

$$+\sum_{\mu=p+1}^{\rho}g_\mu(x_n,\xi;\sigma_{11},\dots,\sigma_{2\rho k_{2\rho}})e^{i\tau_\mu^{+}(\xi;v\pm i0)x_n},$$

where the $g_\mu(x_n,\xi;\sigma_{11},\dots,\sigma_{2\rho k_{2\rho}})$ are polynomials with respect to x_n. By (L.1) we see that

$$\sigma_{j\nu}\neq\sigma_{j'\nu'} \quad \text{if} \quad (j,\nu)\neq(j',\nu').$$

Moreover, from $\sigma_{j\nu}\in\tilde{F}_j^{\pm}(\xi)$ it follows that

$$\tau_\mu^{+}(\xi;v\pm i0)\neq\sigma_{j\nu}.$$

Hence, from the linear independence of the functions of x_n, $e^{ix_n\sigma_{j\nu}}$ and $e^{i\tau_\mu^{+}(\xi;v\pm i0)x_n}$, we have $g_j(\xi,\sigma_{j\nu})=0$, $1\leqq j\leqq 2\rho$, $1\leqq\nu\leqq k_j$. Thus, $g_j(\eta)=0$ holds for almost all $\eta\in F_j^{\pm}$ and, by $\operatorname{Supp} g_j(\eta)\subset F_j^{\pm}$, we obtain $g_j(\eta)=0$ in $L^2(\Xi^n)$, $1\leqq j\leqq 2\rho$, that is, $g(\eta)=0$ in $L^2(\Xi^n)$. This completes the proof of Theorem 6.4. Q.E.D.

§7. Some Remarks and Examples

First we shall consider the case where the condition (L.2) is removed, that is, the non-elliptic case. Let the A_j be $N\times N$ Hermitian matrices and B an $l\times N$ matrix with rank l as stated in §0. In this case N is not necessarily even. Put $p=\operatorname{rank} A_n$. Then (L.1) implies that $\operatorname{rank} A_j=p$, $1\leqq j\leqq n$, and that p is even. Moreover (B.1) implies that $l=p/2$. Under the condition (B.1) it follows from [5] that the operator $\mathscr{A}$ defined in §0 is essentially self-adjoint. For the fundamental solution $E(x-y;\lambda)$ we obtain the same estimates as in the elliptic case. For we deal with only the case where λ is away from zero. Hence we have only to estimate the compensating kernel $E_c(x, y; \lambda)$ $(\tilde{E}_c(\xi, x_n, y; \lambda))$. Recall that $\tilde{E}_c(\xi, x_n, y; \lambda)$ satisfies the equations

$$(1.5)' \qquad \left[\frac{1}{i}A_n\frac{d}{dx_n}-(\lambda I-A(\xi, 0))\right]\tilde{E}_c(\xi, x_n, y; \lambda)=0,\ x_n>0,$$

$$(1.6)' \qquad B\tilde{E}_c(\xi, +0, y; \lambda)=\mathscr{F}_{x'}[BE(x-y;\lambda)|_{x_n=0}].$$

There exists a unitary matrix T such that

$$(7.1) \qquad T^*A_nT=\begin{bmatrix} a_1 & & & & & \\ & \ddots & & & 0 & \\ & & a_{2l} & & & \\ & & & 0 & & \\ & 0 & & & \ddots & \\ & & & & & 0 \end{bmatrix}\equiv\Lambda,$$

where $a_1,\dots, a_{2l}$ are the repeated non-zero eigenvalues of A_n. Replacing $\tilde{E}_c(\xi, x_n, y; \lambda)$ by an N-dimensional vector $u(x_n, \xi; \lambda)$ and $\mathscr{F}_{x'}[BE(x-y;\lambda)|_{x_n=0}]$ by an l-dimensional vector g and putting

$$(7.2) \qquad v(x_n, \xi; \lambda)=T^*u(x_n, \xi; \lambda),$$

we have for non-real λ

$$(1{,}5)'' \qquad \left(\frac{1}{i}\Lambda\frac{d}{dx_n}-T^*(\lambda I-A(\xi, 0))T\right)v(x_n, \xi; \lambda)=0,\ x_n>0,$$

$$(1.6)'' \qquad BTv(0, \xi; \lambda)=g.$$

Put

$$\begin{pmatrix} N_{11}(\xi;\lambda) & N_{12}(\xi) \\ N_{21}(\xi) & N_{22}(\xi;\lambda) \end{pmatrix} = T^*[\lambda I - A(\xi, 0)]T, \tag{7.3}$$

where $N_{11}(\xi;\lambda)$ is a $2l\times 2l$ matrix. It is clear that $N_{11}(\xi;\lambda)^* = N_{11}(\xi;\bar{\lambda})$ and $N_{21}(\xi)^* = N_{12}(\xi)$. Moreover from the condition (L.1) it follows that $N_{22}(\xi;\lambda) = \lambda I_{N-2l}$. Put

$$\begin{pmatrix} v_1(x_n, \xi;\lambda) \\ v_2(x_n, \xi;\lambda) \end{pmatrix} = v(x_n, \xi;\lambda), \tag{7.4}$$

where v_1 is a $2l$-dimensional vector. Finally we obtain

$$\left(\frac{1}{i}\frac{d}{dx_n} - M(\xi;\lambda)\right)v_1(x_n, \xi;\lambda) = 0,\ x_n > 0, \tag{7.5}$$

$$\tilde{B}(\xi;\lambda)v_1(0, \xi;\lambda) = g, \tag{7.6}$$

where

$$M(\xi;\lambda) = \begin{bmatrix} \frac{1}{a_1} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{a_{2l}} \end{bmatrix} \left(N_{11}(\xi;\lambda) - \frac{1}{\lambda}N_{12}(\xi)N_{21}(\xi)\right), \tag{7.7}$$

$$\tilde{B}(\xi;\lambda) = BT \begin{bmatrix} I_{2l} \\ -\frac{1}{\lambda}N_{21}(\xi) \end{bmatrix}, \tag{7.8}$$

Therefore we can discuss the expansion problem in the same way as we did in §§1–6 and obtain the expansion theorem in the same form as Theorem 6.2.

Next we consider

$$\frac{1}{i}\sum_{j=1}^{3} A_j \frac{\partial v}{\partial x_j}(x) - \lambda v(x) = g(x),\ x \in \mathbf{R}^3_+, \tag{7.9}$$

$$Bv(x)|_{x_3=0} = 0, \tag{7.10}$$

where

$$A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\quad A_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\quad A_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\quad B = (1, 1). \tag{7.11}$$

It is easily seen that the above system satisfies the conditions (L.1)–(L.3) and (B.1). However, the condition (B.2) is not satisfied (see, [6]). We observe first that

$$\Delta(\xi;\lambda)=\lambda+\xi_1. \tag{7.12}$$

By Theorem 6.2 we have for $f\in L^2(\mathbf{R}^3_+)$

$$f(x)=\sum_{j=1}^{4}\int_{\Xi^3}\Psi_j^+(x,\eta)\hat{f}_j^+(\eta)d\eta, \tag{7.13}$$

$$\hat{f}_j^+(\eta)=\int_{\mathbf{R}^3_+}\Psi_j^+(x,\eta)^*f(x)dx,\qquad 1\leqq j\leqq 4, \tag{7.14}$$

where

$$\Psi_1^+(x,\eta)=\begin{cases} 0 \qquad (\sigma>0),\\ (2\pi)^{-\frac{3}{2}}\dfrac{1}{2|\eta|}e^{ix'\cdot\xi}\Big[e^{ix_3\sigma}\begin{pmatrix}|\eta|+\sigma, & \xi_1-i\xi_2\\ \xi_1+i\xi_2, & |\eta|-\sigma\end{pmatrix}\\ \quad -\dfrac{e^{-ix_3\sigma}}{|\eta|+\xi_1}\begin{pmatrix}\varphi_1(\eta), & \overline{\varphi_2(\xi,-\sigma)}\\ \varphi_2(\xi,\sigma), & \varphi_1(\eta)\end{pmatrix}\Big] \quad (\sigma<0), \end{cases} \tag{7.15}$$

$$\Psi_2^+(x,\eta)=\begin{cases} 0 \qquad (\sigma<0),\\ (2\pi)^{-\frac{3}{2}}\dfrac{1}{2|\eta|}e^{ix'\cdot\xi}\Big[e^{ix_3\sigma}\begin{pmatrix}|\eta|-\sigma, & -\xi_1-i\xi_2\\ -\xi_1+i\xi_2, & |\eta|+\sigma\end{pmatrix}\\ \quad -\dfrac{e^{-ix_3\sigma}}{|\eta|-\xi_1}\begin{pmatrix}\varphi_1(-\xi_1,\xi_2,\sigma), & \varphi_2(-\xi_1,\xi_2,\sigma)\\ \overline{\varphi_2(-\xi_1,\xi_2,-\sigma)}, & \varphi_1(-\xi_1,\xi_2,\sigma)\end{pmatrix}\Big]\\ \qquad\qquad (\sigma>0), \end{cases} \tag{7.16}$$

$$\Psi_3^+(x,\eta)=\begin{cases} 0 \qquad (\xi_2>0)\\ (2\pi)^{-\frac{3}{2}}\dfrac{i\xi_2}{2|\eta|(\xi_1+|\eta|)}e^{ix'\cdot\xi}e^{x_3\xi_2}\\ \qquad \begin{pmatrix}\varphi_3(\eta), & \overline{\varphi_3(\xi,-\sigma)}\\ \varphi_3(\eta), & -\overline{\varphi_3(\xi,-\sigma)}\end{pmatrix} \quad (\xi_2<0), \end{cases} \tag{7.17}$$

$$\text{(7.18)} \qquad \Psi_4^{+}(x,\eta)=\begin{cases} 0 & (\xi_2>0) \\ (2\pi)^{-\frac{3}{2}}\dfrac{i\xi_2}{2|\eta|(\xi_1-|\eta|)}e^{ix'\cdot\xi}e^{x_3\xi_2} \\ \qquad \begin{pmatrix} \varphi_3(-\eta), & \overline{\varphi_3(-\xi,\sigma)} \\ -\varphi_3(-\eta), & -\overline{\varphi_3(-\xi,\sigma)} \end{pmatrix} & (\xi_2<0), \end{cases}$$

$$\text{(7.19)} \qquad \varphi_1(\eta)=|\xi|^2+|\eta|\xi_1-i\xi_2\sigma,$$

$$\text{(7.20)} \qquad \varphi_2(\eta)=\sigma^2+\xi_1^2+|\eta|\sigma+\xi_1|\eta|+\sigma\xi_1+i(|\eta|\xi_2+\sigma\xi_2+\xi_1\xi_2),$$

$$\text{(7.21)} \qquad \varphi_3(\eta)=|\eta|+\sigma+\xi_1+i\xi_2.$$

Moreover it follows from Theorem 5.10 that the spectrum $\sigma(\mathbf{A})$ of the operator $\mathbf{A}$ defined by (7.9) and (7.10) is equal to $\mathbf{R}^1$ and $\sigma_{ac}(\mathbf{A})$, that is, $\sigma(\mathbf{A})=\sigma_{ac}(\mathbf{A})=\mathbf{R}^1$.

In order to show that the eigenfunction expansion for a single elliptic equation can be obtained in the same way we consider the following example:

$$\text{(7.22)} \qquad (-\Delta-\lambda)v(x)=g(x), \qquad x\in\mathbf{R}_+^n,$$

$$\text{(7.23)} \qquad \left[av(x)+b\frac{\partial v}{\partial x_n}(x)\right]_{x_n=0}=0,$$

where a and b are real and $|a|+|b|\neq 0$. Then

$$\text{(7.24)} \qquad \Delta(\xi;\lambda)=ib\sqrt{\lambda-|\xi|^2}+a,$$

where $\operatorname{Im}\sqrt{\lambda-|\xi|^2}>0$ for $\operatorname{Im}\lambda\neq 0$. If $a\cdot b<0$ or $b=0$, $\Delta(\xi;k)\neq 0$ for $|k|+|\xi|\neq 0$. Thus we have for $f\in L^2(\mathbf{R}_+^n)$

$$\text{(7.25)} \qquad f(x)=\int_{\Xi^n}\Psi_1^{\pm}(x,\eta)\hat{f}_1^{\pm}(\eta)d\eta,$$

$$\text{(7.26)} \qquad \hat{f}_1^{\pm}(\eta)=\int_{\mathbf{R}_+^n}\Psi_1^{\pm}(x,\eta)^*f(x)dx,$$

where

$$(7.27)\qquad \Psi_1^{\pm}(x,\eta)=\begin{cases} 0 & (\pm\sigma>0),\\ (2\pi)^{-\frac{n}{2}}e^{ix'\cdot\xi}\left[e^{ix_n\sigma}-e^{-ix_n\sigma}\dfrac{a+bi\sigma}{a-bi\sigma}\right] & (\pm\sigma<0)\,. \end{cases}$$

If $a\cdot b>0$ or $a=0$, $\Delta(\xi;k)=0$ for $k=|\xi|^2-\frac{a^2}{b^2}$. Therefore putting

$$(7.28)\qquad \Psi_2(x,\eta)=(2\pi)^{-\frac{n}{2}}\frac{2ai}{b\sigma+ai}e^{ix'\cdot\xi}e^{-x_n\frac{a}{b}},$$

we have for $f\in L^2(\mathbf{R}^n_+)$

$$(7.29)\qquad f(x)=\int_{\Xi^n}\Psi_1^{\pm}(x,\eta)\hat{f}_1^{\pm}(\eta)d\eta+\int_{\Xi^n}\Psi_2(x,\eta)\hat{f}_2(\eta)d\eta,$$

$$(7.30)\qquad \hat{f}_2(\eta)=\int_{\mathbf{R}^n_+}\Psi_2(x,\eta)^*f(x)dx\,.$$

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References

[1] Carleman, T., Sur la théorie mathématique de l'équation de Schrödinger, *Arkiv Mat. Astr. o. Fys.*, **24**, (1934) 1–7.

[2] Ikebe, T., Eigenfunction expansions associated with the Schrödinger operators and their applications to scattering theory, *Arch. Rational Mech. Anal.*, **5**, (1960) 1–34.

[3] ———, Orthogonality of the eigenfunctions for the exterior problem connected with $-\Delta$, *Arch. Rational Mech. Anal.*, **19**, (1965) 71–73.

[4] Kato, T., *Perturbation Theory for Linear Operators*, Springer, Berlin (1966).

[5] Lax, P. D. and Phillips, R. S., Local boundary conditions for dissipative symmetric linear differential operators, *Comm. Pure Appl. Math.*, **13**, (1960) 427–455.

[6] ——— and ———, *Scattering Theory*, Academic Press, New York (1967).

[7] Matsumura, M., Comportement asymptotique de solutions de certains problèms mixtes pour des systèmes hyperboliques symétriques à coefficients constants, *Publ. RIMS, Kyoto Univ.*, **5**, (1970) 301–360.

[8] Mochizuki, K., Spectral and scattering theory for symmetric hyperbolic systems in an exterior domain, *Publ. RIMS, Kyoto Univ.*, **5**, (1969) 219–258.

[9] Pólya, G. und Szegö, G., *Aufgaben und Lehrsätze aus der Analysis II*, Springer-Verlag, Berlin (1964).

[10] Povzner, A. Ya., On the expansions of arbitrary functions in terms of the eigenfunctions of the operator $-\Delta u+cu$ (in Russian), *Mat. Sb.*, **32**, (1953) 109–156=*Amer. Math. Soc. Transl., Ser.* 2, **60**, (1967) 1–50.

[11] Schulenberger, J. R. and Wilcox, C. H., Eigenfunction expansions and scattering theory for wave propagation problems of classical physics, *ONR Technical Summary Report, No.* 14, Univ. of Denver (August, 1971).

[12] Shenk II, N. A., Eigenfunction expansions and scattering theory for the wave equation in an exterior region, *Arch. Rational Mech. Anal.,* **21**, (1966) 120–150.

[13] Shizuta, Y., Eigenfunction expansions associated with the operator $-\Delta$ in the exterior domain, *Proc. Japan Acad.,* **39**, (1963) 656–660.

[14] Stone, M. H., Linear Transformations in Hilbert Space and their Applications to Analysis, *Amer. Math. Soc. Colloquium Publ.* vol. **XV** (1932).

[15] Wilcox, C. H., Wave operators and asymptotic solutions of wave propagation problems of classical physics, *Arch. Rational Mech. Anal.,* **22**, (1966) 37–78.

[16] Yosida, K., *Functional Analysis,* Springer, Berlin (1965).