

A Rigid Analytical Regulator for the K_2 of Mumford Curves

by

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Abstract

We construct a rigid analytical regulator for the K_2 of Mumford curves, a non-archimedean analogue of the complex analytical Beilinson–Bloch–Deligne regulator.

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1. Introduction

Motivation 1.1. Suitably normalized, the Beilinson–Bloch–Deligne regulator is a morphism of functors (for definition and properties see [7], pages 18–23):

$$\{\cdot\} : H_{\mathcal{M}}^2(U, \mathbb{Z}(2)) \rightarrow H_{an}^1(U, \mathbb{C}^*) = \text{Hom}(\pi_1^{ab}(U), \mathbb{C}^*),$$

defined on the category of Riemann surfaces such that

- (i) if $U = Y - \{x\}$, where Y is a Riemann surface, $x \in Y$ is a point, and $f, g \in H^0(U, \mathcal{O}^*)$ are meromorphic at x , then the value of $\{f \otimes g\}$ on the positively oriented loop around x is the tame symbol $\{f, g\}_x$ at x ,
- (ii) if $f_n \rightarrow f$ and $g_n \rightarrow g$ converge uniformly on compact sets, where f_n, f, g_n and g are elements of $H^0(U, \mathcal{O}^*)$, then the value of $\{f_n \otimes g_n\}$ on every closed loop in U converges to the value of $\{f \otimes g\}$ on that loop.

The second property follows from the integral representation of the monodromy. We wish to construct a regulator which is analogous to the monodromy of the Beilinson–Bloch–Deligne regulator on the boundary components of Riemann surfaces with boundary in the rigid analytical context. If we want the two properties

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above to hold, then we should define the regulator of two nowhere vanishing holomorphic functions f and g on a connected rational subdomain of the projective line around a complementary disk by approximating f and g by rational functions and set the regulator $\{f, g\}$ as the limit of the product of tame symbols at points inside the disk. This is exactly what we will do. As an application we will construct a rigid analytical regulator for the K_2 of Mumford curves whose properties strongly resemble those of the complex analytical Beilinson–Bloch–Deligne regulator. It is a homomorphism:

$$\{\cdot\} : H_{\mathcal{M}}^2(X, \mathbb{Z}(2)) \rightarrow \mathcal{H}(\Gamma(\mathfrak{X}_0), \mathbb{C}^*),$$

where \mathbb{C} is an algebraically closed field complete with respect to an ultrametric absolute value, X is a Mumford curve over \mathbb{C} and $\mathcal{H}(\Gamma(\mathfrak{X}_0), \mathbb{C}^*)$ denotes the group of \mathbb{C}^* -valued harmonic cochains on the oriented incidence graph of the special fiber \mathfrak{X}_0 of some semi-stable model \mathfrak{X} of X over the valuation ring of \mathbb{C} . The problem of finding rigid analytic analogues of the Beilinson–Bloch–Deligne regulator has already been studied for example in [3] by Coleman. Coleman’s methods are radically different from ours, and they cover different ground. On the other hand in the two papers [8] and [9] Kato constructed a regulator for higher local fields which is essentially the same as ours. Partially to generalize his results to fields whose valuation is not discrete, partially to be self-contained, we develop the foundations of this theory independently of his work.

Contents 1.2. In the second chapter we define the regulator first for connected rational subdomains of the projective line by approximating holomorphic functions on the domain by rational functions and taking the limit of the product of the tame symbols inside a complementary open disk as we already mentioned above. The main result of this section is Theorem 2.2 which gives a complete characterization of our regulator. In the third chapter we first extend the definition of the rigid analytical regulator for the K_2 of connected rational subdomains of the projective line (Theorem 3.2). Then we prove the invariance theorem (Theorem 3.11) which is the functoriality property of the regulator for such subdomains. In order to do so we develop an elementary homology theory for connected rational subdomains of the projective line (Theorems 3.6 and 3.8). The fourth chapter is somewhat technical: it contains two auxiliary results used in the next chapter. In the fifth chapter we reap the fruits of our labours when we define the rigid analytical regulator for Mumford curves. The latter takes values in the group of \mathbb{R}^* -valued harmonic cochains on the oriented incidence graph of the special fiber of a semi-stable model of the Mumford curve where \mathbb{R} is the field of definition. We also show a reciprocity law relating the valuation of the rigid analytical regulator to the (generalized) tame symbol along the special fiber. We also formulate a functoriality property for

this regulator phrased in terms of measures on the ends of the universal covering of the oriented incidence graph of the special fiber (Proposition 5.6). In the last chapter we look at the particular case of the Drinfeld upper half plane and we express the tame symbol at the cusps as a non-archimedean integral of the rigid analytic regulator (Theorem 6.5).

Notation 1.3. In this paper we will use the somewhat incorrect notation $K_2(X)$ to denote $H_{\mathcal{M}}^2(X, \mathbb{Z}(2))$ for various types of spaces X as the latter is rather awkward.

2. The rigid analytical regulator for connected rational subdomains

Notation 2.1. Let \mathbb{C} be an algebraically closed field complete with respect to an ultrametric absolute value which will be denoted by $|\cdot|$. Let $|\mathbb{C}|$ denote the set of values of the latter. Let \mathbb{P}^1 denote the projective line over \mathbb{C} . For any $x \in \mathbb{P}^1$ and any two rational non-zero functions $f, g \in \mathbb{C}((t))$ on the projective line let $\{f, g\}_x$ denote the tame symbol of the pair (f, g) at x . We call a set $D \subset \mathbb{P}^1$ an *open disk* if it is the image of the set $\{z \in \mathbb{C} \mid |z| < 1\}$ under a Möbius transformation. Recall that a subset U of \mathbb{P}^1 is a *connected rational subdomain* if it is non-empty and it is the complement of the union of finitely many pairwise disjoint open disks. Let ∂U denote the set of these complementary open disks. The elements of ∂U are called the *boundary components* of U , by slight abuse of language. Let $\mathcal{O}(U), \mathcal{R}(U)$ denote the algebra of holomorphic functions on U and the subalgebra of restrictions of rational functions, respectively. Let $\mathcal{O}^*(U), \mathcal{R}^*(U)$ denote the group of invertible elements of these algebras. The group $\mathcal{R}^*(U)$ consists of rational functions which do not have poles or zeros lying in U . For each $f \in \mathcal{O}(U)$ let $\|f\|$ denote $\sup_{z \in U} |f(z)|$. This value is finite, and makes $\mathcal{O}(U)$ a Banach algebra over \mathbb{C} . We say that the sequence $f_n \in \mathcal{O}(U)$ converges to $f \in \mathcal{O}(U)$, denoted by $f_n \rightarrow f$, if f_n converges to f with respect to the topology of this Banach algebra, i.e. $\lim_{n \rightarrow \infty} \|f - f_n\| = 0$. For every real number $0 < \epsilon < 1$ we define the sets $\mathcal{O}_\epsilon(U) = \{f \in \mathcal{O}(U) \mid \|1 - f\| \leq \epsilon\}$, and $U_\epsilon = \{z \in \mathbb{C} \mid |1 - z| \leq \epsilon\}$.

The main result of this section is the following

Theorem 2.2. *There is a unique map $\{\cdot, \cdot\}_D : \mathcal{O}^*(U) \times \mathcal{O}^*(U) \rightarrow \mathbb{C}^*$ for every $D \in \partial U$, called the rigid analytic regulator, with the following properties:*

- (i) *For any two $f, g \in \mathcal{R}^*(U)$ their regulator is:*

$$\{f, g\}_D = \prod_{x \in D} \{f, g\}_x,$$

- (ii) the regulator $\{\cdot, \cdot\}_D$ is bilinear in both variables,
- (iii) the regulator $\{\cdot, \cdot\}_D$ is alternating: $\{f, g\}_D \cdot \{g, f\}_D = 1$,
- (iv) if $f, 1 - f \in \mathcal{O}(U)^*$, then $\{f, 1 - f\}_D$ is 1,
- (v) for each $f \in \mathcal{O}_\epsilon(U)$ and $g \in \mathcal{O}^*(U)$ we have $\{f, g\}_D \in U_\epsilon$.

Remark 2.3. It is an immediate consequence of property (v) that the rigid analytic regulator is continuous with respect to the supremum norm topology. Explicitly, if f and g are elements of $\mathcal{O}^*(U)$, $D \in \partial U$ is a boundary component, and $f_n \in \mathcal{O}^*(U)$, $g_n \in \mathcal{O}^*(U)$ are sequences such that $f_n \rightarrow f$ and $g_n \rightarrow g$, then the limit

$$\lim_{n \rightarrow \infty} \{f_n, g_n\}_D$$

exists, and it is equal to $\{f, g\}_D$.

Weil’s Reciprocity Law 2.4. *Let f, g be two non-zero rational functions on \mathbb{P}^1 defined over the field \mathbb{C} . Then the product of all tame symbols of the pair (f, g) is equal to 1:*

$$\prod_{x \in \mathbb{P}^1} \{f, g\}_x = 1.$$

Proof. See [12], Proposition 6, pages 44–46. Although it holds for smooth projective algebraic curves in general, we will only use this result in the case when the curve is \mathbb{P}^1 , when there is a simple direct proof as follows. The multiplicative group of the function field of \mathbb{P}^1 is generated by the elements $c \in \mathbb{C}^*$, and $z - a$, $a \in \mathbb{C}$. Since the tame symbols are bilinear and alternating, we only have to check the identity in the claim for pairs of these elements. This reduces our proof to three cases: $(z - a, z - b)$, when $a \neq b$, $(z - a, z - a)$ and $(c, z - a)$. We compute:

$$\begin{aligned} \prod_{x \in \mathbb{P}^1} \{z - a, z - b\}_x &= \{z - a, z - b\}_a \{z - a, z - b\}_b \{z - a, z - b\}_\infty \\ &= (a - b)^{-1} (b - a) (-1) = 1, \\ \prod_{x \in \mathbb{P}^1} \{z - a, z - a\}_x &= \{z - a, z - a\}_a \{z - a, z - a\}_\infty = (-1) (-1) = 1, \\ \prod_{x \in \mathbb{P}^1} \{c, z - a\}_x &= \{c, z - a\}_a \{c, z - a\}_\infty = cc^{-1} = 1. \quad \square \end{aligned}$$

Definition 2.5. Let U be a connected rational subdomain of \mathbb{P}^1 . For any $D \in \partial U$ and for any two $f, g \in \mathcal{R}^*(U)$ we define the rigid analytical regulator $\{f, g\}_D$ by the formula:

$$\{f, g\}_D = \prod_{x \in D} \{f, g\}_x \in \mathbb{C}^*.$$

Since the product on the right hand side is finite, the regulator is well-defined. For each $D \in \partial U$ and $f \in \mathcal{R}(U)$ let $\text{deg}_D(f)$ denote the number of zeros z of f with $z \in D$ counted with multiplicities minus the number of poles z of f with $z \in D$ counted with multiplicities. For every real number $0 < \epsilon < 1$ we define the set $\mathcal{R}_\epsilon(U)$ as the intersection $\mathcal{R}_\epsilon(U) = \mathcal{O}_\epsilon(U) \cap \mathcal{R}(U)$. We also define the sets $\mathcal{O}_1(U) = \bigcup_{0 < \epsilon < 1} \mathcal{O}_\epsilon(U)$ and $\mathcal{R}_1(U) = \bigcup_{0 < \epsilon < 1} \mathcal{R}_\epsilon(U)$.

- Lemma 2.6.** (i) *The regulator is a bilinear map: $\{\cdot, \cdot\}_D : \mathcal{R}^*(U) \otimes \mathcal{R}^*(U) \rightarrow \mathbb{C}^*$,*
 (ii) *the regulator $\{\cdot, \cdot\}_D$ is alternating: $\{f, g\}_D \cdot \{g, f\}_D = 1$,*
 (iii) *$f, 1 - f \in \mathcal{R}^*(U)$, then $\{f, 1 - f\}_D$ is 1.*
 (iv) *for any $f \in \mathcal{R}^*(U)$, $c \in \mathbb{C}^*$ we have $\{c, f\}_D = c^{\text{deg}_D(f)}$.*

Proof. The first three claims hold because they hold for the tame symbol. Claim (iv) is obvious. □

Lemma 2.7. *The set $\mathcal{R}_\epsilon(U)$ is a subgroup of $\mathcal{R}^*(U)$.*

Proof. Note that $\mathcal{R}_\epsilon(U)$ is a subset of $\mathcal{R}^*(U)$, so we have to show that it is a group with respect to multiplication. If $f \in \mathcal{R}_\epsilon(U)$ then $|f(z)| = 1$ for each $z \in U$ by the ultrametric inequality. Hence

$$\|1 - f^{-1}\| = \sup_{z \in U} |f(z)|^{-1} |f(z) - 1| \leq \epsilon.$$

Similarly for any $f, g \in \mathcal{R}_\epsilon(U)$ we have

$$\begin{aligned} \|1 - fg\| &\leq \sup_{z \in U} \max(|1 - f(z)|, |f(z) - f(z)g(z)|) \\ &\leq \max(\sup_{z \in U} |1 - f(z)|, \sup_{z \in U} |f(z)| \cdot |1 - g(z)|) \leq \epsilon. \end{aligned} \quad \square$$

Definition 2.8. We call a set $D \subset \mathbb{P}^1$ a *closed disk* if it is the image of the set $\{z \in \mathbb{C} \mid |z| \leq 1\}$ under a Möbius transformation. This terminology might be confusing to some as closed disks are open and closed in the natural topology of \mathbb{P}^1 , just like open disks. On the other hand closed disks are rational subdomains while open disks are not. For every open disk D let \overline{D} denote the unique closed disk which contains D and minimal with respect to this property. Moreover let ∂D denote the complement of D in \overline{D} : this notation will not cause confusion because open disks are not rational subdomains. For every $z \in \mathbb{C}$ and positive number $\rho \in |\mathbb{C}|$ let $D(z, \rho)$, $\overline{D}(z, \rho)$, $D(\infty, \rho)$ and $\overline{D}(\infty, \rho)$ denote the following open disks and closed disks:

$$\begin{aligned} D(z, \rho) &= \{z \in \mathbb{C} \mid |z| < \rho\}, & \overline{D}(z, \rho) &= \{z \in \mathbb{C} \mid |z| \leq \rho\}, \\ D(\infty, \rho) &= \{z \in \mathbb{C} \mid |z| > \rho^{-1}\} \cup \{\infty\}, & \overline{D}(\infty, \rho) &= \{z \in \mathbb{C} \mid |z| \geq \rho^{-1}\} \cup \{\infty\}. \end{aligned}$$

Let \mathbb{C}^0 , \mathbb{C}^{00} and \mathbf{k} denote the valuation ring $\{z \in \mathbb{C} \mid |z| \leq 1\}$ of \mathbb{C} , its maximal ideal $\{z \in \mathbb{C} \mid |z| < 1\}$, and its residue field $\mathbb{C}^0/\mathbb{C}^{00}$, respectively. Let $\bar{z} \in \mathbf{k}$ denote the reduction of every $z \in \mathbb{C}^0$ modulo the ideal \mathbb{C}^{00} . For every finite set $S \subset \mathbf{k}$ let $U(S)$ denote the set $\{z \in \mathbb{C}^0 \mid \bar{z} \notin S\}$. This set is a connected rational subdomain of \mathbb{P}^1 , because it is the complement of the pairwise disjoint disks $D(\infty, 1)$ and $D(s, 1)$, where s is an element of $S \subset \mathbb{C}^0$, a set of representatives of the residue classes in S . Finally let S^c denote the complement of every subset S of \mathbb{P}^1 .

Lemma 2.9. *Assume that $0 \in S$. Then every $f \in \mathcal{R}_\epsilon(U(S))$ can be written in the form:*

$$f(z) = c \prod_{a \in D(\infty, 1)} \left(1 - \frac{z}{a}\right)^{v(a)} \cdot \prod_{s \in S} \prod_{a \in D(s, 1)} \left(1 - \frac{a-s}{z-s}\right)^{v(a)},$$

where $c \in U_\epsilon$ and for each $a \in \mathbb{C}$ the integer $v(a)$ is the multiplicity of a in the divisor of f .

Proof. The rational function:

$$g(z) = \prod_{a \in D(\infty, 1)} \left(1 - \frac{z}{a}\right)^{v(a)} \cdot \prod_{s \in S} \prod_{a \in D(s, 1)} \left(1 - \frac{a-s}{z-s}\right)^{v(a)},$$

is in $\mathcal{R}_1(U(S))$, because it is a product of elements of $\mathcal{R}_1(U(S))$, which is a group, since it is the union of a chain of groups. The rational function $f(z)/g(z)$ is also in $\mathcal{R}_1(U(S))$, but it has no zeros or poles in $\mathbb{C} - S$, so it must be equal to the function $c \prod_{s \in S} (z-s)^{n(s)}$ for some $c \in \mathbb{C}^*$ and $n(s) \in \mathbb{Z}$. Since $\|f/g\| = 1$, we have

$$1 = \sup_{z \in U(S)} \left| c \prod_{s \in S} (z-s)^{n(s)} \right| = |c|.$$

Hence $c \in \mathbb{C}^0$, so f/g is a rational function with coefficients on \mathbb{C}^0 . Its reduction modulo the ideal \mathbb{C}^{00} is $r(z) = \bar{c} \prod_{s \in S} (z-s)^{n(s)}$. Since $f(z)/g(z)$ is in $\mathcal{R}_1(U(S))$, the rational function $r(z)$ is identically one on $\mathbf{k} - S$. Hence it is constant as \mathbf{k} is algebraically closed, therefore $f(z) = cg(z)$. By the Proposition of I.1.3 in [4], page 7, we know that every $h(z) \in \mathcal{O}(U(S))$ has a generalized Laurent expansion:

$$h(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{s \in S} \sum_{n=1}^{\infty} b_n^s (z-a)^{-n},$$

and $\|h(z)\| = \max(\max_{n=0}^{\infty} |a_n|, \max_{s \in S} (\max_{n=1}^{\infty} |b_n^s|))$. The constant term a_0 in the generalized Laurent expansion of $g(z)$ is 1, so the constant term of $f(z)$ is c and we have $|1-c| \leq \|1-f(z)\| \leq \epsilon$. \square

Proposition 2.10. *Every $f \in \mathcal{R}_\epsilon(U)$ can be written in the form:*

$$f(z) = \prod_{D \in \partial U} f_D(z),$$

where $f_D(z) \in \mathcal{R}_\epsilon(D^c)$ for all $D \in \partial U$, and these functions are uniquely determined up to a constant factor in U_ϵ .

Proof. It is clear that the functions $f_D(z)$ are uniquely determined up to a factor in U_ϵ , so we only have to show that they exist. Assume first that U is of the form $U(S)$ for some finite set $S \subset \mathbf{k}$. The claim holds trivially for the domain $U(S)$ if the set S is empty. Otherwise we might assume that $0 \in S$ by a linear change of coordinates. In this case we know that

$$f(z) = f_\infty(z) \cdot \prod_{s \in S} f_s(z),$$

where $f_\infty(z) \in \mathcal{R}(\overline{D}(0, 1)) \cap \mathcal{R}_1(U)$, and $f_s(z) \in \mathcal{R}(\overline{D}(\infty, 1)) \cap \mathcal{R}_1(U)$ for all $s \in S$ by Lemma 2.9. Moreover $f_\infty(0) \in U_\epsilon$ and $f_s(\infty) \in U_\epsilon$ for all $s \in S$. However the functions $f_\infty(z)$ and $f_s(z)$ are unique up to a factor in U_ϵ , so $f_\infty(a) \in U_\epsilon$ and $f_s(b) \in U_\epsilon$ for all $a \in D(0, 1)$ and $b \in D(\infty, 1)$ for all $s \in S$, which can be seen by anticipating the automorphisms $z \mapsto z + a$ and $\frac{1}{z} \mapsto \frac{1}{z} + \frac{1}{b}$ of U , respectively. Let $1 - f_\infty(z) = \sum_{n=0}^\infty a_n z^n$. There is an $N(\epsilon) \in \mathbb{N}$ such that $|a_n| < \epsilon$ if $n > N(\epsilon)$. Let $B = \{b \in D(0, 1) \mid |b|^{n-m} \neq |a_m/a_n|, \forall n, \forall m \leq N(\epsilon)\}$. If $b \in B$, then

$$\max_{n \leq N(\epsilon)} |a_n| \cdot |b|^n = \left| \sum_{n \leq N(\epsilon)} a_n b^n \right| \leq \max\left(|1 - f_\infty(b)|, \left| \sum_{n > N(\epsilon)} a_n b^n \right|\right) \leq \epsilon.$$

Since $\sup_{b \in B} |b| = 1$, the inequality above implies that $\max_{n \leq N(\epsilon)} |a_n| \leq \epsilon$, so $\|1 - f_\infty(z)\| = \max_{n \in \mathbb{N}} |a_n| \leq \epsilon$. Hence $f_\infty \in \mathcal{R}_\epsilon(\overline{D}(0, 1))$, and a similar argument shows that $f_s \in \mathcal{R}_\epsilon(\overline{D}(\infty, 1))$ for all $s \in S$, so the claim holds for domains of the form $U(S)$.

In the general case we prove the proposition by induction on the cardinality of ∂U . When ∂U is empty the claim is obvious. Otherwise let D be a boundary component of U . If $\partial D \not\subseteq U$ then there is a $D' \in \partial U$ such that $\emptyset \neq \partial D \cap D'$. We claim that $D' \subset \partial D$ whenever D, D' are two disjoint disks such that $\emptyset \neq \partial D \cap D'$, and $D \cup D' \neq \mathbb{P}^1$. We may assume that $\infty \notin D \cup D'$ by a linear change of coordinates. Let $a \in \partial D \cap D'$: then $\overline{D} = \overline{D}(a, \rho_1)$ and $D' = D(a, \rho_2)$ for some ρ_1, ρ_2 , where \overline{D} denotes the closure of D . If $\rho_2 > \rho_1$, then $D \subset \overline{D}(a, \rho_1) \subset D(a, \rho_2)$ which is impossible. Therefore $\rho_2 \leq \rho_1$, so $D' = D(a, \rho_2) \subseteq \overline{D}(a, \rho_1) - D = \partial D$. We define the relation $D' \leq D$ on the set ∂U by the rule $D' \subset \partial D$. This is clearly a partial ordering. Let $\mathcal{D} \subseteq \partial U$ be an equivalence class of minimal elements with respect to this ordering. This means that all $D \in \mathcal{D}$ are minimal with respect to

this ordering, and any $D' \in \partial U$ is an element of \mathcal{D} if and only if $D \leq D'$ and $D' \leq D$ for some (and hence all) $D \in \mathcal{D}$.

Choose a boundary component $D \in \mathcal{D}$ and make a linear change of coordinates such that $D = D(0, 1)$. Then each $D' \in \mathcal{D}$ is of the form $D(s, 1)$, where $s \in \mathcal{S}$, and the latter is a finite subset of \mathbb{C}^0 . Since these disks are pairwise disjoint, \mathcal{S} injects into \mathbf{k} with respect to the reduction modulo the ideal \mathbb{C}^{00} . If S denotes the image of \mathcal{S} with respect to this map, then the set $E = \bigcap_{D \in \mathcal{D}} \partial D$ is equal to $U(S)$. Therefore E is an affinoid subdomain of U , and we have already proved in the previous paragraph that the claim holds for this domain. Hence

$$f(z) = g_{\mathcal{D}}(z) \cdot \prod_{D \in \mathcal{D}} f_D(z)$$

on this set, where $g_{\mathcal{D}}(z) \in \mathcal{R}_{\epsilon}(\overline{D}(0, 1))$ and $f_D(z) \in \mathcal{R}_{\epsilon}(D^c)$ for all $D \in \mathcal{D}$. Since the set E is infinite, the equation above holds for all $z \in \mathbb{P}^1$. Hence

$$g_{\mathcal{D}}(z) = \frac{f(z)}{\prod_{D \in \mathcal{D}} f_D(z)} \in \mathcal{R}_{\epsilon}(U),$$

so $g_{\mathcal{D}}(z) \in \mathcal{R}_{\epsilon}(Y)$, too, where $Y = U \cup \overline{D}(0, 1)$. Because $|\partial Y| < |\partial U|$, the induction hypothesis implies that $g_{\mathcal{D}}(z) = \prod_{D' \in \partial Y} f_{D'}(z)$ with $f_{D'}(z) \in \mathcal{R}_{\epsilon}(D'^c)$. Since $\partial U = \partial Y \cup \mathcal{D}$, the claim follows. \square

Proposition 2.11. *For every boundary component $D \in \partial U$ and for each $f \in \mathcal{R}_{\epsilon}(U)$ and $g \in \mathcal{R}^*(U)$ we have $\{f, g\}_D \in U_{\epsilon}$.*

Proof. If the cardinality $|\partial U| = 1$ then the claim is clear, since by Weil’s reciprocity $\{f, g\}_D = 1$. Hence we may assume that $\infty \in D^c - U$ by choosing an appropriate coordinate function z . Since U_{ϵ} is a group, it is sufficient to prove the claim for $g(z)$ equal to one of the following generators of $\mathcal{R}^*(U)$: $c \in \mathbb{C}^*$, $z - a$, where $a \in D$, and $z - b$, where $b \in D^c - U$. Then

$$\{f(z), c\}_D = \prod_{D' \in \partial U} \{f_{D'}(z), c\}_D = \{f_D(z), c\}_D = c^{-\deg_D(f_D)}.$$

But $\deg_D(f_D)$ is zero, since f_D has no zeros or poles in D^c , hence $\{f(z), c\}_D = 1$. Also

$$\begin{aligned} \{f(z), z - a\}_D &= \{f_D(z), z - a\}_D \cdot \prod_{D' \in \partial U - \{D\}} \{f_{D'}(z), z - a\}_D \\ &= \prod_{x \in D^c} \{f_D(z), z - a\}_x^{-1} \cdot \prod_{D' \in \partial U - \{D\}} \{f_{D'}(z), z - a\}_a \\ &= \{f_D(z), z - a\}_{\infty}^{-1} \cdot \prod_{D' \in \partial U - \{D\}} f_{D'}(a) = f_D(\infty) \cdot \prod_{D' \in \partial U - \{D\}} f_{D'}(a) \in U_{\epsilon}, \end{aligned}$$

where we used Weil's reciprocity law in the second line. Finally

$$\begin{aligned} \{f(z), z - b\}_D &= \prod_{D' \in \partial U} \{f_{D'}(z), z - b\}_{D'} = \{f_D(z), z - b\}_D \\ &= \prod_{x \in D^c} \{f_D(z), z - b\}_x^{-1} = \{f_D(z), z - b\}_b^{-1} \cdot \{f_D(z), z - b\}_\infty^{-1} \\ &= f_D(b)^{-1} \cdot f_D(\infty) \in U_\epsilon, \end{aligned}$$

using again Weil's reciprocity law. □

Proof of Theorem 2.2. Let f and g be elements of $\mathcal{O}^*(U)$ and let $D \in \partial U$ be a boundary component. By definition (see [4], page 5) there are sequences $f_n \in \mathcal{R}^*(U)$, $g_n \in \mathcal{R}^*(U)$ such that $f_n \rightarrow f$ and $g_n \rightarrow g$. We say that the rigid analytical regulator $\{f, g\}_D \in \mathbb{C}^*$ exists, if the limit

$$\lim_{n \rightarrow \infty} \{f_n, g_n\}_D$$

exists, it is an element of \mathbb{C}^* , and it is independent of the choice of the sequences f_n and g_n . In this case we define $\{f, g\}_D$ to be this limit. We start our proof by showing that the rigid analytical $\{f, g\}_D$ regulator exists. Let $f_n \in \mathcal{R}^*(U)$ and $g_n \in \mathcal{R}^*(U)$ be two sequences such that $f_n \rightarrow f$ and $g_n \rightarrow g$. We will first show that the sequence $\{f_n, g_n\}_D$ converges using Cauchy's criterion. For every $0 < \epsilon < 1$ there is an $n(\epsilon) \in \mathbb{N}$ such that $f_n/f_m \in \mathcal{R}_\epsilon(U)$, $g_n/g_m \in \mathcal{R}_\epsilon(U)$ for any $n, m \geq n(\epsilon)$. By Proposition 2.11 for any $n, m \geq n(\epsilon)$:

$$\{f_n, g_n\}_D / \{f_m, g_m\}_D = \{f_n/f_m, g_n/g_m\}_D \in U_\epsilon,$$

so $\{f_n, g_n\}_D$ is a Cauchy sequence. If $f'_n \in \mathcal{R}^*(U)$ and $g'_n \in \mathcal{R}^*(U)$ are another two sequences such that $f'_n \rightarrow f$ and $g'_n \rightarrow g$, then the sequences f''_n, g''_n also converge to f and g respectively, where $f''_{2n+1} = f_n, f''_{2n} = f'_n$, and $g''_{2n+1} = g_n, g''_{2n} = g'_n$. The limit $\lim_{n \rightarrow \infty} \{f''_n, g''_n\}_D$ also exists, so it must be equal to the limit of its subsequences:

$$\lim_{n \rightarrow \infty} \{f''_n, g''_n\}_D = \lim_{n \rightarrow \infty} \{f_n, g_n\}_D = \lim_{n \rightarrow \infty} \{f'_n, g'_n\}_D,$$

hence the limit is independent of the sequences chosen. On the other hand

$$1 = \lim_{n \rightarrow \infty} \{f_n, g_n\}_D \cdot \{g_n, f_n\}_D = \lim_{n \rightarrow \infty} \{f_n, g_n\}_D \cdot \lim_{n \rightarrow \infty} \{g_n, f_n\}_D,$$

so this limit is non-zero, so the existence is proved.

Incidentally we also proved that the rigid analytical regulator satisfies property (iii). Because $\mathcal{O}^*(U)$ is open in $\mathcal{O}(U)$, and the inverse map is continuous, properties (ii), (iv) and (v) follow from Lemma 2.6 and Proposition 2.11, by continuity. Property (i) follows by taking the sequences $f_n = f$ and $g_n = g$. On the

other hand properties (ii) and (v) imply that any map satisfying these properties must be continuous in both variables, so it is equal to the rigid analytical regulator if it also satisfies (i). \square

3. The invariance theorem

Definition 3.1. In this chapter we will continue to use the notation of the previous chapter. Let U be a connected rational subdomain of \mathbb{P}^1 , and f, g be two meromorphic functions on U . Then for all $x \in U$ the functions f and g have Laurent series expansion around x , in particular their tame symbol $\{f, g\}_x$ at x is defined. Let $\mathcal{M}(U)$ denote the field of meromorphic functions of U . For every $x \in U$ the tame symbol at x extends to a homomorphism $\{\cdot, \cdot\}_x : K_2(\mathcal{M}(U)) \rightarrow \mathbb{C}^*$. We define the group $K_2(U)$ as the kernel of the direct sum of tame symbols:

$$\bigoplus_{x \in U} \{\cdot, \cdot\}_x : K_2(\mathcal{M}(U)) \rightarrow \bigoplus_{x \in U} \mathbb{C}^*.$$

Let $k = \sum_i f_i \otimes g_i \in K_2(\mathcal{M}(U))$, where $f_i, g_i \in \mathcal{M}(U)$, and let $D \in \partial U$. Let moreover Y be a connected rational subdomain of U such that $f_i, g_i \in \mathcal{O}^*(Y)$ for all i and $\partial U \subseteq \partial Y$. Define the rigid analytical regulator $\{k\}_D$ by the formula:

$$\{k\}_D = \prod_i \{f_i|_Y, g_i|_Y\}_D.$$

Theorem 3.2. (i) For each $k \in K_2(\mathcal{M}(U))$ the rigid analytical regulator $\{k\}_D$ is well-defined, and it is a homomorphism $\{\cdot\}_D : K_2(\mathcal{M}(U)) \rightarrow \mathbb{C}^*$,
(ii) for any two functions $f, g \in \mathcal{O}^*(U)$ we have $\{f \otimes g\}_D = \{f, g\}_D$,
(iii) for every $k \in K_2(U)$ the product of all regulators on the boundary components of U is equal to 1:

$$\prod_{D \in \partial U} \{k\}_D = 1,$$

(iv) for every connected subdomain $Y \subseteq U$, boundary component $D \in \partial Y \cap \partial U$ and $k \in K_2(\mathcal{M}(U))$ we have:

$$\{k|_Y\}_D = \{k\}_D.$$

In order to prove this theorem, we will need three lemmas.

Lemma 3.3. Let $f, g \in \mathcal{O}^*(U)$, where U is a connected rational subdomain of \mathbb{P}^1 . Then the product of all rigid analytical regulators of the pair (f, g) on the boundary components of U is equal to 1:

$$\prod_{D \in \partial U} \{f, g\}_D = 1.$$

Proof. Let f_n, g_n be two sequences of rational functions invertible on U which converge to f and g on the domain, respectively. Then

$$\begin{aligned} \prod_{D \in \partial U} \{f, g\}_D &= \prod_{D \in \partial U} \lim_{n \rightarrow \infty} \{f_n, g_n\}_D = \lim_{n \rightarrow \infty} \prod_{D \in \partial U} \prod_{x \in D} \{f_n, g_n\}_x \\ &= \lim_{n \rightarrow \infty} \prod_{x \notin U} \{f_n, g_n\}_x = 1 \end{aligned}$$

by Weil’s reciprocity law. □

Lemma 3.4. *Let $Y \subseteq U$ be two connected rational subdomains. Then for every $f, g \in \mathcal{O}^*(U)$ and $D \in \partial Y$ we have:*

$$\{f|_Y, g|_Y\}_D = \prod_{\substack{D' \in \partial U \\ D' \subseteq D}} \{f, g\}_{D'},$$

where the empty product is understood to be equal to 1.

Proof. We may reduce immediately to the case when $f, g \in \mathcal{R}^*(U)$ by approximation. Then we have:

$$\begin{aligned} \{f|_Y, g|_Y\}_D &= \prod_{x \in D} \{f|_Y, g|_Y\}_x = \prod_{x \in D} \{f, g\}_x \\ &= \prod_{\substack{D' \in \partial U \\ D' \subseteq D}} \left(\prod_{x \in D'} \{f, g\}_x \right) = \prod_{\substack{D' \in \partial U \\ D' \subseteq D}} \{f, g\}_{D'}. \end{aligned} \quad \square$$

Lemma 3.5. *Let U be a connected rational subdomain and let $D \subset U$ be an open disk. Let f, g be two meromorphic functions on U which are holomorphic and invertible on the rational subdomain $Y = U - D$. Then we have:*

$$\{f|_Y, g|_Y\}_D = \prod_{x \in D} \{f, g\}_x.$$

Proof. The meromorphic functions f and g have only finitely many poles and zeros on the disk D . Let S denote the set of these points. For every such a pole or zero $s \in S$ choose a small open disk $D_s \subset D$ containing s such that the pairwise intersections of these disks are empty. Let W denote the rational subdomain $U - \bigcup_{s \in S} D_s$. Then

$$\{f|_Y, g|_Y\}_D = \prod_{s \in S} \{f|_W, g|_W\}_{D_s}$$

by Lemma 3.4. Hence we may assume without loosing generality that $D = D(a, \rho)$ and a is the only pole or zero of f and g on D by letting W, D_s play the role of U and D , respectively, in the claim above. Write $f(z) = (z - a)^k f_0(z)$, $g(z) =$

$(z-a)^l g_0(z)$, where $f_0(z), g_0(z)$ are elements of $\mathcal{O}^*(U)$. Let f_n, g_n be two sequences of rational functions invertible on the rational subdomain U which converge to f_0, g_0 , respectively, with respect to the supremum norm on U . Then

$$\begin{aligned} \{f, g\}_D &= \lim_{n \rightarrow \infty} \{(z-a)^k f_n, (z-a)^l g_n\}_D = \lim_{n \rightarrow \infty} \{(z-a)^k f_n, (z-a)^l g_n\}_a \\ &= \{(z-a)^k, (z-a)^l\}_a \cdot \lim_{n \rightarrow \infty} \{(z-a)^k, g_n\}_a \cdot \lim_{n \rightarrow \infty} \{f_n, (z-a)^l\}_a \\ &= (-1)^{kl} \cdot \lim_{n \rightarrow \infty} g_n(a)^{-k} \cdot \lim_{n \rightarrow \infty} f_n(a)^l = (-1)^{kl} g_0(a)^{-k} f_0(a)^l = \{f, g\}_a. \quad \square \end{aligned}$$

Proof of Theorem 3.2. In order to show that the rigid analytical regulator is well-defined, we have to prove that:

- (a) for each $k \in K_2(U)$ there is an affinoid subdomain Y with the properties required in Definition 3.1,
- (b) the value of the rigid analytical regulator is independent of the choice of the affinoid subdomain Y ,
- (c) the value of the rigid analytical regulator is independent of the choice of the presentation $k = \sum_i f_i \otimes g_i$.

Let $S \subset U$ be a finite set such that none of the functions f_i and g_i has a zero or a pole on $U - S$. For each $s \in S$ take an open disk $D_s \subset U$ containing s such that these disks are pairwise disjoint. The set $Y = U - \bigcup_{s \in S} D_s$ is a connected rational subdomain which satisfies the properties required in Definition 3.1, so claim (a) holds. Let Y' be another such subdomain. It is clear that the set $Y \cup Y'$ is also a connected rational subdomain which satisfies these properties. Lemma 3.4 applied to the inclusions $Y \hookrightarrow Y \cup Y'$ and $Y' \hookrightarrow Y \cup Y'$ implies that (b) is also true. Let $k = \sum_i f'_i \otimes g'_i$ be another presentation of k . By definition

$$\sum_i f_i \otimes g_i - \sum_i f'_i \otimes g'_i = \sum_j r_j$$

in the free group generated by symbols $f \otimes g$, where $f, g \in \mathcal{M}(U)$, and the elements r_j are defining relations of the group $K_2(\mathcal{M}(U))$, i.e. they are of the form:

$$fg \otimes h - f \otimes h - g \otimes h, h \otimes fg - h \otimes f - h \otimes g, \text{ or } f \otimes (1 - f), f \neq 1.$$

It is possible to choose a connected rational subdomain $Y \subset U$ which satisfies the properties required in Definition 3.1 and it does not contain any of the zeros or poles of the functions f_i, g_i, f'_i, g'_i , and the functions appearing in the relations r_j . By (ii) and (iv) of Theorem 2.2 the regulator $\{\cdot, \cdot\}_D$ evaluated on the relations r_j is equal to 1, so the products:

$$\prod_i \{f_i|_Y, g_i|_Y\}_D = \prod_i \{f'_i|_Y, g'_i|_Y\}_D$$

are equal. Therefore (c) holds, too. The same argument also shows that the map $\{\cdot\}_D : K_2(\mathcal{M}(U)) \rightarrow \mathbb{C}^*$ is a homomorphism. Claim (ii) is obvious, if we choose $Y = U$ in the definition. We start the proof of claim (iii) by noting that

$$\prod_{D \in \partial Y} \{k|_Y\}_D = \prod_{D \in \partial U} \{k\}_D \prod_{D \in \partial Y - \partial U} \{k|_Y\}_D$$

by Lemma 3.4, where Y is a connected rational subdomain which satisfies the properties required in Definition 3.1 with respect to some presentation of k . By Lemma 3.5 the factors of the second product on the right hand side are all equal to 1. On the other hand Lemma 3.3 applied to Y implies that the product on the left hand side is equal to 1. Hence the first product on the right hand side is equal to 1, too. We may assume that $k = f \otimes g$ for some $f, g \in \mathcal{M}(U)$ by bilinearity while we prove claim (iv). There is a connected subdomain $Z \subseteq U$ such that $\partial U \subseteq \partial Z$, the intersection $Y \cap Z$ is nonempty and $f, g \in \mathcal{O}^*(Z)$. Clearly $D \in \partial(Y \cap Z)$, so it will be sufficient to prove that

$$\{f|_{Y \cap Z}, g|_{Y \cap Z}\}_D = \{f|_Z, g|_Z\}_D.$$

We may even assume that f and g are in $\mathcal{R}^*(Y)$ by approximation. But in this case the claim is obviously true. \square

Theorem 3.6. *There is a unique set of homomorphisms $\text{deg}_D : \mathcal{M}^*(U) \rightarrow \mathbb{Z}$ where U is any connected rational subdomain and $D \in \partial U$ is a boundary component with the following properties:*

- (i) *the homomorphism deg_D is zero on $\mathcal{O}_1(U)$,*
- (ii) *for every $f \in \mathcal{R}^*(U)$ the integer $\text{deg}_D(f)$ is the quantity defined in 2.5,*
- (iii) *for every $f \in \mathcal{M}^*(U)$ we have $\text{deg}_D(f|_Y) = \text{deg}_D(f)$ where $Y \subseteq U$ is any connected rational subdomain satisfying the property $D \in \partial Y$.*

This map is equal to the integer-valued function introduced in [4], Proposition I.3.1, page 19, in the limited context when the latter is defined.

Proof. By condition (i) it is clear that the homomorphism deg_D restricted to $\mathcal{O}^*(U)$ must be continuous with respect to the supremum norm topology on $\mathcal{O}^*(U)$ and the discrete topology on \mathbb{Z} , if it exists. For every $f \in \mathcal{M}^*(U)$ there is a $Y \subseteq U$ connected rational subdomain such that $D \in \partial Y$ and $f|_Y \in \mathcal{O}^*(Y)$, so the set of homomorphisms $\text{deg}_D : \mathcal{M}^*(U) \rightarrow \mathbb{Z}$ should be unique. Pick an element $c \in \mathbb{C}$ with $|c| > 1$ and define $\text{deg}_D(f)$ as

$$\text{deg}_D(f) = \log_{|c|}(|\{c, f\}_D|)$$

where \log_a is the logarithm with base a for any positive real number a . This homomorphism is well-defined by Theorem 3.2. This function also satisfies condition (ii) of the proclaim above by claim (iv) of Lemma 2.3. Hence the image of $\mathcal{R}^*(Y)$ with respect to this map lies in \mathbb{Z} . But it is also dense in the image of $\mathcal{O}^*(Y)$ via this map, so the latter lies in \mathbb{Z} , too. We may conclude that the homomorphisms \deg_D take integral values. By (v) of Theorem 2.2 the element $\{c, f\}_D$ is a unit in \mathbb{C}^0 when $f \in \mathcal{O}_1(U)$, so condition (i) is also satisfied. Condition (iii) is a consequence of claim (iv) of Theorem 3.2. \square

Definition 3.7. For every $U \subset \mathbb{P}^1$ connected rational subdomain let $\mathbb{Z}\partial U$ denote the free abelian group with the elements of ∂U as free generators. Let $H_1(U)$ denote the quotient of $\mathbb{Z}\partial U$ by the \mathbb{Z} -module generated by $\sum_{D \in \partial U} D$. For every $D \in \partial U$ we let D denote the class of D in $H_1(U)$ as well. Let $\mathcal{A}b$ denote the category of abelian groups. Let $\mathcal{C}rs$ denote the category whose objects are connected rational subdomains of \mathbb{P}^1 and whose morphisms are holomorphic maps between them. Finally for every pair $a \leq b$ of numbers in $|\mathbb{C}|$ let $A(a, b)$ denote the closed annulus $\mathbb{P}^1 - D(0, a) - D(\infty, 1/b)$. Of course it is a connected rational subdomain.

Theorem 3.8. *There is a unique functor $H_1 : \mathcal{C}rs \rightarrow \mathcal{A}b$ with the following properties:*

- (i) *for every $U \subset \mathbb{P}^1$ connected rational subdomain $H_1(U)$ is the group defined in 3.7,*
- (ii) *for every map $U \rightarrow Y$ which is the restriction of a projective linear transformation f and $D \in \partial U$ boundary component we have:*

$$H_1(f)(D) = f(D) \in H_1(Y),$$

- (iii) *for every $f : U \rightarrow D(a, b)$ holomorphic map and $D \in \partial U$ boundary component we have:*

$$H_1(f)(D) = \deg_D(f)D(0, a) \in H_1(A(a, b)).$$

Proof. We are going to prove first that this functor is unique. Let $h : U \rightarrow Y$ be a holomorphic map between two connected rational subdomains. We need to show that $H_1(h)$ is uniquely determined by the conditions above. We may assume that Y has at least two boundary components. Fix a boundary component $F \in \partial Y$. Then for every other boundary component $F \neq E \in \partial Y$ there is a projective linear transformation j_E of \mathbb{P}^1 such that $j_E(E) = D(0, 1)$ and $\infty \in j_E(F)$. Then $j_E \circ h$ maps into $A(a, b)$ for some a and b for every $F \neq E \in \partial Y$. By property (ii) and (iii) we have:

$$H_1(h)(D) = \sum_{F \neq E \in \partial Y} \deg_D(j_E \circ h)E \in H_1(Y)$$

for every boundary component $D \in \partial U$. In particular this class is uniquely determined. Let $H^1(U)$ denote the quotient

$$H^1(U) = \mathcal{O}^*(U)/(\mathbb{C}^*\mathcal{O}_1(U)),$$

where $\mathbb{C}^* \subset \mathcal{O}^*(U)$ is the subgroup of constant functions. For every $f \in \mathcal{O}^*(U)$ let the same letter denote its class in $H^1(U)$ as well. The degree map of Theorem 3.6 induces a bilinear pairing:

$$\text{deg} : H^1(U) \times H_1(U) \rightarrow \mathbb{Z}$$

characterized by the property:

$$\text{deg}(f, D) = \text{deg}_D(f)$$

for every $f \in \mathcal{O}^*(U)$ and $D \in \partial U$.

Proposition 3.9. *The pairing deg is perfect.*

Proof. We need to show the following two claims in order to prove the proposition:

- (i) for every $f \in \mathcal{O}^*(U)$ if $\text{deg}_D(f) = 0$ for every $D \in \partial U$ then $f \in \mathbb{C}^*\mathcal{O}_1(U)$,
- (ii) for every homomorphism $l : H_1(U) \rightarrow \mathbb{Z}$ there is a function $f \in \mathcal{O}^*(U)$ such that $l(h) = \text{deg}(f, h)$ for every $h \in H_1(U)$.

We may assume that $f \in \mathcal{R}^*(U)$ by approximation while we show the first claim. Let $\sum_{x \in \mathbb{P}^1} n(x)x$ be the divisor of f . Let $g(z)$ be the product $\prod_{D \in \partial D} g_D(z)$, where

$$g_D(z) = \prod_{x \in D} (z - x)^{n(x)}$$

if $\infty \notin D$, and

$$g_D(z) = \prod_{x \in D} \left(\frac{1}{z} - \frac{1}{x} \right)^{n(x)},$$

otherwise. The rational functions f and g have the same divisor, so their quotient is constant. Therefore it will be enough to show that $g \in \mathcal{O}_1(U)$. We will prove that the functions $g_D \in \mathcal{R}_1(U)$ which is sufficient by Lemma 2.7. First consider the case when $\infty \notin D$. In this case $D = D(c, d)$ for some $c \in \mathbb{C}$ and $d \in |\mathbb{C}|$. By assumption $\sum_{x \in D} n(x) = \text{deg}_D(f) = 0$, so the function $g_D(z)$ is the product of factors of the form $(z - a)/(z - b)$, where $a, b \in D$. For any $z \notin D$ we have

$$\left| \frac{z - a}{z - b} \right| = \frac{|z - c + c - a|}{|z - c + c - b|} = \frac{|z - c|}{|z - c|} = 1,$$

so $g_D \in \mathcal{R}_1(U)$ as we claimed. In the other case the argument is similar. We may assume that ∂U has at least two elements while we show the second claim.

Fix a boundary component $F \in \partial U$. As $l(F) = -\sum_{F \neq D \in \partial U} l(D)$, there is a rational function $f \in \mathcal{R}^*(U)$ such that $\deg_D(f) = l(D)$ for every other boundary component $F \neq D \in \partial U$. Clearly the function f satisfies the property in the second claim. \square

The proof of the existence of the functor H_1 is now easy: we define it as the \mathbb{Z} -dual of the contravariant functor H^1 . Condition (i) is automatic via the identification between $H_1(U)$ and $\text{Hom}(H^1(U), \mathbb{Z})$ furnished by Proposition 3.9. Properties (ii) and (iii) can be verified by looking at appropriate test functions to compute the effect of $H_1(f)$. In the first case one considers rational functions, in the second case the identity function. \square

Definition 3.10. Let $U \subset \mathbb{P}^1$ be a connected rational subdomain. For every class $c \in H_1(U)$ and element $k \in K_2(U)$ we define the regulator $\{k\}_c$ as

$$\{k\}_c = \prod_{D \in \partial U} \{k\}_D^{c(D)},$$

where $\sum_{D \in \partial U} c(D)D$ is a lift of c in $\mathbb{Z}\partial U$. By claim (iii) of Theorem 3.2 this regulator is well-defined. For every holomorphic map $h : U \rightarrow Y$ between two connected rational subdomains let $h^* : K_2(\mathcal{M}(Y)) \rightarrow K_2(\mathcal{M}(U))$ be the pull-back homomorphism induced by h . By restriction it induces a homomorphism $K_2(Y) \rightarrow K_2(U)$.

Theorem 3.11. For any $k \in K_2(Y)$ and $c \in H_1(U)$ we have:

$$\{h^*(k)\}_c = \{k\}_{H_1(h)(c)}.$$

Proof. Let $k = \sum_i f_i \otimes g_i \in K_2(\mathcal{M}(Y))$, where $f_i, g_i \in \mathcal{M}(U)$. Let moreover Y' be a connected rational subdomain of Y such that $f_i, g_i \in \mathcal{O}^*(Y')$ for all i and $\partial Y \subseteq \partial Y'$. There is a connected rational subdomain U' of U such that $h(U') \subseteq Y'$ and $\partial U \subseteq \partial U'$. The map $H_1(U') \rightarrow H_1(U)$ induced by the inclusion is surjective, so there is a $c' \in H_1(Y')$ whose image is c . We claim that $\{h^*(k)|_{U'}\}_{c'} = \{h^*(k)\}_c$. We may write c' as a sum $c' = c_1 + c_2$ where c_1, c_2 can be represented as the linear combination of boundary components lying in ∂U and $\partial U' - \partial U$, respectively. We have $\{h^*(k)|_{U'}\}_{c_1} = \{h^*(k)\}_c$ by definition, on the other hand $\{h^*(k)|_{U'}\}_{c_2} = 1$ as $h^*(k)$ is an element of $K_2(U)$. The same argument shows that $\{k|_{Y'}\}_{H^1(h|_{U'})(c')} = \{k\}_{H^1(h)(c)}$, so it will be sufficient to prove the claim for $U', Y', h|_{U'}, c'$ and $k|_{Y'}$ instead of U, Y, h, c and k , respectively. In other words we may assume that $k = f \otimes g$ for some $f, g \in \mathcal{O}^*(Y)$. Let $f_n \in \mathcal{R}^*(Y)$ and $g_n \in \mathcal{R}^*(Y)$ be two sequences such that $f_n \rightarrow f$ and $g_n \rightarrow g$. Obviously $\|f \circ h - f_n \circ h\| \leq \|f - f_n\|$

for all $n \in \mathbb{N}$, so $f_n \circ h \rightarrow f \circ h$. The same holds for g , so

$$\{h^*(f \otimes g)\}_c = \lim_{n \rightarrow \infty} \{h^*(f_n \otimes g_n)\}_c \text{ and } \{f \otimes g\}_{H_1(h)(c)} = \lim_{n \rightarrow \infty} \{f_n \otimes g_n\}_{H_1(h)(c)}$$

by Remark 2.3. Therefore it is sufficient to show the claim when $f, g \in \mathcal{R}^*(Y)$. We may also assume that $\infty \notin U$ and $\infty \notin Y$ by shrinking Y and U the same way as above if necessary.

Lemma 3.12. *For every $f \in \mathcal{O}(U)$ the following holds: for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for every $x, y \in U$ with $|x - y| < \delta$.*

Proof. Of course the claim above is just the analogue of the usual uniform continuity property. The reason why it is not completely obvious in this case is that \mathbb{C} is not locally compact. Let $\mathcal{U}(U)$ denote the set of all functions $f \in \mathcal{O}(U)$ which satisfy the property in the claim above. It is clear that $\mathcal{U}(U)$ is a \mathbb{C} -subalgebra of $\mathcal{O}(U)$. Moreover for every $f \in \mathcal{O}^*(U)$ we have the estimate:

$$\left| \frac{1}{f(x)} - \frac{1}{f(y)} \right| = \left| \frac{f(y) - f(x)}{f(x)f(y)} \right| \leq \|f^{-1}\|^2 |f(x) - f(y)|,$$

so for every $f \in \mathcal{O}^*(U) \cap \mathcal{U}(U)$ we have $f^{-1} \in \mathcal{U}(U)$, too. Obviously $z - c \in \mathcal{U}(U)$ for every $c \in \mathbb{C}$, so $\mathcal{R}(U) \subseteq \mathcal{U}(U)$ by the above. On the other hand $\mathcal{U}(U)$ is closed with respect to the supremum norm topology, so it must be equal to the whole algebra $\mathcal{O}(U)$. \square

Let us return to the proof of Theorem 3.11. Since $\infty \notin U$ there is a rational $\epsilon > 0$ such that for every $x \in U$ the disk $D(x, \epsilon) \subset U$. Hence we may choose an infinite sequence $h_n \in \mathcal{R}(U)$ converging to h in the supremum norm topology such that $h_n(U) \subseteq Y$ for all n . By the lemma above $f \circ h_n \rightarrow f \circ h$ and $g \circ h_n \rightarrow g \circ h$ in the supremum norm topology. Therefore it is sufficient to prove the theorem when $h \in \mathcal{R}(U)$, too. We may also assume that $c = C$ for some boundary component $C \in \partial U$ by linearity. Let $F \in \partial Y$ be the unique boundary component which contains ∞ . We may write $H^1(h)(C)$ uniquely in the form:

$$H^1(h)(C) = \sum_{F \neq D \in \partial Y} n(D)D$$

for some $n(D) \in \mathbb{Z}$. There is a closed annulus $A(a, b)$ such that $h(z) - c$ maps U into $A(a, b)$ for every $F \neq D \in \partial Y$ and $c \in D$. Fix a boundary component $F \neq D \in \partial Y$ and for every $c \in D$ let $z(c)$ denote the number of zeros of the rational function $h(z) - c$ lying in the open disk C counted with multiplicities. We claim that $z(c)$ is independent of the choice of c . First note that the number of poles of the rational function $h(z) - c$ lying in the open disk C counted with

multiplicities is independent of the choice of c . This number does not even depend on D , and it will be denoted by $p(C)$. By claim (iii) of Theorem 3.8 we have $z(c) = \deg_C(h(z) - c) + p(D) = n(D) + p(C)$ which is clearly independent of the choice of c . Let $z(D)$ denote this number. We also claim that for every $\infty \neq c \in F$ the number $z(c)$ of zeros of the rational function $h(z) - c$ lying in the open disk C counted with multiplicities is equal to $p(C)$. We have $F = D(\infty, d)$ for some rational number $d > 0$. Hence $\|h(z)\| \leq 1/d$, so we have

$$\left\| 1 - \frac{h(z) - c}{c} \right\| = \left\| \frac{h(z)}{c} \right\| < 1,$$

so $\deg_C(h(z) - c) = 0$. But $z(c) = \deg_D(h(z) - C) + p(C) = p(C)$ by definition. For all $x \in \mathbb{P}^1$ let $v(x) \in \mathbb{N}$ denote the degree of ramification of the map $h : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ at x . Then

$$\{f \circ h, g \circ h\}_x = \{f, g\}_{h(x)}^{v(x)}$$

for all $x \in \mathbb{P}^1$. Therefore

$$\begin{aligned} \{h^*(f \otimes g)\}_C &= \prod_{x \in C} \{f \circ h, g \circ h\}_x = \prod_{x \in C} \{f, g\}_{h(x)}^{v(x)} \\ &= \prod_{D \in \partial Y} \prod_{y \in D} \prod_{\substack{x \in C \\ h(x)=y}} \{f, g\}_y^{v(x)} \\ &= \prod_{F \neq D \in \partial Y} \prod_{y \in D} \{f, g\}_y^{z(D)} \cdot \prod_{y \in F} \{f, g\}_y^{p(C)} \\ &= \prod_{F \neq D \in \partial Y} \prod_{y \in D} \{f, g\}_y^{z(D) - p(C)} = \{f \otimes g\}_{H^1(h)(C)}, \end{aligned}$$

where we used Weil's reciprocity law in the fifth equation. □

Remarks 3.13. The theorem above incorporates two fundamental properties of the rigid analytic regulator which might be called biholomorphic invariance and homotopy invariance based on the analogy explained in 1.1. The first claims that every biholomorphic map $h : U \rightarrow U$, where U is any connected rational subdomain of \mathbb{P}^1 , which induces the identity map on $H^1(U)$ leaves the rigid-analytic regulator invariant. The second claims that whenever we have two connected rational subdomains $Y \subseteq U$ and a boundary component $D \in \partial U$ such that there is a unique boundary component $D' \in \partial Y$ containing D then the rigid analytic regulators taken at D and D' do not differ for elements of $K_2(U)$. Of course the best way to formulate these properties is the way we did, expressing them as a functoriality property via the homology group H_1 . The latter has a more high-brow definition using the étale cohomology of rigid analytic spaces, but for our purposes our elementary definition was more suitable.

4. Relation to the generalized tame symbol

Notation 4.1. Let \mathbb{R} be a closed subfield of \mathbb{C} : it is automatically complete with respect to $|\cdot|$. Let U be a connected rational subdomain of \mathbb{P}^1 defined over \mathbb{R} . This means that

$$U = \{z \in \mathbb{P}^1 \mid |f_i(z)| \leq 1 \ (\forall i = 1, \dots, n)\}$$

for some natural number n and rational functions $f_1, \dots, f_n \in \mathbb{R}((t))$. Let $\mathcal{O}_{\mathbb{R}}(U)$, $\mathcal{R}_{\mathbb{R}}(U)$, $\mathcal{O}_{\mathbb{R}}^*(U)$, $\mathcal{R}_{\mathbb{R}}^*(U)$ and $\mathcal{M}_{\mathbb{R}}(U)$ denote the algebra of holomorphic functions, the subalgebra of restrictions of \mathbb{R} -rational functions, the groups of invertible elements of these algebras and the field of meromorphic functions on the rigid analytic space U , respectively. Let U denote the underlying rational subdomain over \mathbb{C} by slight abuse of notation. An \mathbb{R} -rational boundary component of U is a set $D \in \partial U$ such that D is of the form $D(a, |\rho|)$ or $D(\infty, |\rho|)$ for some $a, \rho \in \mathbb{R}$. Let $K_2(U)_{\mathbb{R}}$ denote the largest subgroup of $K_2(\mathcal{M}_{\mathbb{R}}(U))$ which maps into $K_2(U)$ under the natural homomorphism $K_2(\mathcal{M}_{\mathbb{R}}(U)) \rightarrow K_2(\mathcal{M}_{\mathbb{C}}(U))$.

Proposition 4.2. *Let D be an \mathbb{R} -rational boundary component of U , and let $k \in K_2(\mathcal{M}_{\mathbb{R}}(U))$. Then $\{k\}_D \in \mathbb{R}^*$.*

Proof. We may assume that $k = f \otimes g$ for some $f, g \in \mathcal{O}_{\mathbb{R}}^*(U)$ by linearity. Since $\mathcal{R}_{\mathbb{R}}^*(U)$ is dense in $\mathcal{O}_{\mathbb{R}}^*(U)$ and \mathbb{R} is complete, we may assume that f and g are actually in $\mathcal{R}^*(U)$ by approximation. We may also assume that $\infty \in D$ after an \mathbb{R} -linear change of coordinates. Then by bilinearity we may assume that f and g are irreducible polynomials in $\mathbb{R}[t]$. Assume first that f and g are separable, too. Clearly $\{f, g\}_x$ can be different from 1 only if $x = \infty$ or x is a zero of f or g . In the latter case x is an element of the separable closure $\overline{\mathbb{R}}$ hence $\{f, g\}_x \in \overline{\mathbb{R}}$, too. Moreover if $x \in D$ then $x^\gamma \in D$, too, where γ is any element of $\text{Gal}(\overline{\mathbb{R}}|\mathbb{R})$. Also $\{f, g\}_x^\gamma = \{f, g\}_{x^\gamma}$ for any $x \in \overline{\mathbb{R}}$ and $\gamma \in \text{Gal}(\overline{\mathbb{R}}|\mathbb{R})$. Therefore the product $\prod_{x \in D} \{f, g\}_x$ is an element of $\overline{\mathbb{R}}$ invariant under the action of $\text{Gal}(\overline{\mathbb{R}}|\mathbb{R})$, so it is in \mathbb{R} . If f is not separable, then $f = (f')^{p^n}$ for some separable polynomial f' whose coefficients are in a purely inseparable extension \mathbb{L} of \mathbb{R} such that $\mathbb{L}^{p^n} \subseteq \mathbb{R}$ where p is the characteristic of \mathbb{R} . It is enough to show that $\{f', g\}_D \in \mathbb{L}^*$, since $\{f, g\}_D = \{f', g\}_D^{p^n} \in \mathbb{R}^*$ in this case. The latter follows from applying the same argument to g over \mathbb{L} and what we have just proved above. □

Definition 4.3. Let \mathbf{k} , \mathcal{O} and \mathfrak{m} denote the residue field of \mathbb{R} , the valuation ring of \mathbb{R} and the maximal proper ideal of \mathcal{O} , respectively. A finite subset $S \subset \mathbf{k}$ is called \mathbf{k} -rational if it is the zero set of a polynomial with coefficients in \mathbf{k} . If $S \subset \mathbf{k}$ is \mathbf{k} -rational then the set $U(S)$ introduced in Definition 2.8 is a connected rational subdomain defined over \mathbb{R} .

Lemma 4.4. *Let $S \subset \mathbf{k}$ be an \mathbf{f} -rational subset. Then*

- (i) *for every $f \in \mathcal{M}_{\mathbb{R}}(U(S))^*$ there is an \mathbf{f} -rational subset $S \subseteq S' \subset \mathbf{k}$ such that $f|_{U(S')}$ can be written in the form:*

$$f|_{U(S')} = c(f)f_0,$$

where $c(f) \in \mathbb{R}$ and $f_0 \in \mathcal{O}^(U(S'))$ with $|f_0(z)| = 1$ for all $z \in U(S')$,*

- (ii) *the positive number $|c(f)|$ does not depend on the choice of S' or f_0 ,*
- (iii) *the map $|\cdot| : \mathcal{M}_{\mathbb{R}}(U(S))^* \rightarrow |\mathbb{C}|$ given by the rule $f \mapsto |c(f)|$ is a non-archimedean absolute value on the field $\mathcal{M}_{\mathbb{R}}(U(S))$.*

Proof. The set $Z \subset U(S)$ of zeros and poles of f is finite. Hence the reduction of the elements of Z with respect to \mathbb{C}^{00} is a finite set, too. Since every finite subset of \mathbf{k} is contained in a finite \mathbf{f} -rational subset, we may assume that $f \in \mathcal{O}^*(U(S))$ by enlarging S if necessary. Let $g \in \mathcal{O}^*(U(S))$ be another function such that $\|1 - f/g\| < 1$. We claim that it is sufficient to prove the claim (i) for g in order to prove it for f . We may assume that $g = c(g)g_0$ where $c(g) \in \mathbb{R}$ and $g_0 \in \mathcal{O}^*(U(S))$ with $|g_0(z)| = 1$ for all $z \in U(S)$ by enlarging S if necessary. Write f as $f = c(g)f_0$. Then $\|1 - f_0/g_0\| = \|1 - f/g\| < 1$ so $|f_0(z)| = 1$ for all $z \in U(S)$. Hence we may assume that $f \in \mathcal{R}^*(U(S))$ by approximation. Also note that the elements of $\mathcal{M}_{\mathbb{R}}(U(S))^*$ satisfying claim (i) form a subgroup. Therefore we may assume that f is in fact a polynomial. Then we may write $f(z)$ as:

$$f(z) = c(f) \sum_{n=0}^N a_n z^n$$

with $a_n \in \mathcal{O}$ and $\max_{n=0}^N |a_n| = 1$. There is a finite \mathbf{f} -rational subset S' of \mathbf{k} such that the reduction of the polynomial $f_0 = \sum_{n=0}^N a_n z^n$ is nowhere zero on the complement of S' . Clearly $f_0 \in \mathcal{O}^*(U(S'))$ with $|f_0(z)| = 1$ for all $z \in U(S')$.

This proves claim (i). Assume that $S \subseteq S'' \subset \mathbf{k}$ is another finite \mathbf{f} -rational subset such that $f|_{U(S'')} = c(f)'f'_0$ where $c(f)' \in \mathbb{R}$ and $f'_0 \in \mathcal{O}^*(U(S''))$ with $|f'_0(z)| = 1$ for all $z \in U(S'')$. On the set $U(S' \cup S'') = U(S') \cap U(S'')$ we have $c(f)f_0 = c(f)'f'_0$. As $\|f_0\| = \|f'_0\| = 1$ on this set we must have $|c(f)| = |c(f)'|$ as claim (ii) says. Clearly the map v is a homomorphism, so we only have to show that $|c(f + g)| \leq \max(|c(f)|, |c(g)|)$ for any $f, g \in \mathcal{O}^*(U(S))$ with $f + g \neq 0$ in order to prove claim (iii). There is an \mathbf{f} -rational subset $S \subseteq S' \subset \mathbf{k}$ such that $|c(f)| = \|f|_{U(S')}\|$, $|c(g)| = \|g|_{U(S')}\|$ and $|c(f + g)| = \|(f + g)|_{U(S')}\|$. The last claim now follows from the strong triangle inequality for the spectral norm $\|\cdot\|$. \square

Definition 4.5. For any pair $S \subseteq S' \subset \mathbf{k}$ of finite \mathbf{f} -rational subsets the inclusion $U(S) \rightarrow U(S')$ induces an imbedding $\mathcal{M}_{\mathbb{R}}(U(S)) \rightarrow \mathcal{M}_{\mathbb{R}}(U(S'))$. Under these in-

clusions the fields $\mathcal{M}_{\mathbb{R}}(U(S))$ form an injective system: let \mathcal{M} denote the inductive limit of this system. By part (ii) of the lemma above the absolute value $|\cdot|$ of part (iii) is well-defined on \mathcal{M} and makes the latter a valued field. The residue field of \mathcal{M} is equal to the rational function field $\mathbf{f}(t)$ where t is the reduction of the identity map z with respect to the maximal ideal in the valuation ring of \mathcal{M} . We are going to need a mild extension of the tame symbol. Let F be a field equipped with a valuation $\nu : F^* \rightarrow \mathbb{Q}$ and let O and \mathbf{r} denote its valuation ring and its residue field, respectively. Let moreover $\bar{\cdot} : O \rightarrow \mathbf{r}$ denote the reduction modulo the maximal ideal of O . For any pair of elements $f, g \in F^*$ we are going to define their generalized tame symbol $T(f \otimes g) \in \mathbf{r}^* \otimes \mathbb{Q}$ as follows. There is an element $\pi \in F^*$ such that $f = f_0\pi^{n(f)}$ and $g = g_0\pi^{n(g)}$ for some integers $n(f), n(g) \in \mathbb{Z}$ and elements $f_0, g_0 \in \mathcal{O}^*$. We let

$$T(f \otimes g) = (-1) \otimes \nu(f)\nu(g) \cdot \overline{f_0} \otimes \nu(g) \cdot \overline{g_0} \otimes (-\nu(f)) \in \mathbf{r}^* \otimes \mathbb{Q}.$$

(We only included the first factor in the product above in order to resemble the usual formula, but it is always equal to 1.) One may prove the usual way that this symbol is well-defined and satisfies the Steinberg relation. In particular it induces a homomorphism $T : K_2(F) \rightarrow \mathbf{r}^* \otimes \mathbb{Q}$ which depends on the choice of normalization of the valuation v linearly. Let $\nu : \mathcal{M}^* \rightarrow \mathbb{Q}$ be a valuation corresponding to the absolute value $|\cdot|$ and let $T : K_2(\mathcal{M}) \rightarrow \mathbf{f}(t)^* \otimes \mathbb{Q}$ denote the corresponding generalized tame symbol. For any $k \in K_2(\mathcal{M}(U))$ and $s \in S \cap \mathbf{f}$ let $T_s(k) \in \mathbb{Q}$ denote the value of $T(k)$ with respect to the unique \mathbb{Q} -linear extension of the normalized valuation at the closed point $s \in \mathbb{P}_{\mathbf{f}}^1$.

Proposition 4.6. *We have*

$$\nu(\{k\}_{D(s,1)}) = T_s(k)$$

for every $k \in K_2(\mathcal{M}_{\mathbb{R}}(U(S)))$.

Proof. The linear transformation $z \mapsto z - s$ maps $U(S)$ biholomorphically onto $U(\bigcup_{x \in S} x - s)$ and interchanges $\{\cdot\}_{D(s,1)}$ and T_s with $\{\cdot\}_{D(0,1)}$ and T_0 , respectively. Hence we may assume that $s = 0$. By linearity we may assume that $k = f \otimes g$ for some $f, g \in \mathcal{M}_{\mathbb{R}}(U(S))$. Since for every \mathbf{f} -rational finite set $S \subseteq S' \subset \mathbf{k}$ we have $\{k\}_{D(0,1)} = \{k|_{U(S')}\}_{D(0,1)}$ and $T_0(k) = T_0(k|_{U(S)})$ we may assume that $f = c(f)f_0$ and $g = c(g)g_0$ where $c(f), c(g) \in \mathbb{R}^*$ and $f_0, g_0 \in \mathcal{O}^*(U(S))$ with $|f_0(z)| = |g_0(z)| = 1$ for every $z \in U(S)$ by enlarging S if necessary. Assume that $\|1 - f\| < 1$. In this case $T(f \otimes g) = 1$ by definition and $|\{f \otimes g\}_{D(0,1)}| = 1$ by (v) of Theorem 2.2. Hence we may assume that f and g are rational functions by approximation. We may even assume that f and g are polynomials using the

bilinearity of both sides of the equation we want to prove. In this case we may assume by multiplying $c(f)$ and $c(g)$ by an element of \mathcal{O}^* , if necessary, such that

$$f_0(z) = z^{n(f)}(1 + f_{00}(z)) \quad \text{and} \quad g_0(z) = z^{n(g)}(1 + g_{00}(z)),$$

where $n(f), n(g) \in \mathbb{Z}$ and $f_{00}(z), g_{00}(z) \in \mathcal{O}[z]$ with $\|f_{00}\| < 1$ and $\|g_{00}\| < 1$. Therefore we may assume that f_0 and g_0 are powers of z by applying the argument we used above. In this case the claim is obvious. \square

5. The rigid analytical regulator for Mumford curves

Definition 5.1. Let \mathfrak{X} be a Hausdorff topological space. For any R commutative group let $\mathcal{M}(\mathfrak{X}, R)$ denote the set of R -valued finitely additive measures on the open and compact subsets of \mathfrak{X} . When \mathfrak{X} is compact let $\mathcal{M}_0(\mathfrak{X}, R)$ denote the set of measures of total volume zero, that is the subset of those $\mu \in \mathcal{M}(\mathfrak{X}, R)$ such that $\mu(\mathfrak{X}) = 0$. For every abelian topological group M let $\mathcal{C}_0(\mathfrak{X}, M)$ denote the group of compactly supported continuous functions $f : \mathfrak{X} \rightarrow M$. If M is discrete then every element of $\mathcal{C}_0(\mathfrak{X}, M)$ is locally constant. In this case for every $f \in \mathcal{C}_0(\mathfrak{X}, M)$ and $\mu \in \mathcal{M}(\mathfrak{X}, R)$ we define the *modulus* $\mu(f)$ of f with respect to μ as the \mathbb{Z} -submodule of R generated by the elements $\mu(f^{-1}(g))$, where $0 \neq g \in M$. We also define the *integral* of f on \mathfrak{X} with respect to μ as the sum:

$$\int_{\mathfrak{X}} f(x) d\mu(x) = \sum_{g \in M} g \otimes \mu(f^{-1}(g)) \in M \otimes \mu(f).$$

Lemma 5.2. (a) *If $f, g \in \mathcal{C}_0(\mathfrak{X}, M)$, then $f \times g \in \mathcal{C}_0(\mathfrak{X}, M \times M)$, $\mu(f)$, $\mu(g)$ and $\mu(f + g)$ are contained in $\mu(f \times g)$ and*

$$\int_{\mathfrak{X}} (f(x) + g(x)) d\mu(x) = \int_{\mathfrak{X}} f(x) d\mu(x) + \int_{\mathfrak{X}} g(x) d\mu(x) \quad \text{in } M \otimes \mu(f \times g).$$

(b) *Let $f \in \mathcal{C}_0(\mathfrak{X}, M)$ and $m \in N$, where N is also a \mathbb{Z} -module. Then $m \otimes f$ is in $\mathcal{C}_0(\mathfrak{X}, N \otimes M)$, $\mu(m \otimes f) \subseteq \mu(f)$ and*

$$\int_{\mathfrak{X}} m \otimes f(x) d\mu(x) = m \otimes \int_{\mathfrak{X}} f(x) d\mu(x) \quad \text{in } N \otimes M \otimes \mu(f).$$

(c) *Let $f \in \mathcal{C}_0(\mathfrak{X}, M)$ and let $\phi : M \rightarrow N$ be a homomorphism. Then $\phi \circ f$ is in $\mathcal{C}_0(\mathfrak{X}, N)$, $\mu(\phi \circ f) \subseteq \mu(f)$ and*

$$\int_{\mathfrak{X}} \phi \circ f(x) d\mu(x) = \phi \left(\int_{\mathfrak{X}} f(x) d\mu(x) \right) \quad \text{in } N \otimes \mu(f).$$

(d) *Assume that $f \in \mathcal{C}_0(\mathfrak{X}, \mathbb{R})$ is locally constant and $\mu \in \mathcal{M}(\mathfrak{X}, \mathbb{R})$ is a Borel measure on \mathfrak{X} , which is a positive measure on the Borel sets of \mathfrak{X} . Then the*

image of the integral of f on \mathfrak{X} with respect to μ under the homomorphism $\mathbb{R} \otimes \mu(f) \rightarrow \mathbb{R}$ induced by the product is the usual Lebesgue integral of f on \mathfrak{X} with respect to μ . \square

Definition 5.3. Let \mathbb{R} be a closed subfield of \mathbb{C} and let $K \subset \mathbb{P}^1(\mathbb{R})$ be a non-empty compact subset. For every $\rho \in GL_2(\mathbb{R})$ and $z \in \mathbb{P}^1(\mathbb{C})$ let $\rho(z)$ denote the image of z under the Möbius transformation corresponding to ρ . Let moreover $D(\rho)$ denote the open disk

$$D(\rho) = \{z \in \mathbb{P}^1(\mathbb{C}) \mid 1 < |\rho^{-1}(z)|\}.$$

Let \mathcal{D} denote the set of open disks of the form $D(\rho)$ where $\rho \in GL_2(\mathbb{R})$. For each $D \in \mathcal{D}$ let $D(K)$ denote $D \cap K$. Let $\mathcal{P}(K)$ denote those subsets S of \mathcal{D} such that the sets $D, D \in S$ are pairwise disjoint and the union of the sets $D(K), D \in S$ form a partition of K . For each $S \in \mathcal{P}(K)$ let $\Omega(S)$ denote the unique connected rational subdomain defined over \mathbb{R} with the property $\partial\Omega(S) = S$. Let Ω_K denote the complement of K in $\mathbb{P}^1(\mathbb{C})$. Then Ω_K is equipped naturally with the structure of a rigid analytic space over \mathbb{R} such that the open subsets $\Omega(S), S \in \mathcal{P}(K)$ form an admissible cover by affinoid subdomains. In particular a function $f : \Omega_K \rightarrow \mathbb{R}$ is holomorphic if the restriction of f onto $\Omega(S)$ is holomorphic for every $S \in \mathcal{P}(K)$. Let $\mathcal{O}(\Omega_K)$ and $\mathcal{M}(\Omega_K)$ denote the \mathbb{R} -algebra of holomorphic functions and the field of meromorphic functions on Ω_K , respectively. The latter is of course the quotient field of the former. We define $K_2(\Omega_K)$ as the intersection of the kernels of all the tame symbols $\{\cdot, \cdot\}_x$ inside $K_2(\mathcal{M}(\Omega_K))$ where x runs through the set Ω_K .

Lemma 5.4. For each $k \in K_2(\Omega_K)$ there is a unique measure $\{k\} \in \mathcal{M}_0(K, \mathbb{R}^*)$ such that $\{k\}(D(K)) = \{k|_{\Omega(S)}\}_D$ for every $S \in \mathcal{P}(K)$ and $D \in S$.

Proof. First we are going to show that every open cover \mathcal{U} of K has a subordinate cover of the form $D(K), D \in S$ where $S \in \mathcal{P}(K)$. By the compactness of K there is a finite cover \mathcal{V} of the form $D(K), D \in I$ subordinate to \mathcal{U} where $I \subset \mathcal{D}$ is a finite set. We may assume that the union $\bigcup_{D \in I} D$ is not equal to $\mathbb{P}^1(\mathbb{C})$ by refining the cover \mathcal{V} further. Then any two disks in I are either disjoint or one contains the other, hence the claim is now clear. The same argument works for any compact and open subset L of K . When we apply it to the one element cover of L we get that L can be written as the pairwise disjoint union of sets of the form $D(K)$. In particular the measure $\{k\}$ is unique, if it exists. In order to prove that $\{k\}(L)$ is well-defined we have to show that the product

$$\prod_{D \in I} \{k|_{\Omega(S)}\}_D \in \mathbb{R}^*$$

is independent of the choice of I and S for every $S \in \mathcal{P}(K)$ and $I \subseteq S$ such that $L = \bigcup_{D \in I} D(K)$. Let $T \in \mathcal{P}(K)$ and $J \subseteq T$ be another pair such that $L = \bigcup_{D \in J} D(K)$. Then there is a $V \in \mathcal{P}(K)$ and an $M \subseteq V$ such that $\emptyset \neq \Omega(V)$ contains $\Omega(S) \cup \Omega(T)$ and $L = \bigcup_{D \in M} D(K)$. Clearly we only have to show that

$$\{k|_{\Omega(U)}\}_D = \prod_{\substack{E(K) \subseteq D(K) \\ E \in V}} \{k|_{\Omega(V)}\}_E$$

for every $D \in U$ where U is either S or T . By our assumptions we have either $E(K) \subseteq D(K)$ or $E \subset \Omega_K$ for every $E \subseteq D$ with $E \in V$. As $\{k|_{\Omega(V)}\}_E = 1$ for disks of the latter type the equality above follows from the invariance theorem. Finally note that the product $\{k\}(K) = \prod_{D \in S} \{k|_{\Omega(S)}\}_D$ is equal to one for every $S \in \mathcal{P}(K)$ by (iii) of Theorem 3.2 hence $\{k\}$ is indeed an element of $\mathcal{M}_0(K, \mathbb{R}^*)$. \square

Definition 5.5. Now let K and L be two non-empty compact subsets of $\mathbb{P}^1(\mathbb{R})$ and assume that a non-constant holomorphic map $h : \Omega_K \rightarrow \Omega_L$ of rigid analytic spaces over \mathbb{R} are given. Let $h^* : K_2(\mathcal{M}(\Omega_L)) \rightarrow K_2(\mathcal{M}(\Omega_K))$ be the pull-back homomorphism induced by h . By restriction it induces a homomorphism $K_2(\Omega_L) \rightarrow K_2(\Omega_K)$, also denoted by h^* . For every abelian topological group M and compact Hausdorff topological space \mathfrak{X} let $\tilde{\mathcal{C}}_0(\mathfrak{X}, M)$ denote the quotient of $\mathcal{C}_0(\mathfrak{X}, M)$ by the group of M -valued constant functions. The integration introduced in Definition 5.1 induces a canonical identification between $\text{Hom}(\tilde{\mathcal{C}}_0(\mathfrak{X}, \mathbb{Z}), R)$ and $\mathcal{M}_0(\mathfrak{X}, R)$ for every R commutative group when \mathbb{Z} is discrete. We are going to define a homomorphism $h_* : \tilde{\mathcal{C}}_0(K, \mathbb{Z}) \rightarrow \tilde{\mathcal{C}}_0(L, \mathbb{Z})$, where \mathbb{Z} is equipped with the discrete topology, as follows. Given an element $\tilde{f} \in \tilde{\mathcal{C}}_0(K, \mathbb{Z})$ first choose one of its representatives $f \in \mathcal{C}_0(K, \mathbb{Z})$. Then choose an $S \in \mathcal{P}(K)$ such that f is equal to a constant $f(D)$ on the set $D(K)$ for every $D \in S$. Then there is a $T \in \mathcal{P}(L)$ such that $h(\Omega(S)) \subseteq \Omega(T)$. Choose a $\tilde{g} = \sum_{E \in T} \tilde{g}(E) \in \mathbb{Z}\partial\Omega(T)$ which represents

$$H_1(h) \left(\sum_{D \in S} f(D)D \right) \in H_1(\Omega(T)).$$

Let $g \in \mathcal{C}_0(L, \mathbb{Z})$ be the function given by the rule $g(z) = \tilde{g}(E)$ for every $z \in E(K)$ and $E \in T$. We define $h_*(\tilde{f})$ as the class of g in $\tilde{\mathcal{C}}_0(L, \mathbb{Z})$. One may see that h_* is a well-defined homomorphism in the same way we proved the lemma above. Let $h_* : \mathcal{M}_0(L, R) \rightarrow \mathcal{M}_0(K, R)$ be the homomorphism induced by this h_* via the duality described above. The following proposition is an immediate consequence of the definitions and the invariance theorem:

Proposition 5.6. *We have*

$$\{h^*(k)\} = h^*({k}) \in \mathcal{M}_0(K, \mathbb{R}^*)$$

for every $k \in K_2(\mathcal{M}(\Omega_L))$. □

Definition 5.7. For any graph G let $\mathcal{V}(G)$ and $\mathcal{E}(G)$ denote its set of vertices and edges, respectively. In this paper we will only consider such oriented graphs G which are equipped with an involution $\bar{\cdot} : \mathcal{E}(G) \rightarrow \mathcal{E}(G)$ such that for each edge $e \in \mathcal{E}(G)$ the initial and terminal vertices of the edge $\bar{e} \in \mathcal{E}(G)$ are the terminal and initial vertices of e , respectively. The edge \bar{e} is called the *edge e with reversed orientation*. Let R be a commutative group. A function $\phi : \mathcal{E}(G) \rightarrow R$ is called a *harmonic R -valued cochain* if it satisfies the following conditions:

(i) We have:

$$\phi(e) + \phi(\bar{e}) = 0 \quad (\forall e \in \mathcal{E}(G)).$$

(ii) If for an edge e we introduce the notation $o(e)$ and $t(e)$ for its initial and terminal vertex respectively, then for all but finitely many $e \in \mathcal{E}(G)$ with $o(e) = v$ we have $\phi(e) = 0$ and

$$\sum_{\substack{e \in \mathcal{E}(G) \\ o(e)=v}} \phi(e) = 0 \quad (\forall v \in \mathcal{V}(G)),$$

where by our assumption the sum above is well-defined.

We denote by $\mathcal{H}(G, R)$ the group of R -valued harmonic cochains on G .

Definition 5.8. A path γ on an oriented graph G is a sequence of edges

$$\{e_1, e_2, \dots, e_n, \dots\} \subseteq \mathcal{E}(G)$$

indexed by the set I where $I = \mathbb{N}$ or $I = \{1, \dots, m\}$ for some $m \in \mathbb{N}$ such that $t(e_i) = o(e_{i+1})$ for every $i, i + 1 \in I$. We say that γ is an *infinite path* or a *finite path* whether we are in the first or in the second case, respectively. We say that a path $\{e_1, \dots, e_n, \dots\}$ indexed by the set I on an oriented graph G is *without backtracking* if $\bar{e}_i \neq e_{i+1}$ for every $i, i + 1 \in I$. An oriented graph G is called a *tree* if for every pair of different vertices v and $w \in \mathcal{V}(G)$ there is a unique finite path $\{e_1, \dots, e_n\}$ without backtracking such that $o(e_1) = v$ and $t(e_n) = w$. Recall that a *half-line* γ on an oriented graph G is an infinite path without backtracking. We say that two half-lines on an oriented graph are *equivalent* if they only differ in a finite graph. We define the set $\text{Ends}(G)$ of ends of a tree G as the equivalence classes of half-lines of G . There is a natural topology on $\text{Ends}(G)$ given by the

sub-basis G_e , $e \in \mathcal{E}(G)$ where G_e consists of the equivalence classes of half-lines of the form $\{e_1, e_2, \dots, e_n, \dots\}$ with the property $e_1 = e$.

Definition 5.9. By slightly extending the usual terminology we will call a scheme C defined over a field a *curve* if it is reduced, locally of finite type and of dimension one. A curve C is said to have *normal crossings* if every singular point of C is an ordinary double point in the usual sense. For any curve C with normal crossings let \tilde{C} denote its normalization, and let $\tilde{S}(C)$ denote the pre-image of the set $S(C)$ of singular points of C . We denote by $\Gamma(C)$ the oriented graph whose set of vertices is the set of irreducible components of \tilde{C} , and its set of edges is the set $\tilde{S}(C)$ such that the initial vertex of any edge $x \in \tilde{S}(C)$ is the irreducible component of \tilde{C} which contains x and the terminal vertex of x is the irreducible component which contains the unique other element \bar{x} of $\tilde{S}(C)$ which maps with respect to the normalization map to the same singular point as x . The map $x \mapsto \bar{x}$ is an involution $\bar{\cdot} : \mathcal{E}(\Gamma(C)) \rightarrow \mathcal{E}(\Gamma(C))$ of the type described in Definition 5.7.

Definition 5.10. Let \mathcal{O} , \mathbf{f} denote the valuation ring of \mathbb{R} and its residue field, respectively. Let \mathfrak{U} be an admissible formal scheme of dimension one over \mathcal{O} and let U denote the rigid analytic space we get from \mathfrak{U} by applying Raynaud's functor (for its definition see [2]). Let \mathfrak{U}_0 denote the special fiber of \mathfrak{U} and assume that the curve \mathfrak{U}_0 over \mathbf{f} is totally degenerate. The latter means that \mathfrak{U}_0 has normal crossings and its irreducible components are smooth projective rational curves over \mathbf{f} . Assume that U is biholomorphic to $\Omega_{\partial U}$ for some non-empty $\partial U \subseteq \mathbb{P}^1(\mathbb{R})$. In this case the graph $\Gamma(\mathfrak{U}_0)$ is a tree and the topological space $\text{Ends}(\Gamma(\mathfrak{U}_0))$ is canonically homeomorphic to ∂U (see [6]). We will use this identification without further notice. For every element $k \in K_2(U)$ let $\{k\}$ denote the function $\{k\} : \mathcal{E}(\Gamma(\mathfrak{U}_0)) \rightarrow \mathbb{R}^*$ which is given by the rule $\{k\}(e) = \{k\}(\Gamma(\mathfrak{U}_0)_e)$ for every edge $e \in \mathcal{E}(\Gamma(\mathfrak{U}_0))$ where we use the notation of Definition 5.8 and the symbol $\{k\}$ on the right hand side of the equation above denotes the measure we introduced in Lemma 5.4.

Lemma 5.11. *The function $\{k\}$ lies in $\mathcal{H}(\Gamma(\mathfrak{U}_0), \mathbb{R}^*)$.*

Proof. The claim is purely graph-theoretical in nature. In fact for every tree \mathcal{T} , commutative group R and measure $\mu \in \mathcal{M}_0(\text{Ends}(\mathcal{T}), R)$ the function $c(\mu) : \mathcal{E}(\mathcal{T}) \rightarrow R$, given by the rule $c(\mu)(e) = \mu(\mathcal{T}_e)$ for every edge $e \in \mathcal{E}(\mathcal{T})$, is an R -valued harmonic cochain. Fix a vertex $v \in \mathcal{V}(\mathcal{T})$: then every end of \mathcal{T} has a unique representative of the form $\{e_1, e_2, \dots, e_n, \dots\}$ with the property $o(e_1) = v$. Now it is clear that the sets \mathcal{T}_e , $o(e) = v$ form a pairwise disjoint partition of $\text{Ends}(\mathcal{T})$. Therefore $c(\mu)$ satisfies property (ii) of Definition 5.7. Similarly property (i) of Definition 5.7 follows from the fact that, given an edge $e \in \mathcal{E}(\mathcal{T})$, every

end of \mathcal{T} has a unique representative of the form $\{e_1, e_2, \dots, e_n, \dots\}$ such that either $e_1 = e$ or $e_1 = \bar{e}$. \square

Definition 5.12. By the lemma above we have constructed a regulator

$$\{\cdot\} : K_2(U) \rightarrow \mathcal{H}(\Gamma(\mathfrak{U}_0), \mathbb{R}^*).$$

We are going to recall the definition of a similar invariant

$$\text{Reg} : K_2(U) \rightarrow \mathcal{H}(\Gamma(\mathfrak{U}_0), \mathbb{Q})$$

which is perhaps best called the *tame regulator*. Let $\nu : \mathbb{R}^* \rightarrow \mathbb{Q}$ be a valuation induced by the absolute value $|\cdot|$. The normalization map identifies the irreducible components of \mathfrak{U}_0 and its normalization which we will use without further notice. For every vertex $v \in \mathcal{V}(\Gamma(\mathfrak{U}_0))$ let \mathfrak{U}_v denote the open affine subvariety of \mathfrak{U}_0 consisting of irreducible component v with all singular points removed. Let \mathfrak{U}_v denote also the unique open affine formal subscheme of \mathfrak{U} whose fiber in \mathfrak{U}_0 is equal to \mathfrak{U}_v . Let U_v denote the open affinoid of the rigid analytic space U we get by applying Raynaud’s functor to the admissible formal scheme \mathfrak{U}_v . Then U_v is a connected rational subdomain of $\mathbb{P}^1(\mathbb{C})$ via the embedding of U into the latter which is isomorphic to $U(S)$ for some finite subset $S \subset \mathfrak{f}$. As we saw in Definition 4.5 there is a valuation on the field $\mathcal{M}(U_v)$, hence on the field $\mathcal{M}(U)$, whose restriction to \mathbb{R} is ν . Let T_v denote the corresponding generalized tame symbol from $K_2(U)$ into the multiplicative group of the function field of \mathfrak{U}_v tensored with \mathbb{Q} . For every $k \in K_2(U)$ and every $e \in \mathcal{E}(\Gamma(\mathfrak{U}_0))$ let $\text{Reg}(k)(e) \in \mathbb{Q}$ denote the valuation of $T_{o(e)}(k)$ at the image of e with respect to the normalization map. It is not difficult to see that $\text{Reg}(k)$ is a harmonic cochain but this fact also follows from the following result:

Theorem 5.13. For every $k \in K_2(U)$ we have: $\text{Reg}(k) = \nu(\{k\})$.

Proof. For every $v \in \mathcal{V}(\Gamma(\mathfrak{U}_0))$ there is a bijection b_v from the set

$$B_v = \{e \in \mathcal{E}(\Gamma(\mathfrak{U}_0)) \mid o(e) = v\}$$

to the set ∂U_v such that $\nu(\{k\}_{b_v(e)}) = \text{Reg}(k)(e)$ by Proposition 4.6. Since for every $e \in B_v$ we have $\Gamma(\mathfrak{U}_0) = b_v(e)(\partial U)$ the claim is now obvious. \square

Definition 5.14. Let X be a geometrically connected regular projective curve defined over the field \mathbb{R} and let $\mathcal{R}(X)$ denote the field of rational functions of the curve X . For any $x \in X(\mathbb{C})$ and any two non-zero functions $f, g \in \mathcal{R}(X)$ let $\{f, g\}_x$ denote the tame symbol of the pair (f, g) at x . We define $K_2(X)$ as the intersection of the kernels of all the tame symbols $\{\cdot, \cdot\}_x$ inside $K_2(\mathcal{R}(X))$ where

x runs through the set $X(\mathbb{C})$. By the usual abuse of notation let X denote also the rigid analytic variety associated to the projective curve X as well.

Definition 5.15. Recall that X is called a *Mumford curve* if there is a flat, projective, regular and semistable scheme \mathfrak{X} over the spectrum of \mathcal{O} whose generic fiber over \mathbb{R} is isomorphic to X and whose special fiber \mathfrak{X}_0 over \mathfrak{f} is totally degenerate. Let $\mathfrak{p} : \mathfrak{U} \rightarrow \mathfrak{X}$ be the universal cover of \mathfrak{X} in the category of admissible formal schemes and let Γ denote the group of deck transformations of the cover \mathfrak{p} . According to [6] the formal scheme \mathfrak{U} is of the type considered in Definition 5.10, at least under some assumptions on the base field \mathbb{R} . Let $\mathfrak{p}_0 : \mathfrak{U}_0 \rightarrow \mathfrak{X}_0$ denote the special fiber of \mathfrak{p} over \mathfrak{f} and let $p : U \rightarrow X$ denote the map of rigid analytic spaces we get by applying Raynaud's functor to \mathfrak{p} . The map \mathfrak{p}_0 induces a map of oriented graphs $\Gamma(\mathfrak{U}_0) \rightarrow \Gamma(\mathfrak{X}_0)$ which in turn induces a map:

$$\mathfrak{p}_0^* : \mathcal{H}(\Gamma(\mathfrak{X}_0), \mathbb{R}^*) \rightarrow \mathcal{H}(\Gamma(\mathfrak{U}_0), \mathbb{R}^*)^\Gamma$$

where the superscript Γ denotes the subgroup of Γ -invariant harmonic cochains. The natural action of Γ on the graph $\Gamma(\mathfrak{U}_0)$ is proper and free therefore \mathfrak{p}_0^* is in fact an isomorphism. By the invariance theorem the regulator of the pull-back $p^*(k)$ of any element $k \in K_2(X)$ with respect to p lies in $\mathcal{H}(\Gamma(\mathfrak{U}_0), \mathbb{R}^*)^\Gamma$. Hence we may define the rigid analytic regulator for X as a map:

$$\{\cdot\} : K_2(X) \rightarrow \mathcal{H}(\Gamma(\mathfrak{X}_0), \mathbb{R}^*)$$

given by the rule $\{k\} = (\mathfrak{p}_0^*)^{-1}(\{p^*(k)\})$ for every $k \in K_2(X)$.

Example 5.16. A case of particular interest is when $X = E$ is a Tate elliptic curve. In this case the regulator is uniquely determined by its value on any of the edges of the reduction graph of a minimal model of the elliptic curve E over $\text{Spec}(\mathbb{C}^0)$ so it is really a homomorphism $\{\cdot\} : K_2(E) \rightarrow \mathbb{C}^*$, well-defined up to sign. It is very easy to give an explicit description of this homomorphism in general using the Tate uniformization of the elliptic curve. Recall that an elliptic curve E defined over \mathbb{C} is a Tate curve if its j invariant $j(E)$ is not an element of \mathbb{C}^0 . Under this assumption there is a rigid analytic Tate uniformization $u : \mathbb{C}^* \rightarrow E$. The pull-back $u^*(k)$ of every $k \in K_2(E)$ as an element of $K_2(\mathcal{R}(E))$ in $K_2(\mathcal{M}(U))$ for any $U \subset \mathbb{C}^*$ connected rational subdomain lies in $K_2(U)$ hence the regulator $\{u^*(k)\}_D \in \mathbb{C}^*$ is well-defined for every $D \in \partial U$. The value of this regulator $\{u^*(k)\}_D \in \mathbb{C}^*$ does not depend on the choice of U or D if the disk D contains 0 by the homotopy invariance of the regulator. This value is the regulator $\{k\}$ of the element $k \in K_2(E)$.

Next we present a purely analytical proof of Weil’s Reciprocity Law for Tate elliptic curves.

Theorem 5.17. *Let E be an elliptic curve defined over \mathbb{C} such that its j -invariant $j(E) \notin \mathbb{C}^0$ and let f, g be two non-zero rational functions on E . Then the product of all tame symbols of the pair (f, g) is equal to 1:*

$$\prod_{x \in E(\mathbb{C})} \{f, g\}_x = 1.$$

Proof. This argument can be generalized to Mumford curves using the concept of a fundamental domain for a Schottky group, but for the sake of simplicity we present the argument for Tate curves only. As we already noted there is a rigid analytic Tate uniformization $u : \mathbb{C}^* \rightarrow E$ with Tate period $t \in \mathbb{C}$ such that $|t| > 1$. Let f and g also denote the pull-back of these functions to \mathbb{C}^* via u by abuse of notation. Then the restriction of f, g to the annulus $A(1, |t|)$ is meromorphic. Let $S \subset A(1, |t|)$ be a finite set such that the functions f and g do not have a zero or a pole on $A(1, |t|) - S$. For each $s \in S$ take an open disk $D_s \subset A(1, |t|)$ containing s such that these disks are pairwise disjoint. The set $Y = A(1, |t|) - \bigcup_{s \in S} D_s$ is a connected rational subdomain. If these disks are sufficiently small claim (iii) of Theorem 3.2 reads as follows:

$$\{f, g\}_{D(0,1)} \cdot \{f, g\}_{D(\infty, |t|^{-1})} \cdot \prod_{s \in S} \{f, g\}_s = 1.$$

But the functions f, g are periodic with multiplicative period t , so the regulators $\{f, g\}_{D(0,1)}$ and $\{f, g\}_{D(\infty, |t|^{-1})}^{-1} = \{f, g\}_{D(0, |t|)}$ are equal, because they only depend on the restrictions of f and g to the sets $\partial D(0, 1)$ and $\partial D(0, |t|)$, respectively. Hence the product of the first two terms in the equation above is one, and the claim follows. \square

6. The rigid analytic regulator on Drinfeld’s upper half plane

Example 6.1. Let \mathbb{R} denote again a closed subfield of \mathbb{C} and assume that the valuation on \mathbb{R} induced by $|\cdot|$ is discrete. Also assume that the residue field of \mathbb{R} is a finite field \mathbb{F}_q and let \mathcal{O} denote the valuation ring of \mathbb{R} . Let Ω denote the rigid analytic upper half plane, or Drinfeld’s upper half plane over \mathbb{R} . It is the rigid analytic space Ω_K introduced in Definition 5.3 in the special case when $K = \mathbb{P}^1(\mathbb{R})$. In particular the set of points of Ω is $\mathbb{P}^1(\mathbb{C}) - \mathbb{P}^1(\mathbb{R})$, denoted also by Ω by abuse of notation. We can give a very simple description of the regulator of every $k \in K_2(\Omega)$ as follows. By the invariance theorem the value $\{k\}(\rho) = \{k|_{\Omega(S)}\}_{D(\rho)}$, where $\rho \in GL_2(\mathbb{R})$ and $D(\rho) \in S \in \mathcal{P}$, is independent of the choice of S . We define

the regulator $\{k\} : GL_2(\mathbb{R}) \rightarrow \mathbb{C}^*$ of k as the function given by this rule. The assignment $k \mapsto \{k\}$ is $GL_2(\mathbb{R})$ -equivariant by the invariance theorem.

Definition 6.2. We say that an additive submodule $A \subset \mathbb{R}$ is a lattice if it is discrete and the quotient $A \backslash \mathbb{R}$ is compact. Let $\Gamma(A)$ denote the following subgroup:

$$\Gamma(A) = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \in GL_2(\mathbb{R}) \mid a \in A \right\}.$$

The subgroup $\Gamma(A)$ stabilizes the point ∞ on the projective line via the Möbius action. Also note that $\Gamma(A)$ leaves the set

$$\Omega_c = \{z \in \Omega \mid c < |z|_i\}$$

invariant for any positive $c \in |\mathbb{C}|$ where $|z|_i = \inf_{x \in \mathbb{R}} |z - x|$ is the imaginary absolute value. Let μ be a Haar measure on the additive group of the non-archimedean field \mathbb{R} . This measure induces another measure on the quotient group $A \backslash \mathbb{R}$ which will be denoted by the same symbol. We may normalize μ such that $\mu(A \backslash \mathbb{R}) = 1$. In this case μ will take only rational values. We may and we will assume that the absolute value $|\cdot|$ on \mathbb{R} is normalized such that $\mu(y\mathcal{O}) = |y|\mu(\mathcal{O})$ for every $y \in \mathbb{R}^*$. If $k \in K_2(\Omega)$ is a $\Gamma(A)$ -invariant element then the regulator $\{k\} : GL_2(\mathbb{R}) \rightarrow \mathbb{C}^*$ is also invariant with respect to the left regular action of $\Gamma(A)$. Moreover the regulator $\{k\}$ is left invariant by multiplication on the right by a compact, open subgroup of $GL_2(\mathbb{R})$ hence the integrand of the integral

$$\{k\}_\infty = \int_{A \backslash \mathbb{R}} \{k\} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} d\mu(x) \in \mathbb{R}^* \otimes \mu(\{k\} \left(\begin{pmatrix} 1 & \cdot \\ 0 & 1 \end{pmatrix} \right))$$

is locally constant and the integral itself is well-defined. The modulus above is a subset of \mathbb{Q} as we already remarked.

Definition 6.3. Let A_0 denote the intersection $A \cap \mathcal{O}$ and let p denote the characteristic of the residue field \mathbb{F}_q . The set A_0 is finite and it is also a vector space over \mathbb{F}_p so its cardinality is a power of p . Let $e_A(z) : \Omega \rightarrow \mathbb{C}^*$ denote the classical Carlitz-exponential:

$$e_A(z) = z \prod_{0 \neq \lambda \in A} \left(1 - \frac{z}{\lambda} \right).$$

It is well known (see for example 2.7 of [5], pages 44–45) that the function e_A^{-1} is $\Gamma(A)$ -invariant and it is a biholomorphic map between the quotient $\Gamma(A) \backslash \Omega_c$ and a small open disk around 0 punctured at 0 for a sufficiently large c . We say that a $\Gamma(A)$ -invariant meromorphic function u on Ω is meromorphic at ∞ if the composition of u and the inverse of the biholomorphic map e_A^{-1} is meromorphic

at 0 for some (and hence all) such numbers c . In this case we can speak about its value, order of zero or order of pole at ∞ . Let $\mathcal{M}^A(\Omega)$ denote the field of $\Gamma(A)$ -invariant meromorphic functions on Ω meromorphic at ∞ . Let $K_2^A(\Omega)$ denote the intersection $K_2(\Omega) \cap K_2(\mathcal{M}^A(\Omega))$. For every $k \in K_2(\mathcal{M}^A(\Omega))$ we may speak about its tame symbol at ∞ in the sense introduced above.

Theorem 6.4. *For each element $k \in K_2^A(\Omega)$ we have $\mu(\{k\}(\begin{pmatrix} 1 & \cdot \\ 0 & 1 \end{pmatrix})) \subseteq \mathbb{Z}$ and the integral $\{k\}_\infty$ is equal to the tame symbol of k at ∞ multiplied by $|A_0|$.*

Proof. For every positive $c \in |\mathbb{C}|$ let \mathcal{D}_c denote the set of those disks $D \in \mathcal{D}$ such that there is an element $S \in \mathcal{P}$ such that $D \in S$ and $\Omega(S) \subset \Omega_c$. The set Ω_c has the structure of a rigid analytic space such that a function $f : \Omega_c \rightarrow \mathbb{C}$ is holomorphic if and only if the restriction of f onto $\Omega(S)$ is holomorphic for every $S \in \mathcal{P}$ whenever $\Omega(S) \subset \Omega_c$. Let $\mathcal{M}(\Omega_c)$ denote the field of meromorphic functions on Ω_c . For each $k \in K_2(\mathcal{M}(\Omega))$ the value $\{k\}(D) = \{k|_{\Omega(S)}\}_D$, where $D \in \mathcal{D}_c$, $D \in S \in \mathcal{P}$ and $\Omega(S) \subset \Omega_c$, is independent of the choice of S and defines a function $\{k\} : \mathcal{D}_c \rightarrow \mathbb{C}^*$. For every $y \in \mathbb{R}^*$ and $x \in \mathbb{R}$ we have

$$D\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = \{z \in \mathbb{C} \mid |z - x| > |y|\},$$

so $D(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}) \in \mathcal{D}_c$ if $|y| > c$. In particular the integral

$$\{k\}_\infty^y = \int_{A \setminus \mathbb{R}} \{k\}\left(D\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right)\right) d(|y|^{-1}\mu)(x) \in \mathbb{R}^* \otimes |y|^{-1}\mu(\{k\}\left(\begin{pmatrix} y & \cdot \\ 0 & 1 \end{pmatrix}\right))$$

is well-defined for every $y \in \mathbb{R}^*$ and $k \in K_2(\mathcal{M}(\Omega))$ when $|y| > c$ and k can be represented as a linear combination of symbols of $\Gamma(A)$ -invariant meromorphic functions on Ω_c . Under this notation we have $\{k\}_\infty = \{k\}_\infty^1$. We say that the $\Gamma(A)$ -invariant meromorphic function u on Ω_c is meromorphic at ∞ if it satisfies the same condition as we demanded for the elements of $\mathcal{M}^A(\Omega)$ in Definition 6.3. Let $\mathcal{M}^A(\Omega_c)$ denote the field of $\Gamma(A)$ -invariant meromorphic functions on Ω_c meromorphic at ∞ . Let $K_2^A(\Omega_c)$ denote the intersection $K_2(\Omega_c) \cap K_2(\mathcal{M}^A(\Omega_c))$.

Lemma 6.5. *For each element $k \in K_2^A(\Omega_c)$ we have $|y|^{-1}\mu(\{k\}(\begin{pmatrix} y & \cdot \\ 0 & 1 \end{pmatrix})) \subseteq \mathbb{Z}$ and $\{k\}_\infty^y$ does not depend on the choice of $y \in \mathbb{R}^*$ where $|y| > c$.*

Proof. Fix a uniformizer $\pi \in \mathbb{R}$. As

$$D\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = D\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix}\right)$$

for every $u \in \mathcal{O}^*$ and $v \in \mathcal{O}$, we may assume that $y = \pi^m$ for some integer m using this identity when $v = 0$. On the other hand the identity above also implies when

$v = 1$ that the integrand of the integral $\{k\}_\infty^y$ is translation-invariant with respect to the group $y\mathcal{O}$. Since the measure of the projection of this group into $A \setminus \mathcal{O}$ with respect to the measure $|y|^{-1}\mu$ is an integer we get that $|y|^{-1}\mu(\{k\}(\left(\begin{smallmatrix} y & \\ 0 & 1 \end{smallmatrix}\right))) \subseteq \mathbb{Z}$ as claimed. The function $\{k\}$ satisfies the identity:

$$D\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) = D\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} u & v \\ 0 & 1 \end{pmatrix}\right)$$

for all $g \in GL_2(\mathbb{R})$ because the disks $D(g\left(\begin{smallmatrix} \pi & \epsilon \\ 0 & 1 \end{smallmatrix}\right))$, $\epsilon \in \mathbb{F}_q$ give a pairwise disjoint partition of the disk $D(g)$. An immediate consequence of this identity is the formula:

$$\begin{aligned} \{k\}\left(D\left(\begin{pmatrix} \pi^m & x \\ 0 & 1 \end{pmatrix}\right)\right) q^m \mu(x + \pi^m \mathcal{O}) \\ = \sum_{\epsilon \in \mathbb{F}_q} \{k\}\left(D\left(\begin{pmatrix} \pi^{m+1} & x + \pi^m \epsilon \\ 0 & 1 \end{pmatrix}\right)\right) q^{m+1} \mu(x + \pi^m \epsilon + \pi^{m+1} \mathcal{O}), \end{aligned}$$

which holds for every $x \in \mathbb{R}$. Hence by the translation-invariance of the measure $q^{m+1}\mu$ we have:

$$\begin{aligned} \int_{A \setminus \mathbb{R}} \{k\}\left(D\left(\begin{pmatrix} \pi^m & x \\ 0 & 1 \end{pmatrix}\right)\right) d(q^m \mu)(x) \\ = \int_{A \setminus \mathbb{R}} \{k\}\left(D\left(\begin{pmatrix} \pi^{m+1} & x \\ 0 & 1 \end{pmatrix}\right)\right) d(q^{m+1} \mu)(x) \end{aligned}$$

for every sufficiently small integer m as claimed. □

Let us return to the proof of the theorem. Choose a presentation $k = \sum_i f_i \otimes g_i$ where $f_i, g_i \in \mathcal{M}^A(\Omega)$. There is a positive $c \in \mathbb{R}$ such that the restrictions of the functions f_i and g_i onto the rigid analytic space Ω_c are invertible. For every element of $K_2(\mathcal{M}^A(\Omega_c))$ we may speak about its tame symbol at ∞ in the sense introduced above. By Lemma 6.5 it will be sufficient to prove that the tame symbol of $k|_{\Omega_c}$ at ∞ is equal to the integral $\{k\}_\infty^y$ for some $y \in \mathbb{R}^*$ with the property $|y| > c$. Therefore by bilinearity it will be sufficient to prove that the tame symbol of $f \otimes g$ at infinity is equal to $\{f \otimes g\}_\infty^y$ for every pair of functions $f, g \in \mathcal{O}^*(\Omega_c)$ because of our assumption on c . In fact it will be sufficient to prove this claim in the following three cases:

- (i) the functions f, g are non-zero at ∞ ,
- (ii) the function f is non-zero at ∞ and $g = e_A$,
- (iii) both f and g are equal to e_A .

In the first case we need to show that $\{f \otimes g\}_\infty^y = 1$. We are going to show that for every positive ϵ there is an $y \in \mathbb{R}^*$ with the property $|y| > c$ such that $\{f \otimes g\}_\infty^y \in U_\epsilon$. This is sufficient to prove the claim in the first case by Lemma 6.5. Let $f(\infty)$ and $g(\infty) \in \mathbb{R}^*$ denote the value of the functions f and g at ∞ , respectively. Then the values of the functions $f(z)/f(\infty)$ and $g(z)/g(\infty)$ on the rigid space Ω_d are in the set U_ϵ for a sufficiently large $d > 0$ as the set Ω_d maps to a small neighborhood of 0 with respect to e_A^{-1} . Choose an element $y \in \mathbb{R}^*$ such that $|y| > d$. For every $x \in \mathbb{R}$ let $S(x) \in \mathcal{P}$ be a set such that $D\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) \in \Omega(S(x))$ and $\Omega(S(x)) \subset \Omega_d$. By our assumptions the holomorphic functions $f/f(\infty)$, $g/g(\infty)$ are in $\mathcal{O}_\epsilon(\Omega_{S(x)})$ for any $x \in \mathbb{R}$ hence by Theorem 2.2 we have:

$$\begin{aligned} \{f \otimes g\}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) &= \left\{\frac{f}{f(\infty)} \otimes g\right\}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) \cdot \left\{f(\infty) \otimes \frac{g}{g(\infty)}\right\}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) \\ &\quad \cdot \{f(\infty) \otimes g(\infty)\}\left(\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}\right) \in U_\epsilon, \end{aligned}$$

where we also used that the third factor on the right hand side is equal to 1. Hence the integral $\{k\}_\infty^y$ is an element of U_ϵ , too, since the corresponding modulus is a subset of \mathbb{Z} .

In the second case we need to show that $\{f \otimes e_A\}_\infty^1 = f(\infty)^{-|A_0|}$ where $f(\infty)$ denotes again the value of the function f at ∞ . It is clear that $\{(f/f(\infty)) \otimes e_A\}_\infty^y = 1$ by repeating the argument used in the proof of the claim in the first case, therefore we may assume that $f = f(\infty)$ is constant. By the definition of the degree homomorphism we have

$$\{f(\infty) \otimes e_A\}(D) = f(\infty)^{\deg(e_A|_{\Omega(S)})(D)},$$

where $D \in \mathcal{D}_c$, $D \in S \in \mathcal{P}$ and $\Omega(S) \subset \Omega_c$. In fact for any $u \in \mathcal{O}^*(\Omega_c)$ the expression $\deg(u|_{\Omega(S)})(D)$ is independent of the choice of S and defines a function $\deg(u) : \mathcal{D}_c \rightarrow \mathbb{Z}$. (It is not difficult see that this is the van der Put logarithmic differential when the domain of definition of u is Ω .) Hence it will be sufficient to prove that

$$\int_{A \setminus \mathbb{R}} \deg(e_A) d\mu(x) = -|A_0| \in \mathbb{Z} \otimes \mathbb{Z} = \mathbb{Z}.$$

By (ii) of Theorem 3.6 we have:

$$\deg(e_A)(D(g)) = -|\{\lambda \in A \mid \lambda \notin D(g)\}|$$

for every $g \in GL_2(\mathbb{R})$ such that $\infty \in D(g)$. As

$$\infty \in D\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) = \{z \in \mathbb{P}^1(\mathbb{C}) \mid 1 < |z - x|\}$$

for any $x \in \mathbb{R}$, we get:

$$\deg(e_A) \left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) = -|\{\lambda \in A \mid |\lambda - x| \leq 1\}| = \begin{cases} -|A_0|, & \text{if } x \in A + \mathcal{O}, \\ 0, & \text{otherwise,} \end{cases}$$

so the claim is now clear in the second case. In the last case we need to show that $\{e_A \otimes e_A\}_\infty^1 = (-1)^{-|A_0|}$. Note that

$$\{f \otimes f\}_D = (-1)^{\deg(f)(D)}$$

for every $U \subset \mathbb{P}^1$ rational subdomain, $D \in \partial U$ boundary component and $f \in \mathcal{O}^*(U)$ function. This is obviously true for rational functions and the general case follows from this one by approximation. Therefore we only have to show that

$$\int_{A \setminus \mathbb{R}} (-1)^{\deg(e_A)} d\mu(x) = (-1)^{-|A_0|} \in \mathbb{R}^* \otimes \mathbb{Z} = \mathbb{R}^*.$$

It is clear from the calculations above that the integrand on the left hand side above is equal to $(-1)^{-|A_0|}$, if $x \in A + \mathcal{O}$, and it is equal to 0, otherwise, hence the required identity obviously holds. \square

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