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The Principle of Limit Amplitude for Symmetric Hyperbolic Systems of First Order in the Half-Space $\mathbf{R}^n_+$

By

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§ 1. Introduction

The present paper is concerned with the principle of limit amplitude for symmetric hyperbolic systems in a half-space. Our proof for the validity of the principle is based on the eigenfunction expansion theorem established in the preceding paper [3]. For the notation and terminology in this paper we refer the reader to [3]. We shall consider the following mixed initial-boundary value problem for hyperbolic systems:

$$\frac{\partial}{\partial t}u(t,x)=iAu(t,x)+\frac{1}{i}e^{ikt}f(x),\quad t>0,\quad x\in\mathbf{R}^n_+, \tag{1.1}$$

$$Bu(t,x)|_{x_n=0}=0, \tag{1.2}$$

$$u(0,x)=g(x), \tag{1.3}$$

where k ($\neq 0$) is a real number,

$$A=\frac{1}{i}\sum_{j=1}^{n}A_j\frac{\partial}{\partial x_j}, \tag{1.4}$$

the A_j are $2m\times 2m$ constant Hermitian matrices, B is an $m\times 2m$ constant matrix with rank m and $u(t,x)$, $g(x)$ and $f(x)$ are vector-valued functions whose values lie in $\mathbf{C}^{2m}$. Replacing $u(t,x)$ in (1.1) and (1.2) by $e^{ikt}v(x)$, we obtain the corresponding stationary problem:

$$(A-kI)v(x)=f(x),\qquad x\in\mathbf{R}^n_+, \tag{1.5}$$

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$$(1.6)\qquad Bv(x)|_{x_n=0}=0\,.$$

We assume the following conditions:

(L.1) $L\equiv\left(I\frac{\partial}{\partial t}-iA\right)$ is uniformly propagative.

(L.2) The operator A is elliptic.

(L.3) The multiplicity of the real roots of $Q(\lambda,\eta)|_{\eta=(\xi,\tau)}=0$ with respect to τ is not greater than two for every $\xi\in\Xi^{n-1}$ and real $\lambda\neq 0$. Moreover the equation has at most only one couple of real double roots for every $(\xi,\lambda)\neq(0,0)$.

(B.1) The boundary matrix B is minimally conservative.

Under the above assumptions the following expansion theorem was proved in [3]:

Theorem 1 (cf. Theorem 6.2 in [3]). *Assume that the conditions* (L.1)–(L.3) *and* (B.1) *are satisfied and that* $f\in L^2(\mathbf{R}^n_+)$.

(i) *The expansion formula*

$$(1.7)\qquad Pf(x)=\sum_{j=1}^{2\rho}\int_{\Xi^n}\Psi_j^{\pm}(x,\eta)\hat{f}_j^{\pm}(\eta)d\eta + \sum_{\nu=1}^{s}\sum_{j=1}^{2\rho}\int_{D_\nu\times\Xi}\Psi_{j+2\nu\rho}^{\pm}(x,\eta)\hat{f}_{j+2\nu\rho}^{\pm}(\eta)d\eta$$

holds, where

$$(1.8)\qquad \hat{f}_j^{\pm}(\eta)=\int_{\mathbf{R}^n_+}\Psi_j^{\pm}(x,\eta)^*f(x)dx,\qquad 1\leqq j\leqq 2\rho\,,$$

$$(1.9)\qquad \hat{f}_{j+2\nu\rho}^{\pm}(\eta)=\int_{\mathbf{R}^n_+}\Psi_{j+2\nu\rho}^{\pm}(x,\eta)^*f(x)dx,\quad 1\leqq j\leqq 2\rho,\ 1\leqq\nu\leqq s\,.$$

Here the above integrals are taken in the sense of the limit in the mean and P is the orthogonal projection onto $R(\mathbf{A})^a=N(\mathbf{A})^{\perp}$.

(ii) $f\in D(\mathbf{A})$ *if and only if* $\lambda_j(\eta)\hat{f}_j^{\pm}(\eta)\in P_j(\eta)L^2(\Xi^n)$, $k_\nu(\xi)\hat{f}_{j+2\nu\rho}^{\pm}\in P_j(\eta)L^2(D_\nu\times\Xi)$, $1\leqq j\leqq 2\rho$, $1\leqq\nu\leqq s$. *Then*

$$(1.10)\qquad (\mathbf{A}f)(x)=\sum_{j=1}^{2\rho}\int_{\Xi^n}\lambda_j(\eta)\Psi_j^{\pm}(x,\eta)\hat{f}_j^{\pm}(\eta)d\eta + \sum_{\nu=1}^{s}\sum_{j=1}^{2\rho}\int_{D_\nu\times\Xi}k_\nu(\xi)\Psi_{j+2\nu\rho}^{\pm}(x,\eta)\hat{f}_{j+2\nu\rho}^{\pm}(\eta)d\eta\,,$$

(1.11) $$(\mathbf{A}f)\hat{}_j^{\pm}(\eta)=\lambda_j(\eta)\hat{f}_j^{\pm}(\eta), \qquad 1\leqq j\leqq 2\rho,$$

(1.12) $$(\mathbf{A}f)\hat{}_{j+2\nu\rho}^{\pm}(\eta)=k_\nu(\xi)\hat{f}_{j+2\nu\rho}^{\pm}(\eta), \quad 1\leqq j\leqq 2\rho, \quad 1\leqq \nu\leqq s.$$

Here the $\lambda_j(\eta)$ are distinct roots of $Q(\lambda, \eta)=0$ and the $k_\nu(\xi)$ are non-vanishing zeros of the Lopatinski determinant (see [3]).

Remark. The differential operator A defines an unbounded linear operator $\mathscr{A}$ in $L^2(\mathbf{R}_+^n)$ with domain

$$D(\mathscr{A})=\{v(x)\in C_0^\infty(\overline{\mathbf{R}_+^n});\ Bv(x)|_{x_n=0}=0\}.$$

Thus we denoted by $\mathbf{A}$ the closure of $\mathscr{A}$ in [3], which is a self-adjoint operator in $L^2(\mathbf{R}_+^n)$.

In [3] it was proved that $\sigma(\mathbf{A})=\mathbf{R}^1$. Hence in general there exist no solutions of (1.5) and (1.6) belonging to $L^2(\mathbf{R}_+^n)$ for real k and $f\in L^2(\mathbf{R}_+^n)$. Then the question arises of determining uniquely a solution of (1.5) and (1.6). There are three important approaches of deriving a unique solution of (1.5) and (1.6), i.e., the radiation principle, the limiting absorption principle and the limiting amplitude principle. The limiting absorption principle for (1.5) and (1.6) was justified in M. Matsumura [2] under more restrictive assumptions than ours.

In this paper the following theorems will be proved:

Theorem 2. *Let $f(x)=0$ in* (1.1), *$g(x)\in D(\mathbf{A})$ in* (1.3) *and* $\operatorname{Supp} g(x)$ *be bounded. Suppose that at least one of the following conditions holds*: a) *$E^+(\xi;0)\cap\mathscr{B}=\{0\}$ for $|\xi|\neq 0$,* b) *$g(x)$ belongs to $C^\infty(\overline{\mathbf{R}_+^n})$. Then for the solution $u(t, x)$ of the equations* (1.1)–(1.3) *and any compact set $K\subset\overline{\mathbf{R}_+^n}$*

(1.13) $$\frac{d^l}{dt^l}u(t, x) \longrightarrow \delta_{0,l}g_2(x) \quad in\ L^2(K)\ as\ t\longrightarrow\infty,\ l=0,1,$$

hold, where $\delta_{0,l}=1$ for $l=0$, $=0$ for $l=1$, and $g=g_1+g_2$, $g_1\in N(\mathbf{A})^\perp=R(\mathbf{A})^a$, $g_2\in N(\mathbf{A})$. Moreover if a) *holds, then $g_2=0$.*

Theorem 3 (the limiting amplitude principle). *Let $f\in L^2(\mathbf{R}_+^n)$, $g\in D(\mathbf{A})$ and $k\neq 0$ be real in* (1.1)–(1.3) *and* $\operatorname{Supp} f(x)$ *and* $\operatorname{Supp} g(x)$

be bounded. Suppose that at least one of the following conditions holds: a) $E^+(\xi; 0)\cap\mathscr{B}=\{0\}$ *for* $|\xi|\neq 0$, b) $f(x)$ *and* $g(x)$ *belong to* $C^\infty(\overline{\mathbf{R}^n_+})$. *Then for the solution* $u(t, x)$ *of the equations* (1.1)–(1.3) *and any compact set* $K\subset\overline{\mathbf{R}^n_+}$

$$(1.14)\quad e^{-ikt}\left[\frac{d^l}{dt^l}u(t, x)-\delta_{0,l}\left(\frac{1}{k}f_2+g_2\right)\right]\longrightarrow (ik)^l v(x)\quad in\ L^2(K)$$

$$as\quad t\longrightarrow\infty,\quad l=0, 1,$$

hold, where $v(x)\in L^2_{loc}(\overline{\mathbf{R}^n_+})$[1] *and* $f=f_1+f_2, f_1\in N(\mathbf{A})^\perp, f_2\in N(\mathbf{A})$. *Moreover* $\alpha_R(x)v(x)\in D(\mathbf{A})$ *and* $v(x)$ *satisfies* (1.5) *in the sense of distribution and also satisfies* (1.5) *and* (1.6) *in the following sense*: *There exists a sequence* $\{v_j\}\subset H^1(\mathbf{R}^n_+)$ *such that* $Bv_j|_{x_n=0}=0$, $v_j\to v(x)$ *in* $L^2_{loc}(\overline{\mathbf{R}^n_+})$ *and* $(A-kI)v_j\to f$ *in* $L^2_{loc}(\overline{\mathbf{R}^n_+})$ *as* $j\to\infty$. *Here* $\alpha_R(x)=1$ *for* $|x|\leqq R$, $=0$ *for* $|x|\geqq R+1$ *and belongs to* $C^\infty(\mathbf{R}^n_+)$.

Remark 1. a) $E^+(\xi; 0)\cap\mathscr{B}=\{0\}$ for $|\xi|\neq 0$ implies the coercivity of B for the elliptic operator A. Thus if a) holds, then

$$(1.14)'\quad e^{-ikt}\frac{d^l}{dt^l}u(t, x)\longrightarrow (ik)^l v(x)\quad in\ H^1_{loc}(\overline{\mathbf{R}^n_+})\quad as\quad t\longrightarrow\infty,\ l=0,1,$$

and $v(x)$ is a solution of (1.5) and (1.6) belonging to $H^1_{loc}(\overline{\mathbf{R}^n_+})$.[2]

Remark 2. From Lemma 3 it is easy to see that

$$(\mathbf{A}-\lambda\mathbf{I})^{-1}f(x)\longrightarrow v(x)\qquad \text{in}\quad L^2_{loc}(\overline{\mathbf{R}^n_+})\ \text{as}\ \lambda\longrightarrow k-i0.$$

In order to prove the above theorems we state some lemmas and propositions in § 2, following D. M. Eidus [1]. In § 3 we shall give their proofs.

§ 2. Preliminaries

Let $\mathscr{H}$ be a separable Hilbert space, $\mathscr{H}_0$ a subspace of $\mathscr{H}$, $\tilde{P}$ the orthogonal projection onto $\mathscr{H}_0$, and A a self-adjoint operator in $\mathscr{H}$.

1) $L^2_{loc}(\overline{\mathbf{R}^n_+})$ denotes the space of vector-valued functions f such that $f\in L^2(K)$ for any compact set $K\subset\overline{\mathbf{R}^n_+}$.

2) $H^1_{loc}(\overline{\mathbf{R}^n_+})$ denotes the space of vector-valued functions f such that $\alpha_R(x)\frac{\partial^\alpha}{\partial x^\alpha}f(x)\in L^2(\overline{\mathbf{R}^n_+})$ for $|\alpha|\leqq 1$ and any $R>0$.

Definition 1. An element $f \in \mathscr{H}$ is said to satisfy the condition T_1 if for all real a and b $(a \cdot b > 0, a < b)$ there exists a constant C such that for any complex $\lambda = k + \varepsilon i$ $(\varepsilon \neq 0, k \in [a, b])$ the inequality

$$\|\tilde{P}R(\lambda)f\| \leqq C \tag{2.1}$$

holds.

Lemma 1. *Suppose that f satisfies T_1. Then for almost all $k \in \mathbf{R}^1$ there exist the weak limits*[3]

$$w-\lim_{\lambda \to k \pm i0} \tilde{P}R(\lambda)f = v^{(\pm)}(k). \tag{2.2}$$

This lemma can be proved by Fatou's theorem and Riemann's mapping theorem.

Lemma 2. *Let f satisfy T_1. Suppose that $F(k)$ is a complex-valued function, continuous and bounded on $(-\infty, 0) \cup (0, \infty)$. Then for any real a and b $(a \cdot b > 0, a < b$, including $a = -\infty$ and $b = \infty)$*

$$\tilde{P}\int_b^a F(k)dE(k)f = \frac{1}{2\pi i}\int_b^a F(k)\theta(k)dk \tag{2.3}$$

holds, where $\{E(k)\}_{-\infty<k<\infty}$ denotes the right-continuous spectral family of A and

$$\theta(k) = v^{(+)}(k) - v^{(-)}(k). \tag{2.4}$$

Here the integral on the right-hand side of (2.3) *is taken in the sense of Bochner's integral.*

We first consider the Cauchy problem for $t > 0$

$$\frac{d}{dt}u(t) = iAu(t) + \frac{1}{i}e^{ikt}f, \tag{2.5}$$

$$u(0) = g \in D(A). \tag{2.6}$$

The following proposition can be easily verified.

3) $\lambda \to k \pm i0$ means that $\lambda \to k$ along any path that does not cross the real axis and is not tangent to it.

Proposition 1. *Let* $f=0$ *in* (2.5), g *satisfy* T_1 *and* $g\in D(A)$. *Let* $u(t)$ *be the solution of* (2.5), (2.6). *Then*

$$(2.7)\qquad \tilde{P}\frac{d^l u}{dt^l} \longrightarrow \delta_{0,l}\tilde{P}g_2 \quad \text{in } \mathscr{H} \quad \text{as } t\longrightarrow\infty,\ l=0, 1,$$

hold, where $\delta_{0,l}=1$ *for* $l=0$, $=0$ *for* $l=1$, *and* $g=g_1+g_2$, $g_1\in N(A)^{\perp}$, $g_2\in N(A)$.

Definition 2. An element $f\in\mathscr{H}$ is said to satisfy the condition T_2 if it satisfies T_1 and if $\theta(k)$ determined by (2.4) satisfies a Hölder condition on any interval $[a, b]\subset\mathbf{R}\backslash\{0\}$.

Lemma 3. *Let* $f\in N(A)^{\perp}$ *satisfy* T_2. *Then replacing the weak limits by the strong limits in* (2.2) *Lemma* 1 *holds. Moreover*

$$(2.8)\qquad v^{(\pm)}(k)=\pm\frac{\theta(k)}{2}+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\theta(k_1)}{k_1-k}dk_1,$$

where the integral is taken in the sense of the principal value. $v^{(\pm)}(k)$ *satisfy a Hölder condition on any interval* $[a, b]\subset\mathbf{R}\backslash\{0\}$.

This lemma follows from Lemma 2.

Proposition 2. *Let* f *satisfy* T_2, $g=0$ *and* $k\neq0$ *be real in* (2.5), (2.6), *and* $u(t)$ *be the solution of* (2.5), (2.6). *Then*

$$(2.9)\qquad e^{-ikt}\tilde{P}\left[\frac{d^l}{dt^l}u(t)-\delta_{0,l}\frac{1}{k}f_2\right]\longrightarrow (ik)^l v^{(-)}(k) \quad \text{in } \mathscr{H}$$

$$\text{as}\quad t\to\infty,\quad l=0, 1,$$

hold.

Using Lemmas 2 and 3, we can easily prove the above proposition.

Lemma 4. *Let* $f\in\mathscr{H}$ *be such that for all* a *and* b *with* $a\cdot b>0$, $a<b$, *and any* $h_0\in\mathscr{H}_0$ *there exists* $C_0>0$ *such that for all complex* $\lambda=k+\varepsilon i$ ($k\in[a, b]$, $\varepsilon\neq0$) *the inequality*

$$(2.10)\qquad |(R(\lambda)f, R(\bar{\lambda})h_0)|\leqq C_0$$

holds. Then f satisfies T_2. *Here* $v^{(\pm)}(k)$ *satisfy a Hölder condition with exponent* 1 *on an arbitrary interval* $[a, b]\subset\mathbf{R}\backslash\{0\}$.

The above lemma follows from the equality

$$\frac{d}{d\lambda}(R(\lambda)f, h_0)=(R(\lambda)^2 f, h_0)=(R(\lambda)f, R(\bar{\lambda})h_0). \tag{2.11}$$

§3. Proof of the Theorems

We first state the following

Lemma 5. *Let* $f(r)$ *be a complex-valued function and* $f(r)\in L^1(0, \infty)\cap C^{1+\alpha}(0, \infty)$,[4] $\alpha>0$. *Then for all* a *and* b $(a\cdot b>0, a<b)$ *there exists a constant* C_0 (>0) *such that for any* $k\in[a, b]$

$$\left|\int_0^\infty \frac{f(r)}{(r-\lambda)^2}\,dr\right|\leqq C_0 \tag{3.1}$$

holds, where $\lambda=k+i\varepsilon$, $\varepsilon\neq0$.

Now let $f, h\in L^2(\mathbf{R}^n_+)$, $\operatorname{Supp} f(x)$ and $\operatorname{Supp} h(x)$ be bounded and $\mathbf{A}$ the self-adjoint operator associated with (1.5) and (1.6). Then it follows from Theorem 1 that for $\operatorname{Im}\lambda\neq0$

$$(R(\lambda)\hat{)}_j^{\pm}(\eta)=\frac{1}{\lambda_j(\eta)-\lambda}\hat{f}_j^{\pm}(\eta), \qquad 1\leqq j\leqq 2\rho, \tag{3.2}$$

$$(R(\lambda)f\hat{)}_{j+2\nu\rho}^{\pm}(\eta)=\frac{1}{k_\nu(\xi)-\lambda}\hat{f}_{j+2\nu\rho}^{\pm}(\eta), \qquad 1\leqq j\leqq 2\rho,\ 1\leqq\nu\leqq s, \tag{3.3}$$

hold. We can replace f by h in (3.2) and (3.3). Moreover we have $PR(\lambda)f=R(\lambda)Pf$. Thus by Theorem 5.10 in [3] we obtain

$$\begin{aligned}(R(\lambda)Pf, R(\bar{\lambda})h)=&\sum_{j=1}^{2\rho}\int_{\Xi^n}\frac{1}{(\lambda_j(\eta-\lambda)^2}\hat{f}_j^{\pm}(\eta)\cdot\overline{\hat{h}_j^{\pm}(\eta)}d\eta\\ &+\sum_{j=1}^{2\rho}\sum_{\nu=1}^{s}\int_{D_\nu\times\Xi}\frac{1}{(k_\nu(\xi)-\lambda)^2}\hat{f}_{j+2\nu\rho}^{\pm}(\eta)\cdot\overline{\hat{h}_{j+2\nu\rho}^{\pm}(\eta)}d\eta\,.\end{aligned} \tag{3.4}$$

4) $C^{1+\alpha}(0, \infty)$ consists of continuous functions whose derivatives satisfy a Hölder condition with exponent α on any bounded closed subinterval of $(0, \infty)$.

Let us consider the case $1 \leqq j \leqq \rho$.[5] Put $S_j = \{\eta \in \Xi^n;\ \lambda_j(\eta) = 1\}$, $r = \lambda_j(\eta)$ and $\omega = \frac{\eta}{\lambda_j(\eta)}$, $1 \leqq j \leqq \rho$. Then

$$\text{(3.5)} \qquad \int_{\Xi^n} \frac{1}{(\lambda_j(\eta) - \lambda)^2} \hat{f}_j^{\pm}(\eta) \cdot \overline{\hat{h}_j^{\pm}(\eta)} d\eta$$

$$= \int_0^{\infty} \frac{r^{n-1}}{(r-\lambda)^2} dr \int_{S_j} (\omega \cdot n_j(\omega)) \hat{f}_j^{\pm}(r\omega) \cdot \overline{\hat{h}_j^{\pm}(r\omega)} dS_j$$

holds, where $n_j(\omega)$ denotes the outward unit normal to S_j at ω and dS_j the surface element on S_j. Put

$$\text{(3.6)} \qquad I_j(r) = \int_{S_j} (\omega \cdot n_j(\omega)) \hat{f}_j^{\pm}(r\omega) \cdot \overline{\hat{h}_j^{\pm}(r\omega)} dS_j .$$

Let us prove below that $I_j(r) \in C^{\infty}(0, \infty)$. By Theorem 6.1 in [3] we have a local representation

$$\Psi_j^{\pm}(x, \eta)$$

$$\text{(3.7)} \qquad = \begin{cases} 0 \\ \quad (\text{when there exists } l_j,\ 1 \leqq l_j \leqq \rho, \text{ such that} \\ \qquad \tau_{l_j}^{+}(\xi;\ \lambda_j(\eta) \pm i0) = \sigma), \\ (2\pi)^{-\frac{n}{2}} e^{ix\cdot\eta} P_j(\eta) + (2\pi)^{-\frac{n}{2}} \sum_{\mu=1}^{p} e^{ix'\cdot\xi} e^{i\tau_{\mu}^{\pm}(\xi;\lambda_j(\eta)\pm i0)x_n} \\ \quad \times C_{j\mu}(\eta) + (2\pi)^{-\frac{n}{2}} e^{ix'\cdot\xi} \sum_{p(l)=l_j} \left[\dfrac{\dfrac{\partial \lambda_j(\eta)}{\partial \tau}}{\Delta(\xi;\ \lambda_j(\eta) \mp i0)} \right. \\ \quad \overline{\times \det (Bh_1^{+}(\xi;\ \lambda_j(\eta) \mp i0), \ldots, \underset{l}{\smile}\, Bq_{\nu}(x_n, \xi;\ \lambda_j(\eta) \mp i0), \ldots} \\ \quad \left. \ldots, Bh_m^{+}) h_l^{+}(\xi;\ \lambda_j(\eta) \mp i0)^{*} \right]_{\nu = 1,\ldots,2m \downarrow} \\ \quad (\text{when there exists } l_j,\ 1 \leqq l_j \leqq \rho, \text{ such that} \\ \qquad \tau_{l_j}^{+}(\xi;\ \lambda_j(\eta) \mp i0) = \sigma), \end{cases}$$

5) $\lambda_1(\eta) > \lambda_2(\eta) > \cdots > \lambda_\rho(\eta) > 0 > \lambda_{\rho+1}(\eta) > \cdots > \lambda_{2\rho}(\eta)$ for $\eta \neq 0$.

where the $C_{j\mu}(\eta)$ are $2m\times 2m$ matrices whose elements are positively homogeneous of degree 0. Noting that the $h_l^+(\xi;\lambda_j(\eta)\mp i0)$, $\frac{\partial\lambda_j(\eta)}{\partial\tau}$ and $\Delta(\xi;\lambda_j(\eta)\mp i0)$ are positively homogeneous of degree 0 and that the $\tau_\mu^+(\xi;\lambda_j(\eta)\pm i0)$ are positively homogeneous of degree 1, we obtain

$$\frac{\partial^k\Psi_j^\pm(x,\eta)}{\partial r^k}$$

$$(3.8)\qquad =\begin{cases} 0 \qquad (\tau_{l_j}^+(\xi;\lambda_j(\eta)\pm i0=\sigma), \\ (2\pi)^{-\frac{n}{2}}\left(\frac{ix\cdot\eta}{r}\right)^k e^{ix\cdot\eta}P_j(\eta)+(2\pi)^{-\frac{n}{2}}\sum_{\mu=1}^{p} \\ \left(\frac{ix'\cdot\xi+i\tau_\mu^+(\xi;\lambda_j(\eta)\pm i0)x_n}{r}\right)^k e^{ix'\cdot\xi}e^{i\tau_\mu^\pm(\xi;\lambda_j(\eta)\pm i0)x_n}C_{j\mu}(\eta) \\ +(2\pi)^{-\frac{n}{2}}e^{ix'\xi}\sum_{p(l)=l_j}\left[\frac{\frac{\partial\lambda_j(\eta)}{\partial\tau}}{\Delta(\xi;\lambda_j(\eta)\mp i0)}\sum_{h=0}^{k}\binom{k}{h}\left(\frac{ix'\cdot\xi}{r}\right)^h\right. \\ \overline{\times\det(Bh_1^+,\dots,\overbrace{\frac{\partial^{k-h}}{\partial r^{k-h}}Bq_\nu(x_n,\xi;\lambda_j(\eta)\mp i0)}^{l},\dots,Bh_m^+)} \\ \left. h_l^+(\xi;\lambda_j(\eta)\mp i0)^*\right]_{\nu=1,\dots,2m\downarrow} \qquad (\tau_{l_j}^+(\xi;\lambda_j(\eta)\mp i0)=\sigma), \\ k=1,2,\dots. \end{cases}$$

Here we have used the relation $\frac{\partial}{\partial r}=\sum_{k=1}^{n}\frac{\eta_k}{r}\frac{\partial}{\partial\eta_k}$ and Euler's identity. It follows from the estimates of $\Psi_j^\pm(x,\eta)$ in [3] that for fixed a and b, $0<a<b$, and $r\in[a,b]$

$$(3.9)\qquad \left|\left(\frac{\partial^k\Psi_j^\pm}{\partial r^k}(x,\eta)\right)^*f(x)\right|\leqq F_j^k(x,\omega)$$

holds, where $F_j^k(x,\omega)$ satisfies the inequality

$$(3.10)\qquad \int_{S_j}(\omega\cdot n_j(\omega))\left(\int_{\mathbf{R}_+^n}|F_j^k(x,\omega)|dx\right)^2 dS_j<+\infty,$$

since $\frac{|\eta|}{r} \leqq C$, $\operatorname{Supp} f(x)$ is bounded and $\frac{\partial^i}{\partial r^i} Bq_\nu(x_n, \xi; \lambda_j(\eta) \mp i0)$ is continuous in (x_n, η). Also we can replace $f(x)$ and $F_j^k(x, \omega)$ by $h(x)$ and $H_j^k(x, \omega)$ in (3.9) and (3.10), respectively. Therefore we have

$$\text{(3.11)} \qquad \frac{\partial^k}{\partial r^k} \hat{f}_j^{\pm}(r\omega) = \int_{\mathbf{R}_+^n} \left(\frac{\partial^k \Psi_j^{\pm}}{\partial r^k}(x, \eta) \right)^* f(x) dx\,,$$

$$\frac{\partial^k}{\partial r^k} \hat{h}_j^{\pm}(r\omega) = \int_{\mathbf{R}_+^n} \left(\frac{\partial^k \Psi_j^{\pm}}{\partial r^k}(x, \eta) \right)^* h(x) dx\,.$$

Thus we have

$$\text{(3.12)} \quad \frac{\partial^k}{\partial r^k} I_j(r) = \int_{S_j} (\omega \cdot n_j(\omega)) \sum_{h=0}^{k} \binom{k}{h} \frac{\partial^h}{\partial r^h} \hat{f}_j^{\pm}(r\omega) \cdot \overline{\frac{\partial^{k-h}}{\partial r^{k-h}} \hat{h}_j^{\pm}(r\omega)}\, dS_j\,,$$

since

$$\text{(3.13)} \qquad \left| \frac{\partial^h}{\partial r^h} \hat{f}_j^{\pm}(r\omega) \cdot \overline{\frac{\partial^{k-h}}{\partial r^{k-h}} \hat{h}_j^{\pm}(r\omega)} \right| \leqq C \left\{ \left(\int_{\mathbf{R}_+^n} |F_j^h(x, \omega)|\, dx \right)^2 + \left(\int_{\mathbf{R}_+^n} |H_j^{k-h}(x, \omega)|\, dx \right) \right\}^2.$$

From this it follows that $I_j(r) \in C^\infty[a, b]$, i.e., $I_j(r) \in C^\infty(0, \infty)$. Put $\tilde{S}_\nu = \{\eta \in \Xi^n;\ k_\nu(\xi) = 1,\ \xi \in D_\nu\}$, $D_\nu^+ = \{\xi \in D_\nu;\ k_\nu(\xi) > 0\}$ and $D_\nu^- = D_\nu \backslash D_\nu^+$, $1 \leqq \nu \leqq s$. Moreover put $r = k_\nu(\xi)$ and $\omega = \frac{\eta}{k_\nu(\xi)}$ for $\xi \in D_\nu^+$. Then

$$\text{(3.14)} \quad \int_{D_\nu \times \Xi} \frac{1}{(k_\nu(\xi) - \lambda)^2} \hat{f}_{j+2\nu\rho}^{\pm}(\eta) \cdot \overline{\hat{h}_{j+2\nu\rho}^{\pm}(\eta)}\, d\eta$$

$$= \int_{D_\nu^- \times \Xi} \frac{1}{(k_\nu(\xi) - \lambda)^2} \hat{f}_{j+2\nu\rho}^{\pm}(\eta) \cdot \overline{\hat{h}_{j+2\nu\rho}^{\pm}(\eta)}\, d\eta$$

$$+ \int_0^\infty \frac{r^{n-1}}{(r - \lambda)^2} dr \int_{\tilde{S}_\nu} |\omega \cdot \tilde{n}_\nu(\omega)|\, \hat{f}_{j+2\nu\rho}^{\pm}(r\omega) \cdot \overline{\hat{h}_{j+2\nu\rho}^{\pm}(r\omega)}\, d\tilde{S}_\nu$$

holds, where $\tilde{n}_\nu(\omega)$ denotes the unit normal to $\tilde{S}_\nu$ at ω and $d\tilde{S}_\nu$ denotes the surface element on $\tilde{S}_\nu$. Put

$$\text{(3.15)} \qquad I_{j+2\nu\rho}(r) = \int_{\tilde{S}_\nu} |\omega \cdot \tilde{n}_\nu(\omega)|\, \hat{f}_{j+2\nu\rho}^{\pm}(r\omega) \cdot \overline{\hat{h}_{j+2\nu\rho}^{\pm}(r\omega)}\, d\tilde{S}_\nu\,.$$

By the definition of $\Psi_{j+2\nu\rho}^{\pm}(x, \eta)$ in [3] we see that

(3.16) $$\Psi^{\pm}_{j+2\nu\rho}(x,\eta)=(2\pi)^{-\frac{n}{2}}e^{ix'\cdot\xi}\sum_{\mu=1}^{p}e^{i\tau^{\pm}_{\mu}(\xi;k_\nu(\xi)\pm i0)x_n}C^{\nu}_{j\mu}(\eta)$$

$$+(2\pi)^{-\frac{n}{2}}e^{ix'\cdot\xi}\sum_{l=1}^{m}\Bigg[\Bigg(\frac{\lambda-k_\nu(\xi)}{\Delta(\xi;\bar\lambda)}\overline{\det(Bh_1^{+}(\xi;\bar\lambda),\dots}$$

$$\overline{\dots,\underbrace{Bq_\mu(x_n,\xi;\bar\lambda)}_{l},\dots,Bh_m^{+}(\xi;\bar\lambda))}\big|_{\lambda=k_\nu(\xi)\pm i0}$$

$$\times\gamma^{\nu}_{jl}(\eta)\Bigg]_{\mu=1,\dots,2m\downarrow},\qquad \eta\in D_{j^\nu},$$

where the $C^{\nu}_{j\mu}(\eta)$ are $2m\times 2m$ matrices and positively homogeneous of degree 0, and the $\gamma^{\nu}_{jl}(\eta)$ are $1\times 2m$ matrices and positively homogeneous of degree -1. Thus we have

(3.17) $$\frac{\partial^k\Psi^{\pm}_{j+2\nu\rho}(x,\eta)}{\partial r^k}=(2\pi)^{-\frac{n}{2}}\sum_{\mu=1}^{p}\Bigg(\frac{ix'\cdot\xi+i\tau^{+}_{\mu}(\xi;k_\nu(\xi)\pm i0)x_n}{r}\Bigg)^k$$

$$\times e^{ix'\cdot\xi}e^{i\tau^{\pm}_{\mu}(\xi;k_\nu(\xi)\pm i0)x_n}C^{\nu}_{j\mu}(\eta)+(2\pi)^{-\frac{n}{2}}e^{ix'\cdot\xi}$$

$$\times\sum_{h=0}^{k}\sum_{l=1}^{m}\Bigg[\Bigg(\binom{k}{h}\Bigg(\frac{ix'\cdot\xi}{r}\Bigg)^h\overline{\Delta(\xi;\bar\lambda)}\;\overline{\det(Bh_1^{+}(\xi;\bar\lambda)\dots,}$$

$$\overline{\dots,\underbrace{\frac{\partial^{k-h}}{\partial r^{k-h}}Bq_\mu(x_n,\xi;k_\nu(\xi\mp i0)}_{l},\dots,Bh_m^{+}(\xi;\bar\lambda)\Big)}\Big|_{\lambda=k_\nu(\xi)\pm i0}$$

$$\times\gamma^{\nu}_{jl}(\eta)\Big]_{\mu=1,\dots,2m\downarrow},\qquad \eta\in D_{j\nu},\quad k=0,1,2,\dots.$$

Here we have used the fact that

$$\frac{\lambda-k_\nu(\xi)}{\Delta(\xi;\bar\lambda)}\overline{\det(Bh_1^{+}(\xi;\bar\lambda),\dots,\underbrace{e_h}_{l},\dots,Bh_m^{+}(\xi;\bar\lambda))}\ \big|_{\lambda=k_\nu(\xi)\pm i0}$$

is positively homogeneous of degree 1, where $e_h={}^t(0,\dots,1,\dots,0)$. We put in [3]

(3.18) $q_\mu(x_n,\xi;k_\nu(\xi)\mp i0)$ is equal to the μ-th column vector of

$$\Bigg(\frac{1}{2\pi i}\int_{\gamma_-}e^{-ix_n\tau_1}(\tau_1 I-M(\xi;k_\nu(\xi)))^{-1}A_n^{-1}d\tau_1\Bigg).$$

Thus

(3.19) $\frac{\partial^h}{\partial r^h} q_\mu(x_n, \xi; k_\nu(\xi) \mp i0)$ is equal to the μ-th column vector of

$$\left(\frac{1}{2\pi i}\int_{\gamma_-} \frac{1}{r^h}\left[\left(\tau_1 \frac{d}{d\tau_1}\right)^h e^{-ix_n\tau_1}\right](\tau_1 I - M(\xi; k_\nu(\xi)))^{-1} A_n^{-1} d\tau_1\right).$$

Hence it follows from (3.19) that

$$\left|\frac{\partial^h q_\mu}{\partial r^h}(x_n, \xi; k_\nu(\xi) \mp i0)\right| \leqq \frac{C_h(1+|\xi|x_n)^h}{r^h} \tag{3.20}$$

holds, where the C_h are independent of (x_n, ξ). Let us consider the case where $E^+(\xi; 0) \cap \mathscr{B} = \{0\}$ for $|\xi| \neq 0$. Then there exists $\delta\,(>0)$ such that for all ξ with $|\xi|=1$ and $\xi \in D_\nu$, $|k_\nu(\xi)| > \delta$. Thus $\tilde{S}_\nu$ are bounded, $1 \leqq \nu \leqq s$. Therefore, by the same argument as for $I_j(r)$, $1 \leqq j \leqq \rho$, it follows from the estimates of $\Psi^\pm_{j+2\nu\rho}(x, \eta)$ in [3] that $I_{j+2\nu\rho}(r) \in C^\infty[a, b]$ for any interval $[a, b]$, $0 < a < b < \infty$, i.e., $I_{j+2\nu\rho}(r) \in C^\infty(0, \infty)$. Next let us consider the case where $f(x)$ belongs to $C^\infty(\overline{\mathbf{R}^n_+})$. Then we have

$$\begin{aligned} &|\mathscr{F}_{x'}[p(x)f(x)](\xi, x_n)| = |(1+|\xi|^2)^{-l}\mathscr{F}_{x'}[(1-\Delta_{x'})^l \\ &\qquad (p(x)f(x))](\xi, x_n)| \leqq C_l(1+|\xi|^2)^{-l} \end{aligned} \tag{3.21}$$

for arbitrary non-negative integer l and $p(x) \in C^\infty$. Since we have

$$\begin{aligned} &|C^\nu_{j\mu}(\eta)| \leqq \frac{C(1+|\xi|)}{1+|\sigma|}, \\ &|\gamma^\nu_{jl}(\eta|) \leqq \frac{C}{1+|\sigma|}, \end{aligned} \tag{3.22}$$

for sufficiently large R, $|\eta| > R$ and $k_\nu(\xi) \in [a, b]$,

$$\int_{a \leqq k_\nu(\xi) \leqq b} (1+|\xi|^2)^{-\frac{n}{2}-h-1} \left|\frac{\partial^h \Psi^\pm_{j+2\nu\rho}(x, \eta)}{\partial r^h}\right|^2 d\eta < +\infty, \tag{3.23}$$

$h = 0, 1, 2, \dots,$

hold. Thus for $r \in [a, b]$ and $h = 0, 1, 2, \dots$,

$$\left|\int_{\mathbf{R}^n_+}\left(\frac{\partial^h \Psi^\pm_{j+2\nu\rho}(x, \eta)}{\partial r^h}\right)^* f(x)\,dx\right| \leqq (1+|\xi|^2)^{-l+\frac{h}{2}+\frac{n}{4}+\frac{1}{2}} F_{j+2\nu\rho}(\omega), \tag{3.24}$$

$$\left|\int_{\mathbf{R}^n_+}\left(\frac{\partial^h \Psi^{\pm}_{j+2\nu\rho}(x,\eta)}{\partial r^h}\right)^* f(x)dx\right| \leqq (1+|\xi|^2)^{\frac{h}{2}+\frac{n}{4}+\frac{1}{2}} H_{j+2\nu\rho}(\omega),$$

hold, where

$$\int_{\tilde{S}_\nu} |\omega\cdot\tilde{n}_\nu(\omega)||F^h_{j+2\nu\rho}(\omega)|^2 d\tilde{S}_\nu < +\infty, \tag{3.25}$$

$$\int_{\tilde{S}_\nu} |\omega\cdot\tilde{n}_\nu(\omega)||H^h_{j+2\nu\rho}(\omega)|^2 d\tilde{S}_\nu < +\infty.$$

Choosing l more than $\frac{n}{2}+\frac{k}{2}+1$, we have $I_{j+2\nu\rho}(r)\in C^k(0,\infty)$. Therefore $I_{j+2\nu\rho}(r)\in C^\infty(0,\infty)$.

Proof of Theorems. Define an orthogonal projection $\tilde{P}$ on $L^2(\mathbf{R}^n_+)$ by $\tilde{P}f(x)=f(x)$ for $|x|\leqq R$, $=0$ for $|x|>R$, $f(x)\in L^2(\mathbf{R}^n_+)$. Let $f\in L^2(\mathbf{R}^n_+)$ and $f=f_1+f_2$, $f_1(x)=(Pf)(x)\in N(\mathbf{A})^\perp$, $f_2\in N(\mathbf{A})$. Then f satisfies T_2 if and only if $f_1=Pf$ satisfies T_2. Suppose that $\mathrm{Supp}f(x)$ is bounded and that at least one of the following conditions holds: a) $E^+(\xi;0)\cap\mathscr{B}=\{0\}$ for $|\xi|\neq 0$. b) $f(x)$ belongs to $C^\infty(\overline{\mathbf{R}^n_+})$. The above arguments and Lemmas 4 and 5 imply that f satisfies T_2. Thus Theorem 2 follows from Proposition 1. Similarly the first assertion of Theorem 3 follows from Proposition 2. Here although $\tilde{P}R(\lambda)f\to v^{(-)}(k)$ in $L^2(\mathbf{R}^n_+)$ as $\lambda\to k-i0$ and $\mathrm{Supp}_x v^{(-)}(k)\subset\{x;|x|\leqq R\}$, moving R to $+\infty$, we can define $v^{(-)}(k)$ as follows:

$$R(\lambda)f \longrightarrow v^{(-)}(k) \quad \text{in} \quad L^2_{loc}(\overline{\mathbf{R}^n_+}) \quad \text{as} \quad \lambda\longrightarrow k-i0.$$

Let $\alpha_R(x)\in C^\infty(\mathbf{R}^n_+)$ and $\alpha_R(x)=1$ for $|x|\leqq R$, $=0$ for $|x|\geqq R+1$. Then,

$$\begin{aligned}(\mathbf{A}-k)\alpha_R(x)R(\lambda)f &= \alpha_R(x)f(x)+(\lambda-k)\alpha_R(x)R(\lambda)f(x)\\ &+(A\alpha_R(x))(R(\lambda)f)(x)\longrightarrow \alpha_R(x)f(x)+(A\alpha_R(x))v^{(-)}(x)\\ &\text{in}\quad L^2(\mathbf{R}^n_+)\quad\text{as}\quad \lambda\longrightarrow k-i0.\end{aligned} \tag{3.26}$$

Since $\alpha_R(x)(R(\lambda)f)(x)\to\alpha_R(x)v^{(-)}(k)$ in $L^2(\mathbf{R}^n_+)$ as $\lambda\to k-i0$, the closedness of the operator $\mathbf{A}$ implies that $\alpha_R(x)v^{(-)}(k)\in D(\mathbf{A})$ and that

$$(\mathbf{A}-k)\alpha_R(x)v^{(-)}(k)=\alpha_R(x)f(x)+(A\alpha_R(x))v^{(-)}(k), \tag{3.27}$$

that is,

(3.27)′ $$(A_{loc}-k)v^{(-)}(k)=f(x).$$

Moreover suppose that $E^{+}(\xi; 0)\cap \mathscr{B}=\{0\}$ for $|\xi|\neq 0$. Then $N(\mathbf{A})=\{0\}$ and the coerciveness inequalities for the operator $\mathbf{A}$,

(3.28) $$\left\|\frac{\partial u}{\partial x_j}\right\| \leqq C(\|\mathbf{A}u\|+\|u\|),\ j=1,\dots, n,$$

hold for $u\in D(\mathbf{A})$. Thus from (3.26) it follows that

(3.29) $$R(\lambda)f \longrightarrow v^{(-)}(k) \quad \text{in} \quad H^1_{loc}(\overline{\mathbf{R}^n_+}) \quad \text{as} \quad \lambda \longrightarrow k-i0.$$

This completes the proof of Theorem 3.

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