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On Semi-Free Finite Group Actions on Homotopy Spheres

By

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§0. Introduction

In this paper, we shall study semi-free finite group actions on homotopy spheres. In [8], M. Sebastiani studied semi-free finite group actions on homotopy spheres with two points as the fixed point set. He showed that the collection of equivariant diffeomorphism classes of these semi-free finite group actions is an abelian group under the equivariant connected sum about a fixed point. By the methods analogous to Kervaire-Milnor [4], he proved that this abelian group is a finite group in the case of cyclic group actions.

In this paper, we shall study semi-free finite group actions on homotopy m-spheres with homotopy n-sheres as the fixed point set in the case of $5 \leqq n < m-2$. Our methods are some extensions of M. Sebastiani [8] and using the results obtained by these methods we can generalize G. Bredon's results [2, Chapter VI, Theorem 8.6].

In §1 we shall see that the collection of equivariant diffeomorphism classes $\Theta_m(\alpha)$ (see Definition 1) of a subset of the above semi-free finite group actions is an abelian group under the equivariant connected sum about a fixed point in the case of $5 \leqq n < m-2$. In §2 we shall define our interesting subgroup $I_m(\alpha)$ of $\Theta_m(\alpha)$ and study an equivariant version of Pontrjagin-Thom construction. Then we shall see that $I_m(\alpha)$ contains a subgroup $\Lambda_m(\alpha)$ whose elements bound equivariant π-manifolds (see Definition 2), and $I_m(\alpha)/\Lambda_m(\alpha)$ is a finite group. In §3 we shall define a homomorphism $I: \Lambda_m(\alpha) \to \boldsymbol{Z}$ in the case of $n=4r-1$ for an integer

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$r \geqq 2$, where $\boldsymbol{Z}$ denotes the group of integers. This homomorphism is given by sending an element of $\Lambda_m(\alpha)$, which bounds an equivariant π-manifold (W^{m+1}, Φ), to the signature of the fixed point set of (W^{m+1}, Φ) (see Theorem 3.3). In §4 we shall apply these results to prove that some semi-free finite group actions on Brieskorn spheres are elements of infinite order in $\Theta_m(\alpha)$. G. Bredon [2, Chapter VI, Theorem 8.6] proved analogous results by a different method when the actions are involutions.

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§1. $\Theta_m(\alpha)$ Is a Group

Let G be a finite group and $\alpha: G \to GL(m, \boldsymbol{R})$ be its m-dimensional representation. Throughout this paper we shall assume $5 \leqq n < m-2$.

Definition 1. Let (Σ^m, φ) denote a smooth semi-free G-action on an oriented homotopy m-sphere Σ^m with an oriented n-sphere Σ^n as the fixed point set. Moreover we will assume that the local representation of G at a fixed point is equivalent to α. Let $\Theta_m(\alpha)$ be the collection of equivariant diffeomorphism classes of these semi-free actions (Σ^m, φ).

We can prove that $\Theta_m(\alpha)$ is a commutative semi-group under the equivariant connected sum about a fixed point.

In this section we shall prove that the commutative semi-group $\Theta_m(\alpha)$ is an abelian group. First we shall study semi-free G-actions on a disc.

Let (D^m, ψ) denote a smooth semi-free G-action on an oriented m-disc D^m with an oriented n-disc D^n as the fixed point set. Moreover we will assume that the local representation of G at a fixed point is equivalent to α.

Let $D_m(\alpha)$ denote the collection of equivariant diffeomorphism classes of these semi-free G-actions (D^m, ψ). Let (D^m, ψ) also denote the equivariant diffeomorphism class of (D^m, ψ). We can see $D^m(\alpha)$ is a commutative semi-group under the equivariant boundary connected sum about a fixed point which belongs to ∂D^n and the standard linear action (D^m, α) serves as zero element.

Let (D^m, ψ) be an element of $D_m(\alpha)$. Let x be a point in the interior of the fixed point set D^n. Let $B \subset \operatorname{int} D^m$ be a G-invariant closed 2ε-disc neighborhood of x on which G acts orthogonally, where ε is a sufficiently small positive real number. Let N be a G-invariant closed ε-tubular neighborhood of D^n in D^m. We can identify N with the normal disc bundle of D^n which is a product G-bundle. Let $W = D^m - \operatorname{int}(B \cup N)$ and $W_0 = \partial B - \operatorname{int}(\partial B \cap N)$. Let $W_1 = \partial D^m - \operatorname{int}(\partial D^m \cap N)$ and $V = \partial W - \operatorname{int}(W_0 \cup W_1)$. It is easy to see that V is equivariant diffeomorphic to $\partial W_0 \times [0, 1]$.

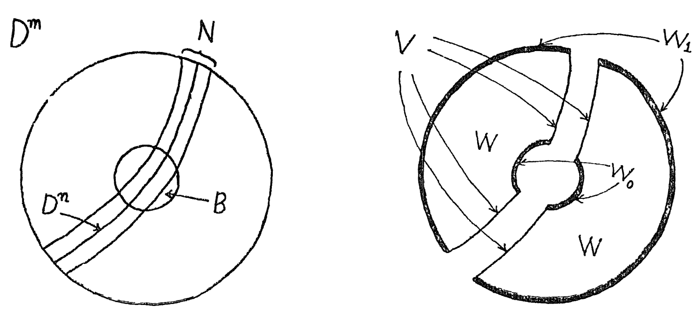


Fig. 1

Since $(\partial W_0 \times [0, 1])/G$ is diffeomorphic to $(\partial W_0/G) \times [0, 1]$ and the inclusions $W_0/G \hookrightarrow W/G$, $W_1/G \hookrightarrow W/G$ are homotopy equivalent in the case of $n < m-2$, $(W/G; W_0/G, W_1/G)$ is an h-cobordism with boundary. Then $(W/G, W_0/G)$ defines the Whitehead torsion $\tau(W/G, W_0/G) \in Wh(G)$. It is not difficult to see that $\tau(W/G, W_0/G)$ depends only on (D^m, ψ). Let $\tau(\psi)$ denote this torsion. This defines a semi-group homomorphism $\tau: D_m(\alpha) \to Wh(G)$.

Lemma 1.1. *τ is isomorphic.*

Proof. Let $(D^{m'}, \psi')$ be an another element of $D_m(\alpha)$. We will describe by V', W'_0, W' manifolds corresponding to the above V, W_0, W respectively. If $\tau(\psi') = \tau(\psi)$, the uniqueness theorem of s-cobordism says that W'/G is diffeomorphic to W/G preserving $W'_0/G = W_0/G$ and $V'/G = V/G$ (see L. Siebenmann [10, §2]). Since $W' \to W'/G$ and $W \to W/G$

are universal coverings, W' is equivariant diffeomorphic to W preserving $W'_0=W_0$ and $V'=V$. Then we have that $(D^{m\prime}, \psi')$ is equivariant diffeomorphic to (D^m, ψ), and τ is injective.

Now we will show that τ is surjective. By the existence theorem of s-cobordism, there exists an h-cobordism $(\overline{W}''; W_0/G, \overline{W}''_1)$ with $\tau(\overline{W}'', W_0/G)=\tau_0$ and $\partial\overline{W}''=W_0/G\cup\overline{W}''_1\cup V/G$, for every $\tau_0\in Wh(G)$ (see J. Milnor [6, Theorem 11.1]. The proof of [6, Theorem 11.1] is valid for the case of h-cobordism with boundary). Let W'' and W''_1 be the universal coverings of $\overline{W}''$ and $\overline{W}''_1$ respectively. Then $\partial W''=W_0\cup W''_1\cup V$, and W'' is a free G-manifold by the covering transformation. Now we obtain an m-dimensional semi-free G-manifold (M, ψ) from the disjoint union $W''+(B\cup N)$ by identifying $W_0\cup V$ in W'' with $W_0\cup V$ in $B\cup N$. Then (M, ψ) is an element of $D_m(\alpha)$, and $\tau(\psi)=\tau_0$. This proves that τ is surjective, and Lemma 1.1 follows.

Let (Σ^m, φ) be a semi-free G-action on Σ^m which represents an element of $\Theta_m(\alpha)$. Let (Σ^m, φ) also denote the equivariant diffeomorphism class of (Σ^m, φ). By removing a G-invariant open disc neighborhood U of a fixed point, on which G acts orthogonally, we have an element $(\Sigma^m-U, \varphi|(\Sigma^m-U))$ in $D_m(\alpha)$, since Σ^m-U and Σ^n-U are diffeomorphic to D^m and D^n respectively in the case of $n\geqq 5$. We put $\tau(\varphi)=\tau(\Sigma^m-U, \varphi|(\Sigma^m-U))$. It can be seen that $\tau(\varphi)$ is independent of U. Note that $\Theta_m(\alpha)$ is a commutative semi-group under the equivariant connected sum about a fixed point and the standard linear action $(S^m, \alpha\oplus\theta)$ serves as zero element, where θ is a trivial 1-dimensional representation of G. Then we have a semi-group homomorphism τ: $\Theta_m(\alpha)\to Wh(G)$.

Now we will prove that (Σ^m, φ) has an inverse. Let W denote $(\Sigma^m-U)\times[0, 1]$, where the action Ψ on W is given by $\varphi|(\Sigma^m-U)$ on the first factor. By straightening the corners equivariantly, (W, Ψ) defines an element in $D_{m+1}(\alpha\oplus\theta)$. By Lemma 1.1 there exists an element (W', Ψ') of $D_{m+1}(\alpha\oplus\theta)$ so that $(W, \Psi)\natural(W', \Psi')\simeq(D^{m+1}, \alpha\oplus\theta)$, where $\natural$ denotes the equivariant boundary connected sum about a fixed point, and $\simeq$ implies equivariant diffeomorphism.

Then we have

$(S^m, \alpha\oplus\theta)=(\partial D^{m+1}, \alpha\oplus\theta)\simeq(\partial W, \Psi)\sharp(\partial W', \Psi')\simeq(\Sigma^m, \varphi)\sharp(-\Sigma^m, \varphi)\sharp(\partial W', \Psi')$,

where $\sharp$ denotes the equivariant connected sum about a fixed point. This implies that (Σ^m, φ) has an inverse $(-\Sigma^m, \varphi)\sharp(\partial W', \Psi')$ in $\Theta_m(\alpha)$. Thus we have:

Theorem 1.2. *If* $5\leqq n<m-2$, $\Theta_m(\alpha)$ *is an abelian group.*

Remark. In G. Bredon [2, Chapter VI], we can find the proof that the equivariant connected sum is well defined and the invariant tubular neighborhood is unique up to equivariantly isotopic. He proves the above results in the case of $G=S^1$ or S^3 (see [2, Chapter VI, Theorem 9.1]).

Let (Σ^m, φ) be an element of $\Theta_m(\alpha)$ with $\tau(\varphi)=0$. Then (Σ^m, φ) is topologically equivalent to $(S^m, \alpha\oplus\theta)$ (see [2, Chapter VI, Corollary 9.3]).

§2. Which Elements of $I_m(\alpha)$ Bound e-π-Manifolds?

Let $\tau: \Theta_m(\alpha)\to Wh(G)$ be the homomorphism defined in §1. Let $\Theta_m^0(\alpha)$ denote the kernel of τ. Let $(S^{m+k}, \alpha\oplus(k+1)\theta)$ denote an orthogonal G-action on S^{m+k} given by an $(m+k+1)$-dimensional representation $\alpha\oplus(k+1)\theta$ for a positive integer k. Define $I_m(\alpha)\subset\Theta_m^0(\alpha)$ as follows. An element (Σ^m, φ) of $\Theta_m^0(\alpha)$ is an element of $I_m(\alpha)$ if and only if (Σ^m, φ) is G-imbeddable in $(S^{m+2}, \alpha\oplus3\theta)$. It is easy to see $I_m(\alpha)$ is a subgroup of $\Theta_m^0(\alpha)$.

Lemma 2.1. *Let* (Σ^m, φ) *be an element of* $I_m(\alpha)$. *Choose a* G-*imbedding* $f: (\Sigma^m, \varphi)\to(S^{m+2}, \alpha\oplus3\theta)$. *Then the normal* G-*bundle* ν_f *of* Σ^m *in* S^{m+2} *is isomorphic to a product* G-*bundle* $\Sigma^m\times\boldsymbol{R}^2$, *where the action on* $\boldsymbol{R}^2$ *is trivial.*

Proof. Let x be a point of the fixed point set Σ^n of (Σ^m, φ). Let B be a G-invariant closed disc neighborhood of x in Σ^m on which G acts orthogonally. Since $\tau(\varphi)=0$, the restricted action $(\Sigma^m-\operatorname{int}B, \varphi|(\Sigma^m-\operatorname{int}B))$ is equivariantly diffeomorphic to (D^m, α). Since B and $\Sigma^m-\operatorname{int}B$ are equivariantly contractible, the restricted G-bundles $\nu_f|B$ and $\nu_f|(\Sigma^m$

$-\operatorname{int} B$) are isomorphic to $D^m \times \boldsymbol{R}^2$ as G-bundles, where the action on $\boldsymbol{R}^2$ is trivial (see E. Biestone [1, Corollary 3.2]). Let ψ_1 and ψ_2 be equivariant normal 2-frames on B and $(\Sigma^m - \operatorname{int} B)$ respectively.

Let $g: \partial B \to SO(2)$ be a G-map so that $\psi_2(p) = g(p) \cdot \psi_1(p)$ for a point p of ∂B, where the action on $SO(2)$ is trivial. Let $\pi: \partial B \to \partial B/G$ denote the natural map. Then g induces a map $\bar{g}: \partial B/G \to SO(2)$ so that $g = \bar{g} \cdot \pi$.

If $\bar{g}$ is extendable over B/G, g can be extended equivariantly on B and Lemma 2.1 follows. Note that B/G is contractible and $\partial B/G$ is homeomorphic to n-fold unreduced suspension of S^{m-n-1}/G, where the action on S^{m-n-1} is given by the non-trivial summand of α. Thus $\bar{g}$ is extendable over B/G since $H^{k+1}(B/G, \partial B/G; \pi_k(SO(2))) = 0$ for any integer k, and Lemma 2.1 follows.

Let (Σ^m, φ) be an element of $I_m(\alpha)$. Let $f: (\Sigma^m, \varphi) \to (S^{m+k}, \alpha \oplus (k+1)\theta)$ be a G-imbedding such that ν_f is isomorphic to $\Sigma^m \times \boldsymbol{R}^k$ as a G-bundle, where the action on $\boldsymbol{R}^k$ is trivial. Let ψ be a specific G-invariant field of normal k-frames. Then the Pontrjagin-Thom construction yields a G-map

$$\bar{p}(\varphi, f, \psi): S^{m+k} \longrightarrow S^k,$$

where the action on S^{m+k} is given by $\alpha \oplus (k+1)\theta$ and on S^k trivial.

Let $[S^{m+k}, S^k]_o^G$ denote the set of all G-homotopy classes of base point preserving G-maps $S^{m+k} \to S^k$, where the base points of S^{m+k} and S^k are the north poles. Then $[S^{m+k}, S^k]_0^G$ is an abelian group.

As shown in the proof of Lemma 2.1, S^{m+k}/G is homeomorphic to $(n+k+1)$-fold suspension space of S^{m-n-1}/G. Put $L = S^{m-n-1}/G$. Then $[S^{m+k}, S^k]_0^G$ is isomorphic to $[S^{n+k+1}L, S^k]_0$ and for a sufficiently large positive integer k this is a stable cohomotopy group $\pi_S^{-n-1}(L)$.

Therefore $\bar{p}(\varphi, f, \psi)$ defines an element $p(\varphi, f, \psi)$ of $\pi_S^{-n-1}(L)$. Allowing the G-imbedding f and the G-invariant normal frame field ψ to vary, we obtain a set of elements

$$p(\Sigma^m, \varphi) = \{p(\varphi, f, \psi)\}_{f,\psi} \subset \pi_S^{-n-1}(L).$$

Definition 2. Let (W, Φ) be a differentiable semi-free G-action on an $(m+1)$-dimensional manifold W. (W, Φ) is called an *e-π-manifold* if there exists a G-imbedding in a linear G-action $(D^{m+k+1}, \alpha\oplus(k+1)\theta)$ such that the normal G-bundle is isomorphic to the product G-bundle $W\times \boldsymbol{R}^k$, where the action on $\boldsymbol{R}^k$ is trivial.

Lemma 2.2. *The subset $p(\Sigma^m, \varphi)\subset\pi_{\$}^{-n-1}(L)$ contains the zero element of $\pi_{\$}^{-n-1}(L)$ if and only if (Σ^m, φ) bounds an e-π-manifold.*

Proof. "If" part is trivial. Conversely if $\bar{p}(\varphi, f, \psi)\colon S^{m+k}\to S^k$ is G-homotopic to zero, $\bar{p}(\varphi, f, \psi)$ can be extended to a G-map $h\colon D^{m+k+1}\to S^k$, and h can be approximated by a differentiable G-map $h'\colon D^{m+k+1}\to S^k$ which is transverse regular on $0\in S^k$, where the action on D^{m+k+1} is given by $\alpha\oplus(k+1)\theta$. Moreover h' may be chosen so that $h'|S^{m+k}=\bar{p}(\varphi, f, \psi)$ (see G. Segal [9], or we can prove in a similar way as Proposition 3.7 of A. Wasserman [11]).

Set $W=h'^{-1}(0)$. The action on W is given by the restricted action Φ on D^{m+k+1}. Clearly (W, Φ) is an e-π-manifold whose boundary is (Σ^m, φ). This completes the proof of Lemma 2.2.

Lemma 2.3. *Let (Σ_0^m, φ_0) and (Σ_1^m, φ_1) are elements of $I_m(\alpha)$. Then*

$$p(\Sigma_0, \varphi_0)+p(\Sigma_1, \varphi_1)\subset p((\Sigma_0, \varphi_0)\#(\Sigma_1, \varphi_1)).$$

Proof. Let $f_i\colon(\Sigma_i, \varphi_i)\to(S^{m+k}, \alpha\oplus(k+1)\theta)$, for $i=0, 1$, be G-imbeddings such that there exist G-invariant fields ψ_i of normal k-frames of Σ_i. We can assume that $f_0(\Sigma_0)\cap f_1(\Sigma_1)=\phi$. It is easy to see that there exists a G-imbedding $h\colon(D^m\times[0, 1], \alpha\oplus\theta)\to(S^{m+k}, \alpha\oplus(k+1)\theta)$ with $h(D^m\times[0, 1])\cap f_i(\Sigma_i)=h(D^m\times i)$ for $i=0, 1$.

Now we obtain W from the disjoint sum $\Sigma_0\times[0, 1]+\Sigma_1\times[0, 1]+D^m\times[0, 1]$ by identifying $(f_i^{-1}(x), 1)$ with $h^{-1}(x)$ for $x\in h(D^m\times i)$, $i=0, 1$. By straightening the corners equivariantly, we have an $(m+1)$-dimensional differentiable G-manifold (W, Φ) so that $\partial W=\Sigma_0\#\Sigma_1+(-\Sigma_0)+(-\Sigma_1)$, where the action Φ on W is given by the actions on the first factors. Moreover we have a G-imbedding

$$F\colon (W, \Phi)\longrightarrow(S^{m+k}\times[0, 1], \alpha\oplus(k+1)\theta\oplus\theta)$$

so that $F(x, t)=(f_i(x), t)$ for $(x, t)\in\Sigma_i\times[0, 1]$, $i=0, 1$.

By pushing the interior of $F(W)$ into the interior of $S^{m+k}\times[0, 1]$, this imbedding can be taken transversely on $\partial(S^{m+k}\times[0, 1])$. It is easy to see that there exists a G-invariant field Ψ of normal k-frames on W with $\Psi|(\Sigma_i\times 0)=\psi_i$. Let ψ denote the restriction of Ψ on $\Sigma_0 \# \Sigma_1$. Then $\bar{p}(\varphi_0, f_0, \psi_0)+\bar{p}(\varphi_1, f_1, \psi_1)$ is G-homotopic to $\bar{p}(\varphi_0 \# \varphi_1, F|\Sigma_0 \# \Sigma_1, \psi)$. This completes the proof of Lemma 2.3.

Theorem 2.4. *The set $p(S^m, \alpha\oplus\theta)$ is a subgroup of the stable cohomotopy group $\pi_{\$}^{-n-1}(L)$. For any element (Σ^m, φ) of $I_m(\alpha)$, the correspondence $(\Sigma, \varphi)\to p(\Sigma, \varphi)$ defines a well-defined homomorphism*

$$p\colon I_m(\alpha) \longrightarrow \pi_{\$}^{-n-1}(L)/p(S^m, \alpha\oplus\theta),$$

and the kernel of p consists exactly of all elements of $I_m(\alpha)$ which bound e-π-manifolds.

Proof. Since $\tau(\varphi)=0$, $(-\Sigma^m, \varphi)$ is an inverse of (Σ^m, φ) in $I_m(\alpha)$. Combining Lemma 2.2 and Lemma 2.3 with the identities

$$(S^m, \alpha\oplus\theta)+(S^m, \alpha\oplus\theta)=(S^m, \alpha\oplus\theta),$$

$$(S^m, \alpha\oplus\theta)+(\Sigma^m, \varphi)=(\Sigma^m, \varphi) \quad \text{and}$$

$$(\Sigma^m, \varphi)+(-\Sigma^m, \varphi)=(S^m, \alpha\oplus\theta),$$

we obtain this theorem.

Remark 1. The above notations $\bar{p}(\varphi, f, \psi)$, $p(\Sigma^m, \varphi)$, $p(S^m, \alpha\oplus\theta)$ and $\pi_{\$}^{-n-1}(L)$ are equivariantly analogous to Kervair-Milnor's notations $p(\Sigma^m, \varphi)$, $p(\Sigma^m)$, $p(S^m)$ and π_m respectively.

2. Let $\Lambda_m(\alpha)$ denote the kernel of the homomorphism $p\colon I_m(\alpha)\to \pi_{\$}^{-n-1}(L)/p(S^m, \alpha\oplus\theta)$. Then any element of $\Lambda_m(\alpha)$ bounds an e-π-manifold. Note that $I_m(\alpha)/\Lambda_m(\alpha)$ is a finite group by Theorem 2.4, since $\pi_{\$}^{-n-1}(L)$ is a finite group (see P. Hilton [3, Theorem 3.18]).

§3. A Homomorphism $\lambda\colon \Lambda_m(\alpha) \longrightarrow Z$

In this section we will assume $n=4r-1$ for some positive integer $r\geqq 2$. Let (W, Φ) be an $(m+1)$-dimensional e-π-manifold whose boundary is $(S^m, \alpha\oplus\theta)$. Let $f\colon (W, \Phi)\to(D^{m+k+1}, \alpha\oplus(k+1)\theta)$ be a G-imbedding such that the normal G-bundle $\nu_{W,D^{m+k+1}}$ is isomorphic to the product G-bundle $W\times R^k$, where the action on R^k is trivial. Here we will assume $k\geqq n+2=4r+1$. Let Ψ be a G-invariant field of normal k-frames of W.

Let F denote the fixed point set of (W, Φ). Then ∂F is the fixed point set of $(S^m, \alpha\oplus\theta)$ and is S^n. Note that f carries F into the fixed point set D^{n+k+1} of $(D^{m+k+1}, \alpha\oplus(k+1)\theta)$. Since $\nu_{F,D^{n+k+1}}\oplus(\nu_{D^{n+k+1},D^{m+k+1}})|F$ is isomorphic to $\nu_{F,W}\oplus(\nu_{W,D^{m+k+1}})|F$ as a G-bundle, $\nu_{F,D^{n+k+1}}$ is isomorphic to $(\nu_{W,D^{m+k+1}})|F$.

Now Ψ defines an element of $[W, SO(k)]^G$, where the action on $SO(k)$ is trivial. Since $\nu_{F,D^{n+k+1}}$ is isomorphic to $(\nu_{W,D^{m+k+1}})|F$, the inclusion $S^n=\partial F\hookrightarrow W$ induces a map $h\colon [W, SO(k)]^G\to\pi_n(SO(k))$.

Lemma 3.1. *$h\colon [W, SO(k)]^G\to\pi_n(SO(k))$ is zero map.*

Proof. There is a commutative diagram

$$\begin{array}{ccc} [W, SO(k)]^G & \xrightarrow{\quad h \quad} & \pi_n(SO(k)) \\ & {\scriptstyle h'}\searrow \quad \nearrow{\scriptstyle h''} & \\ & [S^m, SO(k)]^G & \end{array}$$

where h' and h'' are induced by the inclusions $S^m=\partial W\hookrightarrow W$ and $S^n\hookrightarrow S^m$ respectively. Since G acts on $SO(k)$ trivially, $[S^m, SO(k)]^G$ is identified with $[S^m/G, SO(k)]$. Now S^m/G is homeomorphic to an $(n+1)$-fold suspension space $S^{n+1}L$, so we have that $[S^m, SO(k)]^G$ is identified with $[S^{n+1}L, SO(k)]$.

Therefore h'' is identified with $\bar{h}''\colon [S^{n+1}L, SO(k)]\to\pi_n(SO(k))$. Since $\bar{h}''$ is induced by the inclusion $S^n\hookrightarrow S^{n+1}L$, we have $\bar{h}''$ is zero map. From the above commutative diagram h is zero map, and Lemma 3.1 follows.

Lemma 3.2. *With the above notations, let* Sgn(F) *denote the signature of the $4r$-dimensional manifold F. Then we have* Sgn(F)$=0$.

Proof. Since $\nu_{F,D^{n+k+1}}$ is isomorphic to $(\nu_{W,D^{m+k+1}})|F$, $\Psi|F$ is identified with a field of normal k-frames of F in D^{n+k+1}. Note that $\pi_n(SO(k))=\pi_{4r-1}(SO(k))=Z$ for $k\geqq 4r+1$. Identifying F and D^{4r} along their common boundary, we obtain a closed $4r$-dimensional manifold M.

Note that $h(\Psi)$ is equal to a homotopy class of $\Psi|S^n$ in $\pi_n(SO(k))$. According to Milnor and Kervaire (Lemma 2 in [5]), the r-dimensional Pontrjagin class $p_r(M)=\pm a_r(2r-1)!\cdot h(\Psi)$, where a_r is 2 for r odd and 1 for r even. Thus, by Lemma 3.1, $p_r(M)=0$.

Since F is a parallelizable manifold, so M is an almost parallelizable manifold. Therefore $p_i(M)=0$ for $0<i<r$ (see [5]). Then we have $\mathrm{Sgn}(F)=\mathrm{Sgn}(M)=0$. This completes the proof of Lemma 3.2.

Theorem 3.3. *Let $n=4r-1$ for some positive integer $r\geqq 2$. Let (Σ^m, φ) be an element of $\Lambda_m(\alpha)$, (W, Φ) an $(m+1)$-dimensional e-π-manifold whose boundary is (Σ^m, φ), F the fixed point set of (W, Φ). Then the correspondence $(\Sigma, \varphi)\to \mathrm{Sgn}(F)$ defines a well-defined homomorphism $I: \Lambda_m(\alpha)\to Z$, where Z denotes the group of integers.*

Proof. Let (W', Φ') be an another e-π-manifold whose boundary is (Σ, φ), F' the fixed point set of (W', Φ'). Then the G-invariant boundary connected sum $(W, \Phi)\natural(-W', \Phi')$ is an e-π-manifold whose boundary is equivariant diffeomorphic to $(S^m, \alpha\oplus\theta)$.

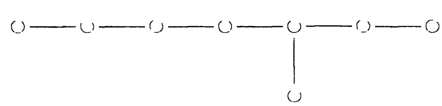
Since the fixed point set of $(W, \Phi)\natural(-W', \Phi')$ is $F\natural(-F')$, we have $\mathrm{Sgn}(F\natural(-F'))=0$ by Lemma 3.2. Therefore $\mathrm{Sgn}(F)=\mathrm{Sgn}(F')$, and I is well-defined. It is clear that I is a homomorphism. This completes the proof of Theorem 3.3.

§4. Semi-Free G-Actions on Brieskorn Spheres

In this section we apply the previous results to semi-free G-actions on Brieskorn spheres.

Let τ_k be the tangent disc bundle of S^k with the total space $E(\tau_k)$. Consider $O(k-1)\subset O(k+1)$ acting on S^k, and hence on τ_k in the standard way. $S^k\times S^k$ is an $O(k-1)$-manifold with the diagonal action. Let $\Delta_S\subset S^k\times S^k$ be the diagonal set which is also an $O(k-1)$-manifold. Let ν_S denote the $O(k-1)$-invariant normal bundle of Δ_S in $S^k\times S^k$, and $E(\nu_S)$ be the total space of its associated disk bundle. Then it is easy to see that $E(\nu_S)$ is isomorphic to $E(\tau_k)$ as an $O(k-1)$-bundle.

Similarly let Δ_D be the diagonal submanifold in $D^{k+1}\times D^{k+1}$ which is a differentiable $O(k-1)$-manifold with the diagonal action by straightening the corners equivariantly. Hereafter, for a given G-manifold with corners, we will think of it as a differentiable G-manifold by straightening the corners equivariantly. Let ν_D denote the $O(k-1)$-invariant normal bundle and $E(\nu_D)$ the total space of its associated disc bundle. Then $E(\nu_D)$ is isomorphic to the tangent disc bundle of D^{k+1} as an $O(k-1)$-bundle. Note that $\partial\Delta_D$ is $O(k-1)$ equivariant diffeomorphic to Δ_S, and hence $E(\nu_D)|\partial\Delta_D$ is isomorphic to the associated disc bundle $E(\nu_S\oplus\theta)$ of $\nu_S\oplus\theta$ as an $O(k-1)$-bundle.

Let $P^{2k}(E_8)$ denote a $2k$-dimensional $O(k-1)$-manifold which is obtained by plumbing 8-copies of $E(\tau_k)$ equivariantly according to the graph E_8:  (see Chapter V, §8 in [2]). Let ρ_{k-1} be the standard $(k-1)$-dimensional representation of $O(k-1)$, and θ the trivial 1-dimensional representation of $O(k-1)$.

Lemma 4.1. *$P^{2k}(E_8)$ can be imbedded in D^{2k+2} equivariantly as an $O(k-1)$-manifold, where the action on D^{2k+2} is given by $2\rho_{k-1}\oplus 4\theta$.*

Proof. Let x be a fixed point of Δ_S, and D_x an $O(k-1)$-invariant closed disc neighborhood of x on which $O(k-1)$ acts linearly. $\partial\Delta_D$ can be thought of as Δ_S, and $S^k\times S^k$ as a submanifold of $D^{k+1}\times D^{k+1}$. Let H_x be an $O(k-1)$-invariant closed half disc neighborhood of x in Δ_D on which $O(k-1)$ acts linearly. Then $E(\nu_S)|D_x$ and $E(\nu_D)|H_x$ are isomorphic to the product $O(k-1)$-bundles $D_x\times D^k$ and $H_x\times D^{k+1}$ as $O(k-1)$-bundles respectively, where the actions on D^k and D^{k+1} are given by $\rho_{k-1}\oplus\theta$ and $\rho_{k-1}\oplus 2\theta$ respectively. Since $\partial E(\nu_D)\supset E(\nu_D)|\Delta_S$, and since $E(\nu_D)|\Delta_S$ is isomorphic to the disc bundle of $\nu_S\oplus\theta$, $E(\nu_S)$ can be thought of as a G-submanifold of $\partial E(\nu_D)$. Hence by the above isomorphisms,

$$H_x\times D^{k+1}\supset H_x\times D^{k+1}\cap\partial E(\nu_D)\supset D_x\times D^{k+1}\supset D_x\times D^{k+1}\cap E(\nu_S)=D_x\times D^k.$$

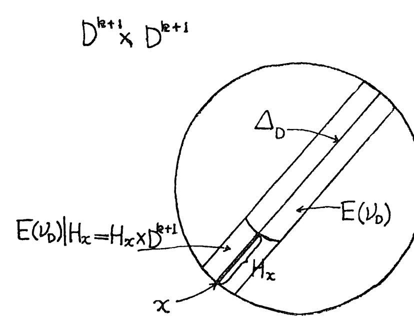


Fig. 2

We will regard D^{k+1} as $D^k \times D^1$. Since D_x is equivariantly diffeomorphic to D^k, we have an equivariant diffeomorphism $\mu: D_x \times D^k \times D^1 \to D_x \times D^k \times D^1$ with $\mu(u, v, t)=(v, u, t)$ for $u \in D_x$, $v \in D^k$ and $t \in D^1$. Now we obtain W from 2-copies of $E(\nu_D)$ by identifying (u, v, t) with $\mu(u, v, t)$ for $(u, v, t) \in D_x \times D^m \times D^1$. By straightening the corners equivariantly, W becomes a $(2k+2)$-dimensional $O(k-1)$-manifold.

Let $P^{2k}(A_2)$ denote a $2k$-dimensional $O(k-1)$-manifold which is obtained by plumbing 2-copies of $E(\tau_k)$ equivariantly according to the graph A_2: ○——○ (see G. Bredon [2, Chapter V, §8]). From the definition, W is obtained by the equivariant boundary connected sum $E(\nu_D)\natural E(\nu_D)$. Since $E(\nu_D)$ is equivariantly diffeomorphic to D^{2k+2}, W is also equivariantly diffeomorphic to D^{2k+2}. Moreover $P^{2k}(A_2)$ is obtained from 2-copies of $E(\nu_S)$ by identifying (u, v) with $\mu(u, v)$ for $(u, v) \in D_x \times D^m$. Thus $P^{2k}(A_2)$ can be imbedded equivariantly in D^{2k+2}.

Iterating this method, $P^{2k}(E_8)$ can be imbedded equivariantly in D^{2k+2}. Note that $P^{2k}(E_8)$ is imbedded equivariantly in ∂D^{2k+2}. By pushing the interior of $P^{2k}(E_8)$ into the interior of D^{2k+2}, this imbedding can be taken transversally on ∂D^{2k+2}, and Lemma 4.1 follows.

Lemma 4.2. *With the notations of Lemma* 4.1, *let* ν *be the* $O(k-1)$-*equivariant normal bundle of* $P^{2k}(E_8)$ *in* D^{2k+2}. *Then* ν *isisomorphic to* $P^{2k}(E_8) \times \boldsymbol{R}^2$ *as an* $O(k-1)$-*bundle, where the action on* $\boldsymbol{R}^2$ *is trivial.*

Proof. Let x be a fixed point of S^k and D_x a closed invariant disc neighborhood of x on which $O(k-1)$ acts linearly. Let $E_1=E(\tau_k)|D_x$ and $E_2=E(\tau_k)|(S^k-\operatorname{int} D_x)$. Since D_x and $S^k-\operatorname{int} D_x$ are equivariantly contractible, E_1 and E_2 are equivariantly diffeomorphic to $D_x \times D^k$ and $(S^k-\operatorname{int} D_x) \times D^k$ respectively, where the action on D^k is given by $\rho_{k-1} \oplus \theta$ (see E. Biestone [1, Corollary 3.2]). Note that $P^{2k}(E_8)$ is obtained by plumbing 8-copies of $E(\tau_k)$ equivariantly, so we can regard $E(\tau_k)$ as a subspace of $P^{2k}(E_8)$.

Since E_1 and E_2 are equivariantly contractible, $\nu|E_1$ and $\nu|E_2$ are isomorphic to $E_1 \times \boldsymbol{R}^2$ and $E_2 \times \boldsymbol{R}^2$ respectively. Note that $E_1 \cap E_2 = E(\tau_k)|\partial D_x$ which is equivariantly diffeomorphic to $\partial D^k \times D^k$. Since $(\partial D^k \times D^k)/O(k-1)$ and $(D_x \times D^k)/O(k-1)$ are contractible, it can be proved that $v|E(\tau_k)$ is isomorphic to $E(\tau_k)\times \boldsymbol{R}^2$ as an $O(k-1)$-bundle by the similar method of Lemma 2.1.

Note that $P^{2k}(A_2)$ is obtained by plumbing 2-copies of $E(\tau_k)$ equivariantly, so we can regard $P^{2k}(A_2)$ as a subspace of $P^{2k}(E_8)$, and $P^{2k}(A_2)-\mathrm{int}\,E(\tau_k)$ is equivariantly diffeomorphic to $D^k\times D^k$. Hence it can be proved that $v|P^{2k}(A_2)$ is isomorphic to $P^{2k}(A_2)\times \boldsymbol{R}^2$ as an $O(k-1)$-bundle, by the above method.

Iterating this method, we can prove v is isomorphic to $P^{2k}(E_8)\times \boldsymbol{R}^2$ as an $O(k-1)$-bundle. This completes the proof of Lemma 4.2.

Consider the Brieskorn sphere $W_{3,5}^{2k-1}$ defined to be the space of all points $(u, v, z_1, z_2,\dots, z_{k-1})\in \boldsymbol{C}^{k+1}$ on the intersection of the space

$$|u|^2+|v|^2+|z_1|^2+\cdots+|z_{k-1}|^2=1$$

and the variety

$$u^3+v^5+z_1^2+z_2^2+\cdots+z_{k-1}^2=0.$$

By the well-known method, $W_{3,5}^{2k-1}$ is an $O(k-1)$-manifold.

Let G be a finite subgroup of $O(k-2r)\subset O(k-1)$ for $k\geqq 2r+2$. $\boldsymbol{R}^{k-2r}$ is an $O(k-2r)$-manifold by the standard orthogonal action, hence it is a G-manifold. We will assume that G acts freely on the unit sphere in $\boldsymbol{R}^{k-2r}$. We have many finite groups G satisfying this condition (see P. Orlik [7, Chapter 6, Theorem 1]). By the restricted action, $W_{3,5}^{2k-1}$ is a semi-free differentiable G-manifold $(W_{3,5}^{2k-1}, \varphi_0)$ whose fixed point set is $W_{3,5}^{4r-1}$. Let $\beta\colon G\to O(k-2r)$ denote a $(k-2r)$-dimensional representation of G given by the inclusion $G\hookrightarrow O(k-2r)$. Then it is easy to see that $(W_{3,5}^{2k-1}, \varphi_0)$ defines an element of $\Theta_{2k-1}(2\beta\oplus(4r-1)\theta)$.

Theorem 4.3. *If* $r\geqq 2$ *and* $k\geqq 2r+2$, $(W_{3,5}^{2k-1}, \varphi_0)$ *is an element of infinite order in* $\Theta_{2k-1}(\alpha)$, *where* $\alpha=2\beta\oplus(4r-1)\theta$.

Proof. Let us consider the homomorphism $\tau\colon \Theta_{2k-1}(\alpha)\to Wh(G)$. If $\tau(\varphi_0)$ is an element of infinite order in $Wh(G)$, we have nothing to prove. Thus we may assume that $q\cdot\tau(\varphi_0)=0$ for some positive integer q. Hence $q\cdot(W, \varphi_0)=(W, \varphi_0)\#\cdots\#(W, \varphi_0)$ is an element of $\Theta_{2k-1}^0(\alpha)$.

Note that $P^{2k}(E_8)$ is an $O(k-1)$-manifold and hence a G-manifold.

Let $(P^{2k}(E_8), \Phi)$ denote this G-manifold. Since $W^{2k-1}_{3,5}$ is equivariantly diffeomorphic to $\partial P^{2k}(E_8)$ as an $O(k-1)$-manifold, $(W^{2k-1}_{3,5}, \varphi_0)$ is equivariantly diffeomorphic to $(\partial P^{2k}(E_8), \Phi')$ as a G-manifold, where Φ' is the restricted action of Φ (see Chapter VI of G. Bredon [2]). By Lemma 4.2 $(P^{2k}(E_8), \Phi)$ is an e-π-manifold and hence $q\cdot(P^{2k}(E_8), \Phi)$ $=(P^{2k}(E_8), \Phi)\natural\cdots\natural(P^{2k}(E_8), \Phi)$ is an e-π-manifold whose boundary is $q\cdot(\partial P^{2k}(E_8), \Phi')\simeq q\cdot(W^{2k-1}_{3,5}, \varphi_0)$. Therefore $q\cdot(W^{2k-1}_{3,5}, \varphi_0)$ is an element of $\Lambda_{2k-1}(\alpha)$ by Theorem 2.4.

Here let us consider the homomorphism $I\colon \Lambda_{2k-1}(\alpha)\to \boldsymbol{Z}$. Since the fixed point set of $(P^{2k}(E_8), \Phi)$ is $P^{4r}(E_8)$, we have

$$I(q\cdot(W^{2k-1}_{3,5}, \varphi_0)) = \mathrm{Sgn}\,(P^{4r}(E_8)\natural\cdots\natural P^{4r}(E_8))$$

$$= q\cdot\mathrm{Sgn}\,(P^{4r}(E_8)) = \pm 8q \neq 0\,.$$

Thus $(W^{2k-1}_{3,5}, \varphi_0)$ must be an element of infinite order in $\Theta_{2k-1}(\alpha)$. This completes the proof of Theorem 4.3.

Remark. In the case of $G=\boldsymbol{Z}_2$, G. Bredon [2, Chapter VI, Theorem 8.6] have proved an analogous theorem to 4.3 by a different method.

References

[1] Biestone, E., The equivariant covering homotopy property for differentiable G-fiber bundles, *J. Differential Geometry* **8**, (1973) 615–622.

[2] Bredon, G., *Introduction to Compact Transformation Groups,* Academic Press, New York and London (1972).

[3] Hilton, P., *General Cohomology Theory and K-Theory,* Cambridge Univ. Press (1971).

[4] Kervaire, M. and Milnor, J., Groups of homotopy spheres I, *Ann. of Math.,* **77** (1963), 504–537.

[5] Milnor, J. and Kervaire, M., Bernoulli numbers, homotopy groups and a theorem of Rohlin, *Proc. Int. Congress of Math.,* Edinburgh (1958), 454–458.

[6] Milnor, J., Whitehead torsion, *Bull. Amer. Math. Soc.*, **72** (1966), 358–426.

[7] Orlic, P., Seifert Manifolds, *Lecture Notes in Math.,* **291**, Springer, Berlin (1972).

[8] Sebastiani, M., Sur les actions à deux points fixes de groupes finis sur les sphères, *Comment. Math. Helv.,* **45** (1970), 405–439.

[9] Segal, G., Equivariant stable homotopy theory, *Proc. Int. Congress of Math.,* Nice (1970), **2**, 59–63.

[10] Siebenmann, L., Disruption of low-dimensional handlebody theory by Rohlin's theorem, *"Topology of Manifolds",* 57–84 Markham Publ., Chicago (1970).

[11] Wassermann, A., Equivariant differential topology, *Topology*, **8** (1969), 127–150.