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The Central Limit Theorem for Piecewise Linear Transformations

By

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§0. Introduction

The purpose of the present paper is to give central limit theorems for piecewise linear transformations ([6]), which are generalizations of β-transformations and which belong to a class of number-theoretical transformations with "dependent digits" (cf. [4]). The central limit theorems for ones with "independent digits" are studied by many authors ([1], [7] etc.). However, the cases of "dependent digits" seem to be not studied. These cases are more complicated than the cases of "independent digits".

In [2] it is shown that the β-transformations have Ornstein's weak Bernoulli property. Then it is easy to see by an analogous way to [2] that our transformations also satisfy the weak Bernoulli condition. Therefore Ornstein and Friedman's theorem implies that the natural extensions of our transformations are isomorphic to the Bernoulli shifts. But we never know how to construct their Bernoulli generators. Hence the classical central limit theorems for the Bernoulli shifts imply no concrete result for our transformations.

We modify the method, which is used in [2] to prove the weak Bernoulli property of β-transformations, to show that the natural generators of piecewise linear transformations satisfy Rosenblatt's strong mixing condition. Thus we obtain central limit theorems. By virtue of the good properties of our generators, we obtain concrete results, namely if f is of bounded variation or Hölder continuous, we get the central limit

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theorems for the process $\{f(T^i x); i=0, 1, 2,\dots\}$. Our results include the central limit theorems for β-transformations as special cases.

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§1. The Piecewise Linear Transformation and Its Symbolical Properties

First of all, we prepare several notations, definitions and properties of piecewise linear transformations ([6]).

Let $\beta=(\beta_0, \beta_1,\dots, \beta_p)$ be a $(p+1)$-tuple of real numbers, satisfying $\beta_k>1$ for $0\leqq k\leqq p$ and $\sum_{k=0}^{p-1}\beta_k^{-1}<1\leqq\sum_{k=0}^{p}\beta_k^{-1}$. We define a partition $R=\{r_i\}_{i=0,1,\dots,p}$ of the interval $[0, 1)$ by

$$r_0=[0, \beta_0^{-1}),$$

$$r_i=[\sum_{k=0}^{i-1}\beta_k^{-1}, \sum_{k=0}^{i}\beta_k^{-1}), \quad i=1, 2,\dots, p-1,$$

$$r_p=[\sum_{k=0}^{p-1}\beta_k^{-1}, 1),$$

and a mapping T: $[0, 1)\to[0, 1)$ by

$$Tx=\beta_0 x, \quad x\in r_0,$$

$$Tx=\beta_i(x-\sum_{k=0}^{i-1}\beta_k^{-1}), \quad x\in r_i, \quad i=1, 2,\dots, p.$$

Then T is called a piecewise linear transformation. If $\beta_0=\beta_1=\cdots=\beta_p$, this is a β-transformation.

It is easy to see that the partition R is a generator in the strict sense, i.e. $\bigvee_{i=0}^{\infty} T^{-i}R=\varepsilon$, where ε denotes the partition into individual points. The transformation T can be represented by a subshift σ on the one-sided infinite product space $A^{\mathbf{N}}$, where $A=\{0, 1,\dots, p\}$. Define a mapping π: $[0, 1)\to A^{\mathbf{N}}$ by

$$(\pi x)(i)=j, \qquad \text{iff } T^i x\in r_j.$$

Let Y be the image $\pi([0, 1))$ and X its closure in the product space $A^{\mathbf{N}}$

with the product topology. It is obvious that Y and X are invariant under the shift σ. We also denote the restrictions of σ to Y or X by the same notation σ.

We define the lexcographical order in $A^{\mathbf{N}}$. For convention, we set

$$T^n 1 \equiv \lim_{t \uparrow 1} T^n t,$$

$$\omega_p \equiv \pi(1) \equiv \max X,$$

where $\max X$ denotes the maximum element of X with respect to this order. We define a mapping ρ from X onto the unit interval $[0, 1]$ by

$$\rho(\omega) = \sum_{i=0}^{\infty} \beta(\omega[0, i))^{-1} \sum_{k=0}^{\omega(i)-1} \beta_k^{-1},$$

where

$$\beta(\omega[0, i)) = \begin{cases} \beta_{\omega(0)}\beta_{\omega(1)}\cdots\beta_{\omega(i-1)}, & (i \geqq 1) \\ 1. & (i=0) \end{cases}$$

In this situation, we can show the following three lemmas. They correspond to Proposition 3.2, Lemma 4.4 and Proposition 3.4 respectively. They can be proved by the same methods as [2], so we omit their proofs.

Lemma 1. *We have the followings.*

1) $\sigma \circ \pi = \pi \circ T$ *on* $[0, 1)$.

2) $\pi\colon [0, 1] \to X$ *is an injection and is strictly order-preserving, i.e.* $t < s$ *implies that* $\pi(t) < \pi(s)$.

3) $\rho \circ \pi$ *is identity on* $[0, 1]$.

4) $\rho \circ \sigma = T \circ \rho$ *on* Y.

5) $\rho\colon X \to [0, 1]$ *is a continuous surjection and is order-preserving, i.e.* $\omega < \omega'$ *implies that* $\rho(\omega) \leqq \rho(\omega')$.

6) *The inverse image* $\rho^{-1}(t)$ *of* $t \in [0, 1]$ *consists either of one point* $\pi(t)$ *or two points* $\pi(t)$ *and* $\sup_{s<t} \pi(s)$. *The latter case occurs only when* $T^n t = 0$ *for some* $n > 0$.

7) $\rho(\omega)$ *is one-to-one except a countable number of points* $\omega \in X$.

By virtue of Lemma 1, we can get enough informations about

$([0, 1), T)$ by studying properties of (X, σ). Let us analyze (X, σ). First the elements of X can be characterized by

Lemma 2. *We have*

$$X=\{\omega \in A^{\mathbf{N}} | \sigma^n \omega \leqq \omega_p \text{ for all } n \geqq 0\}.$$

We call $(a_0, a_1, \dots, a_{n-1}) \in A^n$ a word in X, if there is $\omega \in X$ such that $a_0=\omega(0), \dots, a_{n-1}=\omega(n-1)$. The concatenation of two words $a=(a_0, \dots, a_{n-1})$ and $b=(b_0, \dots, b_{m-1})$ is defined by

$$a*b=(a_0, \dots, a_{n-1}, b_0, \dots, b_{m-1}).$$

For convention, we introduce the empty word ϕ and we define $\phi*a=a*\phi=a$ for any word a.

Let

$$W_n=\{(a_0, \dots, a_{n-1}) | a_0=\omega(0), \dots, a_{n-1}=\omega(n-1) \text{ for some } \omega \in X\},$$

$$W_n^0=\{(a_0, \dots, a_{n-1}) | (a_0, \dots, a_{n-2}, a_{n-1}+1) \in W_n\},$$

and for $u \in W_k$, $k \geqq 0$

$$W_n(u)=\{v \in W_n | u*v \in W_{n+k}\},$$

$$W_n^0(u)=\{v \in W_n^0 | u*v \in W_{n+k}^0\}.$$

We understand $W_0=W_0^0=\{\phi\}$.

Lemma 3. *For any $k \geqq 0$ and any word $u \in W_k$, we have*

$$W_n(u)=\bigcup_{j=1}^{n} W_j^0(u)*\omega_p[0, n-j) \cup \{\max W_n(u)\},$$

where

$$\omega_p[0, j)=\begin{cases} (\omega_p(0), \dots, \omega_p(j-1)) & (j \geqq 1), \\ \phi \text{ (the empty word)} & (j=0), \end{cases}$$

and

$$W_j^0(u)*\omega_p[0, n-j)=\{v*\omega_p[0, n-j) | v \in W_j^0(u)\}.$$

Now, we shall prove the following fundamental estimation, where we use the notations

$$[u]=\{\omega\in X|(\omega(0),\dots,\omega(n-1))=u\}$$

for $u\in W_n$, and

$$R(u)=\text{the length of the interval } \rho([u]).$$

Lemma 4. *For an arbitrary $\alpha>0$, there exists a constant C_α such that*

$$\sup_{k\geqq 0}\sup_{u\in W_k}|(\sum_{v\in W_n^0(u)}\beta(v)^{-1})-R(u)\beta(u)M^{-1}|\leqq C_\alpha n^{-\alpha}$$

for all $n\geqq 1$, where $\beta(v)=\beta_{v(0)}\beta_{v(1)}\cdots\beta_{v(n-1)}$ and

$$M=\sum_{n=0}^{\infty}\beta(\omega_p[0,n))^{-1}T^n 1. \tag{1.1}$$

Proof. Let $u\in W_k$ be fixed. If $v\in W_{n-j}^0(u)*\omega_p[0,j)$ for some $0\leqq j\leqq n-1$, then we have

$$\begin{aligned}R(u*v)&=\rho(\max\{\omega\in X|(\omega(0),\dots,\omega(n+k-1))=u*v\})\\&\quad-\rho((u*v*(0,0,\dots)))\\&=\beta(u)^{-1}\beta(v)^{-1}(\sum_{m=0}^{\infty}\beta(\omega_p[j,j+m))^{-1}\sum_{k=0}^{\omega_p(j+m)-1}\beta_k^{-1})\\&=\beta(u)^{-1}\beta(v)^{-1}T^j 1.\end{aligned}$$

Therefore if $u\in W_{k-m}^0*\omega_p[0,m)$, then using Lemma 3 we obtain

$$\begin{aligned}R(u)&=\sum_{v\in W_n(u)}R(u*v)\\&=\sum_{j=0}^{n-1}\sum_{w\in W_{n-j}^0(u)}\beta(u)^{-1}\beta(w)^{-1}\beta(\omega_p[0,j))^{-1}T^j 1\\&\quad+\beta(u)^{-1}\beta(\omega_p[m,m+n))^{-1}T^{m+n}1.\end{aligned}\tag{1.2}$$

Let us consider a formal power series:

$$\sum_{n=1}^{\infty} t^n \Big(\sum_{j=0}^{n-1} \sum_{w\in W_{n-j}^0(u)} \beta(u)^{-1}\beta(w)^{-1}\beta(\omega_p[0, j))^{-1}T^j 1\Big).$$

This series clearly converges for $|t|<1$ and its value is equal to

$$\sum_{j=0}^{\infty} t^j \beta(\omega_p[0, j))^{-1}T^j 1 \sum_{i=1}^{\infty} t^i \beta(u)^{-1} \sum_{w\in W_i^0(u)} \beta(w)^{-1}.$$

Hence, we can deduce from (1.2) that

$$\sum_{n\geqq 1} t^n \sum_{v\in W_n^0(u)} \beta(v)^{-1} = \frac{t\beta(u)R(u)}{1-\phi(t)} - g_u(t),$$

where

$$g_u(t) = \frac{1-t}{1-\phi(t)} \sum_{n\geqq 1} t^n \beta(\omega_p[m, m+n))^{-1} T^{m+n} 1,$$

and

$$\phi(t) = \sum_{n\geqq 0} t^{n+1} \beta(\omega_p[0, n))^{-1} \sum_{k=0}^{\omega_p(n)-1} \beta_k^{-1}. \tag{1.3}$$

But the series in (1.3) converges in a neighborhood of the unit disk and $1-\phi(t)$ has only one simple root at $t=1$ in a disk $\{t\in \mathbf{C} \mid |t|<1+\varepsilon\}$ for small $\varepsilon>0$, because

$$\frac{1-\phi(t)}{1-t} = \sum_{n\geqq 0} t^n \beta(\omega_p[0, n))^{-1} T^n 1.$$

Noting $\phi'(1)=M$, we can see that

$$f_u(t) \equiv \sum_{n=1}^{\infty} t^n \Big[\Big(\sum_{v\in W_n^0(u)} \beta(v)^{-1}\Big) - \beta(u)R(u)M^{-1}\Big]$$

$$= \frac{\beta(u)R(u)t}{1-\phi(t)} - \frac{\beta(u)R(u)t}{(1-t)\phi'(1)} - g_u(t).$$

Consequently, $f_u(t)$ is analytic in $\{t\in \mathbf{C} \mid t\neq 1,\ |t|<1+\varepsilon\}$ and the singular point $t=1$ is removal. Since $\beta(u)R(u)\leqq 1$ and

$$\beta(\omega_p[m, n+m))^{-1}T^{m+n}1 \leqq (\beta_{\min})^{-n}, \quad \beta_{\min} = \min\{\beta_0, \dots, \beta_p\} > 1,$$

the α-th derivative $f_u^{(\alpha)}(t)$ of $f_u(t)$ is uniformly bounded; precisely speaking we have

$$\sup_{k\geqq 0}\sup_{u\in W_k}\sup_{|t|\leqq 1}|f_u^{(\alpha)}(t)|<+\infty.$$

Using the estimation

$$n(n-1)\dots(n-\alpha+1)|(\sum_{v\in W_n^0(u)}\beta(v)^{-1})-\beta(v)R(u)M^{-1}|$$

$$=\left|(2\pi r^n)^{-1}\int_0^{2\pi}f_u^{(\alpha)}(re^{i\theta})e^{-in\theta}d\theta\right|\leqq r^{-n}\sup_{|t|\leqq 1}|f_u^{(\alpha)}(t)|$$

for $0<r<1$ and $n\geqq 0$, we obtain

$$\sup_{k\geqq 0}\sup_{u\in W_k}\sup_{n\geqq 1}n(n-1)\dots(n-\alpha+1)|(\sum_{v\in W_n^0(u)}\beta(v)^{-1})-\beta(u)R(u)M^{-1}|<+\infty,$$

which proves the lemma.

§2. An Invariant Measure and the Strong Mixing Condition

We shall introduce an invariant measure of a piecewise linear transformation defined in §1. First of all, notice that the Lebesgue measure on $[0, 1)$ is transformed to the measure $d\rho$ on X by the correspondence given by Lemma 1. Let us define an operator S by

$$S\phi(\omega)=\sum_{a\in A;a*\omega\in X}\beta_a^{-1}\phi(a*\omega),$$

where $\phi(\omega)$ is a function on X. Then, we can easily get the following lemma.

Lemma 5. *We have*

$$\int_X\psi(\omega)S\phi(\omega)d\rho(\omega)=\int_X\psi(\sigma\omega)\phi(\omega)d\rho(\omega)$$

for any $\phi(\omega)\in L^1(d\rho)=L^1(X, d\rho)$ *and* $\psi(\omega)\in L^\infty(X, d\rho)$.

We omit the proof, because it can be shown in the same way as the case of β-transformation (c.f. [2], Lemma 5.1). This lemma implies that the measure $\mu(A)\equiv\int_A h(\omega)d\rho(\omega)$ is invariant under σ if and only if $Sh(\omega)=h(\omega)$ (a.e.). Furthermore, we can easily check that

$$h(\omega)\equiv M^{-1}\sum_{n=0}^{\infty}\beta(\omega_p[0,n))^{-1}I_{\{\sigma^n\omega_p\geqq\omega\}}(\omega)$$

fulfils $Sh(\omega)=h(\omega)$ and $\int_X h(\omega)d\rho(\omega)=1$ (c.f. [6]), where M is given by (1.1) and $I_A(\omega)$ denotes the indicator function of the set A. Thus, we get an invariant measure of σ

$$\mu(A)\equiv\int_A h(\omega)d\rho(\omega).$$

Lemma 1 implies that $\mu\circ\rho^{-1}$ is invariant under T. For simplicity, we denote $\mu\circ\rho^{-1}$ by μ and $E_\mu(f)=\int fd\mu$.

In the sequel, $\mathscr{F}(\omega(0),\dots,\omega(n-1))$ stands for the sub-σ-field generated by $\omega(0),\dots,\omega(n-1)$. Now we can prove the key lemma:

Lemma 6. *For any $\delta>0$ and any positive integer k, there exists $\gamma_\delta(k)$ such that*

$$\sum_{k=1}^{\infty}\gamma_\delta(k)^{\frac{\delta}{2+\delta}}<+\infty,$$

and

$$\|S^{k+n}\phi(\omega)-E_\rho(\phi)h(\omega)\|_\infty\leqq\gamma_\delta(k)\|\phi\|_\infty$$

for all non-negative integer n and all $\phi(\omega)\in\mathscr{F}(\omega(0),\dots,\omega(n-1))$.

Proof. Lemma 3 guarantees that

$$S^{k+n}\phi(\omega)=\sum_{w\in W_{k+n}}\beta(w)^{-1}\phi(w*\omega)I_{\{w*\omega\in X\}}(\omega)$$

$$=\sum_{j=0}^{k+n}\sum_{v\in W^0_{k+n-j}}\beta(v)^{-1}\beta(\omega_p[0,j))^{-1}\phi(v*\omega_p[0,j)*\omega)I_{\{\sigma^j\omega_p\geqq\omega\}}(\omega).$$

Let us define

$$S^{k+n}(m)\phi(\omega)\equiv\sum_{j=0}^{m}\sum_{v\in W^0_{k+n-j}}\beta(v)^{-1}\beta(\omega_p[0,j))^{-1}\phi(v*\omega_p[0,j)*\omega)I_{\{\sigma^j\omega_p\geqq\omega\}}(\omega)$$

for $m \leqq k+n$. Then, we have

$$\|S^{k+n}\phi(\omega)-S^{k+n}(m)\phi(\omega)\|_{\infty}$$

$$\leqq \|\phi\|_{\infty} \sum_{j=m+1}^{k+n} \sum_{v \in W^{0}_{k+n-j}} \beta(v)^{-1}\beta(\omega_{p}[0, j))^{-1}$$

$$=\|\phi\|_{\infty} \sum_{j=m+1}^{k+n} \beta(\omega_{p}[0, j))^{-1} \sum_{v \in W^{0}_{k+n-j}} \beta(v)^{-1}.$$

Since $\sum_{v \in W^{0}_{k+n-j}} \beta(v)^{-1}$ is uniformly bounded because of Lemma 4, we have

$$\|S^{k+n}\phi(\omega)-S^{k+n}(m)\phi(\omega)\|_{\infty} \leqq K_{1}\|\phi\|_{\infty}(\beta_{\min})^{-m},$$

where $\beta_{\min}=\min\{\beta_{0}, \beta_{1},\dots, \beta_{p}\}>1$ and K_{1} is an absolute constant.

We now assume $m<k$. Then, since the function $\phi(\omega)$ depends only upon the first n coordinates, we get

$$S^{k+n}(m)\phi(\omega)=\sum_{j=0}^{m} \sum_{u \in W_{n}} \phi(u) \sum_{v \in W^{0}_{k-j}(u)} \beta(u)^{-1}\beta(v)^{-1}\beta(\omega_{p}[0, j))^{-1}I_{\{\sigma^{j}\omega_{p} \geqq \omega\}}(\omega).$$

Let

$$h^{(m)}(\omega)=\frac{1}{M} \sum_{j=0}^{m} \beta(\omega_{p}[0, j))^{-1}I_{\{\sigma^{j}\omega_{p} \geqq \omega\}}(\omega).$$

Using Lemma 4, we have

$$\|S^{k+n}(m)\phi(\omega)-E_{\rho}(\phi)h^{(m)}(\omega)\|_{\infty}$$

$$\leqq \|\phi\|_{\infty} \sum_{j=0}^{m} \beta(\omega_{p}[0, j))^{-1}I_{\{\sigma^{j}\omega_{p} \geqq \omega\}}(\omega) \sum_{v \in W_{n}} \beta(u)^{-1} \times$$

$$\times \Big|\Big(\sum_{v \in W^{0}_{k-j}(u)} \beta(v)^{-1}\Big)-\beta(u)R(u)M^{-1}\Big|$$

$$\leqq \|\phi\|_{\infty} \sum_{j=0}^{m} \beta(\omega_{p}[0, j))^{-1}I_{\{\sigma^{j}\omega_{p} \geqq \omega\}}(\omega) \sum_{u \in W_{n}} \beta(u)^{-1}C_{\alpha}(k-j)^{-\alpha}$$

$$\leqq C_{\alpha}(k-m)^{-\alpha}\|\phi\|_{\infty} \sum_{j=0}^{m} \beta(\omega_{p}[0, j))^{-1} \sum_{u \in W_{n}} \beta(u)^{-1},$$

where C_{α} is given in Lemma 4. Combining Lemmas 3 and 4, we can easily prove that $\sup_{n\geqq 0}\sum_{u\in W_n}\beta(u)^{-1}\leqq K_2<+\infty$.

Consequently,

$$\|S^{k+n}(m)\phi(\omega)-E_\rho(\phi)h^{(m)}(\omega)\|_\infty\leqq K_2C_\alpha(k-m)^{-\alpha}\|\phi\|_\infty.$$

On the other hand, it is clear that

$$\|E_\rho(\phi)h^{(m)}(\omega)-E_\rho(\phi)h(\omega)\|_\infty\leqq K_3(\beta_{\min})^{-m}\|\phi\|_\infty$$

for some $K_3>0$. Taking $m=[k/2]$ and $\alpha>(2+\delta)/\delta$, we get our assertion.

We have prepared enough to show the strong mixing condition ([1], [5]).

Lemma 7. *For any $\delta>0$ and any positive integer k, there exists $\alpha_\delta(k)$ such that*

$$\sum_{k=1}^{\infty}\alpha_\delta(k)^{\frac{\delta}{2+\delta}}<+\infty$$

and

$$|\mu(A\cap T^{-(k+i)}B)-\mu(A)\mu(B)|\leqq\alpha_\delta(k)$$

for all $k\geqq 1$, $i\geqq 1$, $A\in\mathscr{F}(\bigvee_{j=0}^{i-1}T^{-j}R)$ and all measurable set B.

Proof. It is enough to prove our assertion for sufficiently large i, since μ is invariant under T. For an arbitrary positive integer k, there obviously exist a positive integer m_k and a function $h_k(\omega)$ such that $h_k(\omega)$ depends only on $(\omega(0),\dots,\omega(m_k-1))$ and

$$\|h(\omega)-h_k(\omega)\|_{L^1(d\rho)}\leqq\gamma_\delta(k),$$

where $\gamma_\delta(k)$ is given in Lemma 6. Then we have

$$\mu(A\cap T^{-(k+i)}B)-\mu(A)\mu(B)$$

$$=\int_X I_{\rho^{-1}A}(\omega)I_{\rho^{-1}B}(\sigma^{k+i}\omega)d\mu(\omega)-\int_X I_{\rho^{-1}A}(\omega)d\mu(\omega)\int_X I_{\rho^{-1}B}(\omega)d\mu(\omega)$$

$$=\int_X I_{\rho^{-1}A}(\omega)h_k(\omega)I_{\rho^{-1}B}(\sigma^{k+i}\omega)d\rho(\omega)$$

$$+\int_X I_{\rho^{-1}A}(\omega)(h(\omega)-h_k(\omega))I_{\rho^{-1}B}(\sigma^{k+i}\omega)d\rho(\omega)$$

$$-\int_X I_{\rho^{-1}A}(\omega)h_k(\omega)d\rho(\omega)\int_X I_{\rho^{-1}B}(\omega)h(\omega)d\rho(\omega)$$

$$-\int_X I_{\rho^{-1}A}(\omega)(h(\omega)-h_k(\omega))d\rho(\omega)\mu(B)$$

$$=\int_X [S^{k+i}(I_{\rho^{-1}A}(\omega)h_k(\omega))-E_\rho(I_{\rho^{-1}A}(\omega)h_k(\omega))h(\omega)]I_{\rho^{-1}B}(\omega)d\rho(\omega)$$

$$+\int_X I_{\rho^{-1}A}(\omega)[h(\omega)-h_k(\omega)]I_{\rho^{-1}B}(\sigma^{k+i}\omega)d\rho(\omega)$$

$$-\int_X I_{\rho^{-1}A}(\omega)[h(\omega)-h_k(\omega)]d\rho(\omega)\mu(B).$$

If $i\geqq m_k$, then $I_{\rho^{-1}A}(\omega)h_k(\omega)\in\mathscr{F}(\omega(0),\dots,\omega(i-1))$. So we can make use of Lemma 6 and we get

$$|\mu(A\cap T^{-(k+i)}B)-\mu(A)\mu(B)|\leqq 3\gamma_\delta(k)\|h(\omega)\|_\infty,$$

which proves Lemma 7.

§3. Central Limit Theorems

Now we are in the position to state our results. Let

$$\Phi_d(z)=\frac{1}{\sqrt{2\pi}d}\int_{-\infty}^{z}\exp\left[-\frac{t^2}{2d^2}\right]dt$$

for $d>0$ and

$$\Phi_0(z)=\begin{cases}1 & (z>0),\\ 0 & (z\leqq 0).\end{cases}$$

First, combining Lemma 7 in §2 and Theorem 18.6.2 in [1], we get the following

Theorem 1. *If*

(1) $f(t) \in L^{2+\delta}(\mu) = L^{2+\delta}([0, 1), \mu)$ for some $\delta > 0$,

(2) $\sum_{k=1}^{\infty} \|f - E_\mu(f | \bigvee_{i=0}^{k-1} T^{-i}R)\|_{L^\theta(\mu)} < +\infty$,

where $\theta = \dfrac{2+\delta}{1+\delta}$, *then*

$$d^2 = E_\mu(f - E_\mu f)^2 + 2 \sum_{j=1}^{\infty} E_\mu[(f(t) - E_\mu f)(f(T^j t) - E_\mu f)] < +\infty, \tag{3.1}$$

and

$$\lim_{n\to\infty} \mu\left\{\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} (f(T^i t) - E_\mu f) < z\right\} = \Phi_d(z), \tag{3.2}$$

at every continuity point z *of* $\Phi_d(z)$.

Next we shall be concerned with the central limit theorem with respect to Lebesgue measure λ. The following relations between the invariant measure μ and Lebesgue measure λ can be easily shown, using Lemmas 5 and 6, and noticing $1/c \leqq h(\omega) \leqq c$ for some $c > 0$.

Lemma 8. *We have*

$$|\mu(B) - \lambda(T^{-k}B)| \leqq \gamma_\delta(k)\lambda(B),$$

for all measurable set B *in* $[0, 1)$ *and all positive integer* k. *Hence*

$$|E_\mu(g) - E_\lambda(g(T^k t))| \leqq \gamma_\delta(k) \|g\|_{L^1(\lambda)}$$

for all $g \in L^1(\lambda) \equiv L^1([0, 1), \lambda) = L^1(\mu) \equiv L^1([0, 1), \mu)$ *and all* $k \geqq 0$. *Notice that* $\gamma_\delta(k)$ *is given in Lemma* 6.

Let

$$Z_n^{(1)} = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} [f(T^k t) - E_\mu f]$$

$$Z_n^{(2)} = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} [f(T^k t) - E_\lambda(f(T^k t))] .$$

Lemma 9. *For any* $f \in L^1(\lambda) = L^1(\mu)$ *and any real number* τ, *we have*

$$\lim_{n\to\infty} |E_\mu[\exp(i\tau Z_n^{(1)})]-E_\lambda[\exp(i\tau Z_n^{(2)})]|=0.$$

And the convergence is uniform in the wide sense.

Proof of Lemma 9. We have

$$\begin{aligned}
&|E_\mu[\exp(i\tau Z_n^{(1)})]-E_\lambda[\exp(i\tau Z_n^{(2)})]| \\
&\leqq |E_\mu[\exp(i\tau Z_n^{(1)})]-E_\lambda[\exp(i\tau Z_n^{(1)})]| \\
&\quad + |E_\lambda[\exp(i\tau Z_n^{(1)})]-E_\lambda[\exp(i\tau Z_n^{(2)})]| \\
&\leqq E_\mu\left|1-\exp\left\{\frac{i\tau}{\sqrt{n}}\sum_{k=0}^{r}[f(T^k t)-E_\mu f]\right\}\right| \\
&\quad + E_\lambda\left|1-\exp\left\{\frac{i\tau}{\sqrt{n}}\sum_{k=0}^{r}[f(T^k t)-E_\mu f]\right\}\right| \\
&\quad + \left|(E_\mu-E_\lambda)\exp\left\{\frac{i\tau}{\sqrt{n}}\sum_{k=r+1}^{n-1}[f(T^k t)-E_\mu f]\right\}\right| \\
&\quad + E_\lambda\left|1-\exp\left\{\frac{i\tau}{\sqrt{n}}\sum_{k=0}^{r}[f(T^k t)-E_\mu f]\right\}\right| \\
&\quad + E_\lambda\left|1-\exp\left\{\frac{i\tau}{\sqrt{n}}\sum_{k=0}^{r}[f(T^k t)-E_\lambda f(T^k t)]\right\}\right| \\
&\quad + E_\lambda\left|1-\exp\left\{\frac{i\tau}{\sqrt{n}}\sum_{k=r+1}^{n-1}[E_\mu f-E_\lambda f(T^k t)]\right\}\right|.
\end{aligned}$$

Using Lemma 8 and the ergodic theorem, and putting $r=[\log n]$, we get the assertion of Lemma 9.

Thus we get the following

Theorem 2. *Under the conditions* (1) *and* (2) *of Theorem* 1, (3.1) *holds and we have*

$$\lim_{n\to\infty}\lambda\left\{\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}[f(T^k t)-E_\lambda(f(T^k t))]\right\}=\Phi_d(z) \tag{3.3}$$

at every continuity point z of $\Phi_d(z)$.

Remark 1. If ν is an absolutely continuous measure with respect to λ and $d\nu/d\lambda$ is uniformly continuous, then we can prove an analogous assertion to Lemma 8, i.e.

$$|\mu(B)-\nu(T^{-k}B)|\leqq\varepsilon(k)\lambda(B)$$

for all measurable set B and all positive integer k, where $\varepsilon(k)\to 0$ as $k\to\infty$.

If $\sum_{k=1}^{\infty}\varepsilon(k)<+\infty$, then we can get the central limit theorem with respect to ν in the same way as the proof of Theorem 2. Even if $\sum_{k=1}^{\infty}\varepsilon(k)=\infty$, we can prove the central limit theorem

$$\lim_{n\to\infty}\nu\left\{\frac{1}{\sqrt{n}}\sum_{k=0}^{n-1}[f(T^k t)-E_\mu f]<z\right\}=\Phi_d(z)$$

by a little changing of the method.

Finally we get the following concrete result.

Theorem 3. *If either*

(a) *$f(t)$ is a function of bounded variation,*

or

(b) *$f(t)$ is Hölder continuous,*

then the conditions (1) *and* (2) *of Theorem* 1 *are satisfied, and consequently the conclusions of Theorems* 1 *and* 2 *hold.*

Proof. (a) Since $\theta=\dfrac{2+\delta}{1+\delta}<2$, it is clear that $\|\cdot\|_{L^\theta(\mu)}\leqq\|\cdot\|_{L^2(\mu)}$. Therefore we calculate $\|f-E_\mu(f|\bigvee_{i=0}^{k-1}T^{-i}R)\|_{L^2(\mu)}$ in the sequel.

Let $\mathrm{Var}(f;r)$ denote the total variation of $f(t)$ on the interval r, and $\mathrm{Var}f=\mathrm{Var}(f;[0,1))$. Putting $R_k\equiv\bigvee_{i=0}^{k-1}T^{-i}R$, we have

$$\begin{aligned}
&\|f-E_\mu(f|R_k)\|_{L^2(\mu)}\\
&=\left(\sum_{r\in R_k}\int_r\mu(dt)\left[\frac{1}{\mu(r)}\int_r(f(t)-f(s))\,\mu(ds)\right]^2\right)^{\frac{1}{2}}\\
&\leqq\left(\sum_{r\in R_k}\int_r\mu(dt)[\mathrm{Var}(f;r)]^2\right)^{\frac{1}{2}}\\
&\leqq(\mathrm{Var}f)^{\frac{1}{2}}\Big(\sum_{r\in R_k}\mu(r)\,\mathrm{Var}(f;r)\Big)^{\frac{1}{2}},
\end{aligned}$$

where

$$\mu(r) \leqq c\lambda(r) \leqq c(\beta_{\min})^{-k}$$

for every $r \in R_k$, since $1/c \leqq h(\omega) \leqq c$ and the diameters of $r \in R_k$ are less than $(\beta_{\min})^{-k}$. Thus we get

$$\|f - E_\mu(f|R_k)\|_{L^2(\mu)} \leqq c \cdot (\operatorname{Var} f)(\beta_{\min})^{-\frac{k}{2}},$$

which proves our assertion.

(b) Instead of $\|\cdot\|_{L^\theta(\mu)}$, we shall estimate $\|\cdot\|_{L^\infty(\mu)}$. We have

$$\begin{aligned} &\|f - E_\mu(f|R_k)\|_{L^\infty(\mu)} \\ &= \max_{r \in R_k} \operatorname{ess.sup}_{t \in r} \left| f(t) - \frac{1}{\mu(r)} \int_r f(s)\mu(ds) \right| \\ &\leqq \max_{r \in R_k} \operatorname{ess.sup}_{t \in r} \left[\frac{1}{\mu(r)} \int_r |f(t) - f(s)| \mu(ds) \right]. \end{aligned}$$

Since $f(t)$ is Hölder continuous, we have

$$|f(t) - f(s)| \leqq K(\operatorname{diam} r)^\alpha \leqq K[(\beta_{\min})^\alpha]^{-k}$$

for some $\alpha > 0$, $K > 0$ and for all $t, s \in r$.

Consequently, we get

$$\|f - E_\mu(f|R_k)\|_{L^\infty(\mu)} \leqq K[(\beta_{\min})^\alpha]^{-k}.$$

Clearly, this is enough to conclude Theorem 3.

Remark 2. If there exists a positive integer q such that $T^q 1 = 1$, then it is easy to see that $(T, \bigvee_{k=0}^{q-2} T^{-k}R)$ is a mixing Markov endomorphism. Hence, the natural generator satisfies the uniformly mixing condition (c.f. [1]), and so the central limit theorem holds for a wider class of functions. For example, if

$$\int_0^1 |f(t) - f(t+h)|^2 dt = O\left(\log^{-2-\varepsilon} \frac{1}{h}\right)$$

for some $\varepsilon > 0$, then the conclusions of Theorems 1 and 2 hold for f.

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