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Neighborhoods of a Compact Non-Singular Algebraic Curve Imbedded in a 2-Dimensional Complex Manifold

By

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Introduction

Let C be a compact non-singular algebraic curve imbedded in a 2-dimensional complex manifold S. In this paper we consider the following problem:

Under what conditions does there exist a non-constant holomorphic function defined on a small neighborhood of C?

The signature of the normal bundle N_C is not sufficient to solve our problem (see, Table in §6 and Theorem 2 in §5). Hence we have to introduce the concept of *a regularly half pseudoconvex neighborhood system of C* (for definition, see (1.1)). Then the necessary and sufficient condition is given in the following

Main Theorem. *There exists a non-constant holomorphic function defined on a neighborhood of C if and only if either* (1) $N_C<0$ *or* (2) N_C *is of finite order (for difinition, see* §1) *and C has a regularly half pseudoconvex neighborhood system.*

Therefore we can conclude that our problem is completely solved by using the normal bundle and pseudoconvexity of a neighborhood system of C. Detailed results will be summarized in Table in §6.

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§1. Notations

First we recall terminology on complex line bundles. Let E be a complex line bundle on S which is expressed as $E=\{f_{\lambda\mu}\}$ with respect to some open covering $\{U_\lambda\}$ of S. $\mathcal{O}(E)$ (resp. $\mathcal{E}(E)$) denotes the sheaf of germs of holomorphic (resp. C^∞-differentiable) sections of E. A metric $\{a_\lambda\}$ of E is a system of positive C^∞-functions a_λ on U_λ satisfying

$$a_\mu=|f_{\lambda\mu}|^2 a_\lambda \qquad \text{on} \quad U_\lambda\cap U_\mu\neq\phi .$$

E is called positive if there exists a metric $\{a_\lambda\}$ such that the hermitian matrix $(A^\lambda_{\alpha\bar\beta})$ defined by

$$-\partial\bar\partial\log a_\lambda=\sum_{\alpha,\beta=1}^{2} A^\lambda_{\alpha\bar\beta}dz^\alpha_\lambda\wedge d\bar z^\lambda_\beta \qquad \text{on each} \quad U_\lambda$$

is positive definite where $(z^1_\lambda, z^2_\lambda)$ denotes a system of local coordinates. In the case where E is topologically trivial line bundle, E is called of *finite order* if there exists a positive integer k such that $E^k=E\otimes E\otimes\cdots\otimes E$ (k-times tensor product) is analytically trivial. If not, it is called of *infinite order*. A curve C imbedded in S determines a complex line bundle as follows: Let $\{U_\lambda\}$ be a locally finite open covering of S which admits a system of local coordinates (z_λ, R_λ) on U_λ. In the case where $C\cap U_\lambda\neq\phi$ we assume that R_λ is a defining equation of C. Hence letting

$$R_\lambda=f_{\lambda\mu}R_\mu \qquad \text{on} \quad U_\lambda\cap U_\mu,$$

we obtain a 1-cocycle $\{f_{\lambda\mu}\}$ which determines a complex line bundle $[C]$. The normal bundle N_C is defined by $[C]|_C$. A metric $\{a_\lambda\}$ of $[C]$ determines a C^∞-function F and a neighborhood system V_ε of C as follows:

$$F=a_\lambda|R_\lambda|^2,$$

$$V_\varepsilon=\{p\in S\colon F(p)<\varepsilon\} \qquad \text{for small} \quad \varepsilon>0.$$

We make the following definition:

Definition (1.1). *C has a regularly half pseudoconvex neighborhood system $V_\varepsilon=\{F<\varepsilon\}$ if there exist a metric $\{a_\lambda\}$ of $[C]$ and a system of local coordinates (z_λ, R_λ) on U_λ such that*

$$\partial\bar{\partial}F=\beta_\lambda dR_\lambda \wedge d\bar{R}_\lambda \qquad \textit{where} \quad \beta_\lambda>0 .$$

By $\mathcal{O}(V_\varepsilon)$(resp. $\mathcal{M}(V_\varepsilon)$) we indicate the algebra (resp. field) of holomorphic (resp. meromorphic) functions on V_ε. In what follows functions are assumed to be of C^∞-class. A domain D with $D\Subset S$ is called (strongly-)pseudoconvex if at any point $p\in\partial D$ there exist a neighborhood U and a (strongly-)pseudoconvex function φ on U satisfying $D\cap U=\{\varphi<0\}$. By s-pseudoconvex domains (resp. functions) we mean strongly-pseudoconvex domains (resp. functions). Similarly (s-)pseudoconcave domains are also defined.

§2. The Necessity Part of Main Theorem

We fix a metric $\{a_\lambda\}$ of $[C]$ and consider V_ε for $0<\varepsilon\ll 1$ as a small neighborhood of C. Assume that $\mathcal{O}(V_\varepsilon)\not\cong \mathbf{C}$ holds.

At first we consider the following case:

(α) Any non-constant holomorphic function f wihch vanishes on C satisfies

$$\{f=0\}\supsetneqq C .$$

In this case we have the following:

Proposition (2.1). *N_C is negative.*

Proof. Take such a function f. Then $\{f=0\}=C\cup\tilde{C}$ where $\tilde{C}$ is a (possibly reducible) curve. Let k be the order of f at C. Then $\varphi_\lambda=f/R_\lambda^k$ determines a holomorphic section of $[C]^{-k}$ on V_ε. Restricting $\{\varphi_\lambda\}$ to C we get a section of N_C^{-k}, which induces a positive divisor on C. The following Proposition is well known (see, H. Grauert [1] and H. Rossi [8]):

Proposition (2.2). (1) *If N_C is negative, then C has an s-pseudoconvex neighborhood system, and so C is an exceptional curve in the* *sense of H. Grauert* [1]. (2) *If N_C is positive, then C has an s-pseudoconcave neighborhood system.*

Proof. We prove the first part of (1). The proof (2) is similar and may be omitted. By definition there exists a metric $\{\underline{a}_\lambda\}$ of N_C satisfying $\frac{\partial^2 \log \underline{a}_\lambda}{\partial z_\lambda^1 \partial \bar{z}_\lambda^1} > 0$ on $C \cap U_\lambda$. Extending this metric on V_ε, we have a metric $\{a_\lambda\}$ of $[C]$. Set $\tilde{a}_\lambda = a_\lambda e^{\chi(F)}$ with a convex increasing function χ. Choosing $\chi'(0)$ sufficiently large and replacing ε by a smaller constant ε, we get a negative metric $\{a_\lambda\}$ on V_ε. Let $\tilde{F} = \tilde{a}_\lambda |R_\lambda|^2$ and $\tilde{V}_\varepsilon = \{\tilde{F} < \varepsilon\}$. Then we obtain an s-pseudoconvex neighborhood system $\tilde{V}_\varepsilon$. To prove the second part of (1), we shall construct a contracting mapping as follows: Referring to $[C]^{-1} > 0$ and the completeness of $\psi = 1/\left(1 - \frac{F}{\varepsilon}\right)$ on $\tilde{V}_\varepsilon$, by S. Nakano's theorem (see, S. Nakano [6], p. 169, Theorem 1) we have

$$H^q(\tilde{V}_\varepsilon, \mathcal{O}([C]^{-m})) = 0 \qquad \text{for} \quad m \geqq m_0 \quad \text{and} \quad q \geqq 1,$$

where m_0 is a certain positive integer. Then following K. Kodaira [5] there exists a positive integer m_0 such that for any integer m with $m \geqq m_0$, (1) for any pair of points p, q in $\tilde{V}_\varepsilon$ there exists a section $\varphi \in H^0(\tilde{V}_\varepsilon, \mathcal{O}([C]^{-m}))$ satisfying $\varphi(p) \neq 0$ and $\varphi(q) = 0$ and (2) for any point $p \in \tilde{V}_\varepsilon \cap U_\lambda$ and constants c_1, c_2 and c_3, there exists a section $\psi \in H^0(\tilde{V}_\varepsilon, \mathcal{O}([C]^{-m}))$ with

$$\psi_\lambda(p) = c_1, \ \frac{\partial \psi_\lambda}{\partial z_\lambda}(p) = c_2 \quad \text{and} \quad \frac{\partial \psi_\lambda}{\partial R_\lambda}(p) = c_3 .$$

Since $\{\varphi_\lambda\} \in H^0(\tilde{V}_\varepsilon, \mathcal{O}([C]^{-m}))$ determines a holomorphic function $f = \varphi_\lambda R_\lambda^m$ on $\tilde{V}_\varepsilon$, from (1) and (2) we can derive (1′) for any pair of points p, q in $\tilde{V}_\varepsilon - C$ there exists a function f such that $f(p) \neq 0, f(q) = 0$ and $f = 0$ on C and (2′) for any point $p \in C$ there exists f_1 and f_2 such that $\frac{\partial(f_1, f_2)}{\partial(z_\lambda, R_\lambda)} \neq 0$ on $U(p) - C$ where $U(p)$ is a small neighborhood of p. By (1′) we have a system of global holomorphic functions $f_1, f_2, \dots, f_n$ such that $\bigcap_{j=1}^{n} \{f_j = 0\} = C$. Moreover, using (2′) we can find $f_{n+1}, f_{n+2}, \dots, f_r$ such that $\Psi_r = (f_1, f_2, \dots, f_r)\colon \tilde{V}_\varepsilon \to \mathbf{C}^r$ is a proper mapping on $V^* = \Psi_r^{-1}(D_\delta)$ which is also maximal rank on $V_\delta^* - C$, where D_δ is a small polydisc at 0. Adding more functions $f_{r+1}, f_{r+2}, \dots, f_m$ and replacing

Ψ_r by $\Psi_m=(f_1, f_2,\dots, f_m)$, we may assume that in view of (1′) every fibre is connected. Thus we obtain a desired contracting mapping.

Consequently, in the case of (α), putting $\Psi_m(V_\delta^*)=\underline{V}_\delta^*$, we have a 2-dimensional normal Stein space $\underline{V}_\delta^*$ satisfying $\mathcal{O}(V_\delta^*)=\mathcal{O}(\underline{V}_\delta^*)$.

Remark. Conversely if C has an s-pseudoconvex neighborhood system, then N_C is negative (see, H. Grauert [1], p. 355, Satz 9). But in the case of finite order, C may have an s-pseudoconcave neighborhood system (see, Theorem 3).

The remaining case is the following:

(β) There exists a holomorphic function f on V_ε satisfying $\{f=0\}=kC$ with some positive integer k.

Replacing V_ε by $V_\delta^*=\{p\in V_\varepsilon\colon |f(p)|<\delta\}$ with a small constant δ, we may assume that $g=f|_{V_\delta^*}$ is a proper mapping onto a disk D_δ. Using Stein factorization of g we obtain $\mathcal{O}(V_\delta^*)\cong\mathcal{O}(\mathbf{D})$ where $\mathbf{D}$ denotes the unit disk. As in the case (α), set $\{\varphi_\lambda=f/R_\lambda^k\}$. Then $\{\varphi_\lambda\}\in H^0(V_\delta^*, \mathcal{O}([C]^{-m}))$. In this case $\{\varphi_\lambda\}$ vanishes nowhere and so N_C^{-k} is analytically trivial. Any branch of $\varphi_\lambda^{\frac{1}{k}}$ defines a function on U_λ, which is also denoted by the same letter. Defining $z_\lambda^*=z_\lambda$ and $R_\lambda^*=\varphi_\lambda^{\frac{1}{k}}R_\lambda$, we have another system of local coordinates (z_λ^*, R_k^*) on U_λ. Moreover,

$$\partial\bar{\partial}F=dR_\lambda^*\wedge d\bar{R}_\lambda^* \quad \text{where} \quad F=|R_\lambda^*|^2. \tag{2.3}$$

We remark that in (2.3), $\{a_\lambda=1\}$ can be chosen as a metric of $[C]$ and so the necessity part of Main Theorem is hereby proved.

Corollary (2.4). *If there exists a curve C on S such that N_C is positive or N_C is of infinite order, then $\mathcal{O}(S)\cong\mathbf{C}$.*

§3. Holomorphic Foliations and Regularly Half Pseudoconvexity

Let $\mathscr{D}$ be a 1-dimensional holomorphic differential system on S. $\omega=0$ denotes its pfaffian equation. In what follows we assume that $\mathscr{D}$ *is completely integrable.* Then for any $p\in S$ there exists one and only one maximal integral manifold $\mathscr{S}_p$ through p, which is called a leaf of $\mathscr{D}$. We call $\mathscr{F}=\{\mathscr{S}_p\}$ a holomorphic foliation. Given two foliations $\mathscr{F}_1$ and $\mathscr{F}_2$ on S, $\mathscr{F}_1$ is said to be equivalent to $\mathscr{F}_2$ when their leaves coincide completely. $\mathscr{F}$ is called a *closed foliation* if every leaf is a closed submanifold in S. We say that $\mathscr{F}$ is *globally integrable* when $\mathscr{F}$ is equivalent to a foliation defined by a global holomorphic function on S.

Theorem 1. *If C has a regularly half pseudoconvex neighborhood system, then there exists a holomorphic foliation on V_ε for a small ε.*

Proof. From (1.1)

$$F=a_\lambda|R_\lambda|^2, \tag{3.1}$$

$$\partial_{z_\lambda}\bar{\partial}_{z_\lambda}F=0, \tag{3.2}$$

$$\bar{\partial}_{R_\lambda}\partial_{z_\lambda}F=0 \quad \text{and} \quad \bar{\partial}_{z_\lambda}\partial_{R_\lambda}F=0. \tag{3.3}$$

By (3.2) we have

$$F=a_0(R_\lambda)+\sum_{k=1}^{\infty}a_k(R_\lambda)z_\lambda^k+\sum_{k=1}^{\infty}\bar{a}_k(R_\lambda)\bar{z}_\lambda^k. \tag{3.4}$$

Therefore

$$\partial_{z_\lambda}F=\sum_{k=1}^{\infty}ka_k(R_\lambda)z_\lambda^{k-1}$$

which is a holomorphic function of z_λ when R_λ is fixed. On the other hand (3.3) implies that $\partial_{z_\lambda}F$ is also a holomorphic function of R_λ when z_λ is fixed. Hence by Hartogs' theorem it is a holomorphic function of z_λ and R_λ. Thus

$$a_k(R_\lambda)=\sum_{j=0}^{\infty}a_{k,j}R_\lambda^j \qquad (k=1, 2, 3,\dots). \tag{3.5}$$

Put $z_\lambda=0$ in (3.1) we have $a_0(R_\lambda)=a_\lambda(0, R_\lambda)|R_\lambda|^2$. Put $R_\lambda=0$ in (3.4) and taking (3.5) into account, we see that $a_{k,0}=0\,(k=1, 2,\dots)$. This shows that F and the first and the second terms of the right hand of (3.4) can be divided by R_λ and its quotient is a C^∞-function on U_λ. Thus $\Sigma\bar{a}_k(R_\lambda)\bar{z}_\lambda^k$ must be also divided by R_λ and its quotient also must be a C^∞-function on U_λ. Consequently we obtain that $a_{k,j}=0\,(k=1,$ $2,\dots$ and $j=0, 1,\dots)$, which imply

$$F=a_\lambda(0,\ R_\lambda)|R_\lambda|^2\,.$$

This yields

$$\frac{\partial F}{\partial z_\mu}=\frac{\partial F}{\partial R_\lambda}\frac{\partial R_\lambda}{\partial z_\mu}$$

and

$$\frac{\partial^2 F}{\partial z_\mu \partial \bar{z}_\mu}=\frac{\partial^2 F}{\partial R_\lambda \partial \bar{R}_\lambda}\left|\frac{\partial R_\lambda}{\partial z_\mu}\right|^2 \qquad \text{on} \quad U_\lambda \cap U_\mu\,.$$

By the assumption we have $\frac{\partial R_\lambda}{\partial z_\mu}=0$. Hence

$$R_\lambda=f_{\lambda\mu}(R_\mu)R_\mu \qquad \text{on} \quad U_\lambda \cap U_\mu\,.$$

Therefore $dR_\lambda=\xi_{\lambda\mu}dR_\mu$ on $U_\lambda \cap U_\mu$, where $\xi_{\lambda\mu}=f_{\lambda\mu}+\frac{\partial f_{\lambda\mu}}{\partial R_\mu}R_\mu$. Making ε smaller, we may assume that $\xi_{\lambda\mu}\neq 0$ on V_ε. Then we have a holomorphic foliation $\mathscr{F}$ on V_ε by the following equation:

$$\omega=0 \qquad \text{where} \quad \omega=dR_\lambda\,,$$

which is denoted by $\mathscr{F}_F$.

Corollary. *Under the assumption in Theorem* 1, *F and* $\{f_{\lambda\mu}\}$ *can be expressed as follows*:

(3.6) $$F=a_\lambda(R_\lambda)|R_\lambda|^2\,,$$

(3.7) $$R_\lambda=f_{\lambda\mu}(R_\mu)\cdot R_\mu\,.$$

Remark 1. The concept of regularly half pseudoconvexity can be generalized to a compact kähler manifold imbedded as a divisor in another complex manifold.

Remark 2. As for the degeneracy of holomorphic functions and the foliation structure, see A. T. Huckleberry and R. Nirenberg [4].

§4. The Sufficiency Part of Main Theorem

Given a curve C with a regularly half pseudoconvex neighborhood, then (3.6) and (3.7) hold with respect to the coordinates (z_λ, R_λ) on U_λ given in (2.4) and the foliation $\mathscr{F}_F$ is obtained. Let $\{U_\lambda^*\}$ be a finite open covering of V_ε defined by $U_\lambda^*=\{(z_\lambda, R_\lambda): |z_\lambda|<\rho, F<\varepsilon\}$ where ρ is a positive constant. For each λ, we choose a point $p_\lambda^{(0)}=(z_\lambda^{(0)}, R_\lambda^{(0)}) \in U_\lambda^*$ and define $D_\lambda^{(\varepsilon)}$ as follows: $D_\lambda^{(\varepsilon)}=V_\varepsilon \cap \{z_\lambda=z_\lambda^{(0)}\}$, which may be assumed to be a small disk. First from (3.6) and (3.7) we infer that if F attains a value c at a point in $\mathscr{S}$, then $F(p)=c$ for any point $p\in\mathscr{S}$ for each leaf $\mathscr{S}\in\mathscr{F}_F$. This implies that for any point $p\in V_\varepsilon$, the leaf through p is completely contained in V_ε. Fix a point $p\in C \cap U_0^*$ and consider a C^∞-path $\gamma=\{\gamma(t): 0\leqq t\leqq 1\}$ such that $\gamma(0)=p$ and $\gamma\subset C$. By γ^{-1} we denote a path $\{\gamma(1-t): 0\leqq t\leqq 1\}$. Now we take an analytic set $\{R_0=c_0\}$ in $V_\varepsilon\cap U_0^*$ for a constant c_0 and define a continuation of $\{R_0=c_0\}$ along γ as follows: Take t_0 such that $\gamma_0=\{\gamma(t): 0\leqq t<t_0\}\subset U_0^*$ and $\gamma(t_0)\ U\notin{}_0^*$. Choose U_1^* with $t_0\in U_1^*$. Next choose t_1 and U_2^* such that $\gamma_1=\{\gamma(t): t_0\leqq t<t_1\}\subset U_1^*$, $\gamma(t_1)\notin U_1^*$ and $\gamma(t_1)\in U_2^*$. By repeating this process we have $(\gamma_0, U_0^*), (\gamma_1, U_1^*),\dots, (\gamma_m, U_m^*)$, where $\gamma_m=\{\gamma(t): t_{m-1}\leqq t\leqq 1)\}\subset U_m^*$. For each γ_i we define inductively an analytic set $\{R_i=c_i\}$ on U_1^* by the condition

$$c_i=f_{i,i-1}(c_{i-1})c_{i-1} \qquad (i=1, 2,\dots, m).$$

Hence we have an analytic set $S(c_0)$ in $V=\bigcup_{i=1}^m U_i^*$ as follows:

$$S(c_0)=\bigcup_{i=1}^m \{R_i=c_i\}.$$

We call $S(c_0)$ the continuation of $\{R_0=c_0\}$ along γ in V. By the compatibility condition of $\{f_{\lambda\mu}\}$, $S(c_0)$ does not depend on the choices of $\{U_i^*\}$. As for the continuation the following holds:

Proposition (4.1). *For any path $\gamma\subset C$, define $\theta_{m,0}(\gamma): D_0(\varepsilon)\to D_m(\varepsilon)$ by $\theta_{m,0}(\gamma)(c_0)=c_m$. Then $\theta_{m,0}(\gamma)$ is a biholomorphic mapping and $\theta_{m,0}(0)=0$. Moreover, if γ is homotopic to γ', then $\theta_{m,0}(\gamma)=\theta_{m,0}(\gamma')$.*

The proof is easy.

Proposition (4.2). *If C has a regularly half pseudoconvex neighborhood, then there exists a metric $\{\tilde{a}_\lambda\}$ on V_ε and a system of local* *coordinates* $(\tilde{z}_\lambda, \tilde{R}_\lambda)$ *on* U_λ^* *such that* (1) $\tilde{R}_\lambda = \tilde{f}_{\lambda\mu}\tilde{R}_\mu$ *where* $|\tilde{f}_{\lambda\mu}|=1$ *and* (2) $\tilde{F}=\tilde{a}_\lambda|R_\lambda|^2$ *defines a regularly half pseudoconvex neighborhood* $\tilde{V}_\varepsilon=\{F<\varepsilon\}$ *of* C.

Proof. We choose a path $\gamma_\lambda \subset C$ such that $\gamma_\lambda(0)=p_0^{(0)}\in U_0^*$ and $\gamma_\lambda(1)=p_\lambda^{(0)}\in U_\lambda^*$. By (4.1) we have a biholomorphic mapping $\theta_{\lambda,0}(\gamma_\lambda)\colon D_0(\varepsilon)\to D_\lambda(\varepsilon)$, whose inverse is denoted by φ_λ. We can define a new local coordinates $(\tilde{z}_\lambda, \tilde{R}_\lambda)$ on U_λ as follows:

$$\tilde{z}_\lambda = z_\lambda, \quad \tilde{R}_\lambda=\varphi_\lambda(R_\lambda).$$

Now take any point $p\in U_\lambda^* \cap U_\mu^*$. Choosing a path δ_λ (resp. δ_μ) in $U_\lambda^* \cap U_\mu^*$ such that $\delta_\lambda(0)=p$ and $\delta_\lambda(1)=p_\lambda^{(0)}$ (resp. $\delta_\mu(0)=p$ and $\delta_\mu(1)=p_\mu^{(0)}$), we have a closed path $\gamma'_{\lambda\mu}=\gamma_\mu^{-1}\circ\delta_\mu\circ\delta_\lambda^{-1}\circ\gamma_\lambda$. We replace $\gamma'_{\lambda\mu}$ by a path $\gamma_{\lambda\mu}$ in C which is homotopic to $\gamma'_{\lambda\mu}$. We may assume that the homotopy class of $\gamma_{\lambda\mu}$ does not depend on the choices of p and δ_λ, δ_μ. By (4.1) $\theta_{0,0}(\gamma_{\lambda\mu})$ is a biholomorphic automorphism of $D_0(\varepsilon)$, which we write as $\varphi_{\lambda\mu}$. Then with some constant $\alpha_{\lambda\mu}$ it can be written as

$$\varphi_{\lambda\mu}(R_0)=\alpha_{\lambda\mu}\cdot R_0 \qquad \text{where} \quad |\alpha_{\lambda\mu}|=1\,.$$

Accordingly

$$\tilde{R}_\lambda(p)=\alpha_{\lambda\mu}\cdot\tilde{R}_\mu(p) \qquad \text{for} \quad p\in U_\lambda\cap U_\mu\,.$$

We define $\{\phi_\lambda(R_\lambda)\}$ by $\varphi_\lambda(R_\lambda)=\phi_\lambda(R_\lambda)R_\lambda$. Then $\phi_\lambda(R_\lambda)\neq 0$ everywhere. In view of $\tilde{R}_\lambda(p)=\varphi_\lambda(f_{\lambda\mu}(R_\mu)\cdot R_\mu(p))$ for $p\in U_\lambda\cap U_\mu$, we obtain

$$\phi_\lambda(f_{\lambda\mu}(R_\mu(p))R_\mu(p))\cdot f_{\lambda\mu}(R_\mu(p))=\alpha_{\lambda\mu}\cdot R_\mu(p)\,,$$

which implies that $\{\alpha_{\lambda\mu}\}$ is equivalent to $[C]$. It is easily seen that $\tilde{a}_\lambda=a_\lambda|\phi_\lambda|^2$ satisfies the assertion (2) in (4.2).

Now we prove the sufficiency part of Main Theorem. If N_C is negative, then $\mathcal{O}(V_\varepsilon)\not\cong\mathbf{C}$ by (2.2). If C has a regularly half pseudoconvex neighborhood, by (4.2) it may be assumed that $|f_{\lambda\mu}|=1$. Since $N_C^k=1$, $\{f_{\lambda\mu}^k\}$ is analytically trivial on C. Hence there exists a system of non-vanishing holomorphic functions $\{\alpha_\lambda\}$ on $U_\lambda\cap C$ such that

$$f_{\lambda\mu}^k=\alpha_\mu\cdot\alpha_\lambda^{-1}\,. \tag{4.3}$$

In view of $|f_{\lambda\mu}|=1$, we have a harmonic function $h=|\alpha_\lambda|$ on C. Then α_λ must be constant. Therefore (4.3) holds on V_ε. Recalling $f_{\lambda\mu}=R_\lambda/R_\mu$, we get a holomorphic function

$$f=\alpha_\lambda R_\lambda^k \qquad \text{on} \quad V_\varepsilon .$$

We remark that $\{f=0\}=kC$.

§5. Examples

In this section we consider two examples due to H. Grauert [2] and M. Otuki [7] respectively.

Example 1. Let C be a compact Riemann surface of genus g ($g\geqq 1$) and let E be a topologically trivial line bundle on C. π denotes the natural projection of E. We cover C by an open covering $\{V_\lambda\}$ such that $E|V_\lambda$ is analytically trivial. The local coordinate of V_λ and the fiber coordinate of $\pi^{-1}(V_\lambda)$ are denoted by z_λ and ζ_λ respectively. By a well known lemma we may assume

$$|f_{\lambda\mu}|=1 \qquad \text{on} \quad V_\lambda \cap V_\mu .$$

So we have a tubular neighborhood system T_ε of the zero section, $T_\varepsilon=\{(z_\lambda, \zeta_\lambda): |\zeta_\lambda|^2<\varepsilon\}$. We infer that T_ε is always a regularly half pseudoconvex neighborhood system of C. We define a differential system $\mathscr{D}$ by

$$d\zeta_\lambda=0 \qquad \text{on} \quad \pi^{-1}(V_\lambda)\cap T_\varepsilon .$$

Since $|f_{\lambda\mu}|=1$, $\mathscr{D}$ is well defined on T_ε and a holomorphic foliation $\mathscr{F}$ is obtained on T_ε.

Theorem 2. (1) *If E is of finite order, then $\mathscr{F}$ is globally integrable and $\mathcal{O}(T_\varepsilon)\cong\mathcal{O}(\mathbf{D})$ where $\mathbf{D}$ is the unit disk.* (2) *If E is of infinite order, $\mathscr{F}$ can never be a closed foliation and $\mathcal{O}(T_\varepsilon)\cong\mathbf{C}$.*

Proof. By regularly half pseudoconveity of T_ε, (1) is a direct consequence of Main Theorem. For the proof of (2) assume that $\mathscr{F}$ would be a closed foliation. Then a leaf $\mathscr{S}_p$ through $p\in T_\varepsilon\cap\pi^{-1}(V_\lambda)$

can be expressed as $\{\zeta_\lambda = c_\lambda\}$ on $T_\varepsilon \cap \pi^{-1}(V_\lambda)$, where $c_\lambda = \zeta_\lambda(p)$. Since $|f_{\lambda\mu}|=1$, $\mathscr{S}_p$ has finite intersection points with any fiber $\pi^{-1}(p)$, $p \in C$. The connected components of $\mathscr{S}_p$ on $\pi^{-1}(V_\lambda) \cap T_\varepsilon$ are denoted by $\{A_\mu^{(i)}\}$ $(i=1, 2,\dots, k)$ where $A_\mu^{(i)} = \{\zeta_\mu = c_\mu^{(i)}\}$. Note that k does not depend on μ. For each μ we prepare k copies of $\pi^{-1}(V_\lambda) \cap T_\varepsilon$ which are denoted by $W_\mu^{(i)}$ $(i=1, 2,\dots, k)$ whose local coordinates are denoted by $(z_\lambda, \zeta_\mu^{(i)})$. We identify a point $p_\lambda^{(i)} = (z_\lambda, \zeta_\lambda^{(i)}) \in W_\lambda^{(i)}$ with a point $p_\mu^{(j)} = (z_\mu, \zeta_\mu^{(j)}) \in W_\mu^{(j)}$ by the following conditions: (1) $\pi(p_\lambda^{(i)}) = \pi(p_\mu^{(j)})$ (2) $c_\lambda^{(i)} = f_{\lambda\mu} c_\mu^{(j)}$ and (3) $\zeta_\lambda^{(i)} = f_{\lambda\mu}\zeta_\mu^{(j)}$. Then we have a k-fold unramified covering manifold $\tilde{T}_\varepsilon$. The natural projection is denoted by ω. By construction there exists an unramified covering $\tilde{C}$ over C, $\omega' : \tilde{C} \to C$, such that $\tilde{T}_\varepsilon$ coincides with a tubular neighborhood of $\omega'^*(E)$. Then $\omega'^*(E)$ admits a trivial section, which contradicts the assumption of infinite order. For the proof of the second part of (2) we consider $f \in \mathcal{O}(T_\varepsilon)$. Since $|f_{\lambda\mu}|=1, f$ can be expressed as follows:

$$f = \sum_{m=0}^{\infty} a_\lambda^{(m)} \zeta_\lambda^m \qquad \text{on} \quad T_\varepsilon \cap \pi^{-1}(V_\lambda),$$

where the $a_\lambda^{(m)}$ are constants on V_λ. If f were non-constant, $\{f=c\}$ would be an analytic set in T_ε whose irreducible component could be expressed as $\{\zeta_\mu = b\}$ with some constant b on $\pi^{-1}(V_\mu)$. This contradicts the non-closedness of $\mathscr{F}$. Thus we obtain $\mathcal{O}(T_\varepsilon) \cong \mathbf{C}$. This is an alternative proof of the theorem of H. Grauert [2]:

Remark. In Example 1, C has always a regularly half pseudoconvex neighborhood system T_ε. This implies that regularly half pseudoconvexity is not a sufficient condition for the existence problem.

Example 2. Hopf surfaces are defined in the following manner: Let $\mathbf{C}^2 = \{(z_1, z_2)\}$ and $\mathbf{W} = \mathbf{C}^2 - \{(0, 0)\}$ and consider a holomorphic automorphism of $\mathbf{W}$

$$g : (z_1, z_2) \longrightarrow (\alpha_1 z_1 + \lambda z_2^m, \alpha_2 z_2), \tag{5.1}$$

where α_1, α_2 and λ are constants and m is a positive integer with the conditions $(\alpha_1 - \alpha_2^m)\lambda = 0$ and $0 < |\alpha_1| \leqq |\alpha_2| < 1$. Then $G = \{g^n : n \in \mathbf{Z}\}$ acts on $\mathbf{W}$ properly discontinuously, which determines a compact surface

$S=\mathbf{W}/G$. The surface thus obtained is called a *Hopf surface*. π denotes the natural projection, $\pi: \mathbf{W}\to S$. The curve $\{z_2=0\}$ is always invariant by G, which defines a non-singular elliptic curve C on S. Furthermore, in the case where $\lambda=0$, $\{z_1=0\}$ also induces another elliptic curve C_1. The following is essetially due to M. Otuki [5].

Theorem 3. (1) *If* $\lambda=0$, *then* C (*or* C_1) *has a regularly half pseudoconvex neighborhood system* V_ε. *Moreover*, (*i*) *if* $\alpha_1^k=\alpha_2^j$ *with some positive integer* k *and* j, *then* $\mathcal{O}(V_\varepsilon)\cong\mathcal{O}(\mathbf{D})$ *and* (*ii*) *if* $\alpha_1^k\neq\alpha_2^j$ *for any pair of non-zero integers* k *and* j, *then* $\mathcal{O}(V_\varepsilon)\cong\mathbf{C}$. (2) *If* $\lambda\neq 0$, C *has an s-pseudoconcave neighborhood system and* $N_C^m=1$. *Moreover*, $\mathcal{O}(V_\varepsilon)\cong\mathbf{C}$.

Proof. We prove (1) only in the case of C. We write $|\alpha_1|=\rho$, $|\alpha_2|=\sigma$ and define τ by $\sigma=\rho^\tau$. By (5.1) $F=|z_2|^2/|z_1^\tau|^2$ is invariant by G. So F is a function on $S-C_1$ and $V_\varepsilon=\{F<\varepsilon\}$ is a neighborhood system of C. Define a differential system $\mathscr{D}$ on $W_\varepsilon=\pi^{-1}(V_\varepsilon)$ by

$$\omega=0, \quad \omega=d(z_2/z_1^\tau)$$

where one of the branchs of z_1^τ is fixed. Then $\mathscr{D}$ is completely integrable and induces a foliation $\tilde{\mathscr{F}}$ on W_ε. Every leaf is nothing but the analytic continuation of the analytic set $\{z_2=z_1^\tau c\}$ for $|c|^2<\varepsilon$, which is denoted by $\mathscr{S}_c$. Since $g^*\omega=h\omega$ where $h=\alpha_2/\alpha_1^\tau$, we get a foliation $\mathscr{F}=\{\pi(\mathscr{S}_c): |c|^2<\varepsilon\}$ on V_ε from $\tilde{\mathscr{F}}$. Note that $\pi(\mathscr{S}_c)$ has no intersection points with ∂V_ε. In the case of (i) $\tau=j/k$ holds. Then we have a global holomorphic function $f=(z_2/z_1^\tau)^k$ on V_ε. The foliation defined by f is equivalent to $\mathscr{F}$. By the Stein factorization, we obtain $\mathcal{O}(V_\varepsilon)\cong\mathcal{O}(\mathbf{D})$. In the case of (ii), τ is irrational. So we have

$\overline{\pi(\mathscr{S}_c)}=\{F=|c|^2\}$, where $\overline{\pi(\mathscr{S}_c)}$ denotes the closure of $\pi(\mathscr{S}_c)$ in S. Take a holomorphic function f on V_ε, then f must be bounded on $\pi(\mathscr{S}_c)$. Since the universal covering surface of $\pi(\mathscr{S}_c)$ is isomorphic to $\mathbf{C}$, we see that f is constant on $\pi(\mathscr{S}_c)$. Hence by the theorem of identity we see that f is constant on V_ε. To prove (2), it is sufficient to show

$$S^*=\mathbf{C}^*\times\mathbf{C}^* \qquad \text{where} \quad S^*=S-C.$$

Set $\mathbf{C}^2=\{(Z_1, Z_2)\}$, $D=\mathbf{C}^2-\{Z_2=0\}$ and define $\Phi: D\to\mathbf{W}-\{z_2=0\}$ by $\Phi(Z_1, Z_2)=(Z_1Z_2^m, Z_2)$. Then Φ is a biholomorphic mapping. With this identification g can be expressed as $Z_1'=Z_1+\lambda'$, $Z_2'=\alpha_2Z_2$ where $\lambda'=\lambda/\alpha_2^m$. The universal covering manifold of D is isomorphic to $\mathbf{C}^2=(u, v)$ and the covering projection is written as $\omega: (u, v)\to(Z_1, Z_2)=(u, e^v)$. Therefore its covering transformation group is generated by

$$g^{(1)}:\begin{cases} u'=u \\ v'=v+2\pi\sqrt{-1}. \end{cases}$$

Also $g^{(2)}=\omega^*(g)$ can be written as follows:

$$g^{(2)}:\begin{cases} u'=u+\lambda \\ v'=v+\mu,\ \mu=\log\alpha_2. \end{cases}$$

Using a suitable linier transform in $\mathbf{C}^2$ we may assume that $g^{(1)}$ and $g^{(2)}$ are in the following form:

$$g^{(1)}:\begin{cases} u'=u \\ v'=v+a \end{cases} (a\neq 0), \qquad g^{(2)}:\begin{cases} u'=u+b \\ v'=v \end{cases} (b\neq 0).$$

Set $G^{(1)}=\{g^{(1)n}: n\in\mathbf{Z}\}$ and $G^{(2)}=\{g^{(2)n}: n\in\mathbf{Z}\}$, then

$$S^*=[\mathbf{C}^2/G^{(1)}]/G^{(2)}\cong\mathbf{C}^*\times\mathbf{C}^*.$$

$N_C^m=1$ can be proved as follows: $\psi=z_2^{-(m+1)}dz_1\wedge dz_2$ is a meromorphic 2-form on $\mathbf{W}$ which is invariant by G. So ψ can be seen as a form on S. This implies that the canonical line bundle K_S of S can be written as follows:

$$K_S=[C]^{-(m+1)}.$$

Referring to $K_C=1$, we obtain $N_C^m=1$ by the adjunction formula.

Remark 1. The finite order condition is not sufficient for the existence problem.

Remark 2. For another example of a compact curve C such that (1) N_C is of finite order and (2) C has a s-pseudoconcave neighborhood system, see R. Hartshorne [3], Example 3.2, p. 232.

§6. Conclusions and Remarks

By the discussions above we obtain

(1) if N_C is negative, C has an s-pseudoconvex neighborhood system V_ε and there exists a 2-dimensional normal Stein space V_ε^* satisfying $\mathcal{O}(V_\varepsilon)\cong\mathcal{O}(V_\varepsilon^*)$ by (2.2),

(2) if N_C is positive, by (2.2) C has an s-pseudoconcave neighborhood system and $\mathcal{O}(V_\varepsilon)\cong\mathbf{C}$ by (2.5),

(3) in the case where N_C is topologically trivial,

(i) if N_C is of finite order, $\mathcal{O}(V_\varepsilon)\cong\mathcal{O}(\mathbf{D})$ if and only if C has a regularly half pseudoconvex neighborhood system. Moreover, Theorem 3 shows that there exists a $C\subset S$ having an s-pseudoconcave neighborhood system even if N_C satisfies the condition of finite order.

(ii) if N_C is of infinite order, then $\mathcal{O}(V_\varepsilon)\cong\mathbf{C}$ by (2.5). Theorem 2 shows that there exists a curve $C\subset S$ having a regularly half pseudoconvex neighborhood system even if N_C satisfies the condition of infinite order.

We summarize these results in the following Table:

Table

normal bundle		pseudoconvexity of a neighborhood system V_ε	the Remmert reduction of V_ε
negative		s-pseudoconvex	2-dimensional normal Stein space
positive		s-pseudoconcave	p
topologically trivial	finite order	regularly half pseudoconvex	$\mathbf{D}$
		otherwise (s-pseudoconcave cases may occur)	p
	infinite order	(regularly half pseudoconvex cases may occur)	p

Remark 1. When a compact curve C with $N_C>0$ is imbedded in a 2-dimensional compact compex manifold S, S must be a projective algebraic manifold. Moreover, for any neighborhood V of C, $\mathcal{M}(V)\cong\mathcal{M}(S)$.

Remark 2. Let C be a compact curve which has a pseudoconvex neighborhood system $V_\rho=\{F<\rho\}$ where F is a certain pseudoconvex function on a neighborhood of C. Moreover, suppose that C is of infinite order. Then $\mathcal{O}(V_\rho)\cong\mathbf{C}$, but there exists many meromorphic functions on V_ρ for a small ρ. In fact, making ρ smaller, we may assume that there exists a divisor $\tilde{C}$ which is transversal with C. In the similar manner as in (2.2) we see that $[\tilde{C}]>0$ for a small constant ρ. Using Nakano's theorem [4], we see that there exists a positive integer n_0 satisfying $H^q(V_\rho, \mathcal{O}([\tilde{C}]^n))=0$ for $q\geqq 1$ and $n\geqq n_0$. Then V_ρ can be imbedded by sections of $[\tilde{C}]^n$ in a projective space. This shows that V_ρ is meromorphically separable.

Remark 3. In the case of a non-singular non-compact curve imbedded in S, it can be proved that it has always a Stein neighborhood system by using H. Grauert's Lemma (see, H. Grauert [1], p. 340, Satz 5).

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