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Cohomologies of Lie Algebras of Vector Fields with Coefficients in Adjoint Representations Case of Classical Type

By

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Introduction

Let M be a smooth manifold, and $\mathfrak{A}(M)$ the Lie algebra of all smooth vector fields on M. Assume that M admits a volume form τ, a symplectic form ω or a contact form θ. Then we have natural Lie subalgebras of $\mathfrak{A}(M)$ as $\mathfrak{A}_\tau(M)$, $\mathfrak{A}'_\tau(M)$, $\mathfrak{A}_\omega(M)$, $\mathfrak{A}'_\omega(M)$, $\mathfrak{A}_\theta(M)$ (see § 1.1). These Lie algebras including $\mathfrak{A}(M)$ itself are called of classical type. Here we are interested in the cohomology $H^*(\mathfrak{A}; \mathfrak{A})$ of the Lie algebra $\mathfrak{A}$ with coefficients in its adjoint representation.

Calculations of them are not easy in general. But the first cohomology can be calculated rather easily since $H^1(\mathfrak{A}; \mathfrak{A})$ is interpreted in terms of derivations of $\mathfrak{A}$. From this point of view F. Takens [5] calculated $H^1(\mathfrak{A}(M); \mathfrak{A}(M))$ in 1973. Later A. Avez–A. Lichnerowicz–A. Diaz-Miranda [2] and the author [3] calculated $H^1(\mathfrak{A}_\omega(M); \mathfrak{A}_\omega(M))$ of Lie algebra $\mathfrak{A}_\omega(M)$ of hamiltonian vector fields by different methods. In the present paper, we will calculate $H^1(\mathfrak{A}; \mathfrak{A})$ for all $\mathfrak{A}$ of classical type. Our results can be summarized as follows.

Main Theorem.

a) *Let M be a smooth manifold with a volume element τ, a symplectic structure ω or a contact structure θ, and let $\mathfrak{A}$ be one of $\mathfrak{A}(M)$, $\mathfrak{A}'_\tau(M)$, $\mathfrak{A}'_\omega(M)$ and $\mathfrak{A}_\theta(M)$. Then*

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$$H^1(\mathfrak{A};\mathfrak{A})=0.$$

b) *Let M be a connected smooth manifold with a volume element τ or a symplectic structure ω, and $\mathfrak{A}=\mathfrak{A}_\tau(M)$ or $\mathfrak{A}_\omega(M)$ respectively. Then*

$$H^1(\mathfrak{A};\mathfrak{A})\cong\mathbf{R} \quad \textit{or} \quad 0.$$

Moreover, $H^1(\mathfrak{A};\mathfrak{A})\cong\mathbf{R}$ if and only if τ or ω is an exact form on M respectively.

We can reduce the study of derivations of $\mathfrak{A}$ to the case where M is flat. Here the notion of *localizability* of derivations (see § 1.2) is essential. A Euclidean space is furnished with the natural structure: the volume form $\tau=dx_1\cdot\cdots\cdot dx_n$, the symplectic form $\omega=\sum_{i=1}^n dx_i dx_{i+n}$ or the contact form $\theta=dx_0-\sum_{i=1}^n x_{i+n}dx_i$. Then we have the main theorem for flat case:

a) *Let $\mathfrak{A}=\mathfrak{A}(\mathbf{R}^n)$, $\mathfrak{A}'_\tau(\mathbf{R}^n)$, $\mathfrak{A}'_\omega(\mathbf{R}^{2n})$ or $\mathfrak{A}_\theta(\mathbf{R}^{2n+1})$. Then*

$$H^1(\mathfrak{A};\mathfrak{A})=0.$$

b) *Let $\mathfrak{A}=\mathfrak{A}_\tau(\mathbf{R}^n)$ or $\mathfrak{A}_\omega(\mathbf{R}^{2n})$. Then*

$$H^1(\mathfrak{A};\mathfrak{A})\cong\mathbf{R}.$$

The contents of the paper are arranged as follows. In § 1, we explain the notion of Lie algebras of vector fields of classical type, and the localizability of derivations of $\mathfrak{A}$. We also explain the general scheme to prove the main theorem for flat case.

In § 2, the properties of $\mathfrak{A}_\theta(M)$ and its derivations are studied. In § 3, the main theorem for $\mathfrak{A}_\theta(\mathbf{R}^{2n+1})$, the flat case, is proved.

In § 4, the properties of $\mathfrak{A}_\tau(M)$, $\mathfrak{A}'_\tau(M)$ and their derivations are studied. In § 5, the main theorems for $\mathfrak{A}_\tau(\mathbf{R}^n)$ and $\mathfrak{A}'_\tau(\mathbf{R}^n)$, the flat case, are proved.

In § 6, we reproduce briefly the main theorems for $\mathfrak{A}_\omega(\mathbf{R}^{2n})$ and $\mathfrak{A}'_\omega(\mathbf{R}^{2n})$ in this direction.

In § 7, we prove Main Theorem for all Lie algebras of vector fields of classical type.

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§1. Lie Algebras of Vector Fields of Classical Type, and Their Derivation Algebras

1.1. Definition of the Lie Algebras. All manifolds, vector fields, forms etc. are assumed to be of C^∞-class. Denote by $\mathfrak{A}(M)$ the Lie algebra of all vector fields on a manifold M.

Let τ be a volume element on M. A vector field X is called volume preserving or conformally volume preserving if $L_X\tau=0$ or $L_X\tau=c\tau$ for some constant c respectively, where L_X denotes the Lie derivation corresponding to X. We get two natural Lie subalgebras $\mathfrak{A}_\tau(M)$ and $\mathfrak{A}'_\tau(M)$ of $\mathfrak{A}(M)$ defined as

$$\mathfrak{A}_\tau(M)=\{X\in\mathfrak{A}(M);\ L_X\tau=0\},$$

$$\mathfrak{A}'_\tau(M)=\{X\in\mathfrak{A}(M);\ L_X\tau=c\tau \text{ for some constant } c\}.$$

Then $\mathfrak{A}_\tau(M)\subset\mathfrak{A}'_\tau(M)$ obviously.

Assume that a manifold M of even dimension is furnished with the symplectic structure ω. Here the symplectic structure ω is by definition a non-degenerate closed 2-form on M. A vector field X is called hamiltonian or conformally hamiltonian if $L_X\omega=0$ or $L_X\omega=c\omega$ for some constant c respectively. Thus we have the following two natural Lie subalgebras of $\mathfrak{A}(M)$:

$$\mathfrak{A}_\omega(M)=\{X\in\mathfrak{A}(M);\ L_X\omega=0\},$$

$$\mathfrak{A}'_\omega(M)=\{X\in\mathfrak{A}(M);\ L_X\omega=c\omega \text{ for some constant } c\}.$$

Then $\mathfrak{A}_\omega(M)\subset\mathfrak{A}'_\omega(M)$ too.

Assume that a manifold M of odd dimension $2n+1$ is furnished with the contact structure θ, where θ is by definition a 1-form on M such that $\theta\wedge(d\theta)^n$ is a volume form on M. A vector field X is called contact if $L_X\theta=f\theta$ for some function f on M. We denote by $\mathfrak{A}_\theta(M)$ the Lie subalgebra consisting of all contact vector fields on M.

Let $\mathfrak{A}$ be a Lie algebra of vector fields on a manifold M. We call $\mathfrak{A}$ of classical type if it is isomorphic to one of the above six Lie algebras: $\mathfrak{A}(M)$, $\mathfrak{A}_\tau(M)$, $\mathfrak{A}'_\tau(M)$, $\mathfrak{A}_\omega(M)$, $\mathfrak{A}'_\omega(M)$ or $\mathfrak{A}_\theta(M)$. The formal algebras corresponding to them are isomorphic to the classical infinite dimensional Lie algebras of É. Cartan (see Singer-Sternberg [4]).

Let U be an open submanifold of M. Then, replacing M by U, we have naturally the Lie algebra $\mathfrak{A}_U$ according to $\mathfrak{A}$. For instance, $\mathfrak{A}_U=\mathfrak{A}_\tau(U)$ for $\mathfrak{A}=\mathfrak{A}_\tau(M)$. Let r_U be the restriction map on U, then $r_U(\mathfrak{A})\subset\mathfrak{A}_U$, but they do not coincide with each other in general. We say that $\mathfrak{A}$ has *the property* (A) if $r_U(\mathfrak{A})=r_U(\mathfrak{A}_{U'})$ for any two open subsets $U\subset U'$ of M *such that* $\bar{U}\subset U'$.

Proposition 1.1. *The Lie algebras* $\mathfrak{A}(M)$, $\mathfrak{A}_\tau(M)$, $\mathfrak{A}_\omega(M)$ *and* $\mathfrak{A}_\theta(M)$ *have the property* (A).

Proof. Let $\mathfrak{A}$ be any one of the above Lie algebras. Then for any open subset U of M, the Lie algebra $\mathfrak{A}_U$ is a module over $C^\infty(U)$. Q.E.D.

1.2. Derivations of $\mathfrak{A}$. Let $\mathfrak{A}$ be a Lie subalgebra of $\mathfrak{A}(M)$. A mapping $D:\mathfrak{A}\to\mathfrak{A}$ is called a derivation of $\mathfrak{A}$ if D is $\mathbf{R}$-linear and $D([X, Y])=[D(X), Y]+[X, D(Y)]$ for all $X, Y\in\mathfrak{A}(M)$. A derivation D is called inner if $D=\operatorname{ad}W$ for some W in $\mathfrak{A}$. Denote by $\mathfrak{D}(\mathfrak{A})$ the algebra of all derivations of $\mathfrak{A}$, and by $\mathfrak{D}^i(\mathfrak{A})$ its ideal of all inner derivations of $\mathfrak{A}$. Then we know [3, §1] that the first cohomology $H^1(\mathfrak{A};\mathfrak{A})$ of the Lie algebra $\mathfrak{A}$ with coefficients in its adjoint representation is realized as

$$H^1(\mathfrak{A};\mathfrak{A})\cong\mathfrak{D}(\mathfrak{A})/\mathfrak{D}^i(\mathfrak{A}).$$

A derivation D of $\mathfrak{A}$ is called *local* if $D(X)$ vanishes on U for any vector field $X\in\mathfrak{A}$, zero on an open subset U of M. Moreover a local derivation D is called *localizable* if for any open subset U of M, there is a derivation D_U of $\mathfrak{A}_U$ compatible with the restriction map r_U, that is, $D_U\circ r_U=r_U\circ D$. Then we have the following.

Proposition 1.2. *If the subalgebra* $\mathfrak{A}$ *of* $\mathfrak{A}(M)$ *has the property* (A), *then any local derivation of* $\mathfrak{A}$ *is localizable.*

Proof. Let D be a local derivation of $\mathfrak{A}$ and U an open subset of M. For any point p of U and $X \in \mathfrak{A}_U$, by the property (A), there is $\tilde{X} \in \mathfrak{A}$ such that $X = \tilde{X}$ on some neighbourhood U' of p. Define the derivation D_U of $\mathfrak{A}_U$ by $D_U(X)(p) = D(\tilde{X})(p)$, then $D_U(X)(p)$ is well-defined because D is local. Q.E.D.

If all derivations of $\mathfrak{A}$ are localizable, the study of $\mathfrak{D}(\mathfrak{A})$ is reduced in a certain extent to the case where M is flat, that is, M is the Euclidean space $V = \mathbf{R}^n$.

1.3. The Flat Case. Let $\mathfrak{A}$ be a Lie algebra of classical type of vector fields on the Euclidean space V. The main part of our study of the derivation algebra $\mathfrak{D}(\mathfrak{A})$ of $\mathfrak{A} \subset \mathfrak{A}(V)$ is to find the vector field $W \in \mathfrak{A}(V)$ such that $D = \operatorname{ad} W$ on $\mathfrak{A}$, and to clarify the property of W. This will be done according to the following three steps:

(I) To find a good finite-dimensional subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ for which the following differential equation

$$\text{(E)} \qquad [W, X] = D(X) \qquad (X \in \mathfrak{B})$$

has a unique solution $W \in \mathfrak{A}(V)$.

(II) Let $\mathfrak{A}_0$ be the subalgebra of $\mathfrak{A}$ consisting of all elements in $\mathfrak{A}$ whose coefficients are polynomials with respect to the coordinates in V. We wish to show that $[W, X] = D(X)$ for all $X \in \mathfrak{A}_0$.

(III) To show the fact that $D(X)(0) = 0$ if a vector field $X \in \mathfrak{A}$ satisfies $j^r(X)(0) = 0$ for some integer r, independent of X.

Here we apply the following lemma.

Proposition 1.3. *Suppose that (I), (II) and (III) are established for a* $D \in \mathfrak{D}(\mathfrak{A})$, *and that* $\operatorname{ad} W(\mathfrak{A}) \subset \mathfrak{A}$ *where* W *is the vector field obtained in (I). Then* $D = \operatorname{ad} W$ *on* $\mathfrak{A}$.

Proof. Put $D' = D - \operatorname{ad} W$, then D' is a derivation of $\mathfrak{A}$, zero on $\mathfrak{A}_0$. A vector field $X \in \mathfrak{A}$ is decomposed for any point $p \in V$ as $X = X_1 + X_2$ such that $X_1 \in \mathfrak{A}_0$ and $j^r(X_2)(p) = 0$, because there exists a coordinate transformation φ with polynomial coefficients such that $\varphi(p) = 0$ and $\varphi^*(\mathfrak{A}) = \mathfrak{A}$. By (II) and (III), we get

$$D'(X)(p)=D'(X_1)(p)+D'(X_2)(p)=0+0=0 \qquad (p\in V).$$

Hence $D=\operatorname{ad} W$ on $\mathfrak{A}$. Q.E.D.

We also apply the following.

Proposition 1.4. *It is sufficient for (III) to prove the following*:

(III′) *If a vector field $X\in\mathfrak{A}$ satisfies $j^r(X)(0)=0$ for some fixed integer $r\geqq 0$, then there exist a finite number of vector fields $Y_1,\dots, Y_{2q}\in\mathfrak{A}$ such that*

$$X=\sum_{i=1}^{q}[Y_i, Y_{i+q}] \quad \text{and} \quad j^1(Y_i)(0)=0 \qquad (1\leqq i\leqq 2q).$$

Proof. We get

$$\begin{aligned} D(X)(0) &= \sum_{i=1}^{q} D([Y_i, Y_{i+q}])(0) \\ &= \sum_i \{[D(Y_i), Y_{i+q}](0)+[Y_i, D(Y_{i+q})](0)\} \\ &= 0+0=0. \end{aligned}$$

Q.E.D.

1.4. In §2, we shall prove that any $D\in\mathfrak{D}(\mathfrak{A}_\theta(M))$ is localizable (Corollary 2.5), and show (III′), Proposition 2.6, for $\mathfrak{A}_\theta(M)$. In §3, we pass through the steps (I) and (II) in §1.3 above for $\mathfrak{A}_\theta(n)=\mathfrak{A}_\theta(\mathbf{R}^{2n+1})$, Proposition 3.2 and Lemma 3.4. Moreover we obtain the main theorem for $\mathfrak{A}_\theta(n)$, Theorem 3.3.

In §4, we clarify the relations between $\mathfrak{A}_\tau(M)$ and $\mathfrak{A}'_\tau(M)$, and prove that any $D\in\mathfrak{D}(\mathfrak{A}'_\tau(M))$ is local (Proposition 4.4), and any $D\in\mathfrak{D}(\mathfrak{A}_\tau(M))$ is localizable (Proposition 4.5). In §4.4, (III′) for $\mathfrak{A}_\tau(M)$, Proposition 4.6, is proved. In §5, the steps (I) and (II) for $\mathfrak{A}'_\tau(n)=\mathfrak{A}'_\tau(\mathbf{R}^n)$, Proposition 5.6 and Lemma 5.9, are proved. Moreover we obtain the main theorems for $\mathfrak{A}'_\tau(n)$ and $\mathfrak{A}_\tau(n)$, Theorem 5.7 and 5.8 respectively.

In §6, we describe the outline of the proof of the main theorems for $\mathfrak{A}_\omega(\mathbf{R}^{2n})$, $\mathfrak{A}'_\omega(\mathbf{R}^{2n})$ and $\mathfrak{A}(\mathbf{R}^n)$ in this direction.

§2. Contact Vector Fields

2.1. Properties of Contact Vector Flelds. Let (M^{2n+1}, θ) be a contact manifold of dimension $2n+1$. Here we do not need the geometrical meaning of the contact vector fields except the following well-known two lemmata.

Lemma 2.1. *Let* $^{\#}$ *be a mapping from* $\mathfrak{A}_\theta(M)$ *to* $C^\infty(M)$, *which assigns* $X^{\#}=i_X\theta$ *to* $X\in\mathfrak{A}_\theta(M)$, *where* $i_X\theta$ *is the interior product of* X *and* θ. *Then the linear mapping* $^{\#}$ *is bijective.*

By this lemma, the inverse $^\flat: C^\infty(M)\to\mathfrak{A}_\theta(M)$ can be defined, and we can introduce the generalized Poisson bracket $((\ ,\ ))$ in $C^\infty(M)$ as follows:

$$((f, g))^\flat=[f^\flat, g^\flat] \qquad \text{for} \quad f, g\in C^\infty(M).$$

In this way, $C^\infty(M)$ becomes a Lie algebra isomorphic to $\mathfrak{A}_\theta(M)$ under $^{\#}$.

Lemma 2.2 (*Darboux*). *Around any point* p *of a contact manifold* (M^{2n+1}, θ), *there exists a coordinate system* $(z, x_1,\dots, x_n, y_1,\dots, y_n)$ *such that* θ *is expressed as* $\theta=dz-\sum_{i=1}^n y_i dx_i$.

The mapping $^\flat$ and the generalized Poisson bracket are written in this contact coordinate system as

$$f^\flat=(f-\sum_{i=1}^n y_i f_{y_i})\partial_z-\sum_{i=1}^n f_{y_i}\partial_{x_i}+\sum_{i=1}^n (f_{x_i}+y_i f_z)\partial_{y_i}, \tag{2.1}$$

and

$$((f, g))=\{f, g\}_{x,y}-f_z(g-\sum_j y_j g_{y_j})+g_z(f-\sum_j y_j f_{y_j}) \tag{2.2}$$

for any $f, g\in C^\infty(M)$, where $\{\ ,\ \}_{x,y}$ is the usual Poisson bracket in $x_1,\dots, x_n, y_1,\dots, y_n$ variables, that is,

$$\{f, g\}_{x,y}=\sum_{i=1}^n (f_{x_i}g_{y_i}-f_{y_i}g_{x_i}).$$

Here we have the following.

Proposition 2.3. *Let X be a contact vector field on M, and U any open subset of M. Assume that $[X, Y]=0$ on U for any $Y\in\mathfrak{A}_\theta(M)$ with support contained in U. Then $X=0$ on U.*

Proof. Suppose $X(p)\neq 0$ for some point p of U. Let U' be a coordinate neighbourhood of p with contact coordinates $(z, x_1,\dots, x_n, y_1,\dots, y_n)$ around p. Since X is contact, for the function $f=X^\sharp$, one of $f(p), f_{x_i}(p)$ or $f_{y_i}(p)$ $(1\leqq i\leqq n)$ is not zero by (2.1).

Case 1. The case where $f(p)\neq 0$. Let g be a function whose support is contained in U', and equal to z in a smaller neighbourhood U'' of p. Then we have

$$((f, g))=-zf_z+f-\sum_{j=1}^{n} y_j f_{y_j} \qquad \text{in} \quad U''$$

and so $((f, g))(p)=f(p)\neq 0$. Hence we have by (2.1)

$$[X, g^\flat](p)=((f, g))^\flat(p)\neq 0\,.$$

This contracts our assumption that $[X, g^\flat]=0$.

Case 2. The case where $f_{x_i}(p)\neq 0$ or $f_{y_i}(p)\neq 0$. The same arguments as above are also valid here if we take into account the following equalities:

$$((f, y_i))=f_{x_i}, \quad ((x_i, f))=f_{y_i}+x_i f_z\,.$$

Q.E.D.

Proposition 2.4. *Any derivation of $\mathfrak{A}_\theta(M)$ is local.*

Proof. Suppose that $X\in\mathfrak{A}_\theta(M)$ is identically zero on an open subset U of M. For any $Y\in\mathfrak{A}_\theta(M)$ with support contained in U,

$$[D(X), Y]=D([X, Y])-[X, D(Y)]=0-0=0 \qquad \text{on} \quad U\,.$$

By Proposition 2.3, we get $D(X)=0$ on U. Q.E.D.

Corollary 2.5. Any derivation of $\mathfrak{A}_\theta(M)$ is localizable.

Proof. This follows directly from Lemmata 1.1 and 1.2. Q.E.D.

2.2. Proposition 2.6. *Let X be a contact vector field on M such that $j^4(X)(p)=0$ at a point $p\in M$. Then there are a finite number of contact vector fields $Y_1,\dots,Y_{2q}$ on M, and a neighbourhood U of p in M such that*

$$X_{|U}=\sum_{i=1}^{q}[Y_i, Y_{i+q}]_{|U}$$

and

$$j^1(Y_i)(p)=0 \qquad (1\leqq i\leqq 2q).$$

Proof. By means of a contact coordinate system $(z, x_1,\dots, x_n, y_1,\dots, y_n)$ around p, the vector field X and $f=X^\#$ are written as

$$X=h\partial_z+\sum_{i=1}^{n}(h^i\partial_{x_i}+h^{i+n}\partial_{y_i}),$$

$$X^\#=i_X\theta=h-\sum_{i=1}^{n}y^ih_i.$$

We assume that $j^4(h)(0)=0$ and $j^3(h^i)(0)=0$ for all i. Then the assertion follows from the next proposition. Q.E.D.

Proposition 2.7. *Let f be a function on $\mathbf{R}^{2n+1}$ with $j^4(f)(0)=0$. Then there are a finite number of functions $g_1,\dots,g_{2q}$ such that*

$$f=\sum_{i=1}^{q}((g_i, g_{i+q})),$$

and

$$j^1(g_i)(0)=j^1(g_{ix_j})(0)=j^1(g_{iy_j})=0 \qquad (1\leqq i\leqq 2q,\ 1\leqq j\leqq n).$$

Proof. **Case 1.** The case where $f_z=0$. Assume that $j^3(f)(0)=0$. Then by Proposition 2 in [3], there are functions $g_1,\dots,g_{2q}$ such that $g_{iz}=0$, $j^2(g_i)(0)=0$ $(1\leqq i\leqq 2q)$, and $f=\sum_i\{g_i, g_{q+i}\}_{x,y}=\sum_i((g_i, g_{q+i}))$.

Case 2. The case where f is written as $f=z^2h$. Assume that $j^3(f)(0)=0$, that is, $j^1(h)(0)=0$. Put

$$g=\int_0^{y_1} h\, dy_1\,,$$

then $j^2(g)(0)=0$, and

$$\begin{aligned}&((x_1g,\ z^2))-((g,\ x_1z^2))\\&=-z^2x_1g_z+2z(x_1g-x_1\sum_j y_jg_{y_j})-\{-z^2g_{y_1}-x_1z^2g_z+2x_1z(g-\Sigma y_jg_{y_j})\}\\&=z^2g_{y_1}=z^2h=f.\end{aligned}$$

By the above arguments, we may assume that f is expressed as

$$f=zx_1^{p_1}\cdots x_n^{p_n}y_1^{q_1}\cdots y_n^{q_n}h(x,\ y)$$

with $\sum_{i=1}^n (p_i+q_i)\geqq 4$.

Case 3. The case where $\sum_i p_i\geqq 2$.

a) The case where $p_j\geqq 2$ for some j. We may assume that f is written as $f=zx_1^2h(x,\ y)$. Put $g=\int_0^{y_1} h(x,\ y)dy_1$, then $j^2(g)(0)=0$, and

$$((x_1g,\ x_1^2z))-((g,\ x_1^3z))=zx_1^2g_{y_1}=zx_1^2h=f.$$

b) Assume that $p_1=p_2=1$. Then by means of the following contact transformation φ, this case is reduced to a):

$$\varphi:\begin{cases}\bar{x}_1=\sqrt{2^{-1}}(x_1+x_2), & \bar{y}_1=\sqrt{2^{-1}}(y_1+y_2),\\ \bar{x}_2=\sqrt{2^{-1}}(x_1-x_2), & \bar{y}_2=\sqrt{2^{-1}}(y_1-y_2),\\ \bar{x}_i=x_i, & \bar{y}_i=y_i \quad (i\geqq 3),\\ \bar{z}=z. & \end{cases}$$

Case 4. The case where $\Sigma p_i\leqq 1$, that is, $\Sigma q_i\geqq 3$.

a) The case where $q_j\geqq 3$ for some j. We may assume that f is written as $f=zy_1^3h(x,\ y)$. Put $g=\int_0^{x_1} h(x,\ y)dx_1$, then $j^1(g)(0)=0$, and

$$3((zy_1g,\ y_1^3))-2((zg,\ y_1^4))=zy_1^3g_{x_1}=f.$$

b) The case where $q_j=2$ for some j. We may assume that f is written as $f=zy_1^2y_2h(x, y)$. By means of the above transformation φ, this case is reduced to a), because $3y_1^2y_2=\sqrt{2}\,\bar{y}_1^3-\sqrt{2}\,\bar{y}_2^3+y_2^3$

c) Assume that $q_1=q_2=1$. Then by means of φ, this case is reduced to b). Q.E.D.

We have a corollary of Proposition 2.6.

Corollary 2.8. *Let D be a derivation of $\mathfrak{A}_\theta(M)$. If X is a contact vector field on M such that $j^4(X)(p)=0$ for a point p of M, then $D(X)(p)=0$.*

Proof. This follows directly from Proposition 1.4. Q.E.D.

§3. Derivations of $\mathfrak{A}_\theta(n)$

3.1. Structure of $\mathfrak{A}_\theta(n)$. We consider the natural contact structure $\theta=dz-\sum_i y_idx_i$ in the Euclidean space $\mathbf{R}^{2n+1}$. In this section, we will study derivations of the Lie algebra $\mathfrak{A}_\theta(n)=\mathfrak{A}_\theta(\mathbf{R}^{2n+1})$ of contact vector fields on $\mathbf{R}^{2n+1}$. At first, we note the following.

Lemma 3.1. *A vector field $X=h^0\partial_z+\sum_{i=1}^n(h^i\partial_{x_i}+h^{i+n}\partial_{y_i})$ on $\mathbf{R}^{2n+1}$ is contact, if and only if it satisfies the following equalities:*

$$(\sharp)_1\quad h^0_{y_i}=\sum_{j=1}^n y_jh^j_{y_i}\qquad (1\leqq i\leqq n).$$

$$(\sharp)_2\quad y_i(h^0_z-\sum_{j=1}^n y_jh^j_z)=h^{i+n}-h^0_{x_i}+\sum_{j=1}^n y_jh^j_{x_i}\qquad (1\leqq i\leqq n).$$

The coefficient functions h^{i+n} $(1\leqq i\leqq n)$ are determined by $h^0, h^1,\dots, h^n$.

Proof. Since X is contact, $L_X\theta=g\theta$ for some function g. The assertion follows easily from this. Q.E.D.

Let $\mathfrak{B}=\mathfrak{B}_\theta(n)$ be the Lie subalgebra of $\mathfrak{A}=\mathfrak{A}_\theta(n)$ spanned by

$$\begin{cases} Z=\partial_z, \quad X_i=\partial_{x_i}, \quad Y_i=\partial_{y_i}+x_i\partial_z \qquad (1\leqq i\leqq n), \\ I=2z\partial_z+\sum_{i=1}^{n}(x_i\partial_{x_i}+y_i\partial_{y_i}). \end{cases}$$

There hold the following relations among them:

$$[Z, X_i]=[Z, Y_i]=[X_i, X_j]=[Y_i, Y_j]=0, \quad [X_i, Y_j]=\delta_{ij}Z,$$

$$[Z, I]=2Z, \quad [X_i, I]=X_i, \quad [Y_i, I]=Y_i \qquad (1\leqq i, j\leqq n),$$

where δ_{ij} is Kronecker's delta.

For an integer p, we define the subspace $\mathfrak{A}^p$ of $\mathfrak{A}$ as follows:

$$\mathfrak{A}^p=\{X\in\mathfrak{A}_0;\ [I, X]=pX\},$$

where $\mathfrak{A}_0$ is defined in §1.3. We have immediately that $[\mathfrak{A}^p, \mathfrak{A}^q]\subset\mathfrak{A}^{p+q}$, and that $\mathfrak{A}_0$ is an algebraic direct sum of $\mathfrak{A}^p$'s. We remark the following facts which will be applied later:

i) $\mathfrak{A}^p=\{0\} \qquad (p\leqq -3)$,

ii) $\mathfrak{A}^{-2}=\mathbf{R}\cdot Z$,

iii) $\mathfrak{A}^{-1}=\sum_{i=1}^{n}(\mathbf{R}\cdot X_i+\mathbf{R}\cdot Y_i)$.

3.2. Now we will solve the equation (E) for $(\mathfrak{A}_\theta(n), \mathfrak{B}_\theta(n))$.

Proposition 3.2. *Let D be a derivation of $\mathfrak{A}_\theta(n)$. Then there exists a unique vector field W in $\mathfrak{A}_\theta(n)$ such that*

$$\text{(E)} \qquad D(X)=[W, X] \quad \textit{for all} \quad X\in\mathfrak{B}_\theta(n).$$

The proof of this proposition will be given in §3.3. Here we deduce from this proposition the following theorem, a local theorem for contact case.

Theorem 3.3. *Let D be a derivation of $\mathfrak{A}_\theta(n)$. Then there exists a unique vector field W in $\mathfrak{A}_\theta(n)$ such that*

$$D(X)=[W, X] \quad \textit{for all} \quad X\in\mathfrak{A}_\theta(n).$$

In other words, any derivation of $\mathfrak{A}_\theta(n)$ *is inner.*

Proof. To prove this theorem, it is sufficient to show that if D is zero on the subalgebra $\mathfrak{B}_\theta(n)$, then D vanishes on the whole $\mathfrak{A}_\theta(n)$. Its proof is reduced to the next lemma by Proposition 1.3 and Corollary 2.8. Q.E.D.

Lemma 3.4. *If the derivation* D *of* $\mathfrak{A}=\mathfrak{A}_\theta(n)$ *is zero on* $\mathfrak{B}=\mathfrak{B}_\theta(n)$, *then* D *is zero on* $\mathfrak{A}_0$ *for* $\mathfrak{A}$.

Proof. Assume that $X\in\mathfrak{A}^p$, $p\geqq 0$, defined in §3.1. The proof is carried out by induction on p. Let h^i $(0\leqq i\leqq 2n)$ be functions on $\mathbf{R}^{2n+1}$ defined as

$$D(X)=h^0\partial_z+\sum_{i=1}^n(h^i\partial_{x_i}+h^{i+n}\partial_{y_i}).$$

Apply D to $[Z, X]\in\mathfrak{A}^{p-2}$ and $[X_i, X]\in\mathfrak{A}^{p-1}$ $(1\leqq i\leqq n)$. Then by the assumption of induction, $[Z, D(X)]=[X_i, D(X)]=0$, so that $h^i_z=h^i_{x_j}=0$ $(0\leqq i\leqq 2n,\ 1\leqq j\leqq n)$. Hence, by the equalities $(\sharp)_2$ in Lemma 3.1, we get that

$$h^{i+n}=0 \qquad (1\leqq i\leqq n).$$

Apply D to $[Y_i, X]\in\mathfrak{A}^{p-1}$, then

$$\begin{aligned}0=[Y_i, D(X)]&=[\partial_{y_i}+x_i\partial_z,\ h^0\partial_z+\sum_{j=1}^n h^j\partial_{x_j}]\\&=(h^0_{y_i}-h^i)\partial_z+\sum_{j=1}^n h^j_{y_i}\partial_{x_i'},\end{aligned}$$

so that $h^i=h^0_{y_i}$ and $h^j_{y_i}=0$ for $1\leqq i,\ j\leqq n$. Hence, by the equalities $(\sharp)_1$ in Lemma 3.1, we get that

$$h^i=h^0_{y_i}=\sum_{j=1}^n y_j h^j_{y_i}=0 \qquad (1\leqq i\leqq n),$$

and so h^0 is a constant.

Apply D to the both sides of $pX=[I, X]$, then

$$ph^0\partial_z=[I, D(X)]=[I, h^0\partial_z]=-2h^0\partial_z.$$

Since $p \geqq 0$ by assumption, we get $h^0=0$. Hence $D(X)=0$. Q.E.D.

3.3. Proof of Proposition 3.2. We consider the equation (E) for $(\mathfrak{A}_\theta(n), \mathfrak{B}_\theta(n))$. Let us construct the vector field W as a sum of W_1, W_2, W_3, $W_4 \in \mathfrak{A}_\theta(n)$ as follows:

a) $D(Z)=[W_1, Z]$;

b) $D(X_i)=[W_1+W_2, X_i]$, $[W_2, Z]=0$ $(1 \leqq i \leqq n)$;

c) $D(Y_i)=[W_1+W_2+W_3, Y_i]$, $[W_3, Z]=[W_3, X_i]=0$ $(1 \leqq i \leqq n)$;

d) $D(I)=[W_1+W_2+W_3+W_4, I]$, $[W_4, \mathfrak{A}^p]=0$ $(p \leqq -1)$,

where

$$Z=\partial_z,\ X_i=\partial_{x_i},\ Y_i=\partial_{y_i}+x_i\partial_z,\ I=2z\partial_z+\sum_{i=1}^n (x_i\partial_{x_i}+y_i\partial_{y_i}).$$

Then $D=\operatorname{ad} W$ on $\mathfrak{B}_\theta(n)$.

Step I. Construction of W_1. Define the functions h^i on $\mathbf{R}^{2n+1}$ by

$$D(Z)=h^0\partial_z+\sum_{i=1}^n (h^i\partial_{x_i}+h^{i+n}\partial_{y_i}).$$

Put the functions φ_1^i and define the vector field W_1 on $\mathbf{R}^{2n+1}$ as

$$\varphi_1^i=\int_0^z h^i dz \qquad (0 \leqq i \leqq n),$$

$$\varphi_1^{i+n}=\int_0^z h^{i+n}dz+y_i(h^0(0, x, y)-\sum_{j=1}^n y_jh^j(0, x, y)) \quad (1 \leqq i \leqq n);$$

$$W_1=-\varphi_1^0\partial_z-\sum_{i=1}^n (\varphi_1^i\partial_{x_i}+\varphi_1^{i+n}\partial_{y_i}).$$

Then W_1 satisfies a). Moreover,

Lemma 3.5. *W_1 is a contact vector field, or $W_1 \in \mathfrak{A}_\theta(n)$.*

Proof. Let us prove for W_1 the equalities $(\sharp)_1$ and $(\sharp)_2$ in Lemma 3.1.

$(\sharp)_1$. Put $\psi_i=\varphi^0_{1y_i}-\sum_{j=1}^n y_j\varphi^j_{1y_i}$. Then $\psi_i(0, x, y)=0$ and

$$\psi_{iz}=\varphi^0_{1y_iz}-\sum_{j=1}^n y_j\varphi^j_{1y_iz}=h^0_{y_i}-\sum_{j=1}^n y_jh^j_{y_i}=0$$

by the equalities $(\sharp)_1$ for $D(Z)$. Hence $\psi_i=0$ for $1\leqq i\leqq n$.

$(\sharp)_2$. Put $\chi_i=y_i(\sum_{j=1}^n y_j\varphi^j_{1z}-\varphi^0_{1z})-\varphi^0_{1x_i}+\varphi^{i+n}_1+\sum_{j=1}^n y_j\varphi^j_{1x_i}$. Then $\chi_i(0, x, y)=0$. Moreover taking into account $\varphi^j_{1z}=h^j$ $(0\leqq j\leqq 2n)$, we get

$$\chi_{iz}=y_i(\sum_{j=1}^n y_jh^j_z-h^0_z)-h^0_{x_i}+h^{i+n}+\sum_{j=1}^n y_jh^j_{x_i}=0$$

from the equalities $(\sharp)_2$ for $D(Z)$. Hence $\chi_i=0$ for $1\leqq i\leqq n$. Q.E.D.

Step II. Construction of W_2. Put $D_1=D-\mathrm{ad}\,W_1$, then $D_1(Z)=0$. Define the functions f^j_i on $\mathbf{R}^{2n+1}$ as

$$D_1(X_i)=f^0_i\partial_z+\sum_{j=1}^n(f^j_i\partial_{x_j}+f^{j+n}_i\partial_{y_j})\qquad(1\leqq i\leqq n).$$

Apply D_1 to $[Z, X_i]=0$ and $[X_i, X_k]=0$, then we have

$$f^0_{iz}\partial_z+\sum_{j=1}^n(f^j_{iz}\partial_{x_j}+f^{j+n}_{iz}\partial_{y_j})=0,$$

$$(f^0_{kx_i}-f^0_{ix_k})\partial_z+\sum_{j=1}^n\{(f^j_{kx_i}-f^j_{ix_k})\partial_{x_j}+(f^{j+n}_{kx_i}-f^{j+n}_{ix_k})\partial_{y_j}\}=0.$$

Hence

$$f^j_z=0,\quad f^j_{ix_k}=f^j_{kx_i}\qquad(0\leqq j\leqq 2n,\ 1\leqq i,\ k\leqq n).$$

Therefore we see easily that there exist functions φ^j_2 $(0\leqq j\leqq 2n)$ satisfying

$$\varphi^j_{2z}=0,\quad \varphi^j_{2x_i}=f^j_i\qquad(1\leqq i\leqq n).$$

We put

$$\varphi^j_2(0, 0, y)=0\qquad(0\leqq j\leqq n),$$

$$\varphi^{j+n}_2(0, 0, y)=f^0_j(0, 0, y)-\sum_{k=1}^n y_kf^k_j(0, 0, y)\qquad(1\leqq j\leqq n),$$

and

$$W_2 = -\varphi_2^0 \partial_z - \sum_{j=1}^n (\varphi_2^j \partial_{x_j} + \varphi_2^{j+n} \partial_{y_j}).$$

Then the vector field W_2 satisfies b). Furthermore,

Lemma 3.6. *W_2 is a contact vector field, or $W_2 \in \mathfrak{A}_\theta(n)$.*

Proof. Let us check $(\#)_1$ and $(\#)_2$ for W_2.

$(\#)_1$. Put $\psi_i = \varphi_{2y_i}^0 - \sum_{j=1}^n y_j \varphi_{2y_i}^j$ $(1 \leqq i \leqq n)$, then $\psi_i(0, 0, y) = 0$; $\psi_{iz} = 0$,

$$\psi_{ix_k} = \varphi_{2y_i x_k}^0 - \sum_{j=1}^n y_j \varphi_{2y_i x_k}^j = f_{ky_i}^0 - \sum_{j=1}^n y_j f_{ky_i}^j = 0$$

by the equalities $(\#)_1$ for $D_1(X_k)$. Hence $\psi_i = 0$ for $1 \leqq i \leqq n$.

$(\#)_2$. Put $\chi_i = \varphi_2^{i+n} - \varphi_{2x_i}^0 + \sum_{j=1}^n y_j \varphi_{2x_i}^j$. Then

$$\chi_i(0, 0, y) = 0; \quad \chi_{iz} = 0,$$

$$\chi_{ix_k} = f_k^{i+n} - f_{kx_i}^0 + \sum_{j=1}^n y_j f_{kx_i}^j = 0 \qquad (1 \leqq i,\ k \leqq n)$$

by the equalities $(\#)_2$ for $D_1(X_k)$ because $f_{kz}^j = 0$. Hence $\chi_i = 0$ for $1 \leqq i \leqq n$. Q.E.D.

Step III. Construction of W_3. Put $D_2 = D_1 - \operatorname{ad} W_2$, then $D_2(Z) = D_2(X_i) = 0$ for $1 \leqq i \leqq n$. Define the functions g_i^j $(0 \leqq j \leqq 2n)$ on $\mathbf{R}^{2n+1}$ as

$$D_2(Y_i) = g_i^0 \partial_z + \sum_{j=1}^n (g_i^j \partial_{x_j} + g_i^{j+n} \partial_{y_j}) \qquad (1 \leqq i \leqq n).$$

Apply D_2 to $[Z, Y_i] = 0$ and $[X_k, Y_i] = \delta_{ik} Z$, then

$$g_{iz}^j = g_{ix_k}^j = 0 \qquad (0 \leqq j \leqq n,\ 1 \leqq i,\ k \leqq n);$$

$$g_i^{j+n} = 0 \qquad (1 \leqq j \leqq n).$$

Apply D_2 to $[Y_i, Y_k] = 0$, then we have

$$(g_{ky_i}^0 + g_i^k - g_k^i - g_{iy_k}^0)\partial_z + \sum_{j=1}^n (g_{ky_i}^j - g_{iy_k}^j)\partial_{x_j} = 0.$$

Hence,

$$g^0_{ky_i}+g^k_i=g^i_k+g^0_{iy_k},$$

$$g^j_{ky_i}=g^j_{iy_k} \qquad (1\leqq i,\ j,\ k\leqq n).$$

By $(\sharp)_1$ for $D_2(Y_i)$, we get from the second equalities above that

$$g^0_{ky_i}=\sum_{j=1}^n y_j g^j_{ky_i}=\sum_{j=1}^n y_j g^j_{iy_k}=g^0_{iy_k},$$

and so

$$g^k_i=g^i_k \qquad (1\leqq i,\ k\leqq n).$$

By the above equalities, there are unique functions φ^j_3 $(1\leqq j\leqq 2n)$ such that

$$\varphi^j_{3z}=\varphi^j_{3x_i}=0, \quad \varphi^j_{3y_i}=g^j_i \qquad (1\leqq i\leqq n),$$

and

$$\varphi^j_3(0)=-g^0_j(0), \quad \varphi^{j+n}_3(0)=0 \qquad (1\leqq j\leqq n).$$

Finally there is a unique function φ^0_3 such that

$$\varphi^0_{3z}=\varphi^0_{3x_i}=0, \quad \varphi^0_{3y_i}=g^0_i+\varphi^i_3 \qquad (1\leqq i\leqq n),$$

and $\varphi^0_3(0)=0$. Put

$$W_3=-\varphi^0_3\partial_z-\sum_{j=1}^n\varphi^j_3\partial_{x_j},$$

then the vector field W_3 satisfies c). Moreover,

Lemma 3.7. *W_3 is a contact vector field, or $W_3\in\mathfrak{A}_\theta(n)$.*

Proof. W_3 satisfies trivially the equalities $(\sharp)_2$ in Lemma 3.1. Let us prove the equalities $(\sharp)_1$. Put

$$\psi_i=\varphi^0_{3y_i}-\sum_{j=1}^n y_j\varphi^j_{3y_i} \qquad (1\leqq i\leqq n),$$

then

$$\psi_i(0)=g^0_i(0)+\varphi^i_3(0)=0,$$

$$\psi_{iz}=\psi_{ix_k}=0 \qquad (1\leqq i,\ k\leqq n),$$

and by $(\#)_1$ for $D_2(Y_i)$, we get also

$$\psi_{iy_k}=g^0_{iy_k}+\varphi^i_{3y_k}-g^k_i-\sum_{j=1}^n y_j g^j_{iy_k}$$

$$=g^i_k-g^k_i=0\,.$$

Hence $\psi_i=0$ for $1\leqq i\leqq n$. Q.E.D.

Step IV. Construction of W_4. Put $D_3=D_2-\mathrm{ad}\,W_3$, then $D_3(\mathfrak{A}^p)=0$ for $p\leqq -1$. Apply D_3 to the both sides of the equalities

$$[Z,\,I]=2Z,\quad [X_i,\,I]=X_i,\quad [Y_i,\,I]=Y_i\quad (1\leqq i\leqq n),$$

then by the same arguments as in the proof of Lemma 3.4, we get

$$D_3(I)=a\partial_z \qquad \text{for some constant } a\,.$$

Put $W_4=2^{-1}a\partial_z$. Then W_4 is a contact vector field and satisfies d), or

$$[W_4,\,I]=D_3(I),\quad [W_4,\,\mathfrak{A}^p]=0 \qquad (p\leqq -1)\,.$$

Lemma 3.8. *$W_4=2^{-1}a\partial_z$ is a unique solution of the equations above.*

Proof. As in the proof of Lemma 3.4, we see from the fact $[W_4,\,\mathfrak{A}^p]=0\ (p\leqq -1)$ that $W_4\in\mathfrak{A}_\theta(n)$ must be a constant multiple of ∂_z. Put $W_4=c\partial_z$ for some constant c, then $D_3(I)=[W_4,\,I]=2c\partial_z$. Hence $a=2c$. Q.E.D.

The vector field $W=W_1+W_2+W_3+W_4$ is a required one, and the uniqueness of W is guaranteed by the lemma above. This completes the proof of Proposition 3.2.

§4. Volume Preserving Vector Fields

4.1. Lie Algebras $\mathfrak{A}_\tau(M)$ and $\mathfrak{A}'_\tau(M)$. Let M be a connected manifold of dimension n, and τ a volume element on M. Then we get immediately from the definitions of $\mathfrak{A}_\tau(M)$ and $\mathfrak{A}'_\tau(M)$,

$$[\mathfrak{A}'_\tau(M),\ \mathfrak{A}'_\tau(M)]\subset\mathfrak{A}_\tau(M),$$

and $\mathfrak{A}_\tau(M)$ is an ideal of codimension $\leqq 1$ in $\mathfrak{A}'_\tau(M)$. Moreover

Lemma 4.1. *$\mathfrak{A}_\tau(M)$ is of codimension 1 in $\mathfrak{A}'_\tau(M)$, if and only if the volume form τ is exact, that is, $\tau=d\sigma$ for some $(n-1)$-form σ on M.*

Proof. Let τ be exact, that is, $\tau=d\sigma$ for some $(n-1)$-form σ. Then the equality $i_W\tau=\sigma$ determines a vector field W by the non-degeneracy of τ. Hence,

$$L_W\tau=di_W\tau=d\sigma=\tau,$$

so that W lies in $\mathfrak{A}'_\tau(M)$, but not in $\mathfrak{A}_\tau(M)$.

Let $\mathfrak{A}_\tau(M)$ be of codimension 1 in $\mathfrak{A}'_\tau(M)$. Then there is a vector field X such that $L_X\tau=\tau$. Put $\sigma=i_X\tau$, then $\tau=d\sigma$. Q.E.D.

4.2. Properties of Volume Preserving Vector Fields. Let X be a volume preserving vector field on a manifold (M,τ). Then $i_X\tau$ is a closed $(n-1)$-form on M, and so the restriction $r_U(i_X\tau)$ is exact by Poincaré's lemma for a sufficiently small open subsets U of M, that is $r_U(i_X\tau)=d\alpha$ for some $(n-2)$-form α on U. In global, any $(n-2)$-form α on M uniquely determines the vector fields $X=X[\alpha]$ in $\mathfrak{A}_\tau(M)$ by the formula $i_X\tau=d\alpha$.

In a coordinate neighbourhood U with coordinates $(x_1,\dots,x_n)$ such that $\tau=dx_1\wedge\cdots\wedge dx_n$ in U, any $(n-2)$-form α is written as

$$\alpha=\sum_{i<j}f_{ij}\sigma_{ij}$$

where $\sigma_{ij}=dx_1\wedge\cdots\wedge\widehat{dx_i}\wedge\cdots\wedge\widehat{dx_j}\wedge\cdots\wedge dx_n$, and f_{ij} are functions on U for $1\leqq i<j\leqq n$. Then we have the following.

Lemma 4.2. *For any two functions f and g on U,*

$$[X[f\sigma_{ij}],\quad X[g\sigma_{ij}]]=(-1)^{i+j}X[\{f,g\}_{i,j}\sigma_{ij}]\qquad on\quad U.$$

where $\{\ ,\ \}_{i,j}$ *is the Poisson bracket in* x_i *and* x_j, *that is,*

$$\{f, g\}_{i,j}=f_{x_i}g_{x_j}-f_{x_j}g_{x_i} \qquad (1\leqq i<j\leqq n).$$

Proof. We have

$$X[f\sigma_{ij}]=(-1)^{i+j-1}f_{x_j}\partial_{x_i}+(-1)^{i+j}f_{x_i}\partial_{x_j},$$

hence,

$$\begin{aligned}[X[f\sigma_{ij}], X[g\sigma_{ij}]] &= [f_{x_j}\partial_{x_i}-f_{x_i}\partial_{x_j}, g_{x_j}\partial_{x_i}-g_{x_i}\partial_{x_j}] \\ &= -(\{f, g\}_{ij})_{x_j}\partial_{x_i}+(\{f, g\}_{i,j})_{x_i}\partial_{x_j} \\ &=(-1)^{i+j}X[\{f, g\}_{ij}\sigma_{ij}].\end{aligned}$$

Q.E.D.

4.3. Derivations of $\mathfrak{A}'_\tau(M)$.

Proposition 4.3. *Let* X *be a conformally volume preserving vector field on* (M, τ), *and* U *any open subset of* M. *Assume that* $[X, Y]=0$ *on* U *for all* $Y\in\mathfrak{A}_\tau(M)$ *with support contained in* U, *then* $X=0$ *on* U.

Proof. Let $p\in U$ and U' a coordinate neighbourhood of p in U with coordinates $(x_1,\dots, x_n)$ around p such that $\tau=dx_1\wedge\cdots\wedge dx_n$ in U'. Denote ∂_{x_i} by ∂_i $(1\leqq i\leqq n)$. Put $X=\sum_{i=1}^{n} f_i\partial_i$ for some functions f_i on U'. Since the vector fields $\partial_i\in\mathfrak{A}_\tau(U')$,

$$[\partial_i, X]=\sum_{i=1}^{n}\partial_i(f_j)\partial_j=0 \qquad (1\leqq i\leqq n) \quad \text{in} \quad U',$$

and so $\partial_i(f_j)=0$ for all i, j.

Since $x_i\partial_j\in\mathfrak{A}_\tau(U')$ $(i\neq j)$,

$$[X, x_i\partial_j]=f_i\partial_j=0 \qquad \text{in} \quad U',$$

hence all f_i are zero in U'. Therefore $X(p)=0$ for any $p\in U$.

Q.E.D.

Proposition 4.4. *Any derivation of* $\mathfrak{A}_\tau(M)$ *or* $\mathfrak{A}'_\tau(M)$ *is local.*

Proof. By the same arguments as in the proof of Proposition 2.4, the assertions for $\mathfrak{A}'_\tau(M)$ and $\mathfrak{A}_\tau(M)$ follow from Proposition 4.3.

Q.E.D.

Corollary 4.5. *Any derivation of $\mathfrak{A}_\tau(M)$ is localizable.*

Proof. This follows from Lemmata 1.1 and 1.2. Q.E.D.

4.4. Proposition 4.6. *Let X be a volume preserving vector field on M such that $j^2(X)(p)=0$ for some point p of M. Then there are a finite number of volume preserving vector fields $Z_1,\dots, Z_{2q}$ on M and a neighbourhood U of p such that*

$$X_{|U}=\sum_{i=1}^{q}[Z_i, Z_{i+q}]_{|U}$$

and

$$j^1(Z_i)(p)=0 \qquad (1\leqq i\leqq 2q).$$

Proof. Introduce a coordinate system $(x_1,\dots, x_n)$ around p such that $\tau=dx_1\wedge\cdots\wedge dx_n$. Then, by the arguments in §4.2, the assertion follows from the next proposition. Q.E.D.

Proposition 4.7. *Let α be an $(n-2)$-form on $\mathbf{R}^n$ such that $j^3(\alpha)(0)=0$, then there exist a finite number of $(n-2)$-forms $\beta_1,\dots, \beta_{2q}$ on $\mathbf{R}^n$ such that*

$$X[\alpha]=\sum_{i=1}^{q}[X[\beta_i], X[\beta_{i+q}]]$$

and

$$j^2(\beta_i)(0)=0 \qquad \textit{for} \quad 1\leqq i\leqq 2q.$$

Proof. Clearly it is enough to show the assertion for the case

$$\alpha=f(x_1,\dots, x_n)dx_3\wedge\cdots\wedge dx_n=f\sigma_{12}$$

with $j^3(f)(0)=0$. Such a function f can be written as a finite sum of functions of the following type:

$$f=x_1^{r_1}x_2^{r_2}\dots x_n^{r_n}h(x_1,\dots, x_n)$$

with $\sum_{i=1}^{n} r_i \geqq 4$.

Case 1. The case where $r_1 \geqq 2$ or $r_2 \geqq 2$. We may assume that f is written as $f=x_1^2 h(x_1,\dots, x_n)$. Put

$$g=3^{-1}\int_0^{x_2} h(x_1,\dots, x_n)dx_2,$$

then $j^2(g)(0)=0$, and $\{x_1^3, g\}_{1,2}=3x_1^2 g_{x_2}=f$, that is, by Lemma 4.2,

$$X[f\sigma_{12}]=-[X[x_1^3\sigma_{12}],\ X[g\sigma_{12}]].$$

Case 2. The case where r_1 and $r_2 \leqq 1$. Then $\sum_{i=3}^{n} r_i \geqq 2$. We may assume that f is written as $f=x_i x_j h(x_1,\dots, x_n)$ for some $i, j \geqq 3$. Put $g=\int_0^{x_2} h(x_1,\dots, x_n)dx_2$, then $j^2(g)(0)=0$, and $\{x_1x_ix_j, g\}_{12}=x_ix_jg_{x_2}=f$. Then by Lemma 4.2,

$$X[f\sigma_{12}]=-[X[x_1x_ix_j\sigma_{12}], X[g\sigma_{12}]]. \qquad \text{Q.E.D.}$$

We have a corollary of Proposition 4.6.

Corollary 4.8. *Let D be a derivation of $\mathfrak{A}_\tau(M)$. If X is a volume preserving vector field on M such that $j^2(X)(p)=0$ for a point p of M, then $D(X)(p)=0$.*

Proof. This follows directly from Proposition 1.4. Q.E.D.

§5. Derivations of $\mathfrak{A}_\tau(\mathbf{R}^n)$ and $\mathfrak{A}'_\tau(\mathbf{R}^n)$.

5.1. Structure of $\mathfrak{A}'_\tau(n)$. We consider the natural volume element $\tau=dx_1\wedge\cdots\wedge dx_n$ in the Euclidean space $\mathbf{R}^n$. In this section, we will study derivations of the Lie algebras $\mathfrak{A}_\tau(n)=\mathfrak{A}_\tau(\mathbf{R}^n)$ and $\mathfrak{A}'_\tau(n)=\mathfrak{A}'_\tau(\mathbf{R}^n)$ of volume preserving and conformally volume preserving vector fields on $\mathbf{R}^n$ respectively. At first, we note the following.

Lemma 5.1. *Let $X=\sum_{i=1}^{n} f_i\partial_i$ be a vector field on $\mathbf{R}^n$. Then X is* *volume preserving if and only if* $\sum_{i=1}^{n} \partial_i(f_i)=0$, *and is conformally volume preserving if and only if* $\sum_{i=1}^{n} \partial_i(f_i)=c$ *for some constant* c.

Proof. This follows from direct calculations. Q. E. D.

Let $\mathfrak{B}=\mathfrak{B}'_\tau(n)$ be the Lie subalgebra of $\mathfrak{A}=\mathfrak{A}'_\tau(n)$ spanned by

$$I=\sum_{i=1}^{n} x_i\partial_i, \quad X_i=\partial_i \qquad (1\leqq i\leqq n).$$

There hold the following relations among them:

$$[X_i, X_j]=0, \quad [X_i, I]=X_i \qquad (1\leqq i, j\leqq n).$$

Here we note that the vector field I is not volume preserving because $L_I\tau=n\tau$, and that

$$\mathfrak{A}'_\tau(n)=\mathfrak{A}_\tau(n)+\mathbf{R}\cdot I.$$

For an integer p, we define the subspace $\mathfrak{A}^p$ of $\mathfrak{A}$ as follows:

$$\mathfrak{A}^p=\{X\in\mathfrak{A}_0;\ [I, X]=pX\},$$

where $\mathfrak{A}_0$ is defined in § 1.3. We have immediately that $[\mathfrak{A}^p, \mathfrak{A}^q]\subset\mathfrak{A}^{p+q}$, and that $\mathfrak{A}_0$ is an algebraic direct sum of $\mathfrak{A}^p$'s. Moreover,

i) $\mathfrak{A}^p=\{0\} \qquad (p\leqq -2)$,

ii) $\mathfrak{A}^{-1}=\sum_{i=1}^{n}\mathbf{R}\cdot X_i$.

5.2. Relations between $\mathfrak{D}(\mathfrak{A}_\tau(n))$ and $\mathfrak{D}(\mathfrak{A}'_\tau(n))$. First we refer the following results of V. I. Arnold [1].

Lemma 5.2. $[\mathfrak{A}_\tau(n), \mathfrak{A}_\tau(n)]=\mathfrak{A}_\tau(n)$.

Note. This lemma can be also obtained by the analogous arguments as in the proof of Proposition 4.6.

Now, we have the following two lemmata.

Lemma 5.3. $[\mathfrak{A}'_\tau(n), \mathfrak{A}'_\tau(n)]=\mathfrak{A}_\tau(n)$.

Proof. This follows from the inclusion $[\mathfrak{A}'_\tau(n), \mathfrak{A}'_\tau(n)] \subset \mathfrak{A}_\tau(n)$ and the lemma above. Q.E.D.

Lemma 5.4. *Let D be a derivation of $\mathfrak{A}'_\tau(n)$, then $D(\mathfrak{A}_\tau(n)) \subset \mathfrak{A}_\tau(n)$.*

Proof. By Lemma 5.3, a vector field $X \in \mathfrak{A}_\tau(n)$ is written as

$$X = \sum_{i=1}^{q} [Y_i, Y_{i+q}]$$

by means of a finite number of $Y_1, \ldots, Y_{2q} \in \mathfrak{A}'_\tau(n)$. Then we have that

$$D(X) = \sum_{i=1}^{q} ([D(Y_i), Y_{i+q}] + [Y_i, D(Y_{i+q})])$$

is volume preserving, by Lemma 5.3. Q.E.D.

5.3. Now we will solve the equation (E) for $(\mathfrak{A}'_\tau(n), \mathfrak{B}'_\tau(n))$.

Proposition 5.5. *Let D be a derivation of $\mathfrak{A}'_\tau(n)$. Then there exists a unique vector field W in $\mathfrak{A}'_\tau(n)$ such that*

$$\text{(E)} \qquad D(X) = [W, X] \quad \text{for all} \quad X \in \mathfrak{B}'_\tau(n).$$

Proposition 5.6. *Let D be a derivation of $\mathfrak{A}_\tau(n)$. Then there exists a unique vector field W in $\mathfrak{A}'_\tau(n)$ such that*

$$\text{(E)} \qquad D(X) = [W, X] \quad \text{for all} \quad X \in \mathfrak{B}_\tau(n),$$

where $\mathfrak{B}_\tau(n) = \mathfrak{A}^{-1} + (\mathfrak{A}^0 \cap \mathfrak{A}_\tau(n))$ for $\mathfrak{A}^{-1}$ and $\mathfrak{A}^0$ defined in §5.1.

The proof of these two propositions will be given in §5.4. Here we deduce from these propositions the following theorems, local theorems for the volume preserving case.

Theorem 5.7. *Let D be a derivation of $\mathfrak{A}'_\tau(n)$. Then there exists a unique vector field W in $\mathfrak{A}'_\tau(n)$ such that $D(X) = [W, X]$ for all $X \in \mathfrak{A}'_\tau(n)$. In other words, any derivation of $\mathfrak{A}'_\tau(n)$ is inner.*

Theorem 5.8. *Let D be a derivation of $\mathfrak{A}_\tau(n)$. Then there exists a unique vector field W in $\mathfrak{A}'_\tau(n)$ such that*

$$D(X)=[W, X] \qquad \textit{for all} \quad X\in\mathfrak{A}_\tau(n).$$

In other words, the ideal of inner derivations of $\mathfrak{A}_\tau(n)$ *is of codimension* 1 *in the derivation algebra of* $\mathfrak{A}_\tau(n)$.

Proof of Theorem 5.7. It is sufficient to show that if D is zero on the subalgebra $\mathfrak{B}_\tau'(n)$, then D vanishes on the whole $\mathfrak{A}_\tau'(n)$. Its proof is reduced to the next lemma by Proposition 1.3 and Corollary 4.8. Q.E.D.

Lemma 5.9. *If a derivation* D *of* $\mathfrak{A}=\mathfrak{A}_\tau'(n)$ *is zero on* $\mathfrak{B}=\mathfrak{B}_\tau'(n)$, *then* D *is zero on* $\mathfrak{A}_0$ *for* $\mathfrak{A}$.

Proof. Assume that $X\in\mathfrak{A}^p$, $p\geqq 0$, where $\mathfrak{A}^p$ is defined in §5.1. The proof is carried out by induction on p. Define the functions f_i on $\mathbf{R}^n$ as

$$D(X)=\sum_{i=1}^n f_i\partial_i.$$

Apply D to $[X_i, X]\in\mathfrak{A}^{p-1}$ $(1\leqq i\leqq n)$, then we get

$$[X_i, D(X)]=\sum_{j=1}^n \partial_i(f_j)\partial_j=0.$$

Hence all f_i are constants, so that $D(X)\in\mathfrak{A}^{-1}$.

Apply D to the both sides of $pX=[I, X]$, then we get

$$pD(X)=[I, D(X)]=-D(X).$$

Since $p\geqq 0$ by assumption, $D(X)$ must be zero. Q.E.D.

Proof of Theorem 5.8. By Proposition 1.4 and Corollary 4.8, it is sufficient to show that if D is zero on the subalgebra $\mathfrak{B}_\tau(n)$, then D vanishes also on $\mathfrak{A}^1$ (defined in §5.1). Here note that $\mathfrak{A}^1$ consists of all volume preserving vector fields whose coefficients are homogeneous polynomials of degree 2.

As in the proof of Lemma 5.9, we get that $D(X)\in\mathfrak{A}^{-1}$ for $X\in\mathfrak{A}^1$. Moreover we see that $[D(X), Y]=D([X, Y])$ for all $Y\in\mathfrak{A}^0\cap\mathfrak{A}_\tau(n)$. By simple calculations, we get that $D(X)=0$ for all $X\in\mathfrak{A}^1$. Q.E.D.

5.4. Proof of Proposition 5.5. Let us consider the equation (E) for $(\mathfrak{A}'_\tau(n), \mathfrak{B}'_\tau(n))$. We construct the vector fields W_1 and $W_2 \in \mathfrak{A}'_\tau(n)$ as follows:

a) $D(X_i)=[W_1, X_i] \qquad (1 \leqq i \leqq n)$,

b) $D(I)=[W_1+W_2, I], \quad [W_2, X_i]=0 \qquad (1 \leqq i \leqq n)$,

where $X_i=\partial_i\,(1 \leqq i \leqq n)$ and $I=\sum_{i=1}^n x_i\partial_i$. Put $W=W_1+W_2$ then $D=\mathrm{ad}\,W$ on $\mathfrak{B}'_\tau(n)$.

Step I. Construction of W_1. Define the functions f_{ij} on $\mathbf{R}^n$ as

$$D(X_i)=\sum_{j=1}^n f_{ij}\partial_j \qquad (1 \leqq i \leqq n).$$

Apply D to the both sides of $[X_i, X_k]=0$, then we have

$$\sum_{j=1}^n (\partial_i(f_{kj})-\partial_k(f_{ij}))\partial_j=0 \qquad (1 \leqq i,\ k \leqq n),$$

and so

$$\partial_i(f_{kj})=\partial_k(f_{ij}) \qquad (1 \leqq i,\ k \leqq n).$$

Therefore there exist unique functions $\varphi_j\,(1 \leqq j \leqq n)$ such that

$$\partial_i(\varphi_j)=f_{ij} \qquad (1 \leqq i \leqq n)$$

and

$$\varphi_j(0)=0 \qquad (1 \leqq j \leqq n).$$

Put $W_1=-\sum_{i=1}^n \varphi_i\partial_i$, then the vector field W_1 satisfies a). Moreover,

Lemma 5.10. *W_1 is a conformally volume preserving vector field, or $W_1 \in \mathfrak{A}'_\tau(n)$.*

Proof. Since X_k is volume preserving, then by Lemma 5.4, $D(X_k)$ is volume preserving, that is,

$$\sum_{i=1}^{n} \partial_i(f_{ki})=0 \qquad (1\leqq k\leqq n).$$

Put $\psi=\sum_{i=1}^{n} \partial_i(\varphi_i)$, then we have

$$\partial_k(\psi)=\partial_k(\sum_{i=1}^{n} f_{ii})=\sum_{i=1}^{n} \partial_i(f_{ki})=0 \qquad (1\leqq k\leqq n),$$

hence ψ is a constant. Then by Lemma 5.1, W_1 is a conformally volume preserving vector field. Q.E.D.

Step II. Construction of W_2. Put $D'=D-\mathrm{ad}\,W_1$, then $D'(\mathfrak{A}^{-1})=0$. Define the functions g_i on $\mathbf{R}^n$ as

$$D'(I)=\sum_{i=1}^{n} g_i\partial_i.$$

Apply D' to the both sides of $[X_j, I]=X_j$, then we see as in the proof of Lemma 5.9 that all g_i are constants.

Put $W_2=\sum_{i=1}^{n} g_i\partial_i=\sum_{i=1}^{n} g_i(0)\partial_i$. Then W_2 is a volume preserving vector field and satisfies b), or

$$[W_2, I]=D'(I), \quad [W_2, X_i]=0 \qquad (1\leqq i\leqq n).$$

Lemma 5.11. *$W_2=\sum_i g_i\partial_i$ is a unique solution of the equations above.*

Proof. As in the proof of Lemma 5.9, we see from $[W_2, \mathfrak{A}^{-1}]=0$ that W_2 must be a vector field with constant coefficients. Put $W_2=\sum_{i=1}^{n} a_i\partial_i$, then

$$D'(I)=[W_2, I]=\sum_{i=1}^{n} a_i\partial_i.$$

Hence $a_i=g_i$ for $1\leqq i\leqq n$. Q.E.D.

The vector field $W=W_1+W_2$ is a required one, and the uniqueness of W is guaranteed by the lemma above. This completes the proof of Proposition 5.6.

Proof of Proposition 5.6. It is sufficient to construct uniquely the vector fields W_1, $W_2 \in \mathfrak{A}'_\tau(n)$ as follows:

a) $D(X)=[W_1, X]$ $\quad (X \in \mathfrak{A}^{-1})$,

b) $D(Y)=[W_1+W_2, Y]$, $[W_2, X]=0$ $\quad (Y \in \mathfrak{A}^0 \cap \mathfrak{A}_\tau(n))$.

The construction of W_1 exactly the same as in the proof of Proposition 5.5. And one can construct easily a unique W_2 by the similar way as in the hamiltonian case [3]. Q.E.D.

§6. Remarks on Derivations of $\mathfrak{A}'_\omega(M)$ and $\mathfrak{A}'_\omega(n)$

6.1. Hamiltonian Vector Fields. Let (M, ω) be a connected symplectic manifold. By the analogous arguments as in §4, we get the following propositions.

Lemma 6.1. *$\mathfrak{A}_\omega(M)$ is an ideal of codimension $\leqq 1$ in $\mathfrak{A}'_\omega(M)$. Moreover the codimension equals to one, if and only if the symplectic form ω is an exact 2-form.*

Proposition 6.2. *Any derivation of $\mathfrak{A}'_\omega(M)$ is local.*

Proposition 6.3. *Any derivation of $\mathfrak{A}_\omega(M)$ is localizable.*

Since Proposition 1 in [3] is nothing but the assertion (III′) for $\mathfrak{A}_\omega(M)$, we get by Proposition 1.4 the following

Proposition 6.4. *Let D be a derivation of $\mathfrak{A}_\omega(M)$. If X is a hamiltonian vector field on M such that $j^2(X)(p)=0$ at a point $p \in M$, then $D(X)(p)=0$.*

6.2. Derivations of $\mathfrak{A}_\omega(n)$ and $\mathfrak{A}'_\omega(n)$. By the similar method as for the volume preserving case, one can reproduce Theorem 5 in [3], a local theorem for the hamiltonian case. Let us sketch it here for completeness.

We consider the natural symplectic structure $\omega=\sum_i dx_i dx_{i+n}$ on the Euclidean space $\mathbf{R}^{2n}$, then we get the following two lemmata similarly as in §5.2.

Lemma 6.5 (cf. [1]).

$$[\mathfrak{A}'_\omega(n), \mathfrak{A}'_\omega(n)] = [\mathfrak{A}_\omega(n), \mathfrak{A}_\omega(n)] = \mathfrak{A}_\omega(n) = \mathfrak{A}_\omega(\mathbf{R}^{2n}).$$

Lemma 6.6. *Let D be a derivation of $\mathfrak{A}'_\omega(n) = \mathfrak{A}'_\omega(\mathbf{R}^{2n})$. Then $D(\mathfrak{A}_\omega(n)) \subset \mathfrak{A}_\omega(n)$.*

Let $\mathfrak{B} = \mathfrak{B}'_\omega(n)$ be the Lie subalgebra of $\mathfrak{A} = \mathfrak{A}'_\omega(n)$ spanned by

$$I = \sum_{i=1}^{2n} x_i \partial_i, \quad X_i = \partial_i \qquad (1 \leqq i \leqq 2n).$$

Note that $L_I \omega = 2\omega$, then we get

$$\mathfrak{A}'_\omega(n) = \mathfrak{A}_\omega(n) + \mathbf{R} \cdot I.$$

For an integer p, the subspace $\mathfrak{A}^p$ of $\mathfrak{A}$ is defined as

$$\mathfrak{A}^p = \{X \in \mathfrak{A}_0;\ [I, X] = pX\}.$$

We can solve the equation (E) for $(\mathfrak{A}'_\omega(n), \mathfrak{B}'_\omega(n))$.

Proposition 6.7. *Let D be a derivation of $\mathfrak{A}'_\omega(n)$. Then there exists a unique conformally hamiltonian vector field W on $\mathbf{R}^{2n}$ such that*

$$\text{(E)} \qquad D(X) = [W, X] \quad \textit{for all} \quad X \in \mathfrak{B}'_\omega(n).$$

Outline of Proof. The proof is almost the same as the proof of Proposition 5.5. The vector field W is determined by the values of D at X_i $(1 \leqq i \leqq 2n)$ up to constant vector fields (Step I). The value $D(I)$ determines the constant terms of W (Step II). We see similarly as Lemma 5.10 that $W' = W - W_1$ is hamiltonian, where W_1 is the linear term of W, $\mathfrak{A}^0$-component of W. Applying the derivation $D - \mathrm{ad}\, W'$ to $[X_i, \mathfrak{A}^0] \subset \mathfrak{A}^{-1}$ $(1 \leqq i \leqq 2n)$, we see that $D - \mathrm{ad}\, W' = \mathrm{ad}\, W_1$ on $\mathfrak{A}^p$ $(p \leqq 0)$ and that W_1 is conformally hamiltonian.

We get from Propositions 6.4 and 6.7 the following theorem analogously as Theorems 5.7 and 5.8.

Theorem 6.8 (Theorem 5 in [3]). *Let D be a derivation of $\mathfrak{A}_\omega(n)$ or $\mathfrak{A}'_\omega(n)$. Then there exists a unique conformally hamiltonian* *vector field* $W \in \mathfrak{A}'_{\omega}(n)$ *on* $\mathbf{R}^{2n}$ *such that* $D = \mathrm{ad}\, W$.

6.3. The results on the derivations of $\mathfrak{A}(n) = \mathfrak{A}(\mathbf{R}^n)$ in the paper [5] of F. Takens can be obtained more simply in this direction. Let $\mathfrak{B} = \mathfrak{B}(n)$ be the Lie subalgebra of $\mathfrak{A} = \mathfrak{A}(n)$ spanned by

$$I = \sum_{i=1}^{n} x_i \partial_i, \quad X_i = \partial_i \qquad (1 \leqq i \leqq n).$$

For an integer p, define the subspace $\mathfrak{A}^p$ of $\mathfrak{A}(n)$ as

$$\mathfrak{A}^p = \{X \in \mathfrak{A}_0;\ [I, X] = pX\}.$$

Then we get

Theorem 6.9 (Lemma 4 in [5]). *Let* D *be a derivation of* $\mathfrak{A}(n)$. *Then there exists a unique vector field* W *on* $\mathbf{R}^n$ *such that* $D = \mathrm{ad}\, W$ *on* $\mathfrak{A}(n)$.

Key of Proof. The vector field W is determined by the values $D(X_i)$ $(1 \leqq i \leqq n)$ and $D(I)$.

§7. The Cohomology $H^1(\mathfrak{A};\ \mathfrak{A})$

7.1. The Main Theorem for Flat Cases. The following main thecrem for flat cases is obtained immediately from Theorems 3.3, 5.7, 5.8, 6.8 and 6.9 for respective Lie algebras of classical type.

Theorem 7.1. a) *Let* $\mathfrak{A} = \mathfrak{A}(\mathbf{R}^n)$, $\mathfrak{A}'_{\tau}(\mathbf{R}^n)$, $\mathfrak{A}'_{\omega}(\mathbf{R}^{2n})$ *or* $\mathfrak{A}_{\theta}(\mathbf{R}^{2n+1})$. *Then*

$$H^1(\mathfrak{A};\ \mathfrak{A}) = 0.$$

b) *Let* $\mathfrak{A} = \mathfrak{A}_{\tau}(\mathbf{R}^n)$ *or* $\mathfrak{A}_{\omega}(\mathbf{R}^{2n})$. *Then*

$$H^1(\mathfrak{A};\ \mathfrak{A}) \cong \mathbf{R}.$$

Here

$$\tau = dx_1 \dots dx_n,\ \omega = \sum_{i=1}^{n} dx_i dx_{i+n},\ \theta = dx_0 - \sum_{i=1}^{n} x_{i+n} dx_i.$$

7.2. Main Theorem. a) *Let M be a smooth manifold with a volume element τ, a symplectic structure ω or a contact structure θ, and let $\mathfrak{A}$ be one of $\mathfrak{A}(M)$, $\mathfrak{A}'_\tau(M)$, $\mathfrak{A}'_\omega(M)$ and $\mathfrak{A}_\theta(M)$. Then*

$$H^1(\mathfrak{A};\mathfrak{A})=0.$$

b) *Let M be a connected smooth manifold with a volume element τ or a symplectic structure ω, and $\mathfrak{A}=\mathfrak{A}_\tau(M)$ or $\mathfrak{A}_\omega(M)$ respectively. Then*

$$H^1(\mathfrak{A};\mathfrak{A})\cong \mathbf{R} \qquad \textit{or} \quad 0.$$

Moreover, $H^1(\mathfrak{A};\mathfrak{A})\cong\mathbf{R}$ if and only if τ or ω is an exact form on M respectively.

7.3. Proof for $\mathfrak{A}(M)$ and $\mathfrak{A}_\theta(M)$. Let us prove that any derivation D of $\mathfrak{A}$ is inner. Take an atlas $\{U_i, \varphi_i\}_{i\in I}$ such that each U_i are connected and simply connected. Since D is localizable, the derivation D_{U_i} of $\mathfrak{A}_{U_i}$ can be defined for all $i\in I$ in such a way that $r_{U_i}\circ D=D_{U_i}\circ r_{U_i}$. Then by Theorems 3.3 and 6.9 in respective cases, there exists for any $i\in I$ a unique vector field $W_i\in\mathfrak{A}_{U_i}$ such that $D_{U_i}=\mathrm{ad}\,W_i$ on $\mathfrak{A}_{U_i}$. Since $D_{U_i}\circ r_{U_i\cap U_j}=D_{U_j}\circ r_{U_i\cap U_j}$, we get $r_{U_i\cap U_j}(W_i)=r_{U_i\cap U_j}(W_j)$ by the uniqueness of W_U. Hence there exists a vector field $W\in\mathfrak{A}$ such that $r_{U_i}(W)=W_i$ for all $i\in I$ and that $D=\mathrm{ad}\,W$ on $\mathfrak{A}$. Q.E.D.

7.4. Proof for $\mathfrak{A}_\tau(M)$ and $\mathfrak{A}_\omega(M)$. Here we denote $\mathfrak{A}_\tau(M)$ or $\mathfrak{A}_\omega(M)$ by $\mathfrak{A}$, and $\mathfrak{A}'_\tau(M)$ or $\mathfrak{A}'_\omega(M)$ by $\mathfrak{A}'$ respectively.

Lemma 7.2. *For any $X\in\mathfrak{A}'$, $\mathrm{ad}\,X$ is a derivation of $\mathfrak{A}$.*

Proof. Let σ be τ or ω, then

$$L_{[X,Y]}\sigma=L_XL_Y\sigma-L_YL_X\sigma=0 \qquad (Y\in\mathfrak{A}). \qquad \text{Q.E.D.}$$

Let D be a derivation of $\mathfrak{A}$. Since D is localizable, for any open subset U of M, the derivation D_U of $\mathfrak{A}_U$ can be defined in such a way that $r_U\circ D=D_U\circ r_U$. Then by Theorems 5.8 and 6.8 in respective cases, we get a unique vector field W_U of $\mathfrak{A}'_U$ such that $D_U=\mathrm{ad}\,W_U$ on $\mathfrak{A}_U$ for any sufficiently small U. By the arguments in §7.3, there is a vector field $W \in \mathfrak{A}'$ such that $r_U(W) = W_U$ and that $D = \operatorname{ad} W$ on $\mathfrak{A}$. Hence by Lemma 7.2, we get the isomorphism $D(\mathfrak{A}) \cong \mathfrak{A}'$. Therefore the assertion follows from Lemmata 4.1 and 6.1 in respective cases.

Q.E.D.

7.5. Proof for $\mathfrak{A}'_\tau(M)$ and $\mathfrak{A}'_\omega(M)$. Here we use the notations $\mathfrak{A}$ and $\mathfrak{A}'$ as in §7.4. Let D' be a derivation of $\mathfrak{A}'$, then $D = D'_{|\mathfrak{A}}$ is a derivation of $\mathfrak{A}$ with values in $\mathfrak{A}'$. Since D is localizable, for any open subset U of M, the derivation D_U of $\mathfrak{A}_U$ with values in $\mathfrak{A}'_U$ can be defined in such a way that $r_U \circ D = D_U \circ r_U$, as in the proof of Proposition 1.2. If U is sufficiently small, $D_U(\mathfrak{A}_U) \subset \mathfrak{A}_U$ by the same arguments as in the proof of Lemma 5.4. Then by Theorems 5.8 and 6.8 in respective cases, we get a unique vector field $W_U \in \mathfrak{A}'_U$ such that $D_U = \operatorname{ad} W_U$ on $\mathfrak{A}_U$. By the arguments in §7.3, there is a vector field $W \in \mathfrak{A}'$ such that $r_U(W) = W_U$ and that $D = \operatorname{ad} W$ on $\mathfrak{A}$.

For any $Y \in \mathfrak{A}'$ and all $X \in \mathfrak{A}$, we get

$$\begin{aligned} [D'(Y), X] &= D([Y, X]) - [Y, D(X)] \\ &= [W, [Y, X]] - [Y, [W, X]] \\ &= [[W, Y], X]. \end{aligned}$$

By Proposition 4.3 and the similar proposition for the hamiltonian case, we see

$$D'(Y) = [W, Y] \qquad (Y \in \mathfrak{A}').$$

Thus any derivation D' is inner. Q.E.D.

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