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On a Mixed Problem for d'Alembertian with a Mixed Boundary Condition —An Example of Moving Boundary[1)]

By

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§1. Introduction

Let Ω be a domain in $\mathbf{R}^n$ with the smooth boundary $\partial\Omega$.

We consider the following mixed problem

$$(1.1)\quad \begin{cases} \Box u(x,t)=\left(\dfrac{\partial^2}{\partial t^2}-\displaystyle\sum_{j=1}^{n}\dfrac{\partial^2}{\partial x_j^2}\right)u(x,t)=f(x,t) & \text{in } Q=\Omega\times(0,T)\,, \\ u(x,0)=u_0(x),\ \dfrac{\partial u}{\partial t}(x,0)=u_1(x) & \text{on } \Omega, \end{cases}$$

with a mixed boundary condition

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$$(1.2)\qquad \alpha(\tilde{x}, t)\frac{\partial u}{\partial \boldsymbol{\nu}}(\tilde{x}, t)+(1-\alpha(\tilde{x}, t))u(\tilde{x}, t)=0 \qquad \text{on } \Sigma=\partial\Omega\times[0, T],$$

where $\boldsymbol{\nu}=(\nu_1, \nu_2,\dots, \nu_n)$ is the unit exterior normal of $\partial\Omega$ and T is an arbitrary fixed positive number. (We represent the points in Ω and $\partial\Omega$ by x and $\tilde{x}$, respectively.)

We suppose always that $0\leqq\alpha(\tilde{x}, t)\leqq 1$ for $(\tilde{x}, t)\in\Sigma$ and $\Gamma_D=\{(\tilde{x}, t)\in\Sigma;\ \alpha(\tilde{x}, t)=0\}^0\neq\phi$ and $\Gamma_N=\{(\tilde{x}, t)\in\Sigma;\ \alpha(\tilde{x}, t)\neq 0\}\neq\phi$ where A^0 stands for the interior of the set A.

Our problems to the equation (1.1) with (1.2) read as follows:

(I) Under what condition on $\alpha(\tilde{x}, t)$, can we prove the existence of a solution (1.1) with (1.2) for a given data $\{u_0(x), u_1(x), f(x, t)\}$?

(II) How about the 'well-posedness' of this problem? i.e. regularity theorem, existence of the dependence domain and $\mathscr{E}^\infty$-well posedness.

(III) If $f(x, t)\equiv 0$ and $\alpha(\tilde{x}, t)$ converges to $\alpha^\pm(\tilde{x})$ in a suitable sense as t tends to $\pm\infty$, then does there exist functions $u^\pm(x, t)$, solution of $\Box u^\pm(x, t)=0$ in $\Omega\times(-\infty, \infty)$ with the boundary condition

$$\alpha^\pm(\tilde{x})\frac{\partial u^\pm}{\partial \boldsymbol{\nu}}(x, t)+(1-\alpha^\pm(\tilde{x}))u^\pm(\tilde{x}, t)=0 \qquad \text{on } \partial\Omega\times(-\infty, \infty),$$

such that the solution $u(x, t)$ of (1.1) with (1.2) (in $\Omega\times(-\infty, \infty)$ and $\partial\Omega\times(-\infty, \infty)$, respectively) converges to $u^\pm(x, t)$ in the 'energy' norm as t tends to $\pm\infty$?

We give affirmative answers to the problems (I) and (II) under the assumptions below:

Assumption (a). $\alpha(\tilde{x}, t)\in C^\infty(\Sigma)$ and $\partial\Gamma_D$(=the boundary of Γ_D) forms a submanifold of codimension 1 in Σ.

Assumption (b). For each point $(\tilde{x}^0, t^0)\in\partial\Gamma_D$, there exist a neighborhood $V_{(\tilde{x}^0,t^0)}$ and a transformation $\Phi_{(\tilde{x}^0,t^0)}$ *of the class* (E), such that the Jacobian of $\Phi_{(\tilde{x}^0,t^0)}$ does not vanish on $V_{(\tilde{x}^0,t^0)}$ and that the function $\tilde{\alpha}(\tilde{y}, s)\equiv\alpha(\Phi^{-1}_{(\tilde{x}^0,t^0)}(\tilde{y}, s))$ is independent of s for $(\tilde{y}, s)\in\Phi_{(\tilde{x}^0,t^0)}(V_{(\tilde{x}^0,t^0)}\cap\Sigma)$.

Remark. The assumption (b) implies that $\partial\Gamma_D$ is '*time-like*', i.e. the movement of $\partial\Gamma_D$ with respect to t is not so rapid and 'of the class (E)' means that the hyperbolicity of the transformed operator by $\Phi_{(\tilde{x}^0,t^0)}$ is preserved. Precise definitions will be given in §2 and there, we will consider some examples of $\alpha(\tilde{x}, t)$ satisfying above assumptions.

Theorem A. *Under the assumptions* (*a*) *and* (*b*), *the phenomenon governed by* (1.1) *and* (1.2) *has the same dependence domain as that of the Cauchy problem for* $\square$ *in the whole space* $\mathbf{R}^n$.

Theorem B. *We suppose* (*a*) *and* (*b*). *Let the data* $\{u_0(x), u_1(x), f(x, t)\}$ *of* (1.1) *belong to the space* $C^\infty(\bar{\Omega})\times C^\infty(\bar{\Omega})\times C^\infty(\bar{Q})$. *If they* are *compatible*[2] *of order* ∞ *at* $t=0$, *then there exists a solution* $u(x, t)\in C^\infty(\bar{Q})$ *of* (1.1) *with* (1.2).

Theorem C. *If the assumptions* (*a*) *and* (*b*) *are satisfied, then the problem* (1.1) *with* (1.2) *is well posed in* $\mathscr{E}^\infty$ (*or* $\mathscr{E}^\infty$*-well posed*). *That is, for any compact set* K *in* $\bar{Q}$, *an integer* $m\geqq 0$ *and an arbitrary small number* $\varepsilon>0$, *there exist a compact set* $\tilde{K}$, *an integer* $N>0$ *and a constant* $\delta>0$ *such that if the data satisfy*

$$|u_0|_{\mathscr{E}^{N+2}(\tilde{K}_0)}+|u_1|_{\mathscr{E}^{N+1}(\tilde{K}_0)}+|f|_{\mathscr{E}^{N+1}(\tilde{K})}\leqq\delta$$

then we have

$$|u|_{\mathscr{E}^m(K)}\leqq\varepsilon \quad \textit{where} \quad \tilde{K}_0=\tilde{K}\cap(\bar{\Omega}\times\{0\}).$$

(*δ depends only on ε and $\tilde{K}\cap\Sigma$.*)

Summary. In §2, we give the definitions of 'time-like' and 'of the class (E)' and some examples. And we show that the assumption (b) asserts that at least *locally*, the problem (1.1) with (1.2) is reduced to the following problem.

$$(1.3)\quad \begin{cases} \left[\dfrac{\partial^2}{\partial s^2}+a_1(y, s; D)\dfrac{\partial}{\partial s}+a_2(y, s; D)\right]\tilde{u}(y, s)=\tilde{f}(y, s) \\ \qquad\qquad\qquad\qquad \text{in } \Phi_{(\tilde{x}^0,t^0)}(V_{(\tilde{x}^0,t^0)}), \\ \tilde{u}(y, 0)=\tilde{u}_0(y),\ \tilde{u}_s(y, 0)=\tilde{u}_1(y) \\ \qquad\qquad\qquad\qquad \text{on } \Phi_{(\tilde{x}^0,t^0)}(V_{(\tilde{x}^0,t^0)}\cap(\Omega\times\{t^0\})), \\ \left[\tilde{\alpha}(\tilde{y})\dfrac{\partial}{\partial \boldsymbol{n}_s}+(1-\tilde{\alpha}(\tilde{y}))\right]\tilde{u}(\tilde{y}, s)=0 \quad \text{on } \Phi_{(\tilde{x}^0,t^0)}(V_{(\tilde{x}^0,t^0)}\cap\Sigma), \end{cases}$$

2) The definition of compatibility will be given in §7. See also §4 and §6.

where $a_2(y, s; D)$ is an elliptic operator and $\boldsymbol{n}_s$ is the conormal vector associated with $a_2(y, s; D)$.

By extending the data suitably, we may consider the problem above in the domain $\omega\times(0, \tilde{T})$, where ω is a domain in $\mathbf{R}^n$ with the smooth boundary. That is, we consider the following problem in §3∼§6.

$$\text{(1.4)}\quad \begin{cases} \left[\dfrac{\partial^2}{\partial s^2}+a_1(y, s; D)\dfrac{\partial}{\partial s}+a_2(y, s; D)\right]v(y, s)=g(y,s) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{in} \quad \omega\times(0, \tilde{T}), \\ v(y, 0)=v_0(y), \quad v_s(y, 0)=v_1(y) \qquad \text{on} \quad \omega, \end{cases}$$

$$\text{(1.5)}\quad \left[\tilde{\alpha}(\tilde{y})\left(\frac{\partial}{\partial \boldsymbol{n}_s}-\sigma_1(\tilde{y},s)\frac{\partial}{\partial s}+\sigma_2(\tilde{y}, s)\right)+(1-\tilde{\alpha}(\tilde{y}))\right]v(\tilde{y}, s)=0.$$
$$\text{on } \partial\omega\times[0, \tilde{T}].$$

As usual, we want to treat this problem (1.4) in the form of an evolution equation.

$$\text{(1.6)}\quad \frac{d}{dt}\begin{pmatrix}u(t)\\ v(t)\end{pmatrix}=\mathscr{A}(t)\begin{pmatrix}u(t)\\ v(t)\end{pmatrix} \text{ where } \mathscr{A}(t)=\begin{pmatrix}0 & 1\\ -a_2(y,s;D) & -a_1(y, s; D)\end{pmatrix}.$$

With the boundary condition (1.5), the operator $\mathscr{A}(t)$ considered in $H^1(\omega)\times L^2(\omega)$ has the domain $D(\mathscr{A}(t))$, changing with t. For the time being, it seems not so easy to apply the work of T. Kato [16], [17] to our problem (1.6). So, we follow the idea of M. Ikawa [7], [8].

In §§3 and 4, we consider the problem (1.4) with (1.5) under the condition that there exists a constant $\varepsilon>0$ such that

$$\text{(1.7)}_\varepsilon\qquad \sigma_1(\tilde{y}, s)\leqq \sum_{j=1}^{n} h_j(\tilde{y}, s)\tilde{\nu}_j-\varepsilon \qquad \text{on} \quad \partial\omega\times[0, \tilde{T}],$$

where $\tilde{\boldsymbol{\nu}}=(\tilde{\nu}_1, \tilde{\nu}_2,\dots, \tilde{\nu}_n)$ is the unit exterior normal of ω at $\tilde{y}$ and $h_j(y, s)$ are given by $a_1(y, s; D)=2\sum_{j=1}^{n} h_j(y, s)\dfrac{\partial}{\partial y_j}+\cdots$.

In §§5 and 6, we consider the problem (1.4) with (1.5) assuming that

$$\text{(1.8)}\qquad \sigma_1(\tilde{y}, s)\leqq \sum_{j=1}^{n} h_j(\tilde{y}, s)\tilde{\nu}_j \qquad \text{on} \quad \partial\omega\times[0, \tilde{T}],$$

by approximating the solution satisfying the boundary condition with

$(1.7)_\varepsilon$. In these sections, we follow exactly the idea of [7], [8]. But, we must pay careful attentions near the point where $\tilde{\alpha}(\tilde{y})$ changes from zero, and especially, we use the estimate of A. Melin [19] in the form of A. Kaji [15], [15]', K. Taira [24].

In §7, we prove Theorems A, B and C by using the devise of A. Inoue [13].

In the appendix, we give an example of the system of elliptic operators with the boundary condition changes its order on the boundary.

Remark. The mixed problem (1.1) with the boundary condition of the Neumann type was studied fully by R. Agemi [1], M. Ikawa [8] and S. Miyatake [21]. Moreover, if the oblique boundary condition is given, then there exist interesting papers of M. Ikawa [9], [10], [11]. But, the boundary condition of the type (1.2) is not studied quite recently. In A. Inoue [14], he suggests that this problem will be useful to consider the problem with the discontinuous boundary condition. This is the main motivation of studying this problem. And concerning this, we will study in the forth-coming paper.

Finally, we express our thanks to the referee for his kind advices.

§2. Time-like Hypersurface and Change of Variables

Definition 2.1. Let Γ be a submanifold of codimension 1 of the lateral boundary $\Sigma(=\partial\Omega\times[0, T])$ such that $\Gamma(t)=\Gamma\cap P(t)$ is a submanifold of $\partial\Omega$ and $\Gamma(t)$ are diffeomorphic to each other where $P(t^0)=\{(x, t)\in\mathbf{R}^{n+1};\, t=t^0\}$. We say that Γ is *time-like* if it satisfies the following: For any $t^0\in[0, T]$, there exists a positive number ε_1 depending on t^0 such that

$$(2.1)\qquad \bigcup_{|t^0-\tau|\leqq\varepsilon}\Gamma(\tau)\subset\bigcup_{\substack{x^0\in\Gamma(t^0)\\ |t^0-\tau|\leqq\varepsilon}}\{(x, t)\in\mathbf{R}^{n+1};\, |x-x^0|^2\leqq|\tau-t^0|^2\}\cap\Sigma$$

for any ε, $0\leqq\varepsilon\leqq\varepsilon_1$ where we put $\Gamma(t)=\phi$ for $t<0$ and $t>T$.

The condition (2.1) means that the movement of $\Gamma(t)$ near $t=t^0$ is limited by the 'wave front' set starting from $\Gamma(t^0)$.

For the future use, we represent the condition (2.1) in geometrical terms.

Let $\nu(x^0)$ be the unit exterior normal of $\partial\Omega$ at $x^0 \in \partial\Omega$. As $\Gamma(t^0)$ is codimension 2 in $\mathbf{R}^n$, there exists another vector $\boldsymbol{n}(x^0, t^0)$ such that $\nu(x^0)$ and $\boldsymbol{n}(x^0, t^0)$ are orthogonal to each other and they are orthogonal to $\Gamma(t^0)$ at $x^0 \in \partial\Omega$, i.e. the space spanned by $\nu(x^0)$ and $\boldsymbol{n}(x^0, t^0)$ forms the normal bundle of $\Gamma(t^0)$ at x^0. Then, we have the following.

Remark 2.2. Let us consider the trajectory of the point of the intersection of $\Gamma(t)$ with the normal bundle of $\Gamma(t^0)$ at x^0. Then the condition (2.1) implies that (2.2) the speed of the trajectory at $t=t^0$ is smaller than 1 for each $x^0 \in \Gamma(t^0)$. (The propagation speed of $\Box$ equals to 1).

Moreover, if $\partial\Omega$ is compact, then (2.2) implies (2.1).

Let $(\tilde{x}^0, t^0)$ be an arbitrary point of Γ. We may suppose that $(\tilde{x}^0, t^0)=(0, 0)$ without loss of generality because $\Box$ is invariant with respect to the translation of (x, t)-axis. By the rotation of the x-axis, we may take the x_n-axe as the direction of the interior normal of $\partial\Omega$ at $x^0=0$. Other axis $x'=(x_1, x_2,\dots, x_{n-1})$ give the coordinates tangent to $\partial\Omega$ at $\tilde{x}^0=0$. So, in some neighborhood of $x^0=0$, $\partial\Omega$ is represented by $x_n=f(x')$ with a C^∞-function $f(x')$ satisfying $f(0)=0$ and $f_{x_j}(0)=0$ for $j=1, 2,\dots, n-1$. Rotating the x'-axis if necessary, we may suppose also that in some neighborhood of $(\tilde{x}^0, t^0)=(0, 0)$, say $V_{(0,0)}$, Γ is represented by $x_{n-1}=g(x'', t)$ and $x_n=f(x'', x_{n-1})$ with another C^∞-function $g(x'', t)$ satisfying $g(0, 0)=0$ and $g_{x_j}(0, 0)=0$ for $j=1, 2,\dots, n-2$ where $x''=(x_1, x_2,\dots, x_{n-2})$.

Now, we calculate concretely the condition (2.2).

Lemma 2.3. *The condition* (2.2) *implies the inequality*

$$|g_t(0, 0)|<1. \tag{2.3}$$

Proof. At $\tilde{x}^0=0$, we have $\nu(0)=(0, 0,\dots, 0, 1)$ and $\boldsymbol{n}(0; 0)=(0, 0,\dots, 1, 0)$. The trajectory of $\Gamma(t)$ near $t^0=0$ at $\tilde{x}^0=0$ on the space spanned by the vectors $\nu(0)$ and $\boldsymbol{n}(0; 0)$ is represented by $(0,\dots, 0, g(0, t), f(0, g(0, t)))$. So the speed of this trajectory at $t^0=0$ is given by $\sqrt{(1+f_{x_{n-1}}^2(0,g(0, 0)))g_t^2(0, 0)}$. By the choice of the x-axis and the condition (2.2),we have (2.3).

Q.E.D.

Let us consider a level preserving transformation Φ, $(y, s)=\Phi(x, t)$. More precisely, Φ is given by $y_j=\phi_j(x, t), j=1, 2,\dots, n$ and $s=t$. By this transformation Φ, $\square$ is transformed to

(2.4) $L=L(y, s: D_y, D_s)$

$$=\frac{\partial^2}{\partial s^2}+2\sum_{j=1}^{n}\frac{\partial\phi_j}{\partial t}\frac{\partial^2}{\partial y_j\partial s}+\sum_{i,j=1}^{n}\left(\frac{\partial\phi_i}{\partial t}\frac{\partial\phi_j}{\partial t}-\sum_{k=1}^{n}\frac{\partial\phi_i}{\partial x_k}\frac{\partial\phi_j}{\partial x_k}\right)\frac{\partial^2}{\partial y_i\partial y_j}$$
$$+\sum_{j=1}^{n}\left(\frac{\partial^2\phi_j}{\partial t^2}-\sum_{k=1}^{n}\frac{\partial^2\phi_j}{\partial x_k^2}\right)\frac{\partial}{\partial y_j}.$$

Definition 2.4 ([4], [13]). If a transformation Φ satisfies that

(2.5) the matrix$\left(\sum_{k=1}^{n}\frac{\partial\phi_j}{\partial x_k}\frac{\partial\phi_j}{\partial x_k}-\frac{\partial\phi_i}{\partial t}\frac{\partial\phi_j}{\partial t}\right)$ is positive definite for $(x, t)\in$ $\operatorname{dom}\Phi(=\text{the domain of }\Phi)$,

then we say that Φ belongs to the class (E) (or Φ is of the class (E)).

Proposition 2.5. *Suppose that Γ is time like. Then for each $(\tilde{x}^0, t^0)\in\Gamma$, there exists a neighborhood $V_{(\tilde{x}^0,t^0)}$ and a transformation $\Phi_{(\tilde{x}^0,t^0)}\in(\mathrm{E})$ which transform $V_{(\tilde{x}^0,t^0)}$ to a neighborhood $\tilde{V}_{(0,0)}$ of* $(0, 0)$ *such that* (i) $\Phi_{(\tilde{x}^0,t^0)}(V_{(\tilde{x}^0,t^0)}\cap\Omega)=\tilde{V}_{(0,0)}\cap\{y=(y_1, y_2,\dots, y_n); y_n>0\}$, (ii) $\Phi_{(\tilde{x}^0,t^0)}(V_{(\tilde{x}^0,t^0)}\cap\partial\Omega)=\tilde{V}_{(0,0)}\cap\{y; y_n=0\}$ *and* (iii) $\Phi_{(\tilde{x}^0,t^0)}(V_{(\tilde{x}^0,t^0)}\cap\Gamma(t^0))=\tilde{V}_{(0,0)}\cap\{y; y_n=y_{n-1}=0\}$.

Proof. Without losing the generality, we may suppose that $(\tilde{x}^0, t^0)=(0, 0)\in\Gamma$. Moreover, we may suppose that $\partial\Omega$ is represented by $x_n=f(x')$ and Γ is represented by $x_{n-1}=g(x'', t)$, $x_n=f(x')$ in some neighborhood $V'_{(0,0)}$ of $(0, 0)$ satisfying the properties enumerated before. Define a transformation Φ as

$$(2.6)\qquad \Phi:\begin{cases} y''=x'' \quad\text{where}\quad y''=(y_1, y_2,\dots, y_{n-2}),\\ y_{n-1}=x_{n-1}-g(x'', t),\\ y_n=x_n-f(x') \quad\text{and}\\ s=t.\end{cases}$$

Then we have

$$\text{(2.7)}\quad \square \xrightarrow{\Phi} L=L(y,\ s:\ D_y,\ D_s)$$
$$=\frac{\partial^2}{\partial s^2}+2h_{n-1}(y,\ s)\frac{\partial^2}{\partial y_{n-1}\partial s}\sum_{i,j=1}^{n}\frac{\partial}{\partial y_j}\left(a_{ij}(y,\ s)\frac{\partial}{\partial y_j}\right)$$
$$+\sum_{j=1}^{n}b_j(y,\ s)\frac{\partial}{\partial y_j},$$

where the coefficients are given by

$$\text{(2.8)}\quad \begin{cases} h_{n-1}(y,\ s)=-g_t(x'',\ t)|_{(x,t)=\Phi^{-1}(y,s)}, \\ a_{ij}(y,\ s)=\delta_{ij} \qquad 1\leqq i,\ j\leqq n-2, \\ a_{n-1\ n-1}(y,s)=1+\sum_{j=1}^{n-2}g_{x_j}^2(x'',\ t)-g_t^2(x'',\ t)|_{(x,t)=\Phi^{-1}(y,s)}, \\ a_{nn}(y,\ s)=1+\sum_{j=1}^{n-1}f_{x_j}^2(x')|_{(x,t)=\Phi^{-1}(y,s)}, \\ a_{jn}(y,\ s)=a_{nj}(y,\ s)=-f_{x_j}(x')|_{(x,t)=\Phi^{-1}(y,s)} \qquad 1\leqq j\leqq n-2, \\ a_{j\ n-1}(y,\ s)=a_{n-1\ j}(y,\ s)=-g_{x_j}(x'',\ t)|_{(x,t)=\Phi^{-1}(y,s)} \quad 1\leqq j\leqq n-2, \\ a_{n\ n-1}(y,\ s)=a_{n-1\ n}(y,\ s) \\ \qquad =\sum_{j=1}^{n-2}g_{x_j}(x'',\ t)f_{x_j}(x')-f_{x_{n-1}}(x')|_{(x,t)=\Phi^{-1}(y,s)} \\ b_n(y,\ s)=-g_{tt}(x'',\ t)+\sum_{j=1}^{n-2}g_{x_jx_j}(x'',\ t)|_{(x,t)=\Phi^{-1}(y,s)} \\ \qquad +\sum_{j=1}^{n}\frac{\partial}{\partial y_j}a_{jn}(y,s) \\ b_{n-1}(y,\ s)=\sum_{j=1}^{n-1}f_{x_jx_j}(x')|_{(x,t)=\Phi^{-1}(y,s)}+\sum_{j=1}^{n}\frac{\partial}{\partial y_j}a_{jn-1}(y,s), \\ b_j(y,\ s)=\sum_{i=1}^{n}\frac{\partial}{\partial y_i}a_{ij}(y,\ s) \quad 1\leqq j\leqq n-2 \end{cases}$$

with Φ^{-1}=the inverse transformation of Φ. (From (2.6), Φ is a diffeomorphism with its Jacobian=1.) Calculate $\{a_{ij}(y,\ s)\xi_i\xi_j\}$ for any $\xi=(\xi_1,\ \xi_2,\dots,\ \xi_n)\in\mathbf{R}^n$. Then from (2.3), we have

$$\sum_{i,j=1}^{n}a_{ij}(0,\ 0)\xi_i\xi_j\geqq(1-g_t^2(0,\ 0))\sum_{i=1}^{n}\xi_i^2.$$

This means that if we consider the matrix $\{a_{ij}(y,\ s)\}$ in some neighborhood $\tilde{V}_{(0,0)}$ smaller than $\tilde{V}'_{(0,0)}=\Phi(V'_{(0,0)})$, then the matrix is positive definite in $\tilde{V}_{(0,0)}$. Putting $\Phi_{(0,0)}=\Phi$, $V_{(0,0)}=\Phi^{-1}(\tilde{V}_{(0,0)})$, we have the desired result. Q.E.D.

The following proposition is proved by easy calculation. See also the remark in §4 of J. Cooper-C. Bardos [4]. p. 54.

Proposition 2.6. *Let a transformation* Φ *be of the class* (E). *Moreover, we assume that* Φ *has an inverse transformation* Ψ *given by* $(x, t)=\Psi(y, s)$, $x_k=\psi_k(y, s)$ $k=1, 2,\dots, n$ *and* $s=t$. *Then we have always*

$$(2.9)\qquad \sum_{j=1}^{n} h_j(\tilde{y}, s)\tilde{\nu}_j=0 \qquad on \quad (\tilde{y}, s)\in\Phi(\Sigma\cap\operatorname{dom}\Phi),$$

where $\tilde{\boldsymbol{\nu}}=(\tilde{\nu}_1, \tilde{\nu}_2,\dots, \tilde{\nu}_n)$ *is the unit exterior normal of* $\Phi(\Sigma\cap\operatorname{dom}\Phi)$ *and* $h_j(\tilde{y}, s)=\left.\frac{\partial\phi_j}{\partial t}\right|_{(x,t)=\Psi(y,s)}$.

In the following, we consider some examples of the function $\alpha(\tilde{x}, t)$ satisfying the assumptions (a) and (b) of §1.

Example 1. Let Ω be given by $\Omega=\mathbf{R}^2_+=\{(x_1, x_2)\colon x_2>0\}$ and let Γ be represented by $\Gamma=\{(x_1, 0, t); x_1=\gamma(t)\}$ with a C^∞-function $\gamma(t)$. It is clear that Γ is time-like if $\left|\frac{d}{dt}\gamma(t)\right|<1$ for all $0\leqq t\leqq T$. Consider a function $\alpha_\varepsilon(x_1, t)$ (ε is a positive constant) given by

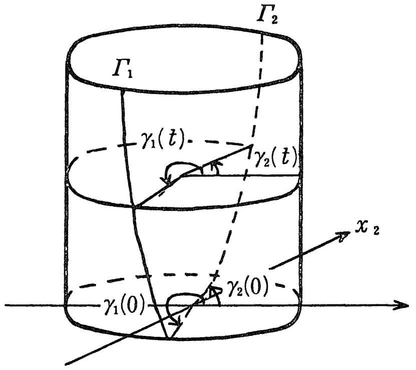
$$\alpha_\varepsilon(x_1,t)=\varepsilon^{-1}\int_0^\infty\rho\left(\frac{x_1-\gamma(t)-s}{\varepsilon}-1\right)ds$$

where $\rho(s)$ is the function satisfying $\rho(s)=\rho(-s)\in C_0^\infty(\mathbf{R})$, $\int_{-\infty}^\infty\rho(s)ds=1$, $\rho(s)\geqq0$ and $\operatorname{supp}\rho=[-1, 1]$. Then clearly, $\{(x_1, t); \alpha_\varepsilon(x_1, t)=0\}=\{(x_1, t); x_1\leqq\gamma(t)-\varepsilon\}$, so $\Gamma_\varepsilon=\{(x_1, t); x_1=\gamma(t)-\varepsilon\}=\partial\{(x_1, t); \alpha_\varepsilon(x_1, t)=0\}$ is time-like. The transformation Φ given by

$$(2.10)\qquad \Phi\colon\begin{cases} y_1=x_1-\gamma(t)\\ y_2=x_2\\ s=t\end{cases}$$

satisfies the conditions of the assumption (b).

Example 2. $\Omega=\{(x_1, x_2); x_1^2+x_2^2<1\}$ or $\{(x_1, x_2); x_1^2+x_2^2>1\}$. $\Gamma=\Gamma_1\cup\Gamma_2$ where $\Gamma_i=\{(x_1, x_2); x_1=\cos\gamma_i(t), x_2=\sin\gamma_i(t)\}$ $i=1.2$. If each $\gamma_i(t)$ satisfies



(2.11) $$1-\gamma_i'(t)^2>\frac{3}{4}\gamma_i'(t)^4 \times(\sin 2\gamma_i(t))^2,$$

then Γ_i is time-like. A sufficient condition of (2.11) is $0\leqq\gamma_i'(t)^2\leqq\frac{2}{3}$. Now, we define a function $\alpha_\varepsilon(\tilde{x}, t)$ by

(2.12) $$\alpha_\varepsilon(\tilde{x}, t)=\alpha_\varepsilon(\cos\theta, \sin\theta, t)$$

$$=\begin{cases} 0 & \gamma_2(t)+\varepsilon\leqq\theta\leqq\gamma_1(t)-\varepsilon \\ \varepsilon^{-1}\int_0^\infty\rho\left(\frac{\theta-\gamma_1(t)-s}{\varepsilon}\right)ds & \gamma_1(t)-\varepsilon\leqq\theta\leqq\gamma_1(t)+\varepsilon \\ 1 & \gamma_1(t)+\varepsilon\leqq\theta\leqq\gamma_2(t)-\varepsilon \\ \varepsilon^{-1}\int_{-\infty}^0\rho\left(\frac{\gamma_2(t)-\theta+s}{\varepsilon}\right)ds & \gamma_2(t)-\varepsilon\leqq\theta\leqq\gamma_2(t)+\varepsilon. \end{cases}$$

Clearly, $\alpha_\varepsilon(\tilde{x}, t)$ is a C^∞-function on $\partial\Omega\times[0, T]$ if ε is taken sufficiently small. We define the transformations Φ_i as the rotation of axis, that is,

(2.13) $$\Phi_i:\begin{cases} y_1=x_1\cos\gamma_i(t)+x_2\sin\gamma_i(t)=r\cos(\theta-\gamma_i(t)) \\ y_2=-x_1\sin\gamma_i(t)+x_2\cos\gamma_i(t)=r\sin(\theta-\gamma_i(t)) \\ s=t \end{cases}$$

where $r=\sqrt{x_1^2+x_2^2}$, $\theta=\arctan\frac{x_2}{x_1}$.

Then, by easy calculation, we have $\Phi_i\in(\mathrm{E})$ when

(2.14) $$1-(x_1^2+x_2^2)\gamma_i'(t)^2>0.$$

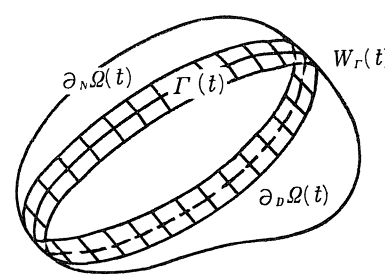
By (2.11) and (2.14), the transformation Φ_i is of the class (E) in each neighborhood V_i of Γ_i. It is clear that $\Gamma^\varepsilon=\Gamma_1^\varepsilon\cup\Gamma_2^\varepsilon$, where $\Gamma_1^\varepsilon=\{(\cos\theta,$

$\sin\theta, t); \theta=\gamma_1(t)-\varepsilon\}\Gamma_2^\varepsilon=\{(\cos\theta, \sin\theta, t); \theta=\gamma_2(t)+\varepsilon\}$ equals to $\partial\{(\tilde{x}, t); \alpha_\varepsilon(\tilde{x}, t)=0\}$ and Γ^ε is time-like. Moreover, by this transformation, the function $\tilde{\alpha}_\varepsilon(\tilde{y}, s)=\alpha_\varepsilon(\Phi^{-1}(\tilde{y}, s))$ is independent of s in $\Phi_i(V_i\cap\Sigma)$, $\Sigma=\partial\Omega\times[0, T]$.

More generally, we have the following proposition.

Proposition 2.7. *Let Ω be a domain in $\mathbf{R}^n$ with the smooth boundary $\partial\Omega$. (For the sake of simplicity, we assume that $\partial\Omega$ is compact). Let Γ by time-like in Ω_T. Then, there exists a function $\alpha_\varepsilon(\tilde{x}, t)$ on Σ satisfying the assumptions (a) and (b) of* § 1.

Proof. As Γ is a submanifold of Σ of codimension 1, there exists a 'collar' neighborhood W_Γ of Γ in Σ. We denote $W_\Gamma(t)=W_\Gamma\cap P(t)$. Moreover, as $\Gamma(t)$ is a submanifold of $\partial\Omega$ of codimension 1, $\Gamma(t)$ divides $\partial\Omega$ into $\partial_D\Omega(t)$ and $\partial_N\Omega(t)$. For the point $(\tilde{x}, t)$ in $W_{\Gamma(t)}$, we define



$$d(\tilde{x}, t)=\pm\operatorname{dis}((\tilde{x}, t), \Gamma(t))$$

where $\operatorname{dis}((\tilde{x}, t), \Gamma(t))$ represents the length of the shortest curve on $\partial\Omega$ from $(\tilde{x}, t)$ to $\Gamma(t)$, and the sign $+(-)$ is taken when $(\tilde{x}, t)$ belongs to $\partial_N\Omega(t)(\partial_D\Omega(t))$. We define a function $\alpha_\varepsilon(\tilde{x}, t)$ as

$$(2.15)\quad \alpha_\varepsilon(\tilde{x}, t)=\begin{cases}0 & (\tilde{x}, t)\in\partial_D\Omega(t)-\{(\tilde{x}, t);\ |\mathrm{d}(\tilde{x}, t)|\leqq\varepsilon\}\\ \varepsilon^{-1}\int_0^\infty\rho\left(\dfrac{\mathrm{d}(\tilde{x}, t)-s}{\varepsilon}\right)ds & (\tilde{x}, t)\in\{(\tilde{x}, t);\ |\mathrm{d}(\tilde{x}, t)|\leqq\varepsilon\},\\ 1 & (\tilde{x}, t)\in\partial_N\Omega(t)-\{(\tilde{x}, t);\ |\mathrm{d}(\tilde{x}, t)|\leqq\varepsilon\}.\end{cases}$$

Clearly, $\partial\{(\tilde{x}, t); \alpha_\varepsilon(\tilde{x}, t)=0\}=\{(\tilde{x}, t); \mathrm{d}(\tilde{x}, t)=-\varepsilon\}$ is time-like. For each point $(\tilde{x}^0, t^0)\in\Gamma$, we consider the transformation $\Phi_{(\tilde{x}^0,t^0)}$ constructed in Proposition 2.5. Then in the neighborhood $V_{(\tilde{x}^0,t^0)}$ of $(\tilde{x}^0, t^0)$, $\operatorname{dis}((\tilde{x}, t), \Gamma(t))$ equals to $x_{n-1}-g(x'', t)$ where the axis x and the point $(\tilde{x}^0, t^0)$ are chosen as in Proposition 2.5. This means that the function $\alpha_\varepsilon(\tilde{x}, t)$

defined as above is transformed by this transformation $\Phi_{(\tilde{x}^0,t^0)}$ in $V_{(\tilde{x}^0,t^0)}$ to the function $\tilde{\alpha}_\varepsilon(\tilde{y}, s)$ independent of s when $(\tilde{y}, s)\in\Phi_{(\tilde{x}^0,t^0)}(V_{(\tilde{x}^0,t^0)}\cap\Sigma)$.

Q. E. D.

§3. Energy Inequalities (I)

Let ω be a domain in $\mathbf{R}^n$ with smooth compact boundary $\partial\omega$, i.e. ω is the interior or exterior domain of $\partial\omega$.

We consider the following mixed problem

$$(3.1)\quad \begin{cases} L(t)\,[u(x,t)]=\left(\dfrac{\partial^2}{\partial t^2}+a_1(x,t;D)\dfrac{\partial}{\partial t}+a_2(x,t;D)\right)u(x,t) \\ \qquad\qquad\qquad\qquad\qquad\qquad =f(x,t) \qquad \text{in } \omega\times(0,T)\,, \\ u(x,0)=u_0(x)\,, \\ u_t(x,0)=u_1(x)\,, \end{cases}$$

$$a_1(x,t;D)=2\sum_{j=1}^n h_j(x,t)\frac{\partial}{\partial x_j}+h(x,t)\,,$$

$$a_2(x,t;D)=-\sum_{j=1}^n\frac{\partial}{\partial x_j}\left(a_{ij}(x,t)\frac{\partial}{\partial x_i}\right)+\sum_{j=1}^n b_j(x,t)\frac{\partial}{\partial x_j}-c(x,t)\,,$$

with a mixed boundary condition

$$(3.2)\quad \alpha(\tilde{x})B(t)u(\tilde{x},t)+(1-\alpha(\tilde{x}))u(\tilde{x},t)=\alpha(\tilde{x})\phi(\tilde{x},t) \qquad \text{on } \partial\omega\times[0,T]\,,$$

$$B(t)=\frac{\partial}{\partial \boldsymbol{n}_t}-\sigma_1(x,t)\frac{\partial}{\partial t}+\sigma_2(x,t)$$

$$\frac{\partial}{\partial \boldsymbol{n}_t}=\sum_{i,j=1}^n a_{ij}(x,t)\nu_j\frac{\partial}{\partial x_i}\bigg|_{\partial\omega}$$

$\boldsymbol{\nu}=(\nu_1,\nu_2,\dots,\nu_n)$: unit exterior normal of ω at $\partial\omega$

where all coefficients belong to $\mathscr{B}^\infty(\overline{\omega\times(0,T)})$ or to $\mathscr{B}^\infty(\partial\omega\times[0,T])$.

We assume that

(a) $0\leqq\alpha(\tilde{x})\leqq 1$,

(b) $\{\tilde{x}\in\partial\omega;\ \alpha(\tilde{x})=0\}\neq\phi$ and $\{\tilde{x}\in\partial\omega;\ \alpha(\tilde{x})\neq 0\}\neq\phi$,

(c) the boundary of the set $\{\tilde{x}\in\partial\omega;\ \alpha(\tilde{x})=0\}$ forms a submanifold of $\partial\omega$ of codimension 1,

(d) $a_2(x, t; D)$ is an uniformly elliptic operator i.e. there exists a constant $d>0$ satisfying

$$(3.3) \qquad \begin{cases} \sum_{i,j=1}^{n} a_{ij}(x, t)\xi_i\xi_j \geqq d \sum_{j=1}^{n} \xi_j^2 \\ a_{ij}(x, t) = a_{ji}(x, t) \end{cases}$$

for all $(x, t) \in \omega \times [0, T]$, $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbf{R}^n$, and

(e) $h_j(x, t)$ and $\sigma_1(x, t)$ are real valued functions and for some constant $\varepsilon_0 > 0$, we have

$$(3.4) \qquad \sigma_1(x, t) \leqq \langle \boldsymbol{h}, \boldsymbol{\nu} \rangle - \varepsilon_0 \qquad \text{on} \quad \partial\omega \times [0, T]$$

where $\langle \boldsymbol{h}, \boldsymbol{\nu} \rangle = \sum_{j=1}^{n} h_j(x, t)\nu_j|_{\partial\omega}$.

For the future use, we introduce the following formulation.

$$(3.5) \qquad \begin{cases} \dfrac{d}{dt} U(t) = \mathscr{A}(t)U(t) \\ U(0) = U_0 \end{cases}$$

$$(3.6) \qquad \mathscr{B}_\alpha(t)U(t) = \alpha(\tilde{x})\phi(\tilde{x}, t)$$

where

$$U(t) = \begin{bmatrix} u\,(x, t) \\ u'(x, t) \end{bmatrix}, \quad F(t) = \begin{bmatrix} 0 \\ f(x, t) \end{bmatrix} \quad \left(' = \frac{\partial}{\partial t}, \quad '' = \frac{\partial^2}{\partial t^2} \right),$$

$$\mathscr{A}(t) = \begin{bmatrix} 0 & 1 \\ -a_2(x, t: D) & -a_1(x, t; D) \end{bmatrix}$$

$$\mathscr{B}_\alpha(t) = \left[\alpha(\tilde{x})\left(\frac{\partial}{\partial \boldsymbol{n}_t} + \sigma_2(\tilde{x}, t) \right) + 1 - \alpha(\tilde{x}) \qquad -\alpha(\tilde{x})\sigma_1(\tilde{x}, t) \right].$$

Here, we introduce some function spaces attached to the operators $\mathscr{A}(t)$ and $\mathscr{B}_\alpha(t)$.

We denote by $\boldsymbol{E}_m$ $(m=1, 2,\ldots)$, the space $H^m(\omega) \times H^{m-1}(\omega)$ whose the norm $|||\cdot|||_m$ is given by

$$|||U|||_m^2 = \|u\|_{H^m(\omega)}^2 + \|v\|_{H^{m-1}(\omega)}^2$$

for $U=\{u, v\}\in H^m(\omega)\times H^{m-1}(\omega)$ ($H^l(\omega)$ is the Sobolev space of order l with the norm $\|\cdot\|_{H^l(\omega)}$. For brevity, we write $\|\cdot\|$ instead of $\|\cdot\|_{L^2(\omega)}$).

Remarking that the form

$$\left(\sum_{i,j=1}^{n}\left(a_{ij}(x, t)\frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_i}\right)+(u, u)\right)^{\frac{1}{2}}$$

gives an equivalent norm in $H^1(\omega)$, we denote by $\mathscr{H}(t)$ the space $H^1(\omega)\times L^2(\omega)$ $(=E_1)$ equipped with the following norm,

$$\|U\|_{\mathscr{H}(t)}^2=\sum_{i,j=1}^{n}\left(a_{ij}(x, t)\frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_j}\right)+(u, u)+(v, v)$$

for $U=\{u, v\}\in H^1(\omega)\times L^2(\omega)$. $\left((u, v)=\int_\omega u(x)\overline{v(x)}dx\right)$.

We denote by $V_\alpha(\omega)$, the completion of all u each of which belongs to $C^\infty(\bar{\omega})$ and satisfies $u(x)=0$ on $\alpha(\tilde{x})=0$ and $\|u\|_{V_\alpha(\omega)}<\infty$ where the norm $\|\cdot\|_{V_\alpha}$ is defined by

$$\|u\|_{V_\alpha(\omega)}^2=\|u\|_{H^1(\omega)}^2+\int_{\partial_N\omega}\frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})}|u(\tilde{x})|^2 d\tilde{x}$$

where $d\tilde{x}$ is a measure on $\partial\omega$ induced from dx and $\partial_N\omega=\{\tilde{x}\in\partial\omega;\ \alpha(\tilde{x})\neq 0\}$.

$\mathscr{V}_\alpha(t)$ stands for the space $V_\alpha(\omega)\times L^2(\omega)$ equipped with the following norm.

$$\|U\|_{\mathscr{V}_\alpha(t)}^2=\|U\|_{\mathscr{H}(t)}^2+\int_{\partial_N\omega}\frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})}|u(x)|^2 dx$$

for $U=\{u, v\}\in V_\alpha(\omega)\times L^2(\omega)$.

Moreover, we denote by $\mathscr{D}_\alpha(t)$, the set of all elements belonging to $H^2(\omega)\times V_\alpha(\omega)$ such that $\mathscr{B}_\alpha(t)U=0$.

First of all, we consider the following elliptic problem

$$\text{(3.7)}\quad\begin{cases}(a_2(x, t; D)+\lambda)u(x)=f(x) & \text{in } \omega,\\ \alpha(\tilde{x})\dfrac{\partial u}{\partial \boldsymbol{n}_t}+(1-\alpha(\tilde{x}))u=\alpha(\tilde{x})\phi(\tilde{x})+(1-\alpha(\tilde{x}))\psi(\tilde{x}) & \text{on } \partial\omega\end{cases}$$

where λ is a parameter and t is also considered as a parameter.

Then, we have

Proposition 3.1. *Let m be a non-negative integer. Let $f(x)\in H^m(\omega)$, $\phi(x)\in H^{m+1/2}(\partial_N\omega)$ and $\psi(x)\in H^{m+3/2}(\partial\omega)$.*

Assuming that there exists a function $u(x)\in H^{m+2}(\omega)$ satisfying (3.7), *we have*

$$\|u\|^2_{H^{m+2}(\omega)}\leqq C(\|f\|^2_{H^m(\omega)}+\|u\|^2+\langle\phi\rangle^2_{H^{m+1/2}(\partial_N\omega)}+\langle\psi\rangle^2_{H^{m+3/2}(\partial\omega)}) \tag{3.8}$$

where C is a constant independent of u and t, and $\langle\cdot\rangle_{H^l(\partial\omega)}$ is the norm of the Sobolev space of order l on $\partial\omega$.

Moreover, if λ is sufficiently large, then there exists a function $u(x)\in H^{m+2}(\omega)$ satisfying (3.7).

Proof. Assume that $u(x)\in H^{m+2}(\omega)$ satisfies (3.7). For sufficiently large λ_1, there exists a function $v(x)\in H^{m+2}(\omega)$ satisfying

$$\begin{cases}(a_2^0(x,t;D)+\lambda_1)v=f-(a_2-a_2^0)u+(\lambda_1-\lambda)u\,,\\ \dfrac{\partial v}{\partial \boldsymbol{n}_t}\Big|_{\partial\omega}=\tilde{\phi}\end{cases} \tag{3.9}$$

where $a_2^0(x,t;D)=-\sum_{i,j=1}^n\dfrac{\partial}{\partial x_i}\Big(a_{ij}(x,t)\dfrac{\partial}{\partial x_j}\Big)$ and $\tilde{\phi}\in H^{m+1/2}(\partial\omega)$ is an extension of ϕ such that $\langle\tilde{\phi}\rangle_{H^{m+1/2}(\partial\omega)}\leqq C\langle\phi\rangle_{H^{m+1/2}(\partial_N\omega)}$. Putting $w(x)=u(x)-v(x)$, we have

$$\begin{cases}(a_2^0(x,t;D)+\lambda_1)w=0,\\ \alpha(\tilde{x})\dfrac{\partial w}{\partial\boldsymbol{n}_t}+(1-\alpha(\tilde{x}))w=(1-\alpha(\tilde{x}))(\psi-v|_{\partial\omega})\,.\end{cases} \tag{3.10}$$

We may assume that λ_1 is chosen large enough such that the following Dirichlet problem is uniquely solvable:

$$\begin{cases}(a_2^0(x,t;D)+\lambda_1)\chi=0,\\ \chi|_{\partial\omega}=g.\end{cases} \tag{3.11}$$

We introduce an operator $P_t(\lambda_1)$ (or, simply P_t) as

$$P_tg=\frac{\partial\chi}{\partial\boldsymbol{n}_t}\Big|_{\partial\omega} \tag{3.12}$$

where χ is the solution of (3.11). We know that P_t is a positive elliptic pseudo-differential operator of order 1 on $\partial\omega$. See, Fujiwara-Uchiyama [5], Inoue [12], Visik [25]. Using this operator, we reduce (3.10) to the problem

$$(3.13)\qquad (\alpha(\tilde{x})P_t+1-\alpha(\tilde{x}))w|_{\partial\omega}=(1-\alpha(\tilde{x}))(\psi-v|_{\partial\omega}).$$

Remarking that $\alpha(\tilde{x})$ vanishes at least of second order near $\{\tilde{x}\in\partial\omega;\ \alpha(\tilde{x})=0\}$, we may apply the theory of Melin [19] to the operator $\Lambda^{2m+3}(\alpha(x)P_t+1-\alpha(x))$, $\Lambda=(1-\Delta')^{1/2}$, Δ': Laplace-Beltrami operator on $\partial\omega$. (This was proved in Kaji [15] and Taira [24]). And, we have

$$(3.14)\quad \langle w|_{\partial\omega}\rangle^2_{H^{m+3/2}(\partial\omega)}\leqq C\{\langle(1-\alpha(x))(\psi-v|_{\partial\omega})\rangle^2_{H^{m+3/2}(\partial\omega)}+\langle w|_{\partial\omega}\rangle^2\},\ \text{where}$$

C is a constant independent of t. (here, we use $\langle\cdot\rangle$ instead if $\langle\cdot\rangle_{L^2(\partial\omega)}$). More precisely, see the estimate (4.1) of Lemma 4.1, [24].

On the other hand, w must satisfy that

$$(3.15)\qquad \begin{cases} (a_2^0(x,t\colon D)+\lambda_1)w=0 & \text{in}\ \ \omega,\\ w=w|_{\partial\omega} & \text{on}\ \ \partial\omega.\end{cases}$$

So, we have

$$(3.16)\qquad \|w\|^2_{H^{m+2}(\omega)}\leqq C(\langle w|_{\partial\omega}\rangle^2_{H^{m+3/2}(\partial\omega)}+\|w\|^2)$$

where C is a constant and it may be chosen independently of t. (See, Miranda [20]).

Remarking that v in (3.9) satisfies

$$(3.17)\quad \|v\|^2_{H^{m+2}(\omega)}\leqq C(\|f-(a_2-a_2^0)u+(\lambda_1-\lambda)u\|^2_{H^m(\omega)}+\|v\|^2+\langle\tilde{\phi}\rangle^2_{H^{m+1/2}(\partial\omega)})$$

where the constant C is independent of t (see also Miranda [20]), we have the desired inequality (3.8).

The existence part of this proposition follows from considering the adjoint operator. More precisely, see [15], [15]', [24]. Q.E.D.

Remark 3.2. (a) From the proof above, it is obvious that the problem (3.7) is 'stable' with respect to suitable lower order perturbations, i.e. adding the operator of order less than one to the equation in ω or adding the operator of the type $\alpha(\tilde{x})\sigma(\tilde{x})I$ to the boundary operator where $\sigma(\tilde{x})\in\mathscr{B}^\infty(\partial\omega)$.

(b) Taking λ sufficiently large in (3.7), we have the inequality (3.8) without the term $\|u\|^2$ in the right hand side when ω is bounded. (See, Agmon-Douglis-Nirenberg [3].)

Using this proposition, we have

Lemma 3.3. *There exists a constant $C>0$ such that for all $U \in H^2(\omega)\times V_\alpha(\omega)$ satisfying $\mathscr{B}_\alpha(t)U=\alpha(\tilde{x})\phi$ with $\phi\in H^{1/2}(\partial_N\omega)$, we have*

$$(3.18) \qquad |||U|||_2^2 \leqq C(\|\mathscr{A}(t)U\|^2_{\mathscr{H}(t)}+\|U\|^2_{\mathscr{H}(t)}+\langle\phi\rangle^2_{H^{1/2}(\partial_N\omega)})$$

Proof. Let us put $\mathscr{A}(t)U=F=\{f, g\}$, i.e.

$$\begin{cases} v=f \\ -a_2(x, t; D)u-a_1(x, t; D)v=g. \end{cases}$$

Rewriting the above relation and $\mathscr{B}_\alpha(t)U=\alpha(\tilde{x})\phi$, we have

$$(3.19) \qquad \begin{cases} a_2(x, t; D)u=-g-a_1(x, t; D)f. \\ \alpha(\tilde{x})\Big(\dfrac{\partial u}{\partial \boldsymbol{n}_t}+\sigma_2(\tilde{x}, t)u\Big)+(1-\alpha(\tilde{x}))u=\alpha(\tilde{x})(\phi+\sigma_1(\tilde{x},t)f)\,. \end{cases}$$

So applying Proposition 3.1 to (3.19), we have

$$\begin{aligned} \|u\|^2_{H^2(\omega)} &\leqq \text{const.}(\|g+a_1(x, t\colon D)f\|^2+\|u\|^2+\langle\phi+\sigma_1 f|_{\partial_N\omega}\rangle^2_{H^{1/2}(\partial_N\omega)}) \\ &\leqq \text{const.}(\|F\|^2_{\mathscr{H}(t)}+\|U\|^2_{\mathscr{H}(t)}+\langle\phi\rangle^2_{H^{1/2}(\partial_N\omega)}). \end{aligned}$$

On the other hand,

$$\|v\|^2_{H^1(\omega)}=\|f\|^2_{H^1(\omega)}\leqq \text{const.}\|F\|^2_{\mathscr{H}(t)}.$$

Combining these estimates, we have the desired inequality.

Lemma 3.4. *There exists a constant $C_0>0$ such that for any $U \in H^2(\omega)\times V_\alpha(\omega)$ satisfying $\mathscr{B}_\alpha(t)U=\alpha(\tilde{x})\phi$ with $\int_{\partial_N\omega}|\phi(\tilde{x})|^2 d\tilde{x}<\infty$, we have*

$$(3.20) \qquad (\mathscr{A}(t)U, U)_{\mathscr{V}_\alpha(t)}+(U, \mathscr{A}(t)U)_{\mathscr{V}_\alpha(t)}\leqq C_0\Big(\|U\|^2_{\mathscr{H}(t)}+\int_{\partial_N\omega}|\phi(\tilde{x})|^2 d\tilde{x}\Big).$$

Proof. $(\mathscr{A}(t)U, U)_{\mathscr{V}_\alpha(t)}+(U. \mathscr{A}(t)U)_{\mathscr{V}_\alpha(t)}$

$$=2\mathrm{Re}\left\{\sum_{i,j=1}^{n}\left(a_{ij}(x,t)\frac{\partial v}{\partial x_j},\ \frac{\partial u}{\partial x_i}\right)+(v,u)\right.$$

$$\left.+(-a_2(x,t;D)u-a_1(x,t;D)v,v)+\int_{\partial_N\omega}\frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})}v(\tilde{x})\overline{u(\tilde{x})}d\tilde{x}\right\}$$

by integration by parts,

$$=\int_{\partial\omega}v\overline{\frac{\partial u}{\partial \boldsymbol{n}_t}}dx+\int_{\partial\omega}\frac{\partial u}{\partial \boldsymbol{n}_t}\bar{v}dx-2\int_{\partial\omega}\langle \boldsymbol{h},\boldsymbol{\nu}\rangle|v|^2dx$$

$$+2\mathrm{Re}\int_{\partial_N\varpi}\frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})}u(\tilde{x})\overline{v(\tilde{x})}d\tilde{x}$$

$$+2\mathrm{Re}\left[(u,v)-\left(\sum_{j=1}^{n}b_j\frac{\partial u}{\partial x_j}+cu,\ v\right)+\left(\left(\sum_{j=1}^{n}\frac{\partial h_j}{\partial x_j}-h\right)v,\ v\right)\right]$$

$$=I_B+I_V$$

$$I_B=\int_{\partial\omega}v\overline{\left(\frac{\partial u}{\partial \boldsymbol{n}_t}-\sigma_1v+\sigma_2u\right)}d\tilde{x}+\int_{\partial\omega}\left(\frac{\partial u}{\partial \boldsymbol{n}_t}-\sigma_1v+\sigma_2u\right)\bar{v}d\tilde{x}$$

$$+2\int_{\partial\omega}(\sigma_1-\langle \boldsymbol{h},\boldsymbol{\nu}\rangle)|v|^2d\tilde{x}-2\mathrm{Re}\int_{\partial\omega}\sigma_2u\bar{v}d\tilde{x}$$

$$+2\mathrm{Re}\int_{\partial_N\omega}\frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})}u(\tilde{x})\overline{v(\tilde{x})}d\tilde{x}$$

since $\sigma_1(x,t)-\langle \boldsymbol{h},\boldsymbol{\nu}\rangle\leqq-\varepsilon_0$ and $v(\tilde{x})=0$ on $\alpha(\tilde{x})=0$,

$$\leqq 2\mathrm{Re}\int_{\partial_N\omega}v\bar{\phi}d\tilde{x}-2\varepsilon_0\int_{\partial_N\omega}|v|^2d\tilde{x}-2\mathrm{Re}\int_{\partial_N\omega}\sigma_2u\bar{v}dx$$

$$\leqq\varepsilon_0\int_{\partial_N\omega}|v|^2d\tilde{x}+\frac{1}{\varepsilon_0}\int_{\partial_N\omega}|\phi|^2d\tilde{x}-2\varepsilon_0\int_{\partial_N\omega}|v|^2d\tilde{x}+\varepsilon_0\int_{\partial_N\omega}|v|^2d\tilde{x}$$

$$+\frac{1}{\varepsilon_0}\int_{\partial_N\omega}|\sigma_2u|^2d\tilde{x}$$

$$\leqq\text{const.}\left(\int_{\partial_N\omega}|\phi|^2d\tilde{x}+\|U\|^2_{\mathscr{H}(t)}\right).$$

On the other hand, we have

$$|I_V|\leqq 2\|u\|\ \|v\|+\text{const.}\ \|u\|_{H^1(\omega)}\|v\|+\text{const.}\ \|v\|^2$$

$$\leqq \text{const.}\ \|U\|^2_{\mathscr{H}(t)}.$$

Combining these estimate, we have (3.20).

Corollary 3.5. *For all* $U \in \mathscr{D}_\alpha(t)$, *we have*

$$\|(\lambda I - \mathscr{A}(t))U\|_{\mathscr{V}_\alpha(t)} \geqq (\lambda - c_0)\|U\|_{\mathscr{V}_\alpha(t)} \quad \text{if} \quad \lambda > c_0. \tag{3.21}$$

Lemma 3.6. *There exists a constant* $\lambda_0 > 0$ *such that for any* $\lambda > \lambda_0$, $\lambda I - \mathscr{A}(t)$ *is a bijective mapping from* $\mathscr{D}_\alpha(t)$ *onto* $\mathscr{V}_\alpha(t)$.

Proof. Consider an equation in U

$$(\lambda I - \mathscr{A}(t))U = F \quad \text{with} \quad U \in \mathscr{D}_\alpha(t), \quad F \in \mathscr{V}_\alpha(t). \tag{3.22}$$

That is,

$$\begin{cases} \lambda u - v = f \\ a_2(x, t; D)u + (\lambda + a_1(x, t; D))v = g \end{cases}$$

where $f \in V_\alpha(\omega)$, $g \in L^2(\omega)$. Substituting the first relation into the second, we have

$$\begin{aligned} a_\lambda(x, t; D)u &= (a_2(x, t; D) + \lambda a_1(x, t; D) + \lambda^2)u \\ &= (\lambda + a_1(x, t; D))f + g \in L^2(\omega) \end{aligned} \tag{3.23}$$

with the boundary condition given by

$$\alpha(\tilde{x})\left(\frac{\partial u}{\partial \boldsymbol{n}_t} - \lambda\sigma_1 u + \sigma_2 u\right) + (1 - \alpha(\tilde{x}))u = -\alpha(\tilde{x})\sigma_1 f \quad \text{on } \partial\omega. \tag{3.24}$$

Conversely, if $u \in H^2(\omega)$ satisfies (3.23) and (3.24) with $f \in V_\alpha(\omega)$, then by defining $v = \lambda u - f$, we see that $U = \{u, v\}$ satisfies (3.22). (It is clear that if $u \in H^2(\omega)$ satisfies (3.24) with $f \in H^1(\omega)$, then $u \in V_\alpha(\omega)$).

Hence, the solvability of (3.22) is equivalent to that of (3.23) with (3.24). Calculating as we have done in Proposition 3.1, we prove the desired result.

Lemma 3.7. *Let to be any fixed point in* $[0, T]$. *Suppose that* $F(t) \in \mathscr{D}_\alpha(t_0)$ *for all* $t \in [0, T]$ *and* $F(t)$, $\mathscr{A}(t_0)F(t)$ *are continuous in*

$\mathscr{V}_\alpha(t_0)$, *then for any* $U_0\in\mathscr{D}_\alpha(t_0)$, *there exists uniquely a solution* $U(t)$ *of the equation*

$$(3.25)\qquad \begin{cases} \dfrac{d}{dt}U(t)=\mathscr{A}(t_0)U(t)+F(t) \\ U(0)=U_0 \end{cases}$$

such that $U(t)\in\mathscr{D}_\alpha(t_0)$ *for all* $t\in[0, T]$ *and* $U(t)\in\mathscr{E}^1_t((0, T); \mathscr{V}_\alpha(t_0))$[3].

Proof. Let us denote by $\tilde{\mathscr{A}}(t_0)$, the operator $\mathscr{A}(t_0)$ with domain $\mathscr{D}_\alpha(t_0)$. Then, the closedness of the operator $\tilde{\mathscr{A}}(t_0)$ in $\mathscr{V}_\alpha(t_0)$ follows from the inequality (3.8). In order to prove the denseness of $\mathscr{D}_\alpha(t_0)$ in $\mathscr{V}_\alpha(t_0)$, it is sufficient to prove that the set $H^2_\alpha(t_0)$ is dense in $V_\alpha(\omega)$, where

$$H^2_\alpha(t_0)=\left\{u\in H^2(\omega)\,;\ \alpha(\tilde{x})\left(\frac{\partial u}{\partial \boldsymbol{n}_{t_0}}+\sigma_2(\tilde{x}, t_0)u\right)+(1-\alpha(\tilde{x}))u=0 \qquad \text{on } \partial\omega\right\}.$$

And the denseness of $H^2_\alpha(t_0)$ in $V_\alpha(\omega)$ is proved by integration by parts and by Proposition 3.1. Moreover, from (3.21) and Lemma 3.6, we have the estimate of

$$\|(\lambda I-\tilde{\mathscr{A}}(t_0))^{-1}\|_{\mathscr{V}_\alpha(t_0)}\leqq(\lambda-\lambda_0)^{-1} \qquad \text{for} \quad \lambda>\lambda_0.$$

So, we may apply the theorem of Hille-Yosida (for example, K. Yosida [26]) to the operator $\tilde{\mathscr{A}}(t_0)$. Q.E.D.

Now, we derive the energy inequalities for the solution with non-homogeneous boundary condition. As $\mathscr{D}_\alpha(t)$ changes with t and we will use the method of Cauchy's polygonal line, these inequalities play an essential role in the existence proof of the solution for (3.1) with (3.2).

Lemma 3.8. *Let* $u(x, t)$ *belong to* $\mathscr{E}^0_t(H^2(\omega))\cap\mathscr{E}^1_t(V_\alpha(\omega))\cap\mathscr{E}^2_t(L^2(\omega))$ *and satisfy*

$$(3.26)\qquad \begin{cases} L[u(x, t)]=f(x, t), \\ \alpha(\tilde{x})\left[\dfrac{\partial u}{\partial \boldsymbol{n}_t}-\sigma_1\dfrac{\partial u}{\partial t}+\sigma_2 u\right]+(1-\alpha(\tilde{x}))u=\alpha(\tilde{x})\phi(\tilde{x}, t)\,, \end{cases}$$

3) $\mathscr{E}^l_t((0, T); X)$ denotes the set of functions in t, l-times continuously differentiable as X-valued function on $(0, T)$. For the sake of brevity, we denote it only by $\mathscr{E}^l_t(X)$ abbreviating to write down explicitly the definition domain.

then we have

$$(3.27)\qquad \|U(t)\|^2_{\mathscr{V}_\alpha(t)} \leqq e^{c_0 t}\Big(\|U(0)\|^2_{\mathscr{V}_\alpha(0)} + \int_0^t \|f(\cdot,\tau)\|^2 d\tau + c_0\int_0^t\int_{\partial_N\omega} |\phi(\tilde{x},\tau)|^2 d\tilde{x}\, d\tau\Big)$$

where $U(t)=\{u(x,t), u'(x,t)\}$.

Proof.

$$\frac{d}{dt}\|U(t)\|^2_{\mathscr{V}_\alpha(t)} = \mathrm{Re}(U'(t), U(t))_{\mathscr{V}_\alpha(t)} + (U(t), U(t))_{\dot{\mathscr{H}}(t)}$$

$$= 2\mathrm{Re}(\mathscr{A}(t)U(t), U(t))_{\mathscr{V}_\alpha(t)} + 2\mathrm{Re}(F(t), U(t))_{\mathscr{V}_\alpha(t)} + (U(t), U(t))_{\dot{\mathscr{H}}(t)}$$

where

$$(U, U)_{\dot{\mathscr{H}}(t)} = \sum_{i,j=1}^n \Big(a'_{ij}(x,t)\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j}\Big) \text{ for } U=\{u, v\}\in\mathscr{H}(t)\,.$$

Evidently, we have

$$|(U(t), U(t))_{\dot{\mathscr{H}}(t)}| \leqq \text{const.}\, \|U(t)\|^2_{\mathscr{H}(t)},$$

$$|(U(t), F(t))_{\mathscr{V}_\alpha(t)}| \leqq \|F(t)\|^2_{\mathscr{V}_\alpha(t)} + \|U(t)\|^2_{\mathscr{V}_\alpha(t)}\,.$$

Using the inequality (3.20) and the above estimates, we have

$$\frac{d}{dt}\|U(t)\|^2_{\mathscr{V}_\alpha(t)} \leqq c_0\Big(\|U(t)\|^2_{\mathscr{V}_\alpha(t)} + \int_{\partial_N\omega} |\phi(\tilde{x},t)|^2 d\tilde{x}\Big) + \|F(t)\|^2_{\mathscr{V}_\alpha(t)}\,.$$

Since $\|F(t)\|^2_{\mathscr{V}_\alpha(t)} = \|f(\cdot,t)\|^2$, we have (3.27) by applying Gronwall's lemma to the differential inequality above.

We prove now the second energy inequality.

Proposition 3.9. *Let* $u(x,t)$ *belong to* $\mathscr{E}^0_t(H^2(\omega))\cap\mathscr{E}^1_t(V_\alpha(\omega))\cap \mathscr{E}^2_t(L^2(\omega))$ *for* $t\in[0, T+\delta_0]$ $(\delta_0>0)$. *If* $u(x,t)$ *satisfies* (3.26) *with* $f(x,t)\in\mathscr{E}^1_t(L^2(\omega))$ *and* $\phi(\tilde{x},t)\in\mathscr{E}^0_t(H^{1/2}(\partial_N\omega))\cap\mathscr{E}^1_t(L^2(\partial_N\omega))$, *then the second energy inequality*

$$(3.28)\qquad \|u(\cdot,t)\|^2_{H^2(\omega)} + \|u'(\cdot,t)\|^2_{H^1(\omega)} + \|u''(\cdot,t)\|^2 + \int_{\partial_N\omega} \frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})}(|u(\tilde{x},t)|^2 + |u'(\tilde{x},t)|^2)d\tilde{x}$$

$$\leqq c(T)\Big[\|u(\cdot,0)\|^2_{H^2(\omega)}+\|u'(\cdot,0)\|^2_{H^1(\omega)}+\|u''(\cdot,0)\|^2 + \int_{\partial_N\omega}\frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})}(|u(\tilde{x},0)|^2$$

$$+|u'(\tilde{x},0)|^2)d\tilde{x}+\|f(\cdot,0)\|^2+\int_0^t(\|f(\cdot,\tau)\|^2+\|f'(\cdot,\tau)\|^2)d\tau$$

$$+\int_0^t\int_{\partial_N\omega}(|\phi(\tilde{x},\tau)|^2+|\phi'(\tilde{x},\tau)|^2)d\tilde{x}d\tau+\int_0^t\langle\phi(\cdot,\tau)\rangle^2_{H^{1/2}(\partial_N\omega)}d\tau\Big]$$

holds for any $t\in[0,T]$ *where* $c(T)$ *is independent of* $u(x,t)$.

Proof. Putting $u_h(x,t)=h^{-1}(u(x,t+h)-u(x,t))$, we have

$$(3.29)\quad\begin{cases} L[u_h(x,t)]=f_h(x,t)-L_h[u(x,t+h)] \\ \alpha(\tilde{x})\Big[\dfrac{\partial u_h}{\partial \boldsymbol{n}_t}-\sigma_1\dfrac{\partial u_h}{\partial t}+\sigma_2 u_h\Big]+(1-\alpha(\tilde{x}))u_h \\ \qquad =\alpha(\tilde{x})\Big\{\phi_h(\tilde{x},t)-\Big[\Big(\dfrac{\partial}{\partial \boldsymbol{n}_t}\Big)_h-\sigma_{1h}\dfrac{\partial}{\partial t}+\sigma_{2h}\Big]u(\tilde{x},t+h)\Big\} \end{cases}$$

where

$$f_h(x,t)=h^{-1}(f(x,t+h)-f(x,t)),\ldots,$$

$$L_h[v(x,t+h)]=-a_{2h}(x,t:D)v(x,t+h)-a_{1h}(x,t:D)\frac{\partial}{\partial t}v(x,t+h),$$

$$a_{2h}(x,t:D)=-\sum_{i,j=1}^{n}\frac{\partial}{\partial x_j}\Big(h^{-1}(a_{ij}(x,t+h)-a_{ij}(x,t))\frac{\partial}{\partial x_j}\Big)+\cdots,$$

$$\Big(\frac{\partial}{\partial \boldsymbol{n}_t}\Big)_h=\sum_{i,j=1}^{n}h^{-1}(a_{ij}(\tilde{x},t+h)-a_{ij}(\tilde{x},t))\nu_j\frac{\partial}{\partial x_i},\ldots \text{ etc}.$$

By applying Lemma 3.8 to the equation (3.29), we get

$$\|U_h(t)\|^2_{\mathscr{V}_{\alpha(t)}}\leqq e^{c_0t}\Big\{\|U_h(0)\|^2_{\mathscr{V}_{\alpha(0)}}+\int_0^t\|f_h(\cdot,\tau)-L_h[u(\cdot,\tau+h)]\|^2d\tau$$

$$+c_0\int_0^t\int_{\partial_N\omega}\Big|\phi_h(\tilde{x},\tau)-\Big[\Big(\frac{\partial}{\partial \boldsymbol{n}_t}\Big)_h-\sigma_{1h}\frac{\partial}{\partial\tau}+\sigma_{2h}\Big]u(\tilde{x},\tau+h)\Big|^2d\tilde{x}d\tau\Big\}.$$

As all functions in the above inequality are sufficiently regular, we may make h tend vers 0 and we have

$$(3.30)\quad \begin{cases} \|U'(t)\|^2_{\mathscr{V}_{\alpha(t)}} \leqq e^{c_0 t}\Big\{\|U'(0)\|^2_{\mathscr{V}_{\alpha(0)}} + \int_0^t \|f'(\cdot,\tau) - L'[u(\cdot,\tau)]\|^2 d\tau \\ \quad + c_0 \int_0^t \int_{\partial_N \omega} \Big|\phi'(\tilde{x},\tau) - \Big[\Big(\frac{\partial}{\partial \boldsymbol{n}_t}\Big)' - \sigma_1' \frac{\partial}{\partial \tau} + \sigma_2'\Big] u(\tilde{x},\tau)\Big|^2 d\tilde{x}\, d\tau\Big\} \end{cases}$$

where

$$L'[u] = a_1'(x, t: D)u' + a_2'(x, t: D)u,$$

$$a_2'(x, t: D) = -\sum_{i,j=1}^n \frac{\partial}{\partial x_j}\Big(a_{ij}'(x,t)\frac{\partial}{\partial x_j}\Big) + \cdots,$$

$$\Big(\frac{\partial}{\partial \boldsymbol{n}_t}\Big)' = \sum_{i,j=1}^n a_{ij}'(x,t)\nu_j \frac{\partial}{\partial x_i}.$$

Remark that

$$\|L'[u]\|^2 \leqq \text{const.}\ |||U|||_2^2,$$

$$\int_{\partial_N \omega} \Big|\Big(\Big(\frac{\partial}{\partial \boldsymbol{n}_t}\Big)' - \sigma_1'\frac{\partial}{\partial t} + \sigma_2'\Big) u(\tilde{x},t)\Big|^2 d\tilde{x} \leqq \text{const.}\ |||U(t)|||_2^2$$

and

$$\|U'(0)\|^2_{\mathscr{H}(0)} \leqq \text{const.}\,(|||U(0)|||_2^2 + \|f(\cdot,0)\|^2).$$

Combining these estimates to (3.30), we have

$$(3.31)\quad \|U'(t)\|^2_{\mathscr{V}_{\alpha(t)}} \leqq c_1 e^{ct}\Big\{|||U(0)|||_2^2 + \|f(\cdot,0)\|^2 + \int_{\partial_N \omega} \frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})} |u'(\tilde{x},0)|^2 d\tilde{x} + \int_0^t (|||U(\tau)|||_2^2 + \|f'(\cdot,\tau)\|^2)\, d\tau + \int_0^t \int_{\partial_N \omega} |\phi'(\tilde{x},\tau)|^2 d\tilde{x}\, d\tau\Big\}.$$

On the other hand, by Lemma 3.3, we have

$$\begin{aligned} &|||U(t)|||_2^2 + \|U'(t)\|^2_{\mathscr{V}_{\alpha(t)}} + \int_{\partial_N \omega} \frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})} |u(\tilde{x},t)|^2 d\tilde{x} \\ &\leqq c(\|\mathscr{A}(t)U(t)\|^2_{\mathscr{H}(t)} + \|U(t)\|^2_{\mathscr{V}_{\alpha(t)}} + \langle \phi(\cdot,t)\rangle^2_{H^{1/2}(\partial_N \omega)}) + \|U'(t)\|^2_{\mathscr{V}_{\alpha(t)}} \\ &\leqq c(\|U'(t)\|^2_{\mathscr{V}_{\alpha(t)}} + \|U(t)\|^2_{\mathscr{V}_{\alpha(t)}} + \|f(\cdot,t)\|^2 + \langle \phi(\cdot,t)\rangle^2_{H^{1/2}(\partial_N \omega)}), \end{aligned}$$

by putting the estimates (3.27) and (3.31) into the above,

$$\leqq c\Big\{ |||U(0)|||_2^2 + \|f(\cdot, 0)\|^2 + \int_{\partial_N\omega} \frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})} |u'(\tilde{x}, 0)|^2 d\tilde{x} + \int_0^t (|||U(\tau)|||_2^2 + \|f'(\cdot, \tau)\|^2) d\tau$$

$$+ \int_0^t \int_{\partial_N\omega} |\phi'(\tilde{x}, \tau)|^2 d\tilde{x} d\tau + \|U(0)\|^2_{\mathscr{V}_\alpha(0)} + \int_0^t \|f(\cdot, \tau)\|^2 d\tau$$

$$+ \int_0^t \int_{\partial_N\omega} |\phi(\tilde{x}, \tau)|^2 d\tilde{x} d\tau + \|f(\cdot, t)\|^2 + \langle \phi(\cdot, t)\rangle^2_{H^{1/2}(\partial_N\omega)} \Big\}.$$

Remarking the obvious estimates below

$$\|f(\cdot, t)\|^2 \leqq 2T\Big(\|f(\cdot, 0)\|^2 + \int_0^t f'(\cdot, \tau)\|^2 d\tau \Big),$$

$$\langle \phi(\cdot, t)\rangle^2_{L^2(\partial_N\omega)} \leqq 2T\Big(\langle \phi(\cdot, 0)\rangle^2_{L^2(\partial_N\omega)} + \int_0^t \langle \phi'(\cdot, \tau)\rangle^2_{L^2(\partial_N\omega)} d\tau \Big),$$

and applying Gronwall's lemma to the above inequality, we have the desired inequality (3.28).

§4. Existence of the Solution and its Regularity

This section is devoted to proving the following theorems by tracing the idea of Ikawa [7] with some modifications.

Theorem 4.1. *Given data* $\{u_0(x), u_1(x), f(x, t), \phi(\tilde{x}, t)\} \in H^2(\omega) \times V_\alpha(\omega) \times \mathscr{E}_t^1(L^2(\omega)) \times (\mathscr{E}_t^0(H^{1/2}(\partial_N\omega)) \cap \mathscr{E}_t^1(L^2(\partial_N\omega)))$. *If the data are compatible of order* 0 *at* $t=0$, *i.e. the data satisfy*

$$\alpha(\tilde{x})\Big(\frac{\partial}{\partial n_0} u_0(\tilde{x}) - \sigma_1(\tilde{x}, 0) u_1(\tilde{x}) + \sigma_2(\tilde{x}, 0) u_0(\tilde{x}) \Big) + (1-\alpha(\tilde{x})) u_0(\tilde{x}) = \alpha(\tilde{x}) \phi(\tilde{x}, 0) \quad on \ \partial\omega, \tag{4.1}$$

then there exists one and only one solution $u(x, t)$ *of* (3.1) *with* (3.2), *belonging to the space* $\mathscr{E}_t^0(H^2(\omega)) \cap \mathscr{E}_t^1(V_\alpha(\omega)) \cap \mathscr{E}_t^2(L^2(\omega))$.

Theorem 4.2. *Suppose that the data* $\{u_0(x), u_1(x), f(x, t), \phi(\tilde{x}, t)\}$

belong to the space $H^{m+2}(\omega)\times\{H^{m+1}(\omega)\cap V_\alpha(\omega)\}\times\left\{\bigcap_{k=0}^{m+1}\mathscr{E}_t^{m+1-k}(H^k(\omega))\right\}$ $\times\left\{\bigcap_{k=0}^{m}\mathscr{E}_t^{m-k}(H^{k+1/2}(\partial_N\omega))\cap\mathscr{E}_t^{m+1}(L^2(\partial_N\omega))\right\}$ *where* m *is a non-negative integer. If they are compatible of order* m *at* $t=0$, *then the solution* $u(x, t)$ *of* (3.1) *with* (3.2) *exists in the space* $\bigcap_{k=0}^{m+1}\mathscr{E}_t^{m+2-k}(H^k(\omega)\cap V_\alpha(\omega))$ $\cap\,\mathscr{E}_t^{m+2}(L^2(\omega))$.

In order to prove Theorem 4.1, we begin by considering the operators $L(t_0)$ and $\mathscr{B}_\alpha(t_0)$ below.

$$L(t_0)=\frac{\partial^2}{\partial t^2}+a_1(x, t_0; D)\frac{\partial}{\partial t}+a_2(x, t_0; D),$$

$$\mathscr{B}_\alpha(t_0)=\left[\alpha(\tilde{x})\left(\frac{\partial}{\partial \boldsymbol{n}_{t_0}}+\sigma_2(\tilde{x}, t_0)\right)+1-\alpha(\tilde{x})\quad -\alpha(\tilde{x})\sigma_1(\tilde{x}, t_0)\right].$$

First of all, we shall treat the existence of the solution for $L(t_0)$ and $\mathscr{B}_\alpha(t_0)$ with non-zero boundary datum.

Proposition 4.3. *Let* $\{u_0(x), u_1(x)\}\in H^2(\omega)\times V_\alpha(\omega), f(x, t)\in\mathscr{E}_t^1(L^2(\omega))$ *and* $\phi(\tilde{x}, t)\in\mathscr{E}_t^0(H^{1/2}(\partial_N\omega))\cap\mathscr{E}_t^1(L^2(\partial_N\omega))$ *be given. If the condition*

$$\begin{aligned}(4.2)\quad \alpha(\tilde{x})\left(\frac{\partial}{\partial \boldsymbol{n}_{t_0}}u_0(\tilde{x})-\sigma_1(\tilde{x}, t_0)u_1(\tilde{x})+\sigma_2(\tilde{x}, t_0)u_0(\tilde{x})\right)&\\ +(1-\alpha(\tilde{x}))u_0(\tilde{x})&\\ =\alpha(\tilde{x})\phi(\tilde{x}, 0)\qquad \text{on}\quad \partial\omega&\end{aligned}$$

is satisfied, then there exists one and only one solution $u(x, t)\in$ $\mathscr{E}_t^0(H^2(\omega))\cap\mathscr{E}_t^1(V_\alpha(\omega))\cap\mathscr{E}_t^2(L^2(\omega))$ *of the problem below:*

$$(4.3)\qquad \begin{cases} L(t_0)[u(x, t)]=f(x, t), & \text{in}\quad \omega\times(0, T)\\ \mathscr{B}_\alpha(t_0)U(t)=\alpha(\tilde{x})\phi(\tilde{x}, t), & \text{on}\quad \partial\omega\times[0, T]\\ u(x, 0)=u_0(x),\quad u_t(x, 0)=u_1(x) & \end{cases}$$

where $U(t)=\{u(x, t), u'(x, t)\}$.

Proof. When $\phi(\tilde{x}, t)\equiv 0$, the condition (4.2) means that $\{u_0(x), u_1(x)\}$ $\in\mathscr{D}_\alpha(t_0)$. So if $f(x, t)\in\mathscr{E}_t^1(V_\alpha(\omega))$, then Lemma 3.7 assures the existence of the solution of (3.25). Denoting the first component of $U(t)$ in (3 .25) as $u(x, t)$, we have a solution of (4.3) in $\mathscr{E}_t^1(V_\alpha(\omega)) \cap \mathscr{E}_t^2(L^2(\omega))$ and $u(x, t) \in H^2(\omega)$. That $u(x, t) \in \mathscr{E}_t^0(H^2(\omega))$ is prove by applying the inequality (3.18) to the problem

$$\begin{cases} \mathscr{A}(t_0)U(t) = \dfrac{d}{dt}U(t) - F(t) \in \mathscr{E}_t^0(\mathscr{V}_\alpha(t_0)), \\ \mathscr{B}_\alpha(t_0)U(t) = 0. \end{cases}$$

Moreover, the additional condition $f(x, t) \in \mathscr{E}_t^1(V_\alpha(\omega))$ is removed by using the fact that $\mathscr{E}_t^1(V_\alpha(\omega))$ is dense in $\mathscr{E}_t^1(L^2(\omega))$ and the second energy inequality (3.28) holds for (4.3).

When $\phi(\tilde{x}, t) \not\equiv 0$, we assume, first of all, that $\phi(\tilde{x}, t)$ is sufficiently smooth so that we may find a function $w(x, t) \in \mathscr{E}_t^3(H^2(\omega))$ satisfying

$$\alpha(\tilde{x})B(t_0)w(\tilde{x}, t) + (1 - \alpha(\tilde{x}))w(\tilde{x}, t) = \alpha(\tilde{x})\phi(\tilde{x}, t). \qquad \text{on} \quad \partial\omega \times [0, T].$$

Then, by the above result, we know that there exists a function $v(x, t) \in \mathscr{E}_t^0(H^2(\omega)) \cap \mathscr{E}_t^1(V_\alpha(\omega)) \cap \mathscr{E}_t^2(L^2(\omega))$ satisfying

$$\begin{cases} L(t_0)[v(x, t)] = f(x, t) - L(t_0)[w(x, t)], \\ v(x, 0) = u_0(x) - w(x, 0), \\ v_t(x, 0) = u_1(x) - w_t(x, 0), \\ \alpha(\tilde{x})B(t_0)v(\tilde{x}, t) + (1 - \alpha(\tilde{x}))v(\tilde{x}, t) = 0, \end{cases}$$

since $\{u_0(x) - w(x, 0), u_1(x) - w_t(x, 0)\} \in \mathscr{D}_\alpha(t_0)$ and $f(x, t) - L(t_0)[w(x, t)] \in \mathscr{E}_t^1(L^2(\omega))$. Putting $u(x, t) = v(x, t) + w(x, t)$, we have a solution of (4.3) belonging to the space $\mathscr{E}_t^0(H^2(\omega)) \cap \mathscr{E}_t^1(V_\alpha(\omega)) \cap \mathscr{E}_t^2(L^2(\omega))$.

We remark that if there exist a sequence of initial data $\{u_{k_0}(x), u_{k_1}(x)\} \in H^2(\omega) \times V_\alpha(\omega)$ and a sufficiently smooth boundary datum $\phi_k(\tilde{x}, t)$ such that the following condition are satisfied, i.e.

(4.4) $\quad \mathscr{B}_\alpha(t_0)\{u_{k0}(x), u_{k1}(x)\} = \alpha(\tilde{x})\phi_k(\tilde{x}, 0),$

(4.5) $\quad \{u_{k0}(x), u_{k1}(x)\} \longrightarrow \{u_0(x), u_1(x)\}$ in $H^2(\omega) \times V_\alpha(\omega)$, and

(4.6) $\quad \phi_k(\tilde{x}, t) \longrightarrow \phi(\tilde{x}, t)$ in $\mathscr{E}_t^0(H^{1/2}(\partial_N\omega)) \cap \mathscr{E}_t^1(L^2(\partial_N\omega)),$

then this proposition is proved. In fact, the solution $u_k(x, t)$ of (4.3) for $\{u_{k0}(x), u_{k1}(x)\}$ and $\phi_k(\tilde{x}, t)$, exists and the sequence $\{u_k(x, t)\}$ forms a Cauchy sequence in $\mathscr{E}_t^0(H^2(\omega)) \cap \mathscr{E}_t^1(V_\alpha(\omega)) \cap \mathscr{E}_t^2(L^2(\omega))$ where we apply the second energy inequality to the function $u_k(x, t) - u_l(x, t)$. Then, it is clear that the limit function $u(x, t)$ of the Cauchy sequence $\{u_k(x, t)\}$ is the required solution of (4.3) for $\{u_0(x), u_1(x)\}$ and $\phi(\tilde{x}, t)$.

Now, let us construct such $\{u_{k0}(x), u_{k1}(x)\}$ and $\phi_k(\tilde{x}, t)$. It is clear that there exists a sequence $\phi_k(\tilde{x}, t)$ of sufficiently smooth functions in $\mathscr{E}_t^1(H^{1/2}(\partial_N\omega))$, which tends to $\phi(\tilde{x}, t)$ in $\mathscr{E}_t^0(H^{1/2}(\partial_N\omega)) \cap \mathscr{E}_t^1(L^2(\partial_N\omega))$. So we have

$$\langle \phi_k(\tilde{x}, 0) - \phi(\tilde{x}, 0) \rangle_{H^{1/2}(\partial_N\omega)} \longrightarrow 0 \quad \text{as} \quad k \to \infty. \tag{4.7}$$

If ω is the interior domain of $\partial\omega$, then by Proposition 3.1, there exists a function $\tilde{u}_k(x) \in H^2(\omega)$ satisfying

$$\begin{cases} (-a_2(x, t_0; D) + \lambda_0)\tilde{u}_k = 0 & \text{in } \omega, \\ \alpha(\tilde{x}) \dfrac{\partial \tilde{u}_k}{\partial \boldsymbol{n}_{t_0}} + (1 - \alpha(\tilde{x}))\tilde{u}_k = \alpha(\tilde{x})(\phi_k(\tilde{x}, 0) - \phi(\tilde{x}, 0)) & \text{on } \partial\omega. \end{cases} \tag{4.8}$$

Moreover, it satisfies, by (b) of Remark 3.2, that

$$\|\tilde{u}_k\|_{H^2(\omega)} \leqq C \langle \phi_k(\tilde{x}, 0) - \phi(\tilde{x}, 0) \rangle_{H^{1/2}(\partial_N\omega)}. \tag{4.9}$$

Putting $u_{k0}(x) = u_0(x) + \tilde{u}_k(x)$, $u_{k1}(x) = u_1(x)$, we construct the desired functions.

If ω is the exterior domain of $\partial\omega$, we take a sufficiently smooth hypersurface $\partial\omega_1$ containing $\partial\omega$ in its interior and we denote by ω_1 the domain surrounded by $\partial\omega$ and $\partial\omega_1$. We solve the problem below.

$$\begin{cases} (-a_2(x, t_0; D) + \lambda_0)\tilde{u}_k = 0 & \text{in } \omega_1, \\ \alpha(\tilde{x}) \dfrac{\partial \tilde{u}_k}{\partial \boldsymbol{n}_{t_0}} + (1 - \alpha(\tilde{x}))\tilde{u}_k = \alpha(\tilde{x})(\phi_k(\tilde{x}, 0) - \phi_k(\tilde{x}, 0)) & \text{on } \partial\omega, \\ \alpha(\tilde{x}) \dfrac{\partial \tilde{u}_k}{\partial \boldsymbol{n}_{t_0}} + (1 - \alpha(\tilde{x}))\tilde{u}_k = 0 & \text{on } \partial\omega_1. \end{cases}$$

Let $\beta(x)$ be a C^∞ function such that $\beta(x) \equiv 1$ near $\partial\omega$ and $\beta(x) \equiv 0$ near and outside of $\partial\omega_1$. Then, putting

$$\begin{cases} u_{k0}(x) = u_0(x) + \beta(x)\tilde{u}_k(x), \\ u_{k1}(x) = u_1(x), \end{cases}$$

we construct the desired functions. Q.E.D.

Proposition 4.4. *Let* $u(x, t)$ *belong to* $\mathscr{E}_t^0(H^2(\omega)) \cap \mathscr{E}_t^1(V_\alpha(\omega)) \cap \mathscr{E}_t^2(L^2(\omega))$ *for* $t \geqq t_0$. *If it satisfies*

$$(4.10)\quad \begin{cases} L(t_0)[u(x, t)] = f(x, t) \in \mathscr{E}_t^1(\mathring{H}^1(\omega)), \\ \alpha(\tilde{x})B(t_0)u(\tilde{x}, t) + (1-\alpha(\tilde{x}))u(\tilde{x}, t) = \alpha(\tilde{x})\phi(\tilde{x}, t), \\ \qquad\qquad \textit{with} \quad \phi(x, t) \in \mathscr{E}_t^1(H^{1/2}(\partial_N\omega)), \end{cases}$$

then for $t \geqq t_0$, *the following estimate holds*

$$(4.11)\quad |||U(t)|||^2_{\mathscr{V}_\alpha(t_0)} \leqq e^{C_0(t-t_0)}\Big[|||U(t_0)|||^2_{\mathscr{V}_\alpha(t_0)} + \int_{t_0}^t (\|F(\tau)\|^2_{\mathscr{H}(t_0)} + \|\mathscr{A}(t_0)F(\tau)\|^2_{\mathscr{H}(t_0)})d\tau + 2C_0\int_{t_0}^t \langle\!\langle \phi(\cdot, \tau)\rangle\!\rangle^2 d\tau\Big]$$

where $\mathring{H}^l(\omega)$ *is the closure of* $C_0^\infty(\omega)$ *in* $H^l(\omega)$,

$$|||U|||^2_{\mathscr{V}_\alpha(t_0)} \equiv \|\mathscr{A}(t_0)U\|^2_{\mathscr{H}(t_0)} + \|U\|^2_{\mathscr{H}(t_0)} + \langle\phi\rangle^2_{H^{1/2}(\partial_N\omega)} + \int_{\partial_N\omega} \frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})}(|u(\tilde{x})|^2 + |v(\tilde{x})|^2)d\tilde{x}$$

for $U = \{u(x), v(x)\}$, $\mathscr{B}_\alpha(t_0)U = \alpha(\tilde{x})\phi(\tilde{x})$ *and*

$$\langle\!\langle \phi(\cdot, t)\rangle\!\rangle^2 = \langle\phi(\cdot, t)\rangle^2_{H^{1/2}(\partial_N\omega)} + \langle\phi'(\cdot, t)\rangle^2_{H^{1/2}(\partial_N\omega)}.$$

Proof. As we have done in Lemma 3.8, we have

$$(4.12)\quad \|U(t)\|^2_{\mathscr{V}_\alpha(t_0)} \leqq e^{C_0(t-t_0)}\Big[|||U(t_0)|||^2_{\mathscr{V}_\alpha(t_0)} + \int_{t_0}^t \|F(\tau)\|^2_{\mathscr{H}(t_0)}d\tau + C_0\int_{t_0}^t\int_{\partial_N\omega} |\phi(\tilde{x}, \tau)|^2 d\tilde{x}d\tau\Big]$$

where $t = t_0$ is an initial plane for the problem (4.10). Now, suppose that $u'(x, t)$ belongs also to $\mathscr{E}_t^0(H^2(\omega)) \cap \mathscr{E}_t^1(V_\alpha(\omega)) \cap \mathscr{E}_t^2(L^2(\omega))$.
Then, remarking that $F(t) \in \mathscr{D}_\alpha(t_0)$ for all t, we have

$$\mathscr{A}(t_0)U(t) = \frac{d}{dt}U(t) - F(t)$$

$$\mathscr{B}_\alpha(t_0)(\mathscr{A}(t_0)U(t)) = \mathscr{B}_\alpha(t_0)\Big(\frac{d}{dt}U(t) - F(t)\Big)$$

$$= \frac{d}{dt}\mathscr{B}_\alpha(t_0)U(t) = \alpha(\tilde{x})\phi'(\tilde{x}, t).$$

Calculating $\frac{d}{dt}\|\mathscr{A}(t_0)U(t)\|^2_{\mathscr{V}_\alpha(t_0)}$ as in proving Lemma 3.8, we have,

$$\text{(4.13)} \quad \|\mathscr{A}(t_0)U(t)\|^2_{\mathscr{V}_\alpha(t_0)} \leqq e^{C_0(t-t_0)}\Big[\|\mathscr{A}(t_0)U(t_0)\|^2_{\mathscr{V}_\alpha(t_0)} + \int_{t_0}^t \|\mathscr{A}(t_0)F(\tau)\|^2_{\mathscr{H}(t_0)}d\tau + C_0\int_{t_0}^t\int_{\partial_N\omega}|\phi'(\tilde{x}, \tau)|^2 d\tilde{x}d\tau\Big]$$

where $\|\mathscr{A}(t_0)F(\tau)\|_{\mathscr{V}_\alpha(t_0)} = \|\mathscr{A}(t_0)F(\tau)\|_{\mathscr{H}(t_0)}$ for any τ because $f(t) \in \mathscr{E}^1_t(\overset{\circ}{H}{}^1(\omega))$. Putting the estimates (4.12) and (4.13) into the definition of $|||\cdot|||_{\mathscr{V}_\alpha(t_0)}$, we have

$$|||U(t)|||^2_{\mathscr{V}_\alpha(t_0)} \equiv \|U(t)\|^2_{\mathscr{V}_\alpha(t_0)} + \|\mathscr{A}(t_0)U(t)\|^2_{\mathscr{V}_\alpha(t_0)} + \langle\phi(\cdot, t)\rangle^2_{H^{1/2}(\partial_N\omega)}$$

$$\leqq e^{C_0(t-t_0)}\Big[|||U(t_0)|||^2_{\mathscr{V}_\alpha(t_0)} + \int_{t_0}^t(\|F(\tau)\|^2_{\mathscr{H}(t_0)} + \|\mathscr{A}(t_0)F(\tau)\|^2_{\mathscr{H}(t_0)})d\tau + C_0\int_{t_0}^t\int_{\partial_N\omega}(|\phi(\tilde{x}, \tau)|^2 + |\phi'(\tilde{x}, \tau)|^2)d\tau + \langle\phi(\cdot, t)\rangle^2_{H^{1/2}(\partial_N\omega)} - \langle\phi(\cdot, t_0)\rangle^2_{H^{1/2}(\partial_N\omega)}\Big]$$

$$\leqq e^{C_0(t-t_0)}\Big[|||U(t_0)|||^2_{\mathscr{V}_\alpha(t_0)} + \int_{t_0}^t(\|F(\tau)\|^2_{\mathscr{H}(t_0)} + \|\mathscr{A}(t_0)F(\tau)\|^2_{\mathscr{H}(t_0)})d\tau + 2C_0\int_{t_0}^t \langle\!\langle\phi(\cdot, \tau)\rangle\!\rangle^2 d\tau\Big].$$

We may remove easily the additional condition that $u'(x, t)$ belongs to space $\mathscr{E}^0_t(H^2(\omega)) \cap \mathscr{E}^1_t(V_\alpha(\omega)) \cap \mathscr{E}^2_t(L^2(\omega))$ by mollifying the function $u(x, t)$ with respect to t. (Here, we use the fact that the operators in (4.10) have the coefficients independent of t.) Q.E.D.

It is clear that there exists a constant $C'_0 > 0$ such that for any $U \in \boldsymbol{E}_1$ and $t, t' \in [0, T]$, we have

$$(4.14)\qquad \|U\|^2_{\mathscr{X}(t')} \leqq (1+C_0'|t-t'|)\|U\|^2_{\mathscr{X}(t)}.$$

Moreover, as $\alpha(\tilde{x})$ is independent of t, we have readily

$$(4.15)\qquad \|U\|^2_{\mathscr{V}_\alpha(t')} \leqq (1+C_0'|t-t'|)\|U\|^2_{\mathscr{V}_\alpha(t)} \quad \text{for any} \quad U\in H^1\times V_\alpha \quad \text{and} \quad t,\ t'\in[0,\ T].$$

Now, we prove the existence of the solution for zero initial data. That is,

Lemma 4.5. *Let* $f(x, t)\in\mathscr{E}^1_t(\overset{\circ}{H}{}^1(\omega))$ *and* $\phi(\tilde{x}, t)\in\mathscr{E}^1_t(H^{1/2}(\partial_N\omega))$. *If* $\phi(\tilde{x}, 0)=0$, *the mixed problem*

$$(4.16)\qquad \begin{cases} L[u(x,t)]=f(x,t), \\ \alpha(\tilde{x})B(t)u(\tilde{x},t)+(1-\alpha(\tilde{x}))u(\tilde{x},t)=\alpha(\tilde{x})\phi(\tilde{x},t), \\ u(x,0)=\dfrac{\partial u}{\partial t}(x,0)=0 \end{cases}$$

has a unique solution $u(x, t)$ *in the space*

$$\mathscr{E}^0_t(H^2(\omega))\cap\mathscr{E}^1_t(V_\alpha(\omega))\cap\mathscr{E}^2_t(L^2(\omega)).$$

Proof. Let $\triangle_k$: $t_0=0<t_1<t_2<\cdots<t_k=T$ be the subdivision of $[0, T]$ into k equal parts. $u_k(x, t)$ is Cauchy's polygonal line for this subdivision, which is constructed as follows: Let $u_{k0}(x, t)$, defined on $[t_0, t_1]$, be the solution of

$$(4.17)\qquad \begin{cases} L(t_0)[u_{k0}(x,t)]=f(x,t) \quad \text{in} \quad \omega\times[t_0,t_1], \\ \alpha(\tilde{x})B(t_0)u_{k0}(\tilde{x},t)+(1-\alpha(\tilde{x}))u_{k0}(\tilde{x},t)=\alpha(\tilde{x})\phi(\tilde{x},t) \ \text{on}\ \partial\omega\times[t_0,t_1], \\ u_{k0}(x,t_0)=u'_{k0}(x,t_0)=0, \end{cases}$$

and for $i\geqq 1$, $u_{ki}(x, t)$, defined on $[t_i, t_{i+1}]$, be the solution of

$$(4.18)\qquad \begin{cases} L(t_i)[u_{ki}(x,t)]=f(x,t) \quad \text{in} \quad \omega\times(t_i,t_{i+1}] \\ \alpha(\tilde{x})B(t_i)u_{ki}(\tilde{x},t)+(1-\alpha(\tilde{x}))u_{ki}(\tilde{x},t)=\alpha(\tilde{x})\Big\{\phi(\tilde{x},t) \\ \quad +\dfrac{t_{i+1}-t}{t_{i+1}-t_i}[(B(t_i)-B(t_{i-1}))u_{k\,i-1}(\tilde{x},t)]_{t=t_i}\Big\} \quad \text{on} \quad \partial\omega\times[t_i,t_{i+1}] \\ u_{ki}(x,t_i)=u_{k\,i-1}(x,t_i), \quad u'_{ki}(x,t_i)=u'_{k\,i-1}(x,t_i). \end{cases}$$

The existence of such $u_{ki}(x, t)$ $(i=0, 1,\dots, k-1)$ is assured by Proposition 4.3, since the condition (4.2) is satisfied at each $t=t_i$. Now, we define $u_k(x, t)$ as

$$u_k(x, t)=u_{ki}(x, t) \qquad \text{if} \quad t\in[t_i, t_{i+1}].$$

Then, we have

$$u_k(x, t)\in\mathscr{E}_t^0(H^2(\omega))\cap\mathscr{E}_t^1(V_\sigma(\omega)) \qquad \text{for} \quad t\in[0, T]$$

and

$$u_k(x, t)\in\mathscr{E}_t^2(L^2(\omega)) \qquad \text{if} \quad t\neq t_i.$$

So, we have

$$u_k(x, t)\in H^2(\omega\times(0, T)).$$

Claim 1. The set of functions $\{u_k(x, t)\}$ $k=1, 2,\dots$ forms a bounded set in $H^2(\omega\times(0, T))$.

In order to prove this, we shall prove the following inequality

$$\text{(4.19)} \quad \left\{ \begin{aligned} &|||U_{ki}(t)|||^2_{\mathscr{V}_\alpha(t_i)}\leqq e^{C_0t}\Big(1+C_1\frac{T}{k}\Big)^{3i}\Big[\int_0^t(\|F(\tau)\|^2_{\mathscr{H}(t_i)} \\ &\qquad\qquad +\|\mathscr{A}(t_i)F(\tau)\|^2_{\mathscr{H}(t_i)})d\tau+2C_0\int_0^t\langle\!\langle\phi(\cdot, \tau)\rangle\!\rangle^2d\tau\Big] \end{aligned}\right.$$

for $t\in[t_i, t_{i+1}]$ $(i=0, 1, 2,\dots, k-1)$ where the constant C_1 is defined in the following. For $i=0$, this is nothing but the inequality (4.11). Suppose that (4.19) holds for $i-1$, then taking t as t_i, we have

$$\begin{aligned} \text{(4.20)} \quad &|||U_{k\,i-1}(t_i)|||^2_{\mathscr{V}_\alpha(t_{i-1})} \\ &\leqq e^{C_0t_i}\Big(1+C_1\frac{T}{k}\Big)^{3(i-1)}\Big[\int_0^{t_i}(\|F(\tau)\|^2_{\mathscr{H}(t_{i-1})} \\ &\qquad +\|\mathscr{A}(t_{i-1})F(\tau)\|^2_{\mathscr{H}(t_{i-1})})d\tau+2C_0\int_0^{t_i}\langle\!\langle\phi(\cdot, \tau)\rangle\!\rangle^2d\tau\Big]. \end{aligned}$$

Remarking that $\mathscr{B}_\alpha(t_i)U_{k\,i}(t_i)=\mathscr{B}_\alpha(t_i)U_{k\,i-1}(t_i)$ and $\mathscr{B}_\alpha(t_{i-1})U_{k\,i-1}(t_i)=\alpha(\tilde{x})\phi(\tilde{x}, t_i)$, we have

$$\text{(4.21)} \qquad |||U_{k\,i}(t_i)|||^2_{\mathscr{V}_\alpha(t_i)}\leqq\Big(1+C_1\frac{T}{k}\Big)|||U_{k\,i-1}(t_i)|||^2_{\mathscr{V}_\alpha(t_{i-1})}.$$

In fact,

$$\begin{aligned}
&|||U_{k\,i}(t_i)|||^2_{\mathscr{V}_\alpha(t_i)} = |||U_{k\,i-1}(t_i)|||^2_{\mathscr{V}_\alpha(t_i)} \\
&\quad = \|U_{k\,i-1}(t_i)\|^2_{\mathscr{V}_\alpha(t_i)} + \|\mathscr{A}(t_i)U_{k\,i-1}(t_i)\|^2_{\mathscr{V}_\alpha(t_i)} \\
&\qquad + \langle \phi(\cdot, t_i) + (B(t_i) - B(t_{i-1}))u_{k\,i-1}(t)|_{t=t_i}\rangle^2_{H^{1/2}(\partial_N \omega)} \\
&\quad \leqq \left(1 + C_0' \frac{T}{k}\right)\{\|U_{k\,i-1}(t_i)\|^2_{\mathscr{V}_\alpha(t_{i-1})} + \|\mathscr{A}(t_{i-1})U_{k\,i-1}(t_i)\|^2_{\mathscr{V}_\alpha(t_{i-1})} \\
&\qquad + 2\|\mathscr{A}(t_{i-1})U_{k\,i-1}(t_i)\|_{\mathscr{V}_\alpha(t_{i-1})}\|(\mathscr{A}(t_i) - \mathscr{A}(t_{i-1}))U_{k\,i-1}(t_i)_{\mathscr{H}(t_{i-1})} \\
&\qquad + \|(\mathscr{A}(t_i) - \mathscr{A}(t_{i-1}))U_{k\,i-1}(t_i)\|^2_{\mathscr{H}(t_{i-1})}\} \\
&\qquad + \langle \phi(\cdot, t_i)\rangle^2_{H^{1/2}(\partial_N \omega)} + 2\langle \phi(\cdot, t_i)\rangle_{H^{1/2}(\partial_N \omega)} \langle (B(t_i) \\
&\qquad\qquad - B(t_{i-1}))u_{k\,i-1}(t)|_{t=t_i}\rangle_{H^{1/2}(\partial_N \omega)} \\
&\qquad + \langle (B(t_i) - B(t_{i-1}))u_{k\,i-1}(t)|_{t=t_i}\rangle^2_{H^{1/2}(\partial_N \omega)} \\
&\quad \leqq \left(1 + C_0' \frac{T}{k}\right)\left(1 + 2C_0'' \frac{T}{k} + C_0''^2 \frac{T^2}{k^2}\right)|||U_{k\,i-1}(t_i)|||^2_{\mathscr{V}_\alpha(t_{i-1})}
\end{aligned}$$

where we use the estimates below combining with (3.18).

$$\begin{aligned}
\|(\mathscr{A}(t_i) - \mathscr{A}(t_{i-1}))U_{k\,i-1}(t_i)\|^2_{\mathscr{H}(t_{i-1})} &\leqq \text{const.}\left(\frac{T}{k}\right)^2 |||U_{k\,i-1}(t_i)|||^2_2 \\
&\leqq C_0''^2 \left(\frac{T}{k}\right)^2 |||U_{k\,i-1}(t_i)|||^2_{\mathscr{V}_\alpha(t_{i-1})}
\end{aligned}$$

$$\begin{aligned}
\langle (B(t_i) - B(t_{i-1}))u_{k\,i-1}(t)|_{t=t_i}\rangle^2_{H^{1/2}(\partial_N \omega)} &\leqq \text{const.}\left(\frac{T}{k}\right)^2 |||U_{k\,i-1}(t_i)|||^2_2 \\
&\leqq C_0''^2 \left(\frac{T}{k}\right)^2 |||U_{k\,i-1}(t_i)|||^2_{\mathscr{V}_\alpha(t_{i-1})}.
\end{aligned}$$

Putting, for example, $C_1 = C_0' + 6((C_0' + C_0'')T)^2$, we have (4.21). C_1 is independent of i, k and t. Applying (4.20) to (4.21), we have

$$\begin{aligned}
(4.22) \quad &|||U_{ki}(t_i)|||^2_{\mathscr{V}_\alpha(t_i)} \\
&\leqq e^{C_0 t_i}\left(1 + C_1 \frac{T}{k}\right)^{3i-2}\left[\int_0^{t_i} (\|F(\tau)\|^2_{\mathscr{H}(t_{i-1})} \right. \\
&\qquad \left. + \|\mathscr{A}(t_{i-1})F(\tau)\|^2_{\mathscr{H}(t_{i-1})}) d\tau + 2C_0 \int_0^{t_i} \langle\!\langle \phi(\cdot, \tau)\rangle\!\rangle^2 d\tau\right].
\end{aligned}$$

Remarking that $\|\mathscr{A}(t)F\|^2_{\mathscr{H}(t)} \leqq (1+C_1|t'-t|)\|\mathscr{A}(t')F\|^2_{\mathscr{H}(t')}$ for any $t, t' \in [0, T]$, $F=\{0, f\}$ and using again (4.14), we have

$$|||U_{ki}(t_i)|||^2_{\mathscr{V}_\alpha(t_i)} \leqq e^{C_0 t_i}\left(1+C_1\frac{T}{k}\right)^{3i-1}$$

$$\left[\int_0^{t_i}(\|F(\tau)\|^2_{\mathscr{H}(t_i)}+\|\mathscr{A}(t_i)F(\tau)\|^2_{\mathscr{H}(t_i)})d\tau+2C_0\int_0^{t_i}\langle\!\langle\phi(\cdot,\tau)\rangle\!\rangle^2 d\tau\right].$$

On the other hand, applying Proposition 4.4 to (4.18), we have

$$|||U_{k\,i}(t)|||^2_{\mathscr{V}_\alpha(t_i)}$$

$$\leqq e^{C_0(t-t_i)}\left[|||U_{k\,i}(t_i)|||^2_{\mathscr{V}_\alpha(t_i)}+\int_{t_i}^t(\|F(\tau)\|^2_{\mathscr{H}(t_i)}+\|\mathscr{A}(t_i)F(\tau)\|^2_{\mathscr{H}(t_i)})d\tau\right.$$

$$\left.+2C_0\int_{t_i}^t\langle\!\langle\phi(\cdot,\tau)+\frac{t_{i+1}-\tau}{t_{i+1}-t_i}(B(t_i)-B(t_{i-1}))u_{k\,i-1}(t)|_{t=t_i}\rangle\!\rangle^2 d\tau\right].$$

Taking some constant $C_0'''>0$ sufficiently large and independent of k, i and t, we have

$$\leqq e^{C_0(t-t_i)}\left(1+\frac{2C_0C_0'''T}{k}\right)\left[|||U_{k\,i}(t_i)|||^2_{\mathscr{V}_\alpha(t_i)}\right.$$

$$+\int_{t_i}^t(\|F(\tau)\|^2_{\mathscr{H}(t_i)}+\|\mathscr{A}(t_i)F(\tau)\|^2_{\mathscr{H}(t_i)})d\tau$$

$$\left.+C_0\int_{t_i}^t\langle\!\langle\phi(\cdot,\tau)\rangle\!\rangle^2 d\tau\right]$$

combining with (4.22),

$$\leqq e^{C_0 t}\left(1+\frac{2C_0C_0'''T}{k}\right)\left(1+C_1\frac{T}{k}\right)^{3i-1}\left[\int_0^t(\|F(\tau)\|^2_{\mathscr{H}(t_i)}\right.$$

$$\left.+\|\mathscr{A}(t_i)F(\tau)\|^2_{\mathscr{H}(t_i)})d\tau+2C_0\int_0^t\langle\!\langle\phi(\cdot,\tau)\rangle\!\rangle^2 d\tau\right].$$

So, redefining the constant C_1 suitably and by mathematical induction, we prove (4.19). Remarking the inequality (3.18), we have, from (4.19),

$$(4.23) \qquad |||U_k(t)|||^2_2 \leqq M'\int_0^t(|||F(\tau)|||^2_2+\langle\!\langle\phi(\cdot,\tau)\rangle\!\rangle^2)d\tau \qquad \text{for any} \quad t\in[0, T]$$

$$(4.24)\qquad \int_{\partial_N\omega} \frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})}(|u_k(\tilde{x},t)|^2+|u_k'(\tilde{x},t)|^2)d\tilde{x}$$

$$\leqq M'\int_0^t(|||F(\tau)|||_2^2+\langle\!\langle\phi(\cdot,\tau)\rangle\!\rangle^2)d\tau \qquad \text{for any} \quad t\in[0,T]$$

where M' is a constant independent of $k, f(x,t)$ and $\phi(\tilde{x},t)$. Combining (4.23) with (4.25) below,

$$(4.25)\qquad \left|\left|\left|\frac{d}{dt}U_k(t)\right|\right|\right|_1^2 \leqq \text{const.}\,(|||U_k(t)|||_2^2+|||F(t)|||_1^2) \qquad \text{except for } t=t_i,$$

we prove our claim.

Define the space $V_\alpha(\omega\times(0,T))$ as

$$V_\alpha(\omega\times(0,T))=\Big\{u(x,t)\in H^1(\omega\times(0,T));\ u(\tilde{x},t)=0 \quad \text{on} \quad \alpha(\tilde{x})=0 \quad \text{and}$$

$$\int_0^T\int_{\partial_N\omega}\frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})}|u(\tilde{x},t)|^2dxdt<\infty\Big\}$$

with the norm given by

$$\|u\|_{V_\alpha(\omega\times(0,T))}^2=\|u\|_{H^1(\omega\times(0,T))}^2+\int_0^T\int_{\partial_N\omega}\frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})}|u(\tilde{x},t)|^2d\tilde{x}.$$

By the weak compactness of the bounded set of Hilbert spaces, there exists a subsequence $k_p(p=1,2,\ldots)$ of k, $u(x,t)\in H^2(\omega\times(0,T))$ and $u'(x,t)\in V_\alpha(\omega\times(0,T))$ such that

$$u_{k_p}\longrightarrow u \quad \text{weakly in } H^2(\omega\times(0,T)) \text{ and}$$

$$u_{k_p}'\longrightarrow u' \quad \text{weakly in } V_\alpha(\omega\times(0,T)).$$

Then, we prove easily that the following equations are satisfied.

$$(4.26)\qquad L[u(x,t)]=f(x,t) \qquad \text{in} \quad \mathscr{D}'(\omega\times(0,T)).$$

$$(4.27)\qquad \alpha(\tilde{x})B(t)u(\tilde{x},t)+(1-\alpha(\tilde{x}))u(\tilde{x},t)=\alpha(\tilde{x})\phi(\tilde{x},t) \quad \text{in} \quad H^{1/2}(\partial\omega\times(0,T)).$$

In fact, (4.26) is proved by integration by parts and we have (4.27) from the inequality below.

$$\langle\alpha(\tilde{x})(B(t)u_{k_p}(\tilde{x},t)-\phi(\tilde{x},t))+(1-\alpha(\tilde{x}))u_{k_p}\rangle_{H^{1/2}(\partial\omega\times(0,T))}^2\leqq\text{const.}\left(\frac{T}{k_p}\right)^2$$

where (4.18) and (4.23) are applied.

As in [7], we have, for some constant $M''>0$,

$$\sum_{\substack{j+|\alpha|\leq 2 \\ j\leq 1}} \left\| \left(\frac{\partial}{\partial x}\right)^{\alpha} \left(\frac{\partial}{\partial t}\right)^{j} u(x, t) \right\|^2_{L^2(\omega\times(0,\tau))} \leqq M''\tau^2 . \tag{4.28}$$

Moreover, by (4.24), we have

$$\int_0^{\tau}\int_{\partial_N\omega} \frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})} |u'(\tilde{x}, s)|^2 d\tilde{x}ds \leqq M''\tau^2. \tag{4.29}$$

Claim 2. By exchanging the values of $u(x, t)$ on a set of measure zero, if necessary, we prove that $u(x, t)$ is a solution of (4.16) belonging to the space $\mathscr{E}^0_t(H^2(\omega)) \cap \mathscr{E}^1_t(V_\alpha(\omega)) \cap \mathscr{E}^2_t(L^2(\omega))$.

Mollify $u(x, t)$ with respect to t as in [7], i.e. let $\rho(t)$ be C^∞ function with support contained in $[-2, -1]$ such that $\rho(t) \geqq 0$ and $\int_{-\infty}^{\infty} \rho(t)dt=1$. We define $\rho_{\delta(t)}{}^*$ by

$$u_\delta(x, t)=(\rho_{\delta(t)}{}^* u)(x, t)=\int_{-\infty}^{\infty} \rho_\delta(t-\tau)u(x, \tau)d\tau$$

for $u(x, t)\in L^2(\omega\times(0, T+\delta_0))$, where

$$\rho_\delta(t)=\frac{1}{\delta}\rho\left(\frac{t}{\delta}\right).$$

Then by (4.28), we have

$$u_\delta(x, 0) \longrightarrow 0 \quad \text{in} \quad H^2(\omega). \tag{4.29}$$

On the other hand, by (4.29) and (4.28), we have

$$\|u'_\delta(\cdot, 0)\|^2_{V_\alpha(\omega)} = \|u'_\delta(\cdot, 0)\|^2_{H^1(\omega)} + \int_{\delta_N\omega} \frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})} |u'_\delta(\tilde{x}, 0)|^2 d\tilde{x}$$

$$= \left\| \int \rho_\delta(-\tau) \frac{\partial}{\partial \tau} u(\cdot, \tau)d\tau \right\|^2_{H^1(\omega)}$$

$$+\int_{\partial_N\omega} \frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})} \left| \int \rho_\delta(-\tau) \frac{\partial}{\partial \tau} u(\tilde{x}, \tau)d\tau \right|^2 d\tilde{x}$$

$$\leqq \left(\int \rho_\delta(-\tau) \left\| \frac{\partial}{\partial \tau} u(\cdot, \tau) \right\|_{H^1(\omega)} d\tau \right)^2$$

$$+\int_{\partial_N\omega}\frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})}\left[\int\rho_\delta(-\tau)d\tau\cdot\int\rho_\delta(-\tau)\left|\frac{\partial}{\partial\tau}u(x,\tau)\right|^2 d\tau\right]d\tilde{x}$$

$$\leqq\int\rho_\delta(-\tau)d\tau\int\rho_\delta(-\tau)\left\|\frac{\partial}{\partial\tau}u(\cdot,\tau)\right\|^2_{H^1(\omega)}d\tau$$

$$+\int\rho_\delta(-\tau)d\tau\left(\iint_{\partial_N\omega}\rho_\delta(-\tau)\frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})}\left|\frac{\partial}{\partial\tau}u(\tilde{x},\tau)\right|^2 d\tilde{x}d\tau\right)$$

$$\leqq\text{const}.\ \frac{1}{\delta}\int_\delta^{2\delta}\left(\left\|\frac{\partial}{\partial\tau}u(\cdot,\tau)\right\|^2_{H^1(\omega)}\right.$$

$$\left.+\int_{\partial_N\omega}\frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})}\left|\frac{\partial}{\partial\tau}u(\tilde{x},\tau)\right|^2 d\tilde{x}\right)d\tau$$

$$\leqq\text{const}.\,\delta.$$

This means that

$$u'_\delta(x,0)\longrightarrow 0\quad\text{in}\quad V_\alpha(\omega) \tag{4.30}$$

Applying ${\rho_{\delta(t)}}^*$ to both sides of (4.26) and (4.27), we have

$$L[u_\delta(x,t)]=f_\delta(x,t)-(C_\delta u)(x,t)$$

$$\alpha(\tilde{x})B(t)u_\delta(\tilde{x},t)+(1-\alpha(\tilde{x}))u_\delta(\tilde{x},t)=\alpha(\tilde{x})\,[\phi(\tilde{x},t)-\Gamma_\delta u)(\tilde{x},t)]$$

where the operators Γ_δ and C_δ are defined by

$$(C_\delta u)(\tilde{x},t)=\left([{\rho_{\delta(t)}}^*,a_1(x,t:D)]\frac{\partial u}{\partial t}+[{\rho_{\delta(t)}}^*,a_2(x,t;D)]\,u\right)(x,t),$$

$$(\Gamma_\delta u)(x,t)=\left(\left[{\rho_{\delta(t)}}^*,\frac{\partial}{\partial t}\right]u-[{\rho_{\delta(t)}}^*,\sigma_1(\tilde{x},t)]\frac{\partial u}{\partial t}\right.$$

$$\left.+[\rho^*_{\delta(t)},\sigma_2(\tilde{x},t)]u\right)(x,t),$$

for all $t\in[0,T-\delta_0]$ if $0<\delta<\frac{\delta_0}{2}$. By applying (3.28) to $u_{\delta'}(x,t)-u_\delta(x,t)$, we have the desired result as same as [7]. Q.E.D.

By the density argument, we have

Proposition 4.6. *Let* $f(x,t)\in\mathscr{E}^1_t(L^2(\omega))$ *and* $\phi(\tilde{x},t)\in\mathscr{E}^0_t(H^{1/2}(\partial_N\omega))$

$\cap\, \mathscr{E}_t^1(L^2(\partial_N\omega))$ *with* $\phi(\tilde{x}, 0)=0$. *Then the mixed problem* (4.16) *has a unique solution belonging to the space* $\mathscr{E}_t^0(H^2(\omega)) \cap \mathscr{E}_t^1(V_\alpha(\omega)) \cap \mathscr{E}_t^2(L^2(\omega))$.

Proof of Theorem 4.1. As $H_\alpha^2(0)$ is dense in $V_\alpha(\omega)$, which was remarked in proving Lemma 3.7, there exists a sequence of functions

$$\{u_{k1}\} \in H_\alpha^2(0)=\left\{u \in H^2(\omega)\,;\, \alpha(\tilde{x})\left(\frac{\partial u}{\partial \boldsymbol{n}_0}+\sigma_2(\tilde{x}, 0)u\right)+(1-\alpha(\tilde{x}))u=0 \text{ on } \partial\omega\right\}$$

converging to u_1 in $V_\alpha(\omega)$. And also, there exists a sequence of functions $\{\phi_k(\tilde{x}, t)\} \in \mathscr{E}_t^1(H^{3/2}(\partial_N\omega))$ converging to $\phi(\tilde{x}, t)$ in $\mathscr{E}_t^0(H^{1/2}(\partial_N\omega)) \cap \mathscr{E}_t^1(L^2(\partial_N\omega))$. As in proving Proposition 4.3, there exists a function $u_{k0} \in H^3(\omega)$ such that it satisfies

$$\alpha(\tilde{x})\left(\frac{\partial u_{k0}}{\partial \boldsymbol{n}_0}+\sigma_2(\tilde{x}, 0)u_{k0}\right)+(1-\alpha(\tilde{x}))u_{k0}=\alpha(\tilde{x})(\sigma_1(\tilde{x}, 0)u_{k1}+\phi_k(\tilde{x}, 0))$$

$$\text{on } \partial\omega$$

and it converges to u_0 in $H^2(\omega)$. Then, by putting $w_k(x, t)=u_{k0}+tu_{k1}$, we solve the following initial-boundary value problem below.

$$(4.31)\quad \begin{cases} L(t)[v_k(x, t)]=f(x, t)-L[w_k(x, t)] & \text{in } \omega\times(0, T) \\ \alpha(\tilde{x})B(t)v_k(\tilde{x}, t)+(1-\alpha(\tilde{x}))v_k(\tilde{x}, t)=\alpha(\tilde{x})\psi_k(\tilde{x}, t) & \text{on } \partial\omega\times[0, T] \\ v_k(x, 0)=v_k'(x, 0)=0 \end{cases}$$

where

$$\psi_k(\tilde{x}, t)=\phi_k(\tilde{x}, t)-\left[\left(\frac{\partial}{\partial \boldsymbol{n}_t}-\frac{\partial}{\partial \boldsymbol{n}_0}\right)w_k(\tilde{x}, t)-(\sigma_1(\tilde{x}, t)-\sigma_1(\tilde{x}, 0))u_{k1}(\tilde{x})\right.$$

$$\left.+(\sigma_2(\tilde{x}, t)-\sigma_2(\tilde{x}, 0))w_k(\tilde{x}, t)\right]-\phi_k(\tilde{x}, 0) \qquad \text{on } \partial_N\omega.$$

Then, by the fact that $f(x, t)-L[w_k(x, t)] \in \mathscr{E}_t^1(L^2(\omega))$ and $\psi_k(\tilde{x}, t) \in \mathscr{E}_t^1(H^{1/2}(\partial_N\omega)) \cap \mathscr{E}_t^0(L^2(\partial_N\omega))$ with $\psi_k(\tilde{x}, 0)=0$, we may apply Proposition 4.6 to (4.31), i.e. there exists a solution $v_k(x, t) \in \mathscr{E}_t^0(H^2(\omega)) \cap \mathscr{E}_t^1(V_\alpha(\omega)) \cap \mathscr{E}_t^2(L^2(\omega))$.

Putting $u_k(x, t)=v_k(x, t)+w_k(x, t)$, we may easily deduce from Proposition 3.9 that the functions $\{u_k(x, t)\}$ form a Cauchy sequence in $\mathscr{E}_t^0(H^2(\omega)) \cap \mathscr{E}_t^1(V_\alpha(\omega)) \cap \mathscr{E}_t^2(L^2(\omega))$. So, the limit function $u(x, t)$ is the desired solution of (3.1) with (3.2). Q. E. D.

In order to prove Theorem 4.2, we begin to define the compatibility condition of order m.

Definition 4.7. For given data $\{u_0(x), u_1(x), f(x, t), \phi(\tilde{x}, t)\} \in H^{m+2}(\omega) \times H^{m+1}(\omega) \times \left(\bigcap_{k=0}^{m+1} \mathscr{E}_t^{m+1-k}(H^k(\omega))\right) \times \left(\bigcap_{k=0}^{m} \mathscr{E}_t^{m-k}(H^{k+1/2}(\partial_N\omega)) \cap \mathscr{E}_t^{m+1}(L^2(\partial_N\omega))\right)$, we say that they satisfy the compatibility condition (or simply, they are compatible) of order m at $t=0$ for the system $\{L(t), \alpha(\tilde{x})B(t)+1-\alpha(\tilde{x})\}$ when the following relations hold on $\partial\omega$.

$$(4.32)\quad \begin{cases} \alpha(\tilde{x}) \sum_{k=0}^{p} \binom{p}{k} \left[\left(\frac{\partial}{\partial \boldsymbol{n}_0}\right)^{(k)} u_{p-k} - \sigma_1^{(k)}(\tilde{x}, 0) u_{p-k+1} + \sigma_2^{(k)}(\tilde{x}, 0) u_{p-k}\right] \\ \qquad + (1-\alpha(\tilde{x}))u_p = \alpha(\tilde{x})\phi^{(p)}(\tilde{x}, 0) \qquad \text{for} \quad p=0, 1, 2, \dots, m, \text{ and} \\ u_{m+1}(x) \in V_\alpha(\omega) \end{cases}$$

where $\{u_p(x)\}$ $p=2, 3, \dots, m+1$ are defined successively by

$$(4.33)\quad u_p(x) = -\sum_{k=0}^{p-2} \binom{p-2}{k} \{a_2^{(k)}(x, 0; D)u_{p-k+2} + a_1^{(k)}(x, 0; D)u_{p-k-1}\} + f^{(p-2)}(x, 0).$$

We may prove Theorem 4.2 by applying 'Taylor series expansion in t', which is employed in [7] without modifications. So we do not reproduce his argument here.

§5. Energy Inequalities (II)

We consider in this and next sections the following problem.

$$(5.1)\quad \begin{cases} \text{(i)} \quad L(t)[u(x, t)] = f(x, t) \qquad \text{in} \quad \omega \times (0, T), \\ \text{(ii)} \quad u(x, 0) = u_0(x), \quad u'(x, 0) = u_1(x), \\ \text{(iii)} \quad \alpha(\tilde{x})B(t)u(\tilde{x}, t) + (1-\alpha(\tilde{x}))u(\tilde{x}, t) = 0 \qquad \text{on} \quad \partial\omega \times [0, T], \end{cases}$$

where the operators $L(t)$, $B(t)$ are defined in §3 but the condition (3.4) is replaced by

(5.2) $\sigma_1(\tilde{x}, t) \leqq \langle \boldsymbol{h}(t), \boldsymbol{\nu} \rangle$ on $\partial\omega \times [0, T]$.

We remark here that the condition (5.2) and others, i.e. (a), (b), (c) and (d) in § 3 are invariant under suitable transformations, for example, a change of space variables and Holmgren's transformation.

Our goal of this section is to prove the following theorem.

Theorem 5.1. *Let* m *be a non-negative integer. There exists a constant* $C_m > 0$ *depending on* T *such that for all* $u(x, t) \in H^{m+3}(\omega \times (0, T))$ *satisfying* (5.1), *we have the energy inequality*

$$(5.3)\quad \sum_{j=0}^{m+2} \left\| \frac{\partial^j u}{\partial t^j}(\cdot, t) \right\|^2_{H^{m+2-j}(\omega)} + \sum_{j=0}^{m+1} \int_{\partial_N \omega} \frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})} \left| \frac{\partial^j u}{\partial t^j}(\tilde{x}, t) \right|^2 d\tilde{x}$$
$$\leqq C_m \left\{ \sum_{j=0}^{m+2} \left\| \frac{\partial^j u}{\partial t^j}(\cdot, 0) \right\|^2_{H^{m+2-j}(\omega)} + \sum_{j=0}^{m+1} \int_{\partial_N \omega} \frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})} \left| \frac{\partial^j u}{\partial t^j}(\tilde{x}, 0) \right|^2 d\tilde{x} \right.$$
$$\left. + \sum_{j=0}^{m} \left\| \frac{\partial^j f}{\partial t^j}(\cdot, 0) \right\|^2_{H^{m-j}(\omega)} + \sum_{j=0}^{m+1} \int_0^t \left\| \frac{\partial^j f}{\partial t^j}(\cdot, s) \right\|^2_{H^{m+1-j}(\omega)} ds \right\}.$$

In order to prove (5.3), we may reduce the problem to the case when $\sigma_2(\tilde{x}, t) \equiv 0$ by taking a sufficiently smooth function $\beta(x, t)$ on $\bar{\omega} \times [0, T]$ such that (i) $\beta(\tilde{x}, t) = 1$ on $\partial\omega \times [0, T]$, (ii) $2 > |\beta(x, t)| > \frac{1}{2}$ for all $(x, t) \in \bar{\omega} \times [0, T]$ and (iii) $\frac{\partial}{\partial \boldsymbol{n}_t} \beta(\tilde{x}, t) + \sigma_2(\tilde{x}, t) = 0$ on $\partial\omega \times [0, T]$. This is the device in [8]. Now, we proceed as Ikawa did in [8].

We denote by Ω an arbitrary domain in $\mathbf{R}^n$. Any function $u(x, t) \in H^{p+1}(\Omega \times (0, T))$ is regarded to belong to $\bigcap_{k=0}^{p} \mathscr{E}_t^{p-k}(H^k(\Omega))$ by being changed its values on a set of measure zero of $\Omega \times (0, T)$ if it is necessary. For simplicity, we denote the space $\bigcap_{k=0}^{p} \mathscr{E}_t^{p-k}(H^k(\Omega))$ by $\mathscr{E}(p, \Omega)$ and for $u(x, t) \in \mathscr{E}(p, \Omega)$, we define $|||u(\cdot, t)|||_{p,\Omega}$ by

$$(5.4)\quad |||u(\cdot, t)|||^2_{p,\Omega} = \sum_{j=0}^{p} \left\| \left(\frac{\partial}{\partial t}\right)^j u(\cdot, t) \right\|^2_{H^{p-j}(\Omega)},$$

and for $u(x, t) \in \mathscr{E}(1, \Omega)$, we define $\|u(\cdot, t)\|_{H(t)}$ by

$$\|u(\cdot, t)\|^2_{H(t)} = \sum_{i,j=1}^{n} \int_\Omega a_{ij}(x, t) \frac{\partial u}{\partial x_i}(x, t) \overline{\frac{\partial u}{\partial x_j}(x, t)} dx$$
$$+ \|u(\cdot, t)\|^2 + \left\| \frac{\partial u}{\partial t}(\cdot, t) \right\|^2.$$

Then, clearly, there exists a constant $M>0$ such that

$$\frac{1}{M}|||u(\cdot, t)|||_{1,\Omega}^2 \leqq \|u(\cdot, t)\|_{H(t)}^2 \leqq M|||u(\cdot, t)|||_{1,\Omega}^2 \tag{5.5}$$

for all $t\in[0, T]$ and $u(x, t)\in\mathscr{E}(1, \Omega)$.

By integration by parts, we have

Lemma 5.2 (Lemma 2.1 of [8]). *Let $u(x, t)\in H^2(\Omega\times(0, T))$ satisfy $L(t)[u(x, t)]=f(x, t)$ in $\Omega\times(0, T)$. Then, we have*

$$\begin{aligned}\|u(\cdot, t)\|_{H(t)}^2 \leqq \|u(\cdot, 0)\|_{H(0)}^2 + c_0\int_0^t \|u(\cdot, s)\|_{H(s)}^2 ds \\ + \int_0^t \|f(\cdot, s)\|^2 ds + 2\mathrm{Re}\int_0^t ds\int_{\partial\Omega} B(s)u(\tilde{x}, s)\overline{\frac{\partial u}{\partial s}(\tilde{x}, s)}d\tilde{x}\end{aligned} \tag{5.6}$$

for all $t\in[0, T]$, where c_0 is a constant determined by $L(t)$.

By $\mathbf{R}_+^n$, we denote the space $\{x=(x', x_n);\ x'\in\mathbf{R}^{n-1}, x_n>0\}$.

Slight modification of Lemma 2.2 of [8] gives us the following.

Lemma 5.3. *Let $p(x', t)$ be a real valued function in $\mathscr{B}^\infty(\mathbf{R}^{n-1}\times(0, T))$. For any $u(x, t)\in H^3(\mathbf{R}_+^n\times(0, T))$ satisfying $u(x'', x_{n-1}, 0, t)=0$ for $x_{n-1}<0$, we have the estimate*

$$\begin{aligned}2\mathrm{Re}\int_0^t ds\int_{\substack{\partial\mathbf{R}_+^n\\ x_{n-1}>0}} p(x', s)\frac{\partial^2 u(x', 0, s)}{\partial s^2}\overline{\frac{\partial u(x', 0, s)}{\partial x_j}}dx' \\ \leqq c_1\Big\{\varepsilon|||u(\cdot, t)|||_{2,\mathbf{R}_+^n}^2 + C(\varepsilon)|||u(\cdot, t)|||_{1,\mathbf{R}_+^n}^2 + |||u(\cdot, 0)|||_{2,\mathbf{R}_+^n}^2 \\ + \int_0^t |||u(\cdot, s)|||_{2,\mathbf{R}_+^n}^2 ds\Big\}\end{aligned} \tag{5.7}$$

for any $t\in[0, T]$ and $j=1, 2,\dots, n-1$ and

$$\begin{aligned}2\mathrm{Re}\int_0^t ds\int_{\substack{\partial\mathbf{R}_+^n\\ x_{n-1}>0}} p(x', s)\frac{\partial^2 u(x', 0, s)}{\partial s^2}\overline{\frac{\partial u(x', 0, s)}{\partial s}}dx' \\ \leqq c_1\Big\{\varepsilon|||u(\cdot, t)|||_{2,\mathbf{R}_+^n}^2 + C(\varepsilon)|||u(\cdot, t)|||_{1,\mathbf{R}_+^n}^2 + |||u(\cdot, 0)|||_{1\mathbf{R}_+^n}^2 \\ + \int_0^t |||u(\cdot, s)|||_{2,\mathbf{R}_+^n}^2 ds\Big\}\end{aligned} \tag{5.8}$$

where c_1 *is a constant determined by* $p(x', t)$, ε *is an arbitrary positive number and* $C(\varepsilon)$ *depends only on* ε.

Lemma 5.4 (Lemma 2.3 of [8]). *For any* $u(x, t) \in H^3(\mathbf{R}_+^n \times (0, T))$, $v(x, t) \in H^2(\mathbf{R}_+^n \times (0, T))$, *we have*

$$(5.9) \qquad 2\mathrm{Re}\int_0^t ds \int_{\mathbf{R}_+^{n-1}} \frac{\partial^2 u}{\partial s^2}(x', 0, s)\overline{v(x', 0, s)}dx'$$

$$\leqq c_2\Big\{\varepsilon |||u(\cdot, t)|||_{2,\mathbf{R}_+^n}^2 + C(\varepsilon)|||u(\cdot, t)|||_{1,\mathbf{R}_+^n}^2 + |||v(\cdot, t)|||_{1,\mathbf{R}_+^n}^2$$

$$+ |||u(\cdot, 0)|||_{2,\mathbf{R}_+^n}^2 + |||v(\cdot, 0)|||_{1,\mathbf{R}_+^n}^2$$

$$+ \int_0^t (|||u(\cdot, s)|||_{2,\mathbf{R}_+^n}^2 + |||v(\cdot, s)|||_{1,\mathbf{R}_+^n}^2) ds\Big\}$$

where c_2 *is a constant independent of* u *and* v.

Remarking the estimate (3.8), we have, as same as Lemma 2.4 of [8], the following lemma.

Lemma 5.5. *Let* p *be an arbitrary integer* $\geqq 1$. *There exists a constant* M_p *such that for any function* $u(x, t) \in \mathscr{E}(p+1, \omega)$ *satisfying* (5.1), *the following estimate holds for all* $t \in [0, T]$.

$$(5.10) \quad |||u(\cdot, t)|||_{p+1,\omega}^2 \leqq M_p(|||u^{(p)}(\cdot, t)|||_{1,\omega}^2 + |||u(\cdot, t)|||_{2,\omega}^p + |||f(\cdot, t)|||_{p-1,\omega}^2)$$

The following proposition corresponds to Proposition 2.6 of [8].

Proposition 5.6. *Let* k *be a non-negative integer and* $\varphi(x)$ *be a real-valued function in* $C_0^\infty(\mathbf{R}^n)$ *with a support contained in an open set* V. *Let* $u(x, t) \in H^{k+2}(\mathbf{R}_+^n \times (0, T))$ *satisfy* (i) *of* (5.1) *in* $(V \cap \mathbf{R}_+^n) \times (0, T)$ *and* (iii) *of* (5.1) *on* $(V \cap \mathbf{R}^{n-1}) \times [0, T]$. *Then, we have*

$$(5.11) \qquad \|(\varphi u)^{(k)}(\cdot, t)\|_{H(t)}^2 + \int_{\substack{\partial \tilde{V} \\ \alpha(\tilde{x}) \neq 0}} \frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})} |(\varphi u)^{(k)}(\tilde{x}, t)|^2 d\tilde{x}$$

$$\leqq C_k\Big\{\|u(\cdot, 0)\|_{H^{k+1}(\tilde{V})}^2 + \|u'(\cdot, 0)\|_{H^k(\tilde{V})}^2 + |||f(\cdot, 0)|||_{H^{k-1}(V)}^2$$

$$+ \int_0^t |||f'(\cdot, s)|||_{H^{k-1}(\tilde{V})}^2 + \varepsilon |||u(\cdot, t)|||_{k+1,\tilde{V}}^2 + C(\varepsilon)|||u(\cdot, t)|||_{k,\tilde{V}}^2$$

$$+\int_0^t\Big(|||u(\cdot,\ s)|||_{k+1,\tilde{V}}^2+\int_{\substack{\partial\tilde{V}\\ \alpha(\tilde{x})\neq 0}}\frac{1-\alpha(\tilde{x})}{\alpha(x)}|u^{(k)}(\tilde{x},\ s)|^2d\tilde{x}\Big)ds\Big\}$$

for all $t\in[0,\ T]$, *where* C_k *depends on* $L,\ B,\ \alpha,\ \varphi$ *and* k *and* $\tilde{V}=V\cap\mathbf{R}_+^n$.

Proof. Putting $v(x,\ t)=\varphi(x)u(x,\ t)$, we have

$$L(t)[v(x,\ t)]=([L,\ \varphi]u)(x,\ t)+\varphi(x)f(x,\ t) \tag{5.12}$$

$$\alpha(\tilde{x})B(t)v(\tilde{x},\ t)+(1-\alpha(\tilde{x}))v(\tilde{x},\ t)=\alpha(\tilde{x})\frac{\partial\varphi}{\partial\boldsymbol{n}_t}\cdot u(\tilde{x},\ t)\,. \tag{5.13}$$

Differentiating these equations k-times with respect to t, we have

$$L(t)[v^{(k)}(x,\ t)]=-\sum_{j=1}^{k}L^{(j)}(t)[v^{(k-j)}(x,\ t)]+([L,\ \varphi]u)^{(k)}(x,\ t)+\varphi(x)f^{(k)}(x,\ t)\,, \tag{5.14}$$

$$\begin{aligned}&\alpha(\tilde{x})B(t)v^{(k)}(\tilde{x},\ t)+(1-\alpha(\tilde{x}))v^{(k)}(\tilde{x},\ t)\\ &=\alpha(\tilde{x})\Big\{-k\Big(\frac{\partial}{\partial\boldsymbol{n}_t}\Big)'v^{(k-1)}(\tilde{x},\ t)+k\sigma_1'(\tilde{x},\ t)v^{(k)}(\tilde{x},\ t)\\ &\quad-\sum_{j=2}^{k}\binom{k}{j}B^{(j)}(t)v^{(k-j)}(\tilde{x},\ t)+\frac{\partial}{\partial\boldsymbol{n}_t}u^{(k)}(\tilde{x},\ t)\\ &\quad+\sum_{j=1}^{k}\binom{k}{j}\Big(\frac{\partial}{\partial\boldsymbol{n}_t}\Big)^{(j)}u^{(k-j)}(\tilde{x},\ t)\Big\}\,.\end{aligned} \tag{5.15}$$

Applying Lemma 5.2 to $v^{(k)}(x,\ t)$, we have

$$\begin{aligned}\|v^{(k)}(\cdot,\ t)\|_{H(t)}^2\leqq{}&\|v^{(k)}(\cdot,\ 0)\|_{H(0)}^2+c_0\int_0^t\|v^{(k)}(\cdot,\ s)\|^2ds\\ &+\int_0^t\|-\sum_{j=1}^{k}L^{(j)}[v^{(k-j)}]+([L,\ \varphi]u)^{(k)}+f^{(k)}\|^2ds\\ &+2\mathrm{Re}\int_0^t ds\int_{\partial\tilde{V}}B(s)v^{(k)}(\tilde{x},\ s)\overline{v^{(k+1)}(\tilde{x},\ s)}d\tilde{x}\,.\end{aligned} \tag{5.16}$$

Remarking that $\varphi(x)\in C_0^\infty(V)$ and $v^{(k)}$ satisfies (5.15), we have

$$2\mathrm{Re}\int_0^t ds\int_{\partial V}B(s)v^{(k)}(\tilde{x},\ s)\overline{v^{(k+1)}(\tilde{x},\ s)}d\tilde{x}$$

$$=2\mathrm{Re}\int_0^t ds\int_{\substack{\partial\mathbf{R}^n_+\cap V\\ \alpha(x')\neq 0}} -\frac{1-\alpha(x')}{\alpha(x')}v^{(k)}(x',0,s)\overline{v^{(k+1)}(x',0,s)}dx'$$

$$+2\mathrm{Re}\int_0^t ds\int_{\substack{\partial\mathbf{R}^n_+\cap V\\ \alpha(x')\neq 0}} \Big\{-k\Big(\frac{\partial}{\partial \boldsymbol{n}_s}\Big)'v^{(k-1)}+k\sigma'_1 v^{(k)}+\frac{\partial\varphi}{\partial\boldsymbol{n}_s}u^{(k)}$$

$$-\sum_{j=2}^k\binom{k}{j}B^{(j)}(s)v^{(k-j)}$$

$$+\sum_{j=1}^k\binom{k}{j}\Big(\frac{\partial\varphi}{\partial\boldsymbol{n}_s}\Big)^{(j)}u^{(k-j)}\Big\}\overline{v^{(k+1)}}dx'.$$

Here, we remark that $d\tilde{x}=dx'=dx_1dx_2\dots dx_{n-1}$.

Clearly, we have

$$\|v^{(k)}(\cdot,0)\|^2_{H(0)}$$

$$\leqq\text{const.}(\|u^{(k)}(\cdot,0)\|^2_{H^1(\tilde{V})}+\|u^{(k+1)}(\cdot,0)\|^2_{L^2(\tilde{V})}),$$

$$\|-\sum_{j=1}^k\binom{k}{j}L^{(j)}(t)[v^{(k-j)}]+([L,\varphi]u)^{(k)}\|^2_{L^2(\tilde{V})}$$

$$\leqq\text{const.}\ |||u(\cdot,t)|||^2_{k+1,\tilde{V}}$$

and

$$\Big|\Big|\Big|-\sum_{j=2}^k\binom{k}{j}B^{(j)}(t)v^{(k-j)}+\sum_{j=1}^k\binom{k}{j}\Big(\frac{\partial\varphi}{\partial\boldsymbol{n}_t}\Big)^{(j)}u^{(k-i)}\Big|\Big|\Big|^2_{i,\tilde{V}}$$

$$\leqq\text{const.}\ |||u(\cdot,t)|||^2_{k-1+i,\tilde{V}}.$$

By applying Lemma 5.4, we have

$$2\mathrm{Re}\int_0^t\int_{\substack{\partial\mathbf{R}^n_+\cap\tilde{V}\\ \alpha(x')\neq 0}}\Big(-\sum_{j=2}^k\binom{k}{j}B^{(j)}(s)v^{(k-j)}+\sum_{j=1}^k\binom{k}{j}\Big(\frac{\partial\varphi}{\partial\boldsymbol{n}_s}\Big)^{(j)}u^{(k-j)}\Big)$$

$$\overline{\frac{\partial^2}{\partial t^2}v^{(k-1)}}dx'ds$$

$$\leqq c\Big(\varepsilon|||u(\cdot,t)|||^2_{k+1,\tilde{V}}+C(\varepsilon)|||u(\cdot,t)|||^2_{k,\tilde{V}}+|||u(\cdot,0)|||^2_{k+1,\tilde{V}}$$

$$+\int_0^t|||u(\cdot,s)|||^2_{k+1,\tilde{V}}ds\Big).$$

Hereafter, we use same letter C or c to represent various constants.

From (5.13), we have

$$\frac{\partial v}{\partial x_n} = -\frac{1}{a_{nn}(x', 0, t)}\Big(\sum_{j=0}^{n-1} a_{nj}(x', 0, t)\frac{\partial v}{\partial x_j} + \sigma_1(x', t)\frac{\partial v}{\partial t} + \frac{\partial \varphi}{\partial \boldsymbol{n}_t} u$$

$$- \frac{1-\alpha(x')}{\alpha(x')} v(x', 0, t)\Big) \text{ on } \partial\mathbf{R}^n_+ \cap V \cap \{x' : \alpha(x') \neq 0\}.$$

Differentiating both sides $(k-1)$-times with respect to t, we have

$$\frac{\partial v^{(k-1)}}{\partial x_n} = -\frac{1}{a_{nn}}\Big(\sum_{j=1}^{n-1} a_{nj}\frac{\partial v^{(k-1)}}{\partial x_j} + \sigma_1 v^{(k)} - \frac{\partial \varphi}{\partial \boldsymbol{n}_t} u^{(k-1)}$$

$$- \frac{1-\alpha(x')}{\alpha(x')} v^{(k-1)}\Big) + B_{k-1} u,$$

where B_{k-1} is a boundary operator of order less than $k-1$. So

$$= 2\mathrm{Re}\int_0^t ds \int_{\substack{\partial\mathbf{R}^n_+ \cap \tilde{V} \\ \alpha(x') \neq 0}} -\Big(\frac{\partial}{\partial \boldsymbol{n}_s}\Big)' v^{(k-1)} \cdot \overline{v^{(k+1)}} dx'$$

$$= 2\mathrm{Re}\int_0^t ds \int_{\substack{\partial\mathbf{R}^n_+ \cap \tilde{V} \\ \alpha(x') \neq 0}} \sum_{j=1}^{n-1}\Big(a'_{nj} - a'_{nn}\frac{a_{nj}}{a_{nn}}\Big)\frac{\partial v^{(k-1)}}{\partial x_j}\overline{v^{(k+1)}} dx'$$

$$+ 2\mathrm{Re}\int_0^t ds \int_{\substack{\partial\mathbf{R}^n_+ \cap \tilde{V} \\ \alpha(x') \neq 0}} \Big[\frac{1}{a_{nn}}\Big(-\sigma_1 v^{(k)} + \frac{\partial \varphi}{\partial \boldsymbol{n}_t} u^{(k-1)} + \frac{1-\alpha(x')}{\alpha(x')} v^{(k-1)}\Big)$$

$$+ B_{k-1} u\Big]\overline{v^{(k+1)}} dx'.$$

By applying Lemmas 5.3 and 5.4, we have

$$\leqq 2\mathrm{Re}\int_0^t ds \int_{\substack{\partial\mathbf{R}^n_+ \cap \tilde{V} \\ \alpha(x') \neq 0}} \frac{1}{a_{nn}}\frac{1-\alpha(x')}{\alpha(x')} v^{(k-1)}\overline{v^{(k+1)}} dx'$$

$$+ c\Big\{\varepsilon |||u(\cdot, t)|||^2_{k+1,\tilde{V}} + C(\varepsilon)|||u(\cdot, t)|||^2_{k,\tilde{V}} + |||u(\cdot, 0)|||^2_{k+1,\tilde{V}}$$

$$+ \int_0^t |||u(\cdot, s)|||^2_{k+1,\tilde{V}}\Big\}.$$

By integration by parts with respect to t, we have

$$2\mathrm{Re}\int_0^t ds\int_{\substack{\partial\mathbf{R}_+^n\cap\tilde{V}\\ \alpha(x')\neq 0}} \frac{1}{a_{nn}}\frac{1-\alpha(x')}{\alpha(x')}v^{(k-1)}\overline{v^{(k+1)}}dx'$$

$$\leqq c\int_{\substack{\partial\mathbf{R}_+^n\cap\tilde{V}\\ \alpha(x')\neq 0}} \frac{1-\alpha(x')}{\alpha(x')}\Big(\varepsilon|v^{(k)}(x',0,t)|^2+\frac{1}{\varepsilon}|v^{(k-1)}(x',0,t)|^2$$

$$+|v^{(k)}(x',0,0)|^2+|v^{(k-1)}(x',0,0)|^2\Big)dx'$$

$$+c\int_0^t ds\int_{\substack{\partial\mathbf{R}_+^n\cap\tilde{V}\\ \alpha(x')\neq 0}} \frac{1-\alpha(x')}{\alpha(x')}|v^{(k)}(x',0,s)|^2dx'\,.$$

On the other hand, we have

$$2\mathrm{Re}\int_0^t ds\int_{\substack{\partial\mathbf{R}_+^n\cap\tilde{V}\\ \alpha(x')\neq 0}} \Big(k\sigma'_1v^{(k)}+\frac{\partial\varphi}{\partial\boldsymbol{n}_s}u^{(k)}\Big)\overline{v^{(k+1)}}dx'$$

$$\leqq c\Big(\varepsilon|||u(\cdot,t)|||^2_{k+1,\tilde{V}}+C(\varepsilon)|||u(\cdot,t)|||^2_{k,\tilde{V}}+|||u(\cdot,0)|||^2_{k+1,\tilde{V}}$$

$$+\int_0^t|||u(\cdot,s)|||^2_{k+1,\tilde{V}}ds\Big).$$

Putting these estimates into (5.16), we have

$$(5.17)\quad \|v^{(k)}(\cdot,t)\|^2_{H(t)}+\int_{\substack{\partial\mathbf{R}_+^n\cap\tilde{V}\\ \alpha(x')\neq 0}} \frac{1-\alpha(x')}{\alpha(x')}|v^{(k)}(x',0,t)|^2dx'$$

$$\leqq \|v^{(k)}(\cdot,0)\|^2_{H(0)}+\int_{\substack{\partial\mathbf{R}_+^n\cap\tilde{V}\\ \alpha(x')\neq 0}} \frac{1-\alpha(x')}{\alpha(x')}|v^{(k)}(x',0,0)|^2dx'$$

$$+c\int_0^t\|f^{(k)}(\cdot,s)\|^2ds+c\int_0^t ds\int_{\substack{\partial\mathbf{R}_+^n\cap\tilde{V}\\ \alpha(x')\neq 0}} \frac{1-\alpha(x')}{\alpha(x')}|v^{(k)}(x',0,s)|^2ds$$

$$+c\int_{\substack{\partial\mathbf{R}_+^n\cap\tilde{V}\\ \alpha(x')\neq 0}} \frac{1-\alpha(x')}{\alpha(x')}\Big(\varepsilon|v^{(k)}(x',0,t)|^2+\frac{1}{\varepsilon}|v^{(k-1)}(x',0,t)|^2$$

$$+|v^{(k)}(x',0,0)|^2+|v^{(k-1)}(x',0,0)|^2\Big)dx'$$

$$+c\Big(\varepsilon|||u(\cdot,t)|||^2_{k+1,\tilde{V}}+C(\varepsilon)|||u(\cdot,t)|||^2_{k,\tilde{V}}+|||u(\cdot,0)|||^2_{k+1,\tilde{V}}$$

$$+\int_0^t|||u(\cdot,s)|||^2_{k+1,\tilde{V}}ds\Big).$$

By differentiating (5.13) $(k-1)$-times with respect to t, we have easily

$$(5.18)\qquad \int_{\substack{\partial\mathbf{R}_+^n\cap\tilde{V}\\ \alpha(x')\neq 0}} \frac{1-\alpha(x')}{\alpha(x')}\,|v^{(k-1)}(x', 0, t)|^2 dx' \leqq \text{const.}\|v^{(k-1)}(\cdot, t)\|^2_{H^2(\tilde{V})}$$

for any $t\in[0, T]$.

Remarking that

$$|||u(\cdot, 0)|||^2_{k+1,V}$$
$$\leqq \text{const.}(\|u(\cdot, 0)\|^2_{H^{k+1}(\tilde{V})} + \|u'(\cdot, 0)\|^2_{H^k(\tilde{V})} + |||f(\cdot, 0)|||^2_{k-1,\tilde{V}}),$$

we have the desired result by taking ε sufficiently small.

Proof of Theorem 5.1. Let $\{\varphi_j(x)\}_{j=1}^N$ be a partition of unity in a neighborhood of $\partial\omega$, namely, $\varphi_j(x)\in C_0^\infty(\mathbf{R}^n)$ such that

$$\sum_{j=1}^N \varphi_j^2(x)=1 \qquad \text{in a neighborhood of } \partial\omega.$$

Assume that the support of φ_j is contained in a sufficiently small neighborhood U_j of some $\tilde{x}_j\in\partial\omega$ and there exists a smooth transformation $\Psi_j=(\psi_{j1}(x), \psi_{j2}(x),\dots, \psi_{jn}(x))$ from U_j onto V_j in $\mathbf{R}^n$ such that

$$\begin{cases} \Psi_j(U_j\cap\omega)=V_j\cap\mathbf{R}_+^n, \\ \Psi_j(U_j\cap\partial\omega)=V_j\cap\mathbf{R}^{n-1}, \\ \Psi_j(\tilde{x}_j)=0. \end{cases}$$

For a function $w(x)$ defined in a domain containing some $U_j\cap\omega$, we denote by $\tilde{w}_j(y)$, the function defined in $V_j\cap\mathbf{R}_+^n$ by $\tilde{w}_j(y)=\tilde{w}_j(\Psi_j(x))=w(x)$.

Considering that the equations of (5.1) hold in $(U_j\cap\omega)\times(0, T)$ or on $(U_j\cap\partial\omega)\times[0, T]$, we have

$$(5.19)\qquad L_j(t)[\tilde{u}_j(y, t)]=\tilde{f}_j(y, t) \qquad \text{in } (V_j\cap\mathbf{R}_+^n)\times(0, T),$$

$$(5.20)\qquad \tilde{\alpha}(y')B_j(t)[\tilde{u}_j(y', t)]+(1-\tilde{\alpha}(y'))\tilde{u}_j(y', t)=0$$
$$\text{on } (V_j\cap\mathbf{R}^{n-1})\times[0, T],$$

where

$$L_j(t)=\frac{\partial^2}{\partial t^2}+2\sum_{k=1}^{n}\Big(\sum_{l=1}^{n}h_l\frac{\partial\psi_{jk}}{\partial x_l}\Big)(y,t)\frac{\partial^2}{\partial y_k\partial t}$$

$$-\sum_{i,k=1}^{n}\Big(\sum_{p,q=1}^{n}a_{pq}\frac{\partial\psi_{ji}}{\partial x_p}\frac{\partial\psi_{jk}}{\partial x_q}\Big)(y,t)\frac{\partial^2}{\partial y_i\partial y_k}$$

$$+(\text{first order term}),$$

$$B_j(t)=-\sum_{i=1}^{n}\Big(\sum_{p,q=1}^{n}a_{pq}\frac{\partial\psi_{ji}}{\partial x_p}\frac{\partial\psi_{jn}}{\partial x_q}\Big)\frac{\partial}{\partial y_i}-\tilde{\sigma}_1(y',t)\frac{\partial}{\partial t}.$$

Applying Proposition 5.6 to (5.19) with (5.20), we have

$$\|\tilde{\varphi}_j(\cdot)\tilde{u}_j^{(m+1)}(\cdot,t)\|^2_{H(t)}+\int_{\substack{V_j\cap\mathbf{R}^{n-1}\\ \tilde{\alpha}(y')\neq 0}}\frac{1-\tilde{\alpha}(y')}{\tilde{\alpha}(y')}|\tilde{\varphi}_j(y',0)\tilde{u}_j^{(m+1)}(y',0,t)|^2dy'$$

$$\leqq C_{jm}\Big\{\|\tilde{u}_j(\cdot,0)\|^2_{H^{m+2},\tilde{V}_j}+\|\tilde{u}'_j(\cdot,0)\|^2_{H^{m+1}(\tilde{V}_j)}$$

$$+\int_{\substack{V_j\cap\mathbf{R}^{n-1}\\ \tilde{\alpha}(y')\neq 0}}\frac{1-\tilde{\alpha}(y')}{\tilde{\alpha}(y')}|\tilde{\varphi}_j(y',0)\tilde{u}_j^{(m+1)}(y',0,0)|^2dy'$$

$$+|||\tilde{f}_j(\cdot,0)|||^2_{m,\tilde{V}_j}+\int_0^t|||\tilde{f}'_j(\cdot,s)|||^2_{m,\tilde{V}_j}ds+\varepsilon|||\tilde{u}_j(\cdot,t)|||^2_{m+2,\tilde{V}_j}$$

$$+C(\varepsilon)|||\tilde{u}_j(\cdot,t)|||^2_{m+1,\tilde{V}_j}+\int_0^t(|||\tilde{u}_j(\cdot,s)|||^2_{m+2,\tilde{V}_j}$$

$$+\int_{\substack{V_j\cap\mathbf{R}^{n-1}\\ \tilde{\alpha}(y')\neq 0}}\frac{1-\tilde{\alpha}(y')}{\tilde{\alpha}(y')}|\tilde{\varphi}_j(y',0)\tilde{u}_j^{(m+1)}(y',0,s)|^2dy')ds\Big\}.$$

Therefore, we have

$$|||\varphi_j(\cdot)u^{(m+1)}(\cdot,t)|||^2_{1,\omega}+\int_{\partial_N\omega}\frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})}|\varphi_j(\tilde{x})u^{(m+1)}(\tilde{x},t)|^2d\tilde{x} \tag{5.21}$$

$$\leqq C_{jm}\Big\{\|u(\cdot,0)\|^2_{H^{m+2}(\omega)}+\|u'(\cdot,0)\|^2_{H^{m+1}(\omega)}$$

$$+\int_{\partial_N\omega}\frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})}|\varphi_j(\tilde{x})u^{(m+1)}(\tilde{x},0)|^2d\tilde{x}+|||f(\cdot,0)|||^2_{m,\omega}$$

$$+\int_0^t|||f'(\cdot,0)|||^2_{m,\omega}+\varepsilon|||u(\cdot,t)|||^2_{m+2,\omega}+C(\varepsilon)|||u(\cdot,t)|||^2_{m+1,\omega}$$

$$+\int_0^t\Big(|||u(\cdot,s)|||^2_{m+2,\omega}+\int_{\partial_N\omega}\frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})}|\varphi_j(\tilde{x})u^{(m+1)}(\tilde{x},s)|^2d\tilde{x}\Big)ds\Big\}.$$

On the other hand, we have easily that

$$(5.22)\quad |||(1-\sum_{j=1}^{N}\varphi_j(\cdot)^2)^{1/2}u^{(m+1)}(\cdot,t)|||_{1,\omega}^2$$

$$\leqq c_m\Big(\|u(\cdot,0)\|_{H^{m+2}(\omega)}^2+\|u'(\cdot,0)\|_{H^{m+1}(\omega)}^2+|||f(\cdot,0)|||_{m,\omega}^2$$

$$+\int_0^t|||f'(\cdot,s)|||_{m,\omega}^2ds+\int_0^t|||u(\cdot,s)|||_{m+2,\omega}^2ds\Big).$$

And also, we have for some constant $c>0$,

$$(5.23)\quad |||v(\cdot,t)|||_{1,\omega}^2\leqq\sum_{j=1}^{N}|||\varphi_j(\cdot)v(\cdot,t)|||_{1,\omega}^2+|||(1-\sum_{j=1}^{N}\varphi_j(\cdot)^2)^{1/2}v(\cdot,t)|||_{1,\omega}^2$$

$$+c\|v(\cdot,t)\|^2$$

for any $v(x,t)\in\mathscr{E}(1,\omega)$. Combining these estimates (5.21), (5.22) and (5.23) with Lemma 5.5, we have

$$|||u(\cdot,t)|||_{m+2,\omega}^2+\int_{\partial_N\omega}\frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})}|u^{(m+1)}(\tilde{x},t)|^2d\tilde{x}$$

$$\leqq C_m\Big\{\|u(\cdot,0)\|_{H^{m+2}(\omega)}^2+\|u'(\cdot,0)\|_{H^{m+1}(\omega)}^2$$

$$+\int_{\partial_N\omega}\frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})}|u^{(m+1)}(\tilde{x},0)|^2d\tilde{x}+|||f(\cdot,0)|||_{m,\omega}^2$$

$$+\int_0^t|||f'(\cdot,0)|||_{m,\omega}^2$$

$$+\int_0^t\Big(|||u(\cdot,s)|||_{m+2,\omega}^2+\int_{\partial_N\omega}\frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})}|u^{(m+1)}(\tilde{x},s)|^2d\tilde{x}\Big)ds$$

$$+\varepsilon|||u(\cdot,t)|||_{m+2,\omega}^2+C(\varepsilon)|||u(\cdot,t)|||_{m+1,\omega}^2\Big\}.$$

Using the relation

$$|||u(\cdot,t)|||_{m+1,\omega}^2\leqq\text{const.}\Big(|||u(\cdot,0)|||_{m+1,\omega}^2+\int_0^t|||u(\cdot,s)|||_{m+2,\omega}^2ds\Big),$$

and choosing ε sufficiently small, we have the desired result by Gronwall's inequality. Here, we use (5.18) to estimate the term $\sum_{j=0}^{m}\int_{\partial_N\omega}\frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})}|u^{(j)}(\tilde{x},t)|^2d\tilde{x}$, Q.E.D.

§6. Existence and Regularity of the Solution

Our aim is to prove the following theorem.

Theorem 6.1. *For any given data* $\{u_0(x), u_1(x), f(x, t)\} \in H^{m+2}(\omega) \times H^{m+1}(\omega) \times H^{m+1}(\omega \times (0, T))$ *satisfying the compatibility condition of order* m *at* $t=0$, *there exists a solution* $u(x, t)$ *of* (5.1) *belonging to the space* $\mathscr{E}(m+2, \omega) \cap \mathscr{E}_t^{m+1}(V_\alpha(\omega))$ *and it is unique in the space* $\mathscr{E}_t^2(L^2(\omega)) \cap \mathscr{E}_t^1(V_\alpha(\omega)) \cap \mathscr{E}_t^0(H^2(\omega))$.

We will prove this theorem by following the idea of Ikawa [8]. The modification will be necessary to construct a solution of the boundary value problem for a system of elliptic operators.

We give, first of all, the definition of the compatibility condition.

Definition 6.2. The data $\{u_0(x), u_1(x), f(x, t)\} \in H^{m+2}(\omega) \times H^{m+1}(\omega) \times H^{m+1}(\omega \times (0, T))$ are said to satisfy the compatibility condition (or simply, to be compatible) of order m at $t=0$ for the system $\{L(t), \alpha(\tilde{x})B(t)+1-\alpha(x)\}$ where the following relations hold on $\partial\omega$.

$$(6.1)\quad \begin{cases} \alpha(\tilde{x}) \sum_{k=0}^{p} \binom{p}{k} \left[\left(\dfrac{\partial}{\partial \boldsymbol{n}_0} \right)^{(k)} u_{p-k} - \sigma_1^{(k)}(\tilde{x}, 0) u_{p-k+1} + \sigma_2^{(k)}(\tilde{x}, 0) u_{p-k} \right] \\ \qquad\qquad + (1-\alpha(\tilde{x})) u_p = 0 \qquad \text{for} \quad p=0, 1, 2, \dots, m, \\ \text{and} \\ u_{m+1}(x) \in V_\alpha(\omega)^{4)} \end{cases}$$

where $u_p(x) \in H^{m+2-p}(\omega)$ $(p=2, 3, \dots, m+1)$ are defined successively by (4.33).

Definition 6.3. $S_\alpha^m(L, B)$ is the space of all data $\Phi = \{u_0, u_1, f\} \in H^{m+2}(\omega) \times H^{m+1}(\omega) \times H^{m+1}(\omega \times (0, T))$ satisfying the compatibility condition of order m, equipped with the following norm.

$$(6.2)\quad |\Phi|^2_{S_\alpha^m(L,B)} = \|u_0\|^2_{H^{m+2}(\omega)} + \|u_1\|^2_{H^{m+1}(\omega)} + \int_{\partial_N \omega} \frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})} |u_{m+1}(\tilde{x})|^2 d\tilde{x} + |||f(\cdot, 0)|||^2_{m,\omega} + \int_0^T \left|\left|\left| \frac{\partial f}{\partial s}(\cdot, s) \right|\right|\right|^2_{m,\omega} ds.$$

4) This condition is more restrictive than the definition of Ikawa.

Clearly, $S_\alpha^m(L, B)$ forms a Hilbert space and $S_\alpha^{m+1}(L, B) \subset S_\alpha^m(L, B)$.

For the future use, we consider the following boundary value problem for a system of elliptic operators.

$$
(6.3)\quad \begin{cases} (\lambda - a_2(x, 0; D))w_p(x) = f_p(x) \quad \text{in } \omega \\ \alpha(\tilde{x}) \sum_{k=0}^{p} \binom{p}{k} \Big[\Big(\frac{\partial}{\partial \boldsymbol{n}_0}\Big)^{(k)} w_{p-k} - \sigma_1^{(k)}(\tilde{x}, 0) w_{p-k+1} + \sigma_2^{(k)}(\tilde{x}, 0) w_{p-k} \Big] \\ \qquad + (1-\alpha(\tilde{x}))w_p = \alpha(\tilde{x})\phi_p(\tilde{x}) + (1-\alpha(\tilde{x}))\psi_p(\tilde{x}) \quad \text{on } \partial\omega \\ \qquad\qquad \text{for } p = 0, 1, 2, \dots, m, \end{cases}
$$

where $w_{m+1}(x)$ is given arbitrary in $H^1(\omega)$.

Proposition 6.4. *Assume that ω is bounded. Let $f_p(x) \in H^{m-p}(\omega)$, $\phi_p(\tilde{x}) \in H^{m+1/2-p}(\partial\omega)$ and $\psi_p(\tilde{x}) \in H^{m+3/2-p}(\partial\omega)$. Assuming that there exist functions $w_p(x) \in H^{m+2-p}(\omega)$ $(p = 0, 1, 2, \dots, m)$ satisfying* (6.3), *we have*

$$
(6.4)\quad \sum_{p=0}^{m} \|w_p\|^2_{H^{m+2-p}(\omega)} \leqq C\Big\{\sum_{p=0}^{m} (\|f_p\|^2_{H^{m-p}(\omega)} + \langle\phi_p\rangle^2_{H^{m+1/2-p}(\partial\omega)} + \langle\psi_p\rangle^2_{H^{m+3/2-p}(\partial\omega)} + \|w_p\|^2_{L^2(\omega)}) + \|w_{m+1}\|^2_{H^1(\omega)}\Big\}
$$

where C is a constant independent of w_p.

Moreover, if λ is taken sufficiently large, then there exist functions $w_p(x) \in H^{m+2-p}(\omega)$ satisfying (6.3).

The proof will be given in the appendix.

The following lemma corresponds to Lemma 3.1 of [8].

Lemma 6.5. *Any element of $S_\alpha^m(L, B)$ can be approximated by smooth elements of $S_\alpha^m(L, B)$.*

Proof. Let $\Phi = \{u_0, u_1, f\} \in S_\alpha^m(L, B)$. There exist sequences of smooth functions v_{j0}, v_{j1} and g_j converging to u_0, u_1 and f in $H^{m+2}(\omega)$, $H^{m+1}(\omega)$ and $H^{m+1}(\omega \times (0, T))$, respectively. We define v_{jp} $(p = 2, 3, \dots, m \pm 1)$ by (4.33) from v_{j0}, v_{j1} and g_j, and we put

$$
\gamma_{jl}(\tilde{x}) = \sum_{k=0}^{l} \binom{l}{k} \Big[\Big(\frac{\partial}{\partial \boldsymbol{n}_0}\Big)^{(k)} v_{jl-k} - \sigma_1^{(k)}(\tilde{x}, 0) v_{jl-k+1} + \sigma_2^{(k)}(\tilde{x}, 0) v_{jl-k} \Big]
$$

$$
l = 0, 1, 2, \dots, m.
$$

Then, $\gamma_{jl}(\tilde{x})$ $(l=0, 1,\dots, m)$ are smooth functions and $\gamma_{jl}(\tilde{x})$ converges to $\gamma_l(\tilde{x})=\sum_{k=0}^{l}\binom{l}{k}\left[\left(\frac{\partial}{\partial \boldsymbol{n}_0}\right)^{(k)}u_{l-k}-\sigma_1^{(k)}(\tilde{x}, 0)u_{l-k+1}+\sigma_2^{(k)}(\tilde{x}, 0)u_{l-k}\right]$ in $H^{m+1/2-l}(\partial\omega)$. $\Phi\in S_\alpha^m(L, B)$ means that $\alpha(\tilde{x})\gamma_l(\tilde{x})+(1-\alpha(\tilde{x}))u_l(\tilde{x})=0$ for $l=0, 1,\dots, m$. Moreover, there exists a sequence of functions $\tilde{v}_{jm+1}(x)$ converging to $u_{m+1}(x)$ in $V_\alpha(\omega)$.

Let ω be the interior domain of $\partial\omega$ and consider the boundary value problem (6.3) with putting $f_p(x)\equiv 0$, $\phi_p(\tilde{x})=\gamma_{jp}(\tilde{x})$ and $\psi_p(\tilde{x})=v_{jp}(\tilde{x})$ for $p=0, 1,\dots, m$ and setting $w_{j\,m+1}(x)=v_{j\,m+1}(x)-\tilde{v}_{j\,m+1}(x)$.[5] Then, by Proposition 6.4, for sufficiently large λ, there exists functions $w_{jp}(x)\in H^{m+2-p}(\omega)$ $(p=0, 1,\dots, m)$ satisfying (6.3) and

$$\sum_{k=0}^{m}\|w_{jp}\|^2_{H^{m+2-p}(\omega)}$$
$$\leqq C\left\{\sum_{j=0}^{m}\left(\langle\gamma_{jp}-\gamma_p\rangle^2_{H^{m+1/2-p}(\partial\omega)}+\langle v_{jp}-u_p\rangle^2_{H^{m+3/2-p}(\partial\omega)}\right)\right.$$
$$\left.+\|v_{j\,m+1}-\tilde{v}_{j\,m+1}\|^2_{H^1(\omega)}\right\}.$$

Here, we use that $\alpha(\tilde{x})\gamma_p(\tilde{x})+(1-\alpha(\tilde{x}))u_p(\tilde{x})=0$ for $p=0, 1,\dots, m$. So, $\sum_{k=0}^{m}\|w_{jp}\|^2_{H^{m+2-p}(\omega)}$ tends to 0 when j tends to ∞. We put $\{u_{j0}, u_{j1}, f_j\}$ as

$$\begin{cases} u_{j0}=v_{j0}-w_{j0}\\ u_{j1}=v_{j1}-w_{j1}\\ f_j=g_j-\sum_{l=2}^{m+1}\left\{w_{jl}+\sum_{k=0}^{l-2}\binom{l-2}{k}(a_2^{(k)}(x, 0; D)w_{jl-k-2}\right.\\ \qquad\qquad\left.+a_1^{(k)}(x, 0; D)w_{jl-k-1})\right\}\frac{t^{l-2}}{(l-2)!}\end{cases}$$

where $w_{j\,m+1}=v_{j\,m+1}-\tilde{v}_{j\,m+1}$. Then, u_{jp} $(p=2, 3,\dots, m+1)$ constructed from $\Phi_j=\{u_{j0}, u_{j1}, f_j\}$ equal to $v_{jp}-w_{jp}$. So, the smooth data Φ_j belong to $S_\alpha^m(L, B)$ and by $u_{j\,m+1}=\tilde{v}_{j\,m+1}$, Φ_j converges to Φ in $S_\alpha^m(L, B)$ when j tends to ∞.

When ω is the exterior domain of $\partial\omega$, by the same device of Proposition 4.3, the existence of an approximating sequence is deduced to the case with a compact domain. Q.E.D.

We denote by B_ε the boundary operator defined by

5) In Lemma 3.1 of Ikawa [8], he puts $w_{m+1}\equiv 0$ without mention it clearly.

$$B_\varepsilon=\frac{\partial}{\partial \boldsymbol{n}_t}-(\sigma_1-\varepsilon)\frac{\partial}{\partial t}+\sigma_2$$

where ε is an arbitrary positive constant.

Lemma 6.6. *For any element $\Phi=\{u_0, u_1, f\}\in S_\alpha^m(L, B)$, there exists a sequence $\Phi_j=\{u_{j0}, u_{j1}, f_j\}\in S_\alpha^m(L, B_{1/j})$ $(j=1, 2,\dots)$ such that $|\Phi_j-\Phi|_{S_\alpha^m}$ tends to 0 when j tends to ∞.*

Proof. Replace the function $\sigma_1^{(k)}(\tilde{x}, 0)$ of (6.3) by $\left(\sigma_1-\frac{1}{j}\right)^{(k)}(\tilde{x}, 0)$. When ω is the interior domain of $\partial\omega$, we solve the boundary value problem (6.3) modified as above, for $w_{m+1}(x)=0, f_p(x)=0, \phi_p(\tilde{x})=0$ and $\psi_p(\tilde{x})=\frac{1}{j}u_{p+1}(\tilde{x})$. Then by Proposition 6.4, there exist the solution $\{w_{jp}\}_{p=0,1,\dots,m}$ of (6.3) satisfying

$$\sum_{p=0}^{m}\|w_{jp}\|^2_{H^{m+2-p}(\omega)}\longrightarrow 0 \qquad \text{when} \quad j\longrightarrow\infty\,.$$

Put $\Phi_j=\{u_{j0}, u_{j1}, f_j\}$ as

$$\begin{cases} u_{j0}=u_0-w_{j0} \\ u_{j1}=u_1-w_{j1} \\ f_j=f-\sum_{l=2}^{m+1}\left\{w_{jl}+\sum_{k=0}^{l-2}\binom{l-2}{k}(a_2^{(k)}w_{jl-k-1}+a_1^{(k)}w_{jl-k-1})\right\}\frac{t^{l-2}}{(l-2)!}, \end{cases}$$

then $\Phi_j\in S_\alpha^m(L, B_{1/j})$ and $|\Phi_j-\Phi|_{S_\alpha^m}\to 0$ when $j\to\infty$. Q.E.D.

Lemma 6.7. *$S_\alpha^{m+1}(L, B)$ is dense in $S_\alpha^m(L, B)$.*

We prove this as same as Lemma 3.3 of [8], so the proof is omitted here.

Proposition 6.8. *For any $\Phi\in S_\alpha^{m+1}(L, B)$, there exists a solution $u(x, t)$ of (5.1) in $H^{m+2}(\omega\times(0, T))$ and $u^{(m+1)}(x, t)$ belongs to $V_\alpha(\omega\times(0, T))$.*

Proof. By Lemma 6.6, there exists a sequence $\Phi_j\in S_\alpha^{m+1}(L, B_{1/j})$ converging to Φ. For each Φ_j, there exists a unique solution $u_j(x, t)$

$\in \bigcap_{k=0}^{m+2} \mathscr{E}_t^{m+3-k}(H^k(\omega) \cap V_\alpha(\omega)) \cap \mathscr{E}_t^{m+3}(L^2(\omega))$ of $P(L, B_{1/j})$ by Theorem 4.2. Here we denote by $P(L, B_{1/j})$ the problem (5.1) with replacing $B(t)$ by $B_{1/j}(t)$.

Therefore from Theorem 5.1, we have

$$|||u_j(\cdot, t)|||_{m+2,\omega}^2 + \int_{\partial_N \omega} \frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})} |u_j^{(m+1)}(\tilde{x}, t)|^2 d\tilde{x}$$

$$\leqq C_m\Big(\|u_{j0}\|_{H^{m+2}(\omega)}^2 + \|u_{j1}\|_{H^{m+1}(\omega)}^2 + \int_{\partial_N \omega} \frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})} |u_{jm+1}(\tilde{x})|^2 d\tilde{x}$$

$$+ |||f_j(\cdot, 0)|||_{m,\omega}^2 + \int_0^t \Big|\Big|\Big| \frac{\partial f_j}{\partial s}(\cdot, s) \Big|\Big|\Big|_{m,\omega}^2 ds \Big)$$

where C_m is independent of j. So, $\{u_j(x, t)\}$ forms a bounded set in $H^{m+2}(\omega \times (0, T))$ and $\{u_j^{(m+1)}(x, t)\}$ is also bounded in $V_\alpha(\omega \times (0, T))$ with respect to j. By the weak compactness of the bounded set, there exists a subsequence $\{u_{jp}(x, t)\}$ converging weakly to some $u(x, t)$ in $H^{m+2}(\omega \times (0, T))$ and $\{u_{jp}^{(m+1)}(x, t)\}$ converging weakly to $u^{(m+1)}(x, t)$ in $V_\alpha(\omega \times (0, T))$. It is easily proved that the function $u(x, t)$ so constructed satisfies $P(L, B)$. Q.E.D.

Proof of Theorem 6.1. Let Φ belong to $S_\alpha^m(L, B)$. There exists a sequence $\Phi_j \in S_\alpha^{m+2}(L, B)$ converging to Φ by Lemma 6.7. Proposition 6.8 assures the existence of the solution $u_j(x, t) \in H^{m+3}(\omega \times (0, T))$ of $P(L, B)$ for the data Φ_j.

By applying Theorem 5.1 to the function $u_k - u_j$, we have

$$|||u_k(\cdot, t) - u_j(\cdot, t)|||_{m+2,\omega}^2 + \int_{\partial_N \omega} \frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})} |(u_k - u_j)^{(m+1)}(\tilde{x}, t)|^2 d\tilde{x}$$

$$\leqq C_m |\Phi_k - \Phi_j|_{S_\alpha^m}^2.$$

This means the convergence of $u_j(x, t)$ to some element $u(x, t)$ in $\mathscr{E}(m+2, \omega) \cap \mathscr{E}_t^{m+1}(V_\alpha(\omega))$. This function $u(x, t)$ is a required solution. It is clear that the solution $u(x, t)$ satisfies the energy inequality

$$|||u(\cdot, t)|||_{m+2,\omega}^2 + \sum_{k=0}^{m+1} \int_{\partial_N \omega} \frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})} |u^{(k)}(\tilde{x}, t)|^2 d\tilde{x}$$

$$\leqq C_m \Big\{ \|u_0\|_{H^{m+2}(\omega)}^2 + \|u_1\|_{H^{m+1}(\omega)}^2 + \sum_{k=0}^{m+1} \int_{\partial_N \omega} \frac{1-\alpha(\tilde{x})}{\alpha(\tilde{x})} |u_k(\tilde{x})|^2 d\tilde{x}$$

$$+ |||f(\cdot,\ 0)|||^2_{m,\omega} + \int_0^t \left|\left|\left| \frac{\partial f}{\partial s}(\cdot,\ s) \right|\right|\right|^2_{m,\omega} ds \Big\} .$$

The uniqueness follows from Lemma 5.2, immediately. Q. E. D.

By the invariance of the condition (5.2) by the Holmgren transformation, we prove easily that the problem (5.1) has a finite velocity. More precisely, let $\lambda_i(x, t; \xi)$ $(i=1, 2)$ be the roots of the characteristic equation of L,

$$\lambda^2 + 2 \sum_{j=1}^n h_j(x, t)\xi_j\lambda - \sum_{i,j=1}^n a_{ij}(x, t)\xi_i\xi_j = 0 ,$$

for $(x, t) \in \bar{\omega} \times [0, T]$ and $\xi \in \mathbf{R}^n$. Denote

$$\lambda_{\max} = \sup_{\substack{i=1,2,|\xi|=1 \\ (x,t)\in\bar{\omega}\times[0,T]}} |\lambda_i(x, t; \xi)|$$

and $\tilde{\Lambda}(x_0, t_0) = \{(x, t);\ |x - x_0| \leqq \lambda_{\max}(t_0 - t)\}$. Then we have

Proposition 6.9. *Let $u(x, t)$ be C^2-function defined on $\tilde{\Lambda}(x_0, t_0) \cap (\bar{\omega} \times [0, T])$ satisfying $L[u]=0$ in $\tilde{\Lambda}(x_0, t_0) \cap (\omega \times (0, T))$ and $\alpha(\tilde{x})B(t)u + (1-\alpha(\tilde{x}))u = 0$ on $\tilde{\Lambda}(x_0, t_0) \cap (\partial\omega \times [0, T])$. If $u_0(x)$, $u_1(x)$ are zero in $\tilde{\Lambda}(x_0, t_0) \cap (\bar{\omega} \times \{0\})$, $u(x, t)$ is identically zero in $\tilde{\Lambda}(x_0, t_0) \cap (\bar{\omega} \times [0, T])$.*

The proof is standard, so omitted here. See, for example [13].

§7. Proofs of Theorems A, B and C

We give another description of Theorem A.

Theorem A. *Let (x^0, t^0) be an arbitrary point of $\bar{\Omega}_T$. Let $u(x, t)$ be a twicely continuously defferentiable function defined on $\Lambda(x^0, t^0) \cap \bar{\Omega}_T$ and satisfying*

$$(7.1) \quad \begin{cases} \Box u = 0 & \text{in } \Lambda(x^0, t^0) \cap \Omega_T, \\ \alpha(\tilde{x},\ t)\dfrac{\partial u}{\partial \boldsymbol{\nu}} + (1 - \alpha(\tilde{x}, t))u = 0 & \text{on } \Lambda(x^0, t^0) \cap \Sigma , \end{cases}$$

where $\Lambda(x^0, t^0) = \{(x, t) \in \mathbf{R}^{n+1};\ |x - x^0| \leqq t^0 - t,\ t \geqq 0\}$.

If the assumptions (a) *and* (b) *are satisfied and the initial data* $\{u_0, u_1\}$ *are zero in* $\Lambda(x^0, t^0) \cap (\bar{\Omega} \times \{0\})$, *then* $u(x, t)$ *vanishes identically on* $\Lambda(x^0, t^0) \cap \bar{\Omega}_T$.

Proof. If $\Lambda(x^0, t^0) \cap \Gamma = \phi$, then the boundary condition satisfied by $u(x, t)$ on $\Lambda(x^0, t^0) \cap \Sigma$ is always of the Neumann type or the Dirichlet type. So in this case, the above statement was already proved in [6], [8], for example.

Assume that $\Lambda(x^0, t^0) \cap \Gamma \neq \phi$. We define the number τ_0 as

$$\tau_0 = \max\{t; (\tilde{x}, t) \in \Lambda(x^0, t^0) \cap \Gamma\}.$$

It is clear from Proposition 6.9 that if $u(x, t)$ is zero in $\Lambda(x^0, t^0) \cap (\bar{\Omega} \times [0, \tau_0])$, then $u(x, t)$ is zero also in $\Lambda(x^0, t^0) \cap (\bar{\Omega} \times [\tau_0, t^0])$. To prove that $u(x, t)$ vanishes in $\Lambda(x^0, t^0) \cap (\bar{\Omega} \times [0, \tau_0])$, we prove as we did in [13]$'$. Define a subset I_N of $[0, \tau_0]$ as

$$I_N = \{t' \in [0, \tau_0];\ \ u(x, t) = 0 \quad \text{in} \quad \Lambda(x^0, t^0) \cap (\bar{\Omega} \times [0, t'])\}.$$

Clearly, $I_N \neq \phi$. Let $(\tilde{x}^1, 0)$ belong to $\Lambda(x^0, t^0) \cap (\Gamma(0) \times \{0\})$. By the assumption (b), there exist a neighborhood $V_{(\tilde{x}^1,0)}$ and a transformation $\Phi_{(\tilde{x}^1,0)} \in (\mathrm{E})$ such that the problem (7.1) in $V_{(\tilde{x}^1,0)}$ is transformed by $\Phi_{(\tilde{x}^1,0)}$ to the following problem (7.2)

$$\text{(7.2)} \quad \begin{cases} \dfrac{\partial^2}{\partial s^2}\tilde{u} + a_1(y, s; D)\dfrac{\partial}{\partial s}\tilde{u} + a_2(y, s; D)\tilde{u} = 0 & \text{in } \tilde{V} \\ \tilde{\alpha}(\tilde{y})\dfrac{\partial u}{\partial \boldsymbol{n}_s} + (1 - \tilde{\alpha}(\tilde{y}))\tilde{u} = 0 & \text{on } \tilde{V} \cap \Phi_{(\tilde{x}^1,0)}(V_{(\tilde{x}^1,0)} \cap \Sigma), \end{cases}$$

where $\tilde{u}(y, s) = u(\Phi^{-1}_{(\tilde{x}^1,0)}(y, s))$, $\tilde{\alpha}(\tilde{y}) = \alpha(\Phi^{-1}_{(\tilde{x}^1,0)}(\tilde{y}, s))$ and the operators $a_1(y, s; D)$ and $a_2(y, s; D)$ are defined in (2.4). By Proposition 2.6, all the requirements in §3~§6 are satisfied if we consider the equations (7.2) in a smaller domain than $\tilde{V}$. Then applying the same argument as in [13]$'$, we prove that I_N is an open, closed and connected set in $[0, \tau_0]$. So $I_N = [0, \tau_0]$. Q.E.D.

From this, we have

Corollary 7.1. *The solution* $u(x, t)$ *of* (1.1) *with* (1.2) *belonging* *to* $C^2(\bar{\Omega}_T)$ *is unique.*

Before proceeding to prove Theorem B, we give the definition of compatibility of the problem (1.1) with (1.2).

Definition 7.2. Let the data $\{u_0(x), u_1(x), f(x, t)\}$ belong to the space $H^{m+2}(\Omega)\times H^{m+1}(\Omega)\times H^{m+1}(\Omega_T)$. We say that the data $\{u_0, u_1, f\}$ satisfies the condition of compatibility (or simply, they are compatible) of order m at $t=0$, when the following condition holds.

$$(7.3)\quad \begin{cases} \sum_{k=0}^{p}\binom{p}{k}\left\{\alpha^{(k)}(\tilde{x}, 0)\dfrac{\partial u_{p-k}}{\partial \nu}+(1-\alpha(\tilde{x}, 0))^{(k)}u_{p-k}\right\}=0 \quad \text{on } \partial\Omega \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for } p=0, 1, 2,\dots, m \\ \text{and} \\ u_{m+1}(x)\in V_{\alpha(\tilde{x},0)}(\Omega) \quad \text{i.e.} \displaystyle\int_{\substack{\partial\Omega \\ \alpha(\tilde{x},0)\neq 0}} \frac{1-\alpha(\tilde{x}, 0)}{\alpha(\tilde{x}, 0)}\,|u_{m+1}(\tilde{x})|^2 d\tilde{x}<\infty \end{cases}$$

where $u_p(x)$ is defined by

$$u_p(x)=u_{p-2}(x)+f^{(p-2)}(x, 0) \qquad \text{for} \quad p=2, 3,\dots, m+1.$$

Remark 7.3. Let $v(x, t)\in H^{m+2}(\Omega_T)$ satisfy the boundary condition

$$\alpha(\tilde{x}, t)\frac{\partial v}{\partial \nu}+(1-\alpha(\tilde{x}, t))v=0 \qquad \text{on } \Sigma.$$

Then, for any vector field $X(\tilde{x}, t)$ tangential to Σ, we have

$$(7.4)\quad X(\tilde{x}, t)^l\left\{\alpha(\tilde{x}, t)\frac{\partial v}{\partial \nu}+(1-\alpha(\tilde{x}, t))v\right\}=0 \quad \text{on} \quad \Sigma \quad \text{for} \quad 0\leqq l\leqq m.$$

So, if $u(x, t)\in H^{m+2}(\Omega_T)$ is a solution of (1.1) with (1.2), then from (7.4) for $u(x, t)$ and the equation $\square u(x, t)=f(x, t)$, by putting $t=0$, we have a certain relation between the data. This is the condition of compatibility. And it is easily proved that the condition of compatibility is independent of the choice of the vector field $X(\tilde{x}, t)$ provided that the inner product of $X(\tilde{x}, t)$ and $\dfrac{\partial}{\partial t}$ never vanishes near $t=0$.

Definition 7.4. We say that the data $\{u_0(x), u_1(x), f(x, t)\}\in C^\infty(\Omega)$

$\times C^\infty(\Omega)\times C^\infty(\Omega_T)$ are compatible of order ∞ at $t=0$, if (7.3) is saitsfied for any m.

Proof of Theorem B. Let (x^0, t^0) be an arbitrary point in $\bar{\Omega}_T$. If $\Lambda(x^0, t^0)\cap\Gamma=\phi$, then we can assign uniquely the value $u(x^0, t^0)$ immediately by Theorem A. Moreover, $u(x, t)$, so defined, is of class C^∞ in a neighborhood of (x^0, t^0) provided that $\Lambda(x, t)\cap\Gamma=\phi$. Let us assume that $\Lambda(x^0, t^0)\cap\Gamma\neq\phi$. Then, by the assumption (a), $\Lambda(x^0, t^0)\cap(\Gamma(0)\times\{0\})\neq\phi$. Let $\tilde{x}^1$ be an arbitrary point of $\Lambda(x^0, t^0)\cap(\Gamma(0)\times\{0\})$. Then, by the assumption (b), there exist a neighborhood $V_{(\tilde{x}^1,0)}$ and a transformation $\Phi_{(\tilde{x}^1,0)}\in(\mathrm{E})$ such that the problem (1.1) with (1.2) in $V_{(\tilde{x}^1,0)}$ is reduced to the following problem.

$$(7.5)\quad \begin{cases} v_{ss}(y, s)+a_1(y, s; D)v_s(y, s)+a_2(y, s; D)v(y, s)=\tilde{f}(y, s), \\ v(y, 0)=\tilde{u}_0(y),\ v_s(y, 0)=\tilde{u}_1(y), \\ \tilde{\alpha}(\tilde{y})\dfrac{\partial v}{\partial \boldsymbol{n}_s}+(1-\tilde{\alpha}(\tilde{y}))v=0, \end{cases}$$

where the operators are defined in (2.4) and

$$\tilde{u}_0(y)=u_0(\psi_k(y, 0), 0),$$

$$\tilde{u}_1(y)=u_1(\psi_k(y, 0), 0)+\sum_{k=1}^{n}\frac{\partial\psi_k}{\partial s}(y, 0)\frac{\partial u_0}{\partial x_k}(\psi_k(y, 0), 0).$$

By easy calculation, we prove that if $\{u_0, u_1, f\}$ are compatible of order ∞ at $t=0$ for $\left\{\square, \alpha(\tilde{x}, t)\frac{\partial}{\partial \boldsymbol{\nu}}+1-\alpha(\tilde{x}, t)\right\}$, then $\{\tilde{u}_0, \tilde{u}_1, \tilde{f}\}$ are compatible of order ∞ at $t=0$ for $\{L(y, s; D_y, D_s), \tilde{\alpha}(\tilde{y})\frac{\partial}{\partial \boldsymbol{n}_s}+1-\tilde{\alpha}(\tilde{y})\}$.

Applying the same argument as we did in §7 of [13]′, we may construct a solution $u(x, t)$ satisfying (1.1) with (1.2). Q.E.D.

Proof of Theorem C. As the problem is linear, we have the desired result by applying Theorem A and the energy inequalities in §5. Q.E.D.

Appendix

Let Ω be a *bounded* domain in $\mathbf{R}^n$ with the smooth boundary $\partial\Omega$.

We consider here the following problem

$$\text{(A.1)}\quad \begin{cases} (\lambda I-\mathscr{A}_0)U=F & \text{in}\quad \Omega\,, \\ \alpha(\tilde{x})\mathscr{B}U+(1-\alpha(\tilde{x}))U=\alpha(\tilde{x})\Phi+(1-\alpha(\tilde{x}))\Psi & \text{on}\quad \partial\Omega\,, \end{cases}$$

where

$$\mathscr{A}_0=\begin{pmatrix} a_0(x;D) & 0 \\ 0 & a_1(x;D)\end{pmatrix},\quad \mathscr{B}=\begin{pmatrix} \dfrac{\partial}{\partial \boldsymbol{n}_0}+b_0(\tilde{x}) & c(\tilde{x}) \\ d_0(\tilde{x})\dfrac{\partial}{\partial \boldsymbol{n}_0}+\sum_{j=1}^{n} d_j(\tilde{x})\dfrac{\partial}{\partial x_j} & \dfrac{\partial}{\partial \boldsymbol{n}_1}+b_1(\tilde{x})\end{pmatrix},$$

$$U=\begin{pmatrix} u_0 \\ u_1\end{pmatrix},\quad F=\begin{pmatrix} f_0 \\ f_1\end{pmatrix},\quad \Phi=\begin{pmatrix} \phi_0 \\ \phi_1\end{pmatrix} \text{ and } \Psi=\begin{pmatrix} \psi_0 \\ \psi_1\end{pmatrix},$$

$a_i(x:D)$ $(i=0, 1)$ are uniformly elliptic operators of order 2 such that $(a_i(x:D)v, v)\leqq 0$ for any $v\in C_0^\infty(\Omega)$, all coefficients are smooth and $\sum_{j=1}^{n} d_j(\tilde{x})n_{0j}=0$ on $\partial\Omega$. Here, we denote by $\boldsymbol{n}_i=(n_{i1}, n_{i2},\dots, n_{in})$ $(i=0, 1)$ the conormal vector of $a_i(x; D)$. As before, we assume that $\alpha(\tilde{x})\in C^\infty(\partial\Omega)$ satisfying (i) $0\leqq\alpha(\tilde{x})\leqq 1$, (ii) $\{\tilde{x}\in\partial\Omega; \alpha(\tilde{x})=0\}$ has a non-empty interior, not equal to $\partial\Omega$, and (iii) $\{\tilde{x}\in\partial\Omega; \alpha(\tilde{x})=0\}$ forms a submanifold of $\partial\Omega$ of codimension 1.

Lemma A.1. *Let* $U=\{u_0, u_1\}\in H^3(\Omega)\times H^2(\Omega)$ *satisfy* (A. 1) *for* $F=\{f_0, f_1\}\in H^1(\Omega)\times L^2(\Omega)$, $\Phi=\{\phi_0, \phi_1\}\in H^{3/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)$ *and* $\Psi=\{\psi_0, \psi_1\}\in H^{5/2}(\partial\Omega)\times H^{3/2}(\partial\Omega)$. *Then, for sufficiently large* λ, *there exists a constant* C *independent of* U, F, Φ *and* Ψ *such that we have*

$$\text{(A.2)}\quad \|u_0\|_3^2+\|u_1\|_2^2\leqq C(\|f_0\|_1^2+\|f_1\|_0^2+\langle\phi_0\rangle_{3/2}^2+\langle\phi_1\rangle_{1/2}^2 +\langle\psi_0\rangle_{5/2}^2+\langle\psi_1\rangle_{3/2}^2)$$

where $\|\cdot\|_l=\|\cdot\|_{H^l(\Omega)}$, $\langle\cdot\rangle_m=\langle\cdot\rangle_{H^m(\partial\Omega)}$.

Proof. From (A. 1), u_0 satisfies

$$\begin{cases} (\lambda-a_0(x:D))u_0=f_0\,, \\ \alpha(\tilde{x})\Big(\dfrac{\partial}{\partial\boldsymbol{\nu}_0}+b_0(\tilde{x})\Big)u_0+(1-\alpha(\tilde{x}))u_0=\alpha(\tilde{x})(\phi_0-c(\tilde{x})u_1|_{\partial\Omega}) \\ \qquad\qquad +(1-\alpha(\tilde{x}))\psi_0(\tilde{x})\,. \end{cases}$$

So, applying the estimate (3.8) to the above problem, we have

$$\|u_0\|_3^2 \leqq C(\|f_0\|_1^2 + \langle \phi_0 - c(\tilde{x})u_1|_{\partial\Omega}\rangle_{3/2}^2 + \langle \psi_0\rangle_{5/2}^2 + \|u_0\|_0^2).$$

Combining this estimate with that of u_1, we have

$$\text{(A.3)} \quad \|u_0\|_3^2 + \|u_1\|_2^2 \leqq C(\|f_0\|_1^2 + \|f_1\|_0^2 + \langle \phi_0\rangle_{3/2}^2 + \langle \phi_1\rangle_{1/2}^2 + \langle \psi_0\rangle_{5/2}^2 + \langle \psi_1\rangle_{3/2}^2 + \|u_0\|_0^2 + \|u_1\|_0^2).$$

We define the operator $\tilde{\mathscr{A}}_0$ by

$$\tilde{\mathscr{A}}_0 U = \mathscr{A}_0 U \qquad \text{for} \quad U \in \mathscr{D}(\tilde{\mathscr{A}}_0),$$

where $\mathscr{D}(\tilde{\mathscr{A}}_0) = \{U \in H^3(\Omega) \times H^2(\Omega);\ \alpha(\tilde{x})\mathscr{B}U + (1-\alpha(\tilde{x}))U = 0 \text{ on } \partial\Omega\}$. Then, using the technique of S. Agmon [2], we have, for sufficiently large λ,

$$\text{(A.4)} \quad \|u_0\|_3^2 + \|u_1\|_2^2 \leqq C|\lambda|^{-1}(\|(\lambda - a_0(x: D))u_0\|_1^2 + \|(\lambda - a_1(x; D))u_1\|_0^2) \quad \text{for} \quad U = \{u_0, u_1\} \in \mathscr{D}(\tilde{\mathscr{A}}_0).$$

As Ω is bounded, applying the method of J. Peetre [23] (see also, J. Lions–E. Magenes [18]), we have the estimate (A.2). Q.E.D.

Remark A.2. The above estimate (A. 2) holds for the operator $\mathscr{A}_0 + \mathscr{A}_1$ where $\mathscr{A}_1 = \begin{pmatrix} 0 & b_1(x, D) \\ b_2(x, D) & 0 \end{pmatrix}$, $b_j(x, D)$ is the operator of order j ($j=1, 2$).

As the same argument as above, we have

Proposition A.3. *Let* $w_p \in H^{m+2-p}(\Omega)$ $(p=0, 1, 2,\dots, m)$ *satisfy* (6.3) *with* $w_{m+1} \in H^1(\Omega)$. *That is,*

$$\text{(6.3)} \quad \begin{cases} (\lambda - a_2(x, 0: D))w_p = f_p & \text{in } \Omega \\ \alpha(\tilde{x}) \sum_{k=0}^{p} \binom{p}{k} \left(\left(\frac{\partial}{\partial \boldsymbol{n}_0} \right)^{(k)} w_{p-k} - \sigma_1^{(k)} w_{p-k+1} + \sigma_2^{(k)} w_{p-k} \right) \\ \qquad + (1-\alpha(\tilde{x}))w_p = \alpha(\tilde{x})\phi_p + (1-\alpha(\tilde{x}))\psi_p & \text{on } \partial\Omega. \end{cases}$$

If Ω is bounded, then we have, for sufficiently large λ,

$$\text{(A.5)}\qquad \sum_{k=0}^{m} \|w_p\|^2_{m+2-p} \leqq C\Big\{\sum_{k=0}^{m} (\|f_p\|^2_{m-p} + \langle\phi_p\rangle^2_{m+1/2-p} + \langle\psi_p\rangle^2_{m+3/2-p}) + \|w_{m+1}\|^2_1\Big\}.$$

Lemma A.4. *Let* $F=\{f_0, f_1\}\in H^1(\Omega)\times L^2(\Omega)$, $\Phi=\{\phi_0, \phi_1\}\in H^{3/2}(\partial\Omega)\times H^{1/2}(\partial\Omega)$ *and* $\Psi=\{\psi_0, \psi_1\}\in H^{5/2}(\partial\Omega)\times H^{3/2}(\partial\Omega)$. *Then, for sufficiently large* λ, *there exists a function* $U=\{u_0, u_1\}\in H^3(\Omega)\times H^2(\Omega)$ *satisfying* (A. 1).

Proof. As $\{\mathscr{A}_0, \mathscr{B}\}$ is 'coercive' in the sense of [3], there exists a function $V=\{v_0, v_1\}\in H^3(\Omega)\times H^2(\Omega)$ satisfying

$$\text{(A.6)}\qquad \begin{cases} (\lambda I-\mathscr{A}_0)V=F & \text{in} \quad \Omega, \\ \mathscr{B}V=\Phi & \text{on} \quad \partial\Omega. \end{cases}$$

If there exists a function U satisfying (A. 1), by putting $W=U-V$, we have

$$\text{(A.7)}\qquad \begin{cases} (\lambda I-\mathscr{A}_0)W=0 & \text{in} \quad \Omega, \\ \alpha(\tilde{x})\mathscr{B}W+(1-\alpha(\tilde{x}))W=(1-\alpha(\tilde{x}))(\Psi-V) & \text{on} \quad \partial\Omega. \end{cases}$$

So, the solvability of (A. 1) is equivalent to that of (A. 7). Moreover, the problem (A. 7) is reduced to the problem on the boundary by introducing the pseudo-differential operators $P_j(\lambda)$ defined as in §3 for each $a_j(x: D)$.

$$\text{(A.8)}\qquad T_\alpha(\lambda)\begin{pmatrix} \tilde{w}_0 \\ \tilde{w}_1 \end{pmatrix}=(1-\alpha(\tilde{x}))\begin{pmatrix} \psi_0-\tilde{v}_0 \\ \psi_1-\tilde{v}_1 \end{pmatrix}.$$

Here, we denote $w_i|_{\partial\Omega}$ by $\tilde{w}_i$ and

$$T_\alpha(\lambda)=\begin{pmatrix} \alpha(\tilde{x})(P_0(\lambda)+b_0(\tilde{x}))+1-\alpha(\tilde{x}) & \alpha(\tilde{x})c(\tilde{x}) \\ \alpha(\tilde{x})\Big(d_0(\tilde{x})P_0(\lambda)+\sum_{j=1}^{n} d_j(\tilde{x})\dfrac{\partial}{\partial x_j}\Big) & \alpha(\tilde{x})(P_1(\lambda)+b_1(\tilde{x})) +1-\alpha(\tilde{x}) \end{pmatrix}.$$

We define the operator $\mathscr{T}_\alpha(\lambda)$: $\mathscr{D}(\mathscr{T}_\alpha(\lambda))=\{\Phi=\{\phi_0, \phi_1\}\in H^{5/2}(\partial\Omega)\times H^{3/2}(\partial\Omega);\ T_\alpha(\lambda)\Phi\in H^{5/2}(\partial\Omega)\times H^{3/2}(\partial\Omega)\}\to H^{5/2}(\partial\Omega)\times H^{3/2}(\partial\Omega)$ by $\mathscr{T}_\alpha(\lambda)\Phi$

$=T_\alpha(\lambda)\Phi$ for $\Phi\in\mathscr{D}(\mathscr{T}_\alpha(\lambda))$. Then, the solvability of the problem (A. 1) is derived from that of the problem $\mathscr{T}_\alpha(\lambda)\Phi=\Psi$. Using the estimate (A. 2), we have readily that $\mathscr{T}_\alpha(\lambda)$ is a 1–1, closed operator with the closed range in $H^{5/2}(\partial\Omega)\times H^{3/2}(\partial\Omega)$. (See, [15]', [24]). So, it is sufficient to claim that the adjoint operator $\mathscr{T}^*_\alpha(\lambda)$ is also 1–1. As A. Kaji did in [15], we consider the following boundary value problem

$$\text{(A.9)}\qquad \begin{cases} (\lambda I-\mathscr{A}^*_0)V=0 & \text{in } \Omega\,,\\ CV=\Psi & \text{on } \partial\Omega\,,\end{cases}$$

where $\mathscr{A}^*_0=\begin{pmatrix} a^*_0(x, D) & 0\\ 0 & a^*_1(x; D)\end{pmatrix}$, $C=\begin{pmatrix} c^\mu_{11}(\tilde{x}, D) & c^\mu_{12}(\tilde{x}, D)\\ c^\mu_{21}(\tilde{x}, D) & c^\mu_{22}(\tilde{x}, D)\end{pmatrix}$, $c^\mu_{ij}(\tilde{x}, D)$ is the pseudo-differential operator with parameter μ defined by

$$\begin{cases} c^\mu_{11}(\tilde{x}, D)=\alpha(\tilde{x})\Big(\dfrac{\partial}{\partial\boldsymbol{\nu}_0}+b^*_0(\tilde{x})\Big)+(\alpha(\tilde{x})P_0(\mu))^*-\alpha(\tilde{x})P^*_0(\mu)+1-\alpha(\tilde{x})\,,\\ c^\mu_{12}(\tilde{x}, D)=\alpha(\tilde{x})d_0(\tilde{x})\dfrac{\partial}{\partial\boldsymbol{\nu}_0}+\Big(\alpha(\tilde{x})\sum\limits_{j=1}^n d_j(x)\dfrac{\partial}{\partial x_j}\Big)^*+(\alpha(\tilde{x})d_0(\tilde{x})P_0(\mu))^*\\ \qquad\qquad -\alpha(\tilde{x})d_0(\tilde{x})P^*_0(\mu)\,,\\ c^\mu_{21}(\tilde{x}, D)=\alpha(\tilde{x})c^*(\tilde{x})\,,\\ c^\mu_{22}(\tilde{x}, D)=\alpha(\tilde{x})\Big(\dfrac{\partial}{\partial\boldsymbol{\nu}_1}+b^*_1(\tilde{x})\Big)+(\alpha(\tilde{x})P_1(\mu))^*-\alpha(\tilde{x})P^*_1(\mu)+1-\alpha(\tilde{x})\,.\end{cases}$$

Here $P(x, D)^*$ stands for the formal adjoint of the pseudo-differential operator $P(x, D)$. $P^*_j(\lambda)$ corresponds to the operator defined by $a^*_j(x: D)$. As is remarked in [15], there exists a positive constant c_s independent of μ such that

$$\langle((\alpha(\tilde{x})P_j(\mu))^*-\alpha(\tilde{x})P^*_j(\mu))\phi\rangle_s\leqq c_s\langle\phi\rangle_s \qquad \text{for any}\quad \phi\in H^s(\partial\Omega)\,.$$

So, we may treat the problem (A. 9) as before. The problem (A. 9) is equivalent to

$$\text{(A.10)}\qquad T^*_{\alpha,\mu}(\lambda)\Phi=\Psi$$

where

$$T^*_{\alpha,\mu}(\lambda)=$$

$$\begin{pmatrix} \alpha(P_0^*(\lambda)+b_0^*)+(\alpha P_0(\mu))^*-\alpha P_0^*(\mu)+1-\alpha & \alpha d_0 P_0^*(\lambda)+\left(\alpha\Sigma d_j\frac{\partial}{\partial x_j}\right)^*+(\alpha d_0 P_0(\mu))^*-\alpha d_0 P_0^*(\mu) \\ \alpha c^* & \alpha(P_1^*(\lambda)+b_1^*)+(\alpha P_1(\mu))^*-\alpha P_1^*(\mu)+1-\alpha \end{pmatrix}.$$

Clearly $T^*_{\alpha,\lambda}(\lambda)=T^*_\alpha(\lambda)$. We define the operator $\mathscr{T}^*_{\alpha 1}(\lambda)$: $\mathscr{D}(\mathscr{T}^*_{1\alpha}(\lambda))=\{\Phi=\{\phi_0, \phi_1\}\in H^{-5/2}(\partial\Omega)\times H^{-3/2}(\partial\Omega);\ T^*_\alpha(\lambda)\Phi\in H^{-5/2}(\partial\Omega)\times H^{-3/2}(\partial\Omega)\}\to H^{-5/2}(\partial\Omega)\times H^{-3/2}(\partial\Omega)$ by $\mathscr{T}^*_{1\alpha}(\lambda)\Phi=T^*_\alpha(\lambda)\Phi$ for $\Phi\in\mathscr{D}(\mathscr{T}^*_{1\alpha}(\lambda))$. Then, by repeating the same argument as before to the problem (A. 9), we have that $\mathscr{T}^*_{1\alpha}(\lambda)$ is a 1–1 operator. Denoting by $\mathscr{T}^*_\alpha(\lambda)$ the adjoint operator of $\mathscr{T}_\alpha(\lambda)$ with respect to the pairing $H^{5/2}(\partial\Omega)\times H^{3/2}(\partial\Omega)$ and $H^{-5/2}(\partial\Omega)\times H^{-3/2}(\partial\Omega)$, we have $\mathscr{T}^*_{1\alpha}(\lambda)\supset\mathscr{T}^*_\alpha(\lambda)$. (Here we use the standard argument of the elliptic boundary value problem. See [15]'). This means that $\mathscr{T}^*_\alpha(\lambda)$ is 1–1. Q. E. D.

The proof of Proposition 6.4 will be carried out by applying the same argument as above.

References

[1] Agemi, R., On energy inequality of mixed problems for hyperbolic equation of second order, *J. Fac. Sci. Hokkaido Univ.*, **21** (1971), 221–236.

[2] Agmon, S., On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems, *Comm. Pure Appl. Math.*, **15** (1962), 119–147.

[3] Agmon, S., Douglis, D. and Nirenberg, L., Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, I. *Comm. Pure Appl. Math.*, **12** (1959), 623–727. II. *Comm. Pure Appl. Math.*, **17** (1964), 35–92.

[4] Cooper, J. and Bardos, C., A non-linear wave equation in a time dependent domain, *J. Math. Anal. Appl.*, **42** (1973), 29–60.

[5] Fujiwara, D. and Uchiyama, K., On some dissipative boundary value problems for the Laplacian, *J. Math. Soc. Japan*, **23** (1971), 625–635.

[6] Ikawa, M., Mixed problems for hyperbolic equations of second order, *J. Math. Soc. Japan*, **20** (1968), 580–608.

[7] ————., A mixed problem for hyperbolic equations of second order with non-homogeneous Neumann type boundary condition, *Osaka J. Math.*, **6** (1969), 339–374.

[8] Ikawa, M., On the mixed problem for hyperbolic equations of second order with the Neumann boundary condition, *Osaka J. Math.*, **7** (1970), 203–223.
[9] ————., Mixed problem for the wave equation with an oblique derivative boundary condition, *Osaka J. Math.*, **7** (1970), 495–525.
[10] ————., Remarques sur les problèmes mixtes pour l'équation des ondes, *Colloque International du C. N. R. S.* (1973), 217–221.
[10]′ ————., ———————————————————— à paraître.
[11] ————., Sur les problèmes mixtes pour l'équation des ondes, à paraîtres.
[12] Inoue, A., On the mixed problem for the wave equation with an oblique boundary condition, *J. Fac. Sci. Univ. Tokyo Sec. I*, **16** (1970), 313–329.
[13] ————., Sur $\Box u+u^3=f$ dans un domaine non-cylindrique, *C. R. Acad. Sci. Paris Ser A*, **275** (1972), 659–662.
[13]′ Inoue, A., ————————————————————, *J. Math. Anal. Appl.*, **46** (1973), 777–819.
[14] ————., On a mixed problem for $\Box$ with discontinuous boundary condition (I), *J. Fac. Sci. Univ. Tokyo Sec. I*, **21** (1974), 85–92.
[15] Kaji, A., On the degenerate oblique derivative problems, *Proc. Japan Acad.*, **50** (1974), 1–5.
[15]′ Kaji, A., On the non-coercive boundary value problems, *Master thesis* University of Tokyo (in Japanese).
[16] Kato, T., Linear evolution of 'hyperbolic' type, *J. Fac. Sci. Univ. Tokyo Sec. I*, **17** (1970), 241–258.
[17] ————., ———————————————————— II, *J. Math. Soc. Japan*, **25** (1973), 648–666.
[18] Lions, J. L. and Magenes, E., *Problèmes aux limites non-homogènes et applications*, Vol. 1, Dunod, Paris, 1971.
[19] Melin, A., Lower bounds for pseudo-differential operators, *Ark. Math.*, **9** (1971), 117–140.
[20] Miranda, C., *Partial differential cquations of elliptic type*, Springer, Berlin-Heidelberg-New York, 1970.
[21] Miyatake, S., Mixed problem for hyperbolic equation of second order, *J. Math. Kyoto Univ.*, **13** (1973), 435–487.
[22] Mizohata, S., *Theory of partial differential equations*, Iwanami, Tokyo (1965), (in Japanese).
[23] Peetre, J., Another approach to elliptic boundary problems, *Comm. Pure Appl. Math.*, **14** (1961), 711–731.
[24] Taira, K., On some degenerate oblique derivative problems, *J. Fac. Sci. Univ. Tokyo* (to appear).
[25] Visik, I. M., On general boundary problems for elliptic differential equations, *Trudy Moskov. Mat. Obšč.*, **1** (1952), 187–246, *Translation A. M. S.* (2), **24** (1963), 107–172.
[26] Yosida, K., *Functional Analysis*, Springer, Berlin-Heidelberg-New York, 1965.