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Singularities of the Riemann Functions of Hyperbolic Mixed Problems in a Quarter-Space

By

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1. Introduction

Singularities of Riemann functions of hyperbolic mixed problems with constant coefficients in a quarter-space have been investigated, for example, by Duff [3], Deakin [2], Matsumura [5] and others. In his pioneering work [3], Duff studied the location and structures of singularities of reflected Riemann functions making use of the stationary phase method. Deakin [2] treated first order hyperbolic systems by the same method. However, it seems that it is difficult to apply the stationary phase method to the study of Riemann functions of more general hyperbolic mixed problems. Matsumura [5] gave an inner estimate of the location of singularities of reflected Riemann functions which correspond to reflected waves making use of the localization method developed by Atiyah, Bott and Gårding [1] and Hörmander [4]. A localization theorem describing the location of singularities of reflected Riemann functions which correspond to lateral waves was obtained by the author [8] under some restrictive assumptions.

In this paper we shall deal with hyperbolic mixed problems in a quarter-space under more general assumptions and prove a localization theorem describing the location of singularities of reflected Riemann functions which correspond to reflected waves, lateral waves and boundary waves. Tsuji [7] studied the same problem in the cases where $P(\xi)$, $B_j(\xi)$ are homogeneous and obtained similar results. We originally

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formulated and proved our localization theorem making use of the representations of reflected Riemann functions given in [5], [8]. However we can give a simpler proof if we use the representation of reflected Riemann functions given in Tsuji [7]. So we shall give our proof using the representation.

Now let us state our problems, assumptions and main results. Let $\mathbf{R}^n$ denote the n-dimensional Euclidean space and Ξ^n its real dual space and write $x'=(x_1,\dots, x_{n-1})$, $x''=(x_2,\dots, x_n)$ for the coordinate $x=(x_1,\dots, x_n)$ in $\mathbf{R}^n$ and $\xi'=(\xi_1,\dots, \xi_{n-1})$, $\xi''=(\xi_2,\dots, \xi_n)$, $\xi'''=(\xi_2,\dots, \xi_{n-1})$ for the dual coordinate $\xi=(\xi_1,\dots, \xi_n)$. The variable x_1 will play the role of "time", the variables $x_2,\dots, x_n$ will play the role of "space". We shall also denote by $\mathbf{R}^n_+$ the half-space $\{x=(x', x_n)\in\mathbf{R}^n; x_n>0\}$. For differentiation we will use the symbol $D=i^{-1}(\partial/\partial x_1,\dots, \partial/\partial x_n)$. Let $P=P(\xi)$ be a hyperbolic polynomial of order m of n variables ξ with respect to $\vartheta=(1, 0,\dots, 0)\in\Xi^n$ in the sense of Gårding, i.e. $P^0(\vartheta)\neq 0$ and $P(\xi+s\vartheta)\neq 0$ when ξ is real and $\operatorname{Im} s<-\gamma_0$, where P^0 denotes the principal part of P. We consider the mixed initial-boundary value problem for the hyperbolic operator $P(D)$ in a quarter-space

$$(1.1)\qquad P(D)u(x)=f(x),\quad x\in\mathbf{R}^n_+,\quad x_1>0,$$

$$(1.2)\qquad (D_1^k u)(0, x'')=0,\quad 0\leqq k\leqq m-1,\quad x_n>0,$$

$$(1.3)\qquad B_j(D)u(x)|_{x_n=0}=0,\quad 1\leqq j\leqq l,\quad x_1>0.$$

Here the $B_j(D)$ are boundary operators with constant coefficients. The number l of boundary conditions will be determined later on. We assume that the hyperplane $x_n=0$ is non-characteristic for $P(D)$.

Let us denote by $\operatorname{Re} A$ be the real hypersurface $\{\xi\in\Xi^n; P^0(\xi)=0\}$ and by $\Gamma=\Gamma(P, \vartheta)\,(\subset\Xi^n)$ the component of $\Xi^n\backslash\operatorname{Re} A$ which contains ϑ. When $\xi'\in\Xi^{n-1}-i\gamma_0\vartheta'-i\Gamma_0$, we can denote the roots of $P(\xi', \lambda)=0$ with respect to λ by $\lambda_1^+(\xi'),\dots, \lambda_l^+(\xi')$, $\lambda_1^-(\xi'),\dots, \lambda_{m-l}^-(\xi')$, which are enumerated so that

$$(1.4)\qquad \begin{aligned}&\operatorname{Im}\lambda_k^+(\xi')>0,\qquad 1\leqq k\leqq l,\\ &\operatorname{Im}\lambda_k^-(\xi')<0,\qquad 1\leqq k\leqq m-l,\end{aligned}$$

Here Γ_0 denotes the set $\{\eta' \in \Xi^{n-1}; (\eta', 0) \in \Gamma\}$. The number l in (1.4) determines that of boundary conditions. Put

$$(1.5) \qquad P_+(\xi', \lambda) = \prod_{j=1}^{l} (\lambda - \lambda_j^+(\xi')), \quad \xi' \in \Xi^{n-1} - i\gamma_0 \vartheta' - i\Gamma_0.$$

We now define the Lopatinski determinant for the system $\{P, B_j\}$ by

$$(1.6) \qquad R(\xi') = \det\left(\frac{1}{2\pi i} \oint \frac{B_j(\xi', \lambda)\lambda^{k-1}}{P_+(\xi', \lambda)} d\lambda\right)_{j,k=1,\dots,l},$$

$$\xi' \in \Xi^{n-1} - i\gamma_0 \vartheta' - i\Gamma_0.$$

We remark that

$$(1.7) \qquad R(\xi') = \det(B_j(\xi', \lambda_k^+(\xi'))) / \prod_{1 \leq j < k \leq l} (\lambda_j^+(\xi') - \lambda_k^+(\xi')),$$

$$\xi' \in \Xi^{n-1} - i\gamma_0 \vartheta' - i\Gamma_0.$$

We state the assumptions that we impose on $\{P, B_j\}$:

$$(\text{A.1}) \qquad P(\xi) = p_1(\xi)^{\nu_1} \dots p_q(\xi)^{\nu_q},$$

where the $p_j(\xi)$ are distinct strictly hyperbolic polynomials with respect to ϑ and irreducible over the complex number field $\mathbf{C}$.

(A. 2) The system $\{P, B_j\}$ is $\mathscr{E}$-well posed, i.e.

$$R(\xi' + s\vartheta') \neq 0 \quad \text{for} \quad \xi' \in \Xi^{n-1} \quad \text{and} \quad \operatorname{Im} s < -\gamma_1,$$

$$\tilde{R}_0(\vartheta') \neq 0,$$

where $\tilde{R}_0(\xi')$ denotes the principal part of $R(\xi')$ defined by (2.6) (see Sakamoto [6]).

Now we can construct the Riemann function $G(x, y)$ for $\{P, B_j\}$ which describes the propagation of waves produced by unit impulse given at position $y = (0, y_2, \dots, y_n) \in \mathbf{R}_+^n$ (see [6], [7]). Write

$$(1.8) \qquad G(x, y) = E(x - y) - F(x, y), \quad x \in \mathbf{R}_+^n, \quad x_1 > 0,$$

$$y = (0, y_2, \dots, y_n) \in \mathbf{R}_+^n,$$

where $E(x)$ is the fundamental solution represented by

$$(1.9)\qquad E(x)=(2\pi)^{-n}\int_{\Xi^n}\exp[ix\cdot(\xi+i\eta)]P(\xi+i\eta)^{-1}d\xi,\quad \eta\in-\gamma_0\vartheta-\Gamma.$$

Then the reflected Riemann function $F(x, y)$ is written in the form

$$(1.10)\qquad F(x, y)=(2\pi)^{-(n+1)}\int_{\Xi^{n+1}}\frac{1}{i}\sum_{j,k=1}^{l}\exp[i\{(x'-y')\cdot(\xi'+i\eta')$$

$$+\lambda x_n-y_n(\xi_n+i\eta_n)\}]\frac{R_{jk}(\xi'+i\eta')B_k(\xi+i\eta)\lambda^{j-1}}{R(\xi'+i\eta')P_+(\xi'+i\eta',\lambda)P(\xi+i\eta)}d\xi d\lambda,$$

where $\eta\in-\gamma_0\vartheta-\Gamma$ and $\eta'\in-\gamma_0\vartheta'-\Gamma_0$. Here $R_{jk}(\xi')=(k, j)$-cofactor of $\left(\frac{1}{2\pi i}\oint\frac{B_j(\xi',\lambda)\lambda^{k-1}}{P_+(\xi',\lambda)}d\lambda\right)_{j,k=1,\dots,l}$. $F(x, y)$ has to be interpreted in the sense of distribution with respect to $(x, y)\in\mathbf{R}_+^n\times\mathbf{R}_+^n$.

Let $\mu_1^+(\xi'),\dots,\mu_l^+(\xi'),\mu_1^-(\xi'),\dots,\mu_{m-l}^-(\xi')$ be the roots of $P^0(\xi',\mu)=0$. Since

$$(1.11)\qquad t^{-m}P(t\xi', t\mu)\longrightarrow P^0(\xi',\mu)\quad \text{as}\quad t\longrightarrow\infty,$$

it follows that,

$$(1.12)\qquad t^{-1}\lambda_k^{\pm}(t\xi')\longrightarrow\mu_k^{\pm}(\xi')\quad \text{as}\quad t\longrightarrow\infty,$$

if the μ's are labelled suitably. Then our main result is stated as follows:

Theorem 1.1. *Let $\xi^0\in\Xi^n$ and μ^0 be real. Then we have*

$$(1.13)\qquad t^{p_0}\exp[-it\{(x'-y')\cdot\xi^{0\prime}+x_n\mu^0-y_n\xi_n^0\}]F(x, y)$$

$$\sim\sum_{j=0}^{\infty}F_j(x, y)t^{-j/L},$$

where p_0 is a rational number and L is a positive integer. Here (1.13) implies that

$$(1.14)\qquad t^{N/L}\{t^{p_0}\exp[-it\{(x'-y')\cdot\xi^{0\prime}+x_n\mu^0-y_n\xi_n^0\}]F(x, y)$$

$$-\sum_{j=0}^{N-1}F_j(x, y)t^{-j/L}\}\longrightarrow F_N(x, y)\quad \text{as}\quad t\longrightarrow\infty,\quad N=0, 1, 2,\dots,$$

in the sense of distribution with respect to $(x, y)\in\mathbf{R}_+^n\times\mathbf{R}_+^n$. Moreover, for $\xi^0\neq0$ we have

(1.15) $$\cup_{j=0}^{\infty} \operatorname{supp} F_j(x, y) \subset \operatorname{sing\,supp} F(x, y)$$

and

(1.16) $$\operatorname{supp} F_j(x, y) \subset \{(x, y) \in \mathbf{R}_+^n \times \mathbf{R}_+^n;$$
$$(x'-y') \cdot \eta' - y_n \eta_n + x_n \lambda \geqq 0 \quad \text{for all} \quad (\eta, \lambda) \in \Gamma_{\xi^0, \mu^0}\},$$

where Γ_{ξ^0,μ^0} will be defined by (4.4).

Remark 1. The corresponding result for the fundamental solution $E(x)$ was obtained in [1], i.e.

(1.17) $$t^{m-p} \exp[-itx \cdot \xi^0] E(x) \longrightarrow E_{\xi^0}(x) \qquad \text{as} \quad t \longrightarrow \infty,$$

(1.18) $$E_{\xi^0}(x) = (2\pi)^{-n} \int_{\Xi^n} \exp[ix \cdot (\xi + i\eta)] P_{\xi^0}(\xi + i\eta)^{-1} d\xi,$$
$$\eta \in -\gamma_0 \vartheta - \Gamma,$$

where the localization P_{ξ^0} of P at ξ^0 is defined by

(1.19) $$v^m P(v^{-1} \xi^0 + \eta) = v^p P_{\xi^0}(\eta) + O(v^{p+1}) \qquad \text{as} \quad v \longrightarrow 0.$$

Remark 2.

(1.20) $$\{(x, y) \in \mathbf{R}_+^n \times \mathbf{R}_+^n; (x'-y') \cdot \eta' - y_n \eta_n + x_n \lambda \geqq 0 \quad \text{for all} \quad (\eta, \lambda) \in \Gamma_{\xi^0,\mu^0}\}$$
$$= \cup_{\substack{\theta_1 + \cdots + \theta_{r_0} = 1 \\ \theta_1, \dots, \theta_{r_0} \geqq 0}} \{(x, y) \in \mathbf{R}_+^n \times \mathbf{R}_+^n; [(x'-y')$$
$$+ x_n(\theta_1 \operatorname{grad} \mu_1^+(\xi^{0\prime}) + \cdots + \theta_{r_0} \operatorname{grad} \mu_{r_0}^+(\xi^{0\prime}))] \cdot \eta'$$
$$- y_n \eta_n \geqq 0 \qquad \text{for all} \quad \eta \in \Gamma(P_{\xi^0}, \vartheta) \cap (\dot{\Sigma}_{\xi^{0\prime}} \times \Xi)\},$$

where $\mu_k^+(\xi')$, $1 \leqq k \leqq r_0$, are the simple roots of $p_j^0(\xi', \mu) = 0$ for $|\xi' - \xi^{0\prime}| < \varepsilon$ such that $\mu_k^+(\xi^{0\prime}) = \mu^0$, and $\dot{\Sigma}_{\xi^{0\prime}}$ will be defined by (3.5). Using the representations of reflected Riemann functions given in [8], we originally obtained

(1.21) $$\operatorname{supp} F_j(x, y) \subset \cup_{\substack{\theta_1 + \cdots + \theta_{r_0} = 1 \\ \theta_1, \dots, \theta_{r_0} \geqq 0}} \{(x, y) \in \mathbf{R}_+^n \times \mathbf{R}_+^n;$$

$$[(x'-y')+x_n(\theta_1 \operatorname{grad} \mu_1^+(\xi^{0\prime})+\cdots+\theta_{r_0} \operatorname{grad} \mu_{r_0}^+(\xi^{0\prime}))]\cdot\eta'$$

$$-y_n\eta_n \geqq 0 \qquad \text{for all} \quad \eta \in \Gamma(P_{\xi^0}, \vartheta) \cap (\dot{\Sigma}_{\xi^{0\prime}} \times \Xi)\}.$$

Remark 3. Tsuji [7] also proved (1.13)–(1.15) in the cases where $P(\xi)$, $B_j(\xi)$ are homogeneous.

The remainder of this paper is organized as follows. In §2 we shall study some properties of the roots $\lambda_j(\xi')$. In §3 the localization of the Lopatinski determinant will be defined and its properties will be studied. In §4 we shall prove the localization theorem (Theorem 1.1). Some examples will be given in §5.

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§2. Algebraic Considerations and Lopatinski Determinant

Put

$$\sigma(\xi'')=\max_{P^0(\xi)=0} \xi_1 \qquad \text{for} \quad \xi'' \in \Xi^{n-1}, \tag{2.1}$$

$$\tilde{\sigma}(\xi''')=\inf_{\xi_n \in \Xi} \sigma(\xi'') \qquad \text{for} \quad \xi''' \in \Xi^{n-2}, \tag{2.2}$$

$$\dot{\Gamma}=\{\xi' \in \Xi^{n-1};\ \xi_1>\tilde{\sigma}(\xi''')\}. \tag{2.3}$$

Then it is obvious that

$$\Gamma=\{\xi \in \Xi^n;\ \xi_1>\sigma(\xi'')\}, \tag{2.4}$$

$$\Gamma_0 \subset \dot{\Gamma}. \tag{2.5}$$

Lemma 2.1 ([6]). *$R(\xi')$ is holomorphic in $\Xi^{n-1}-i\gamma_0\vartheta'-i\dot{\Gamma}$.*

Lemma 2.2 ([6]). *Let K be a compact set in $\Xi^{n-1}-i\dot{\Gamma}$, then there exists $T_K>0$ such that*

$$R(t\xi')=t^{h_0}\{\tilde{R}_0(\xi')+t^{-1}\tilde{R}_1(\xi')+t^{-2}\tilde{R}_2(\xi')+\cdots\} \tag{2.6}$$

whose convergence is uniform in $K\times\{t>T_K\}$, where

(i) *$\{\tilde{R}_j(\xi')\}$ are holomorphic in $\dot{\boldsymbol{\Gamma}}=\cup_{z\in\mathbf{C}\setminus\{0\}} z(\Xi^{n-1}-i\dot{\Gamma})$,*

(ii) $\tilde{R}_j(t\xi')=t^{h_0-j}\tilde{R}_j(\xi')$ *for* $\xi'\in\dot{\Gamma}$, $t\in\mathbf{C}\backslash\{0\}$,

(iii) $\tilde{R}_0(\xi')\not\equiv 0$ *and* h_0 *is an integer.*

Put

$$\dot{\sigma}(\xi''')=\begin{cases}\tilde{\sigma}(\xi''') & \text{if } \tilde{R}_0(\xi')\neq 0 \text{ for } \xi'\in\dot{\Gamma},\\ \sup_{\substack{\tilde{R}_0(\xi')=0\\ \xi'\in\dot{\Gamma}}}\xi_1 & \text{otherwise,}\end{cases}\tag{2.7}$$

$$\dot{\Sigma}=\{\xi'\in\Xi^{n-1};\ \xi_1>\dot{\sigma}(\xi''')\}.\tag{2.8}$$

Then it follows from (A.2) that $\vartheta'\in\dot{\Sigma}$.

Lemma 2.3 ([6]). *There exists a positive constant* γ_1 *such that*

$$R(\xi')\neq 0 \qquad \textit{for} \quad \xi'\in\Xi^{n-1}-i\gamma_1\vartheta'-i\dot{\Sigma}.\tag{2.9}$$

Moreover, for any compact set K *in* $\gamma_1\vartheta'+\dot{\Sigma}$ *there exist positive constants* c_K *and* a_K *such that*

$$|R(\xi')|\geqq c_K|\xi'|^{-a_K} \qquad \textit{for} \quad \xi'\in\Xi^{n-1}-i\Lambda_K,\tag{2.10}$$

where $\Lambda_K=\{t\xi',\ \xi'\in K,\ t\geqq 1\}$.

Let $p(\xi)$ be a strictly hyperbolic polynomial with respect to ϑ and assume that $p^0(0,1)\neq 0$, $p(\xi)\neq 0$ for $\xi\in\Xi^n-i\gamma_0\vartheta-i\Gamma$. We consider the real roots of $p(\xi',\lambda)=0$ in a neighborhood of $\xi'=\xi^0$, where $\xi^{0\prime}\in\Xi^{n-1}\backslash\{0\}$ is arbitrarily fixed. Put

$$p(\xi',\lambda;\nu)=p^0(\xi',\lambda)+\nu p^1(\xi',\lambda)+\cdots+\nu^{\tilde{m}}p^{\tilde{m}}(=\nu^{\tilde{m}}p(\nu^{-1}\xi',\nu^{-1}\lambda))\tag{2.11}$$

where $\deg p=\tilde{m}$. We can assume without loss of generality that $\lambda=0$ is an l-ple root of $p^0(\xi^{0\prime},\lambda)=0$. Therefore we have

$$p(\xi',\lambda;\nu)=(\lambda^l+a_1(\xi';\nu)\lambda^{l-1}+\cdots+a_l(\xi';\nu))q(\xi',\lambda;\nu)\tag{2.12}$$
$$\text{for} \quad |\xi'-\xi^{0\prime}|<\varepsilon \quad \text{and} \quad |\nu|<\varepsilon,$$

where the $a_j(\xi';\nu)$ and $q(\xi',\lambda;\nu)$ are holomorphic for $|\xi'-\xi^{0\prime}|<\varepsilon$ and $|\nu|<\varepsilon$ and $a_j(\xi^{0\prime};0)=0$ and $q(\xi',\lambda;\nu)\neq 0$ for $|\xi'-\xi^{0\prime}|<\varepsilon$, $|\nu|<\varepsilon$ and $|\lambda|<\varepsilon$. Then

(2.13) $a_j(\xi'; \nu)=\sum_{k+|\alpha|\geqq 1} a_{jk\alpha}\nu^k(\xi'-\xi^{0\prime})^\alpha, \quad |\nu|<\varepsilon, \quad |\xi'-\xi^{0\prime}|<\varepsilon,$

and, therefore,

(2.14) $$a_j(\xi^{0\prime}+\nu\eta'; \nu)=\sum_{k=1}^{\infty} a_{jk}(\eta')\nu^k,$$

where

(2.15) $$a_{jk}(\eta')=\sum_{i+|\alpha|=k} a_{ji\alpha}\eta'^\alpha.$$

Lemma 2.4. *Let* $\tau(\xi'')$ *be a root of* $p^0(\tau, \xi'')=0$ *such that* $\tau(\xi^{0\prime\prime\prime}, 0)=\xi_1^0$. *Then*

(2.16) $$a_{l1}(\eta')=\frac{\partial}{\partial\nu}a_l(\xi^{0\prime}+\nu\eta'; \nu)\,|_{\nu=0}$$
$$=\text{const.}^{1)}\left(\frac{\partial}{\partial\nu}p^0(\xi^{0\prime}+\nu\eta', 0)\,|_{\nu=0}+p^1(\xi^{0\prime}, 0)\right)$$
$$=\text{const.}\left[\text{const.}\left(\eta_1-\sum_{j=2}^{n-1}\frac{\partial\tau}{\partial\xi_j}(\xi^{0\prime\prime\prime}, 0)\eta_j\right)+p^1(\xi^{0\prime}, 0)\right].$$

Moreover if $l>1$,

(2.17) $$a_{l1}(\eta')=\text{const.}\, p_{(\xi^{0\prime},0)}(\eta),$$

where $p_{(\xi^{0\prime},0)}$ *is the localization of* p *at* $(\xi^{0\prime}, 0)$ *defined by* (1.19). *Therefore*

(2.18) $$a_{l1}(\eta')\neq 0 \qquad \textit{for} \quad \eta'\in\Xi^{n-1}-i\gamma_0\vartheta'-i\dot{\Gamma}.$$

Proof. (2.16) is obvious. Since $p(\xi)$ is a strictly hyperbolic polynomial,

$$p_{(\xi^{0\prime},0)}(\eta)=\sum_{j=1}^{n}\frac{\partial p^0}{\partial\xi_j}(\xi^{0\prime}, 0)\eta_j+p^1(\xi^{0\prime}, 0).$$

If $\lambda=0$ is a multiple root of $p^0(\xi^{0\prime}, \lambda)=0$, $\frac{\partial p^0}{\partial\xi_n}(\xi^{0\prime}, 0)=0$. This proves (2.17). Since $\Gamma\subset\Gamma(p_{(\xi^{0\prime},0)}, \vartheta)$, (2.18) can be obtained (see [1]). Q.E.D.

1) Here and in sequal const. denotes a non-zero constant.

Let $\zeta(\xi'; \nu)$ be a root of $p(\xi', \lambda; \nu)=0$ such that $\zeta(\xi^{0\prime}; 0)=0$. Then we have the following

Lemma 2.5. *For any compact set K in $\Xi^{n-1}-i\gamma_0\vartheta'-i\dot{\Gamma}$ and any positive integer N there exists $\varepsilon>0$ such that*

$$\zeta(\xi^{0\prime}+\nu\eta'; \nu)=\sum_{j=1}^{N}c_j(\eta')\nu^{j/l}+O(\nu^{(N+1)/l}) \tag{2.19}$$

if $\eta'\in K$ and $|\nu|<\varepsilon$. If $l=1$, the $c_j(\eta')$ are polynomials of η', and if $l>1$, the $c_j(\eta')$ are equal to (polynomial of η')$\times c_1(\eta')^{-n_j}$, where the n_j are integers. In particular,

$$c_1(\eta')=\{-a_{l1}(\eta')\}^{1/l}. \tag{2.20}$$

Remark. We can also prove the following assertion: For any compact set K' in $\gamma_0\vartheta'+\dot{\Gamma}$ and any positive integer N there exist positive constants α_N, β_N, ε and d such that

$$\zeta(\xi^{0\prime}+\nu\eta'; \nu)=\sum_{j=1}^{N}c_j(\eta')\nu^{j/l}+O(\nu^{N/l+\beta_N}) \tag{2.21}$$

if $\eta'\in\Xi^{n-1}-iK'$, $|\eta'|\leqq d|\nu|^{-\alpha_N}$ and $|\nu|<\varepsilon$.

Proof.[2] (i) If $l=1$, then $\zeta(\xi'; \nu)=-a_1(\xi'; \nu)=-\sum_{j=1}^{\infty}a_{1j}(\eta')\nu^j$. Thus the assertions of Lemma 2.5 are obvious.

(ii) Let us prove Lemma 2.5 when $l>1$. From (2.14) and (2.18) $\zeta(\xi^{0\prime}+\nu\eta'; \nu)$ can be represented for fixed $\eta'\in K$ by a development in a Puiseux series of the form

$$\zeta(\xi^{0\prime}+\nu\eta'; \nu)=\sum_{j=1}^{\infty}c_j(\eta')\nu^{j/l}.$$

Then we have

$$c_1(\eta')=\{-a_{l1}(\eta')\}^{1/l}, \quad d_0\leqq|c_1(\eta')|,$$

2) It is possible to simplify this part, according to the referee's remark: $w(\eta'; \nu)=\nu^{-1/l}\zeta(\xi^{0\prime}+\nu\eta'; \nu)$ is a root of the equation

$$\lambda^l+a_1(\xi^{0\prime}+\nu\eta'; \nu)\nu^{-1/l}\lambda^{l-1}+\cdots+a_l(\xi^{0\prime}+\nu\eta'; \nu)\nu^{-1}=0.$$

On the other hand the $b_j(\eta'; \mu)\equiv a_j(\xi^{0\prime}+\mu^l\eta'; \mu^l)\mu^{-j}$ are holomorphic in $(\eta', \mu)\in K\times\{|\mu|<\varepsilon'\}$ and $b_j(\eta'; 0)=0$, $1\leqq j\leqq l-1$ and $b_l(\eta'; 0)\neq 0$ for $\eta'\in K$. Thus we see that $w(\eta'; \mu^l)$ is holomorphic in $(\eta', \mu)\in K\times\{|\mu|<\varepsilon'\}$ and that $\zeta(\xi^{0\prime}+\nu\eta'; \nu)=\sum_{j=1}^{\infty}c_j(\eta')\nu^{j/l}$ is convergent if $(\eta', \nu)\in K\times\{|\nu|<\varepsilon\}$.

$$|c_j(\eta')|\leqq d_j, \qquad j=1, 2,\dots. \tag{2.22}$$

In fact, from the equation

$$lc_1(\eta')^{l-1}c_j(\eta')+\sum_{\substack{1\leqq j_1,\dots,j_l\leqq j-1\\ j_1+\cdots+j_l=l-1+j}}c_{j_1}(\eta')\dots c_{j_l}(\eta')$$

$$+\sum_{k=0}^{l-1}\sum_{\substack{1\leqq j_1,\dots,j_k\leqq j-1\\ lh+j_1+\cdots+j_k=l-1+j}}c_{j_1}(\eta')\dots c_{j_k}(\eta')a_{l-k,h}(\eta')=0,$$

we determine $c_j(\eta')$, $j=2, 3,\dots$. Thus we obtain the estimates (2.22). Moreover we have

$$|c_j(\eta')|\leqq d'_j(1+|\eta'|)^{j/l+r_j}, \qquad j=1, 2,\dots.$$

where $r_j=-1/l+l^{j-2}$ $(\geqq 0)$, which is used in order to prove (2.21). Putting

$$\zeta(\xi^{0\prime}+v\eta';\ v)=\sum_{j=1}^{N}c_j(\eta')v^{j/l}+\zeta_{N+1}(\eta';\ v),$$

we shall estimate $\zeta_{N+1}(\eta';\ v)$ in the remainder of this proof. Write $\lambda^l+a_1(\xi^{0\prime}+v\eta';\ v)\lambda^{l-1}+\cdots+a_l(\xi^{0\prime}+v\eta';\ v)=0$ in the form

$$\lambda^l=-\sum_{j=1}^{l}\left[\sum_{k=1}^{N_j}a_{jk}(\eta')v^k+O(v^{N_j+1})\right]\lambda^{l-j}, \tag{2.23}$$

where $N_{l-j}=1+[(N-j-1)/l]$, $0\leqq j\leqq l-1$. Substitution of $\lambda=\zeta(\xi^{0\prime}+v\eta';\ v)$ into (2.23) gives

$$\sum_{k=0}^{l-1}\binom{l}{k}\{c_1(\eta')^k v^{k/l}+\sum_{\substack{1\leqq j_1,\dots,j_k\leqq N\\ j_1+\cdots+j_k\geqq k+1}}c_{j_1}(\eta')\dots c_{j_k}(\eta')v^{(j_1+\cdots+j_k)/l}\}$$

$$\times\zeta_{N+1}(\eta';\ v)^{l-k}+\sum_{\substack{1\leqq j_1,\dots,j_l\leqq N\\ j_1+\cdots+j_l\geqq l+N}}c_{j_1}(\eta')\dots c_{j_l}(\eta')v^{(j_1+\cdots+j_l)/l}$$

$$=-\sum_{h=1}^{l-1}\sum_{k=0}^{h-1}\sum_{1\leqq j_1,\dots,j_k\leqq N}\binom{h}{k}c_{j_1}(\eta')\dots c_{j_k}(\eta')$$

$$\times\{\sum_{q=1}^{N_{l-h}}a_{l-h,q}(\eta')v^{(j_1+\cdots+j_k)/l+q}+O(v^{(j_1+\cdots+j_k)/l+N_{l-h}+1})\}$$

$$\times\zeta_{N+1}(\eta';\ v)^{h-k}-\sum_{h=0}^{l-1}\sum_{\substack{1\leqq j_1,\dots,j_h\leqq N\\ j_1+\cdots+j_h+lq\geqq l+N\\ 1\leqq q\leqq N_{l-h}}}c_{j_1}(\eta')\dots c_{j_h}(\eta')$$

$$\times a_{l-h,q}(\eta')v^{(j_1+\cdots+j_h)/l+q}-\sum_{h=0}^{l-1}\sum_{1\leqq j_1,\dots,j_h\leqq N}c_{j_1}(\eta')\dots c_{j_h}(\eta')$$

$$\times O(v^{(j_1+\cdots+j_h)/l+N_{l-h}+1}).$$

Thus we have

$$(2.24)\qquad \sum_{k=0}^{l-1}\binom{l}{k}\{c_1(\eta')^k\nu^{k/l}+O(\nu^{(k+1)/l})\}\zeta_{N+1}(\eta';\nu)^{l-k}+O(\nu^{N/l+1})=0.$$

Substituting $\zeta_{N+1}(\eta';\nu)=c_1(\eta')\nu^{1/l}z(\eta';\nu)$ into (2.24), we have

$$\sum_{k=0}^{l-1}\binom{l}{k}\{c_1(\eta')^l\nu+c_1(\eta')^{l-k}O(\nu^{1+1/l})\}z(\eta';\nu)^{l-k}+O(\nu^{N/l+1})=0.$$

Therefore,

$$\sum_{k=0}^{l-1}\binom{l}{k}\{1+c_1(\eta')^{-k}O(\nu^{1/l})\}z(\eta';\nu)^{l-k}+O(\nu^{N/l})=0.$$

Since the roots of

$$\sum_{k=0}^{l-1}\binom{l}{k}z^{l-k}=z((z+1)^{l-1}+(z+1)^{l-2}+\cdots+1)=0$$

are 0, $e^{2\pi ki/l}-1$ $(1\leqq k\leqq l-1)$, we obtain

$$|\zeta_{N+1}(\eta';\nu)|\geqq c|\nu|^{1/l}\quad\text{or}\quad \leqq c'|\nu|^{(N+1)/l}.$$

On the other hand

$$|\zeta_{N+1}(\eta';\nu)|\leqq c(\eta')\nu^{(N+1)/l}$$

holds for fixed $\eta'\in K$ and sufficiently small ν. Thus we have

$$|\zeta_{N+1}(\eta';\nu)|\leqq c'|\nu|^{(N+1)/l}\qquad\text{for}\quad \eta'\in K\quad\text{and}\quad |\nu|<\varepsilon.$$

Q.E.D.

By Lemma 2.5 the following lemma can be easily proved.

Lemma 2.6. *For any compact set K in $\Xi^{n-1}-i\gamma_0\vartheta'-i\dot{\Gamma}$ and any non-negative integer N there exists $\varepsilon>0$ such that if $\eta'\in K$ and $0<\nu<\varepsilon$,*

$$(2.25)\qquad \nu^{h_1}R(\nu^{-1}\xi^{0\prime}+\eta')=\sum_{j=0}^{N}Q_j(\eta')\nu^{j/L}+O(\nu^{(N+1)/L}),$$

where $Q_0(\eta')\not\equiv 0$, L is a positive integer and h_1 is a rational number. Moreover the $Q_j(\eta')$ are equal to $\sum_{\text{finite sum}}$(polynomial of η')$\times$

$\prod_{k=1}^{r_1} c_{1j_k}(\eta')^{-n_{j_k}}$ *and holomorphic in* $\Xi^{n-1}-i\gamma_0\vartheta'-i\dot{\Gamma}_{\xi^{0'}}$. *Here* $\{j_k\}_{1\leqq k\leqq r_1}$ *is the set of suffixes such that* $p^0_{j_k}(\xi^{0'},\mu)=0$ *has a real multiple root* μ_{j_k} *with multiplicity* l_{j_k} *and*

$$c_{1j_k}(\eta')=[\text{const.}\, p_{j_k(\xi^{0'},\mu_{j_k})}(\eta)]^{1/l_{j_k}}, \tag{2.26}$$

$$\dot{\Gamma}_{\xi^{0'}}\times\Xi=\cap_{k=1}^{r_1}\Gamma(p_{j_k(\xi^{0'},\mu_{j_k})},\vartheta). \tag{2.27}$$

Remark. $\dot{\Gamma}_{\xi^{0'}}\supset\dot{\Gamma}$. Moreover $L=$the L. C. M. of $\{l_{j_k}\}_{1\leqq k\leqq r_1}$.

§3. Localization

Definition 3.1. Let Δ be an open connected cone in Ξ^n such that $\vartheta+\Delta\subset\Delta$ and let $f(\xi)$ be a holomorphic function in $\Xi^n-i\gamma_0\vartheta-i\Delta$. Then we say that $f_{\xi^0}(\eta)$ is the localization of $f(\xi)$ at ξ^0 in Ξ^n if $f_{\xi^0}(\eta)$ does not vanish identically and is holomorphic in $\Xi^n-i\gamma_0\vartheta-i\Delta$ and

$$\nu^{h_1}f(\nu^{-1}\xi^0+\eta)\longrightarrow f_{\xi^0}(\eta)\quad\text{as}\quad \nu\longrightarrow+0\quad\text{for each fixed}\quad \eta\in\Xi^n-i\gamma_0\vartheta-i\Delta, \tag{3.1}$$

where h_1 is a rational number and depends on ξ^0.

Remark. $R_{\xi^{0'}}(\eta')=Q_0(\eta')$.

Lemma 3.1. $Q_0(\eta')\neq 0$ *for* $\eta'\in\Xi^{n-1}-i\gamma_1\vartheta'-i\dot{\Sigma}$.

Proof. Assume that there exists $\eta^{0'}$ in $\Xi^{n-1}-i\gamma_1\vartheta'-i\dot{\Sigma}$ such that $Q_0(\eta^{0'})=0$. Since $Q_0(\eta')\not\equiv 0$, there exists $\zeta^{0'}$ in $\Xi^{n-1}+i\Xi^{n-1}$ such that $\mu\zeta^{0'}+\eta^{0'}\in\Xi^{n-1}-i\gamma_1\vartheta'-i\dot{\Sigma}$ for $|\mu|\leqq 1$ and $Q_0(\eta^{0'}+\zeta^{0'})\neq 0$. Thus $Q_0(\eta^{0'}+\mu\zeta^{0'})\not\equiv 0$ in μ. $Q_0(\eta^{0'}+\mu\zeta^{0'})$ is holomorphic in μ, $|\mu|\leqq 1$, and vanishes at $\mu=0$. Therefore there exist $\varepsilon(>0)$, $\delta(1\geqq\delta>0)$ such that

$$|Q_0(\eta^{0'}+\mu\zeta^{0'})|>2\varepsilon\qquad\text{for}\quad|\mu|=\delta.$$

On the other hand, it follows from Lemma 2.6 that

$$|\nu^{h_1}R(\nu^{-1}\xi^{0'}+\eta^{0'}+\mu\zeta^{0'})-Q_0(\eta^{0'}+\mu\zeta^{0'})|<\varepsilon\quad\text{for }0<\nu<\tilde{\delta}\text{ and }|\mu|=\delta.$$

Rouché's theorem implies that $\nu^{h_1}R(\nu^{-1}\xi^{0'}+\eta^{0'}+\mu\zeta^{0'})$ has zeros within

$|\mu|<\delta$, which is a contradiction to $R(\xi')\neq 0$ for $\xi'\in\Xi^{n-1}-i\gamma_1\vartheta'-i\dot{\Sigma}$.
Q.E.D.

Now let us define the principal part of $Q_0(\eta')$. Let K be a compact set in $\Xi^{n-1}-i\gamma_0\vartheta'-i\dot{\Gamma}_{\xi^{0\prime}}$. Then there exists $T_K>0$ such that

$$Q_0(t\eta')=t^{k_0}\sum_{j=0}^{\infty}Q_0^j(\eta')t^{-j/L},\quad Q_0^0(\eta')\not\equiv 0, \tag{3.2}$$

whose convergence is uniform in $K\times\{t>T_K\}$. It is easy to see that the $Q_0^j(\eta')$ are positively homogeneous and equal to $\sum_{\text{finite sum}}$ (polynomial of η')$\times\prod_{k=1}^{r_1}c_{1\,j_k}^0(\eta')^{-q_{j_k}}$, where

$$c_{1\,j_k}^0(\eta')=\{\text{const.}\,p_{j_k(\xi^{0\prime},\mu_{j_k})}^0(\eta)\}^{1/l_{j_k}}. \tag{3.3}$$

We can prove the following lemma in the same way as in Lemma 3.1.

Lemma 3.2. $Q_0^0(\eta')\neq 0$ *for* $\eta'\in\Xi^{n-1}-i\dot{\Sigma}$.

Define

$$\dot{\sigma}_0(\eta''')=\begin{cases}\sup_{\eta'\in\dot{\Gamma}_{\xi^{0\prime}}}\eta_1 & \text{if } Q_0^0(-i\eta')\neq 0 \text{ for } \eta'\in\dot{\Gamma}_{\xi^{0\prime}},\\ \sup_{\substack{Q_0^0(-i\eta')=0\\ \eta'\in\dot{\Gamma}_{\xi^{0\prime}}}}\eta_1 & \text{otherwise},\end{cases} \tag{3.4}$$

$$\dot{\Sigma}_{\xi^{0\prime}}=\{\eta'\in\Xi^{n-1};\ \eta_1>\dot{\sigma}_0(\eta''')\}. \tag{3.5}$$

Then it follows that $\dot{\Gamma}_{\xi^{0\prime}}\supset\dot{\Sigma}_{\xi^{0\prime}}\supset\dot{\Sigma}$.

Lemma 3.3.

$$Q_0(\eta')\neq 0\qquad \textit{for}\quad \eta'\in\Xi^{n-1}-i\gamma_1\vartheta'-i\dot{\Sigma}_{\xi^0}, \tag{3.6}$$

$$Q_0^0(\eta')\neq 0\qquad \textit{for}\quad \eta'\in\Xi^{n-1}-i\dot{\Sigma}_{\xi^{0\prime}}. \tag{3.7}$$

Proof. It is easily shown that

$$Q_0^0(\lambda\eta')=\lambda^{k_0}Q_0^0(\eta')\qquad\text{for}\quad \eta',\ \lambda\eta'\in\Xi^{n-1}-i\dot{\Gamma}_{\xi^{0\prime}},$$

where λ^{k_0} denotes the branch satisfying $1^{k_0}=1$ if k_0 is not an integer. Thus, modifying the proof of Lemma 2.2 in [6], one may prove the above lemma. Q.E.D, We can also prove the following lemma by Seidenberg's lemma.

Lemma 3.4. For any compact set K in $\gamma_1\vartheta'+\mathring{\Sigma}_{\xi^{0'}}$ there exist positive constants a_K and c_K such that

$$|Q_0(\eta')|\geqq c_K(1+|\eta'|)^{-a_K} \quad \text{for} \quad \eta'\in\Xi^{n-1}-i\Lambda_K. \tag{3.8}$$

Next let us consider the localization of $P_+(\xi', \lambda)$ at $(\xi^{0'}, \mu^0)$, where $\xi^{0'}\in\Xi^{n-1}\backslash\{0\}$ and μ_0 is real. We can assume without loss of generality that the roots with positive imaginary part of $P(\xi', \lambda)=0$ is enumerated as follows:

$$\lambda_k^+(t\xi^{0'}+\eta')=t\mu^0+c_k(\eta')+O(t^{-1}), \qquad 1\leqq k\leqq r_0, \tag{3.9}$$

$$\lambda_k^+(t\xi^{0'}+\eta')=t\mu_k^+(\xi^{0'})+t^{1-1/l_k}c_k(\eta')^{3)}+O(t^{1-1/l_k}), \tag{3.10}$$

$$r_0+1\leqq k\leqq r_0+r_1,$$

$$\lambda_k^+(t\xi^{0'}+\eta')-t\mu^0=O(t^{q_k}), \qquad r_0+1\leqq k<l, \tag{3.11}$$

where $\eta'\in\Xi^{n-1}-i\gamma_0\vartheta'-i\Gamma_0$ and t is large enough, $\mu_k^+(\xi^{0'})$, $r_0+1\leqq k\leqq r_0+r_1$, are real and the q_k are positive rational numbers. The above enumeration implies that $\mu_k^+(\xi^{0'})=\mu^0$, $1\leqq k\leqq r_0$, are real simple roots of $p_j^0(\xi^{0'}, \mu)=0$ and that $\mu_k^+(\xi^{0'})$, $r_0+1\leqq k\leqq r_0+r_1$, are real multiple roots with multiplicity l_k of $p_j^0(\xi^{0'}, \mu)=0$. Then we have the following

Lemma 3.5. *Let K be a compact set in $(\Xi^{n-1}-i\gamma_0\vartheta'-i\mathring{\Gamma}_{\xi^{0'}})\times\Xi$. For any non-negative integer N there exists $T>0$ such that*

$$t^{-h_2}P_+(t\xi^{0'}+\eta', t\mu^0+\lambda)=\textstyle\prod_{j=1}^{r_0}(\lambda-c_j(\eta'))S(\eta') \tag{3.12}$$

$$+\textstyle\sum_{k=1}^{N}P_{+k}(\eta', \lambda)t^{-k/L}+O(t^{-(N+1)/L}) \quad \text{for} \quad (\eta', \lambda)\in K \text{ and } t>T,$$

where $h_2=l-r_0-\sum_{j=r_0+1}^{r_0+r_2}1/l_j$,

$$S(\eta')=\textstyle\prod_{j=r_0+1}^{r_0+r_2}\{-c_j(\eta')\}\prod_{j=r_0+r_2+1}^{l}(\mu^0-\mu_j^+(\xi^{0'})) \tag{3.13}$$

and the $P_{+k}(\eta', \lambda)$ are polynomials of λ whose coefficients are equal to $\sum_{\text{finite sum}}(\text{polynomial of } \eta')\times\prod_{j=r_0+1}^{r_0+r_1}c_j(\eta')^{p_j}$. *Here we have assumed*

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3) $\{c_k(\eta')\}_{r_0+1\leqq k\leqq r_0+r_1}$ is equal to $\{c_{1j_k}(\eta')\}_{1\leqq k\leqq r_1}$ defined by (2.26).

that $\mu_k^+(\xi^{0\prime})=\mu^0$ *for* $1\leqq k\leqq r_0+r_2$ *and that* $\mu_k^+(\xi^{0\prime})\neq\mu^0$ *for* $r_0+r_2+1\leqq k\leqq l$.

Proof. From Lemma 2.5 we have

$$t\mu^0+\lambda-\lambda_k^+(t\xi^{0\prime}+\eta')=\lambda-c_k(\eta')-\sum_{j=1}^{\infty}c_{kj}(\eta')t^{-j}, \qquad 1\leqq k\leqq r_0$$

where $c_k(\eta')$ and the $c_{kj}(\eta')$ are polynomials of η', and

$$t\mu^0+\lambda-\lambda_k^+(t\xi^{0\prime}+\eta')=t(\mu^0-\mu_k^+(\xi^{0\prime}))+t^{1-1/l_k}c_k(\eta')$$

$$+\lambda-\sum_{j=2}^{\infty}c_{kj}(\eta')t^{1-j/l_k}, \qquad r_0+1\leqq k\leqq r_0+r_1,$$

where the $c_{kj}(\eta')$ are equal to (polynomial of η')$\times c_k(\eta')^{-n_{kj}}$. Further we have

$$t^{-l+r_0+r_1}\prod_{k=r_0+r_1+1}^{l}(t\mu^0+\lambda-\lambda_k^+(t\xi^{0\prime}+\eta'))$$

$$=\prod_{k=r_0+r_1+1}^{l}(\mu^0-\mu_k^+(\xi^{0\prime}))+\sum_{j=1}^{\infty}q_j(\eta',\lambda)t^{-j},$$

where the $q_j(\eta',\lambda)$ are polynomials of η' and λ. This completes the proof. Q.E.D.

Let $\{s_k\}_{1\leqq k\leqq r_0}$ denote the set of suffixes such that $\mu_k^+(\xi')$ are simple roots of $p_{s_k}^0(\xi',\mu)=0$ for $|\xi'-\xi^{0\prime}|<\varepsilon$ and $\mu_k^+(\xi^{0\prime})=\mu^0$.

Lemma 3.6. *For* $1\leqq k\leqq r_0$

$$c_k(\eta')=\operatorname{grad}\mu_k^+(\xi^{0\prime})\cdot\eta'+\alpha_k(\xi^{0\prime}), \tag{3.14}$$

where

$$\alpha_k(\xi^{0\prime})=(2\pi i)^{-1}\int_{|z-\mu_k^+(\xi^{0\prime})|=\delta}z\left\{p_{s_k}^0(\xi^{0\prime},z)\frac{\partial p_{s_k}^1}{\partial z}(\xi^{0\prime},z)\right. \tag{3.15}$$

$$\left.-\frac{\partial p_{s_k}^0}{\partial z}(\xi^{0\prime},z)p_{s_k}^1(\xi^{0\prime},z)\right\}/p_{s_k}^0(\xi^{0\prime},z)^2dz.$$

Moreover, for $1\leqq k\leqq r_0$

$$\lambda-\operatorname{grad}\mu_k^+(\xi^{0\prime})\cdot\eta'-\alpha_k(\xi^{0\prime})=\text{const.}\,p_{s_k(\xi^{0\prime},\mu^0)}(\eta',\lambda). \tag{3.16}$$

Remark.

$$P_{+(\xi^{0\prime},\mu^0)}(\eta',\lambda)=\prod_{j=1}^{r_0}\text{const.}\,p_{s_j(\xi^{0\prime},\mu^0)}(\eta',\lambda)S(\eta'). \tag{3.17}$$

Proof. Since $\mu_k^+(\xi^{0\prime})=\mu^0$ is a real simple root of $p_{s_k}^0(\xi^{0\prime},\mu)=0$, $\lambda_k^+(t\xi^{0\prime}+\eta')$ is a simple root of $p_{s_k}(t\xi^{0\prime}+\eta',\lambda)=0$ for t large enough. Thus we have

$$\lambda_k^+(t\xi^{0\prime}+\eta')-t\mu_k^+(\xi^{0\prime})=(2\pi i)^{-1}\int_{|z-\mu_k^+(\xi^{0\prime})|=\delta} z$$

$$\times\Big[p_{s_k}^0(\xi^{0\prime},z)\Big\{\sum_{j=1}^{n-1}\frac{\partial^2 p_{s_k}^0}{\partial\xi_j\partial z}(\xi^{0\prime},z)\eta_j+\frac{\partial p_{s_k}^1}{\partial z}(\xi^{0\prime},z)\Big\}$$

$$-\frac{\partial p_{s_k}^0}{\partial z}(\xi^{0\prime},z)\Big\{\sum_{j=1}^{n-1}\frac{\partial p_{s_k}^0}{\partial\xi_j}(\xi^{0\prime},z)\eta_j+p_{s_k}^1(\xi^{0\prime},z)\Big\}\Big]$$

$$\times p_{s_k}^0(\xi^{0\prime},z)^{-2}dz+O(t^{-1}).$$

This implies (3.14). (3.16) is obvious. Q.E.D.

§4. Proof of Theorem 1.1

From the results in §3 we have the following

Lemma 4.1. *Let $\tilde K$ be a compact set in $(\Xi^n-is\vartheta)\times\Xi$, where s is sufficiently large. Then for any non-negative integer N there exists $T>0$ such that*

$$\sum_{j,k=1}^{l}\frac{R_{jk}(t\xi^{0\prime}+\eta')B_k(t\xi^0+\eta)(t\mu^0+\lambda)^{j-1}}{R(t\xi^{0\prime}+\eta')P_+(t\xi^{0\prime}+\eta',t\mu^0+\lambda)P(t\xi^0+\eta)} \tag{4.1}$$

$$=t^{-p_0}\sum_{j=0}^{N}\tilde F_j(\eta,\lambda)t^{-j/L}+O(t^{-(N+1)/L})$$

for (η,λ) in $\tilde K$ and $t>T$, where the $\tilde F_j(\eta,\lambda)$ are equal to $\sum_{\text{finite sum}}$ (polynomial of (η,λ)) $\times\, Q_0(\eta')^{-n_1}\prod_{j=1}^{r_0}p_{s_j(\xi^{0\prime},\mu^0)}(\eta',\lambda)^{-n_2}\times P_{\xi^0}(\eta)^{-n_3}\prod_{k=1}^{r_1}\{p_{j_k(\xi^{0\prime},\mu_{j_k})}(\eta)\}^{n_{4k}/l_{j_k}}$,[4] *$p_0$ is a rational number, n_1, n_2 and n_3 are positive integers and the n_{4k} are integers.*

4) The j_k and l_{j_k} have been defined in §2.

Proof. The localizations $Q_0(\eta')$, $p_{s_j(\xi^{0},\mu^0)}(\eta', \lambda)$, $P_{\xi^0}(\eta)$ and $p_{jk(\xi^{0'},\mu_{J_k})}(\eta)$ do not vanish for (η, λ) in $(\Xi^n - is\vartheta)\times\Xi$. From Lemmas 2.5, 2.6, 3.5 and 3.6 the assertions of Lemma 4.1 follows. Q.E.D.

By Seidenberg's lemma we have the following

Lemma 4.2. *For any positive integer N there exist positive constants a and c such that*

$$\left|t^{p_0+N/L}\left\{\sum_{j,k=1}^{l}\frac{R_{jk}(t\xi^{0'}+\eta')B_k(t\xi^0+\eta)(t\mu^0+\lambda)^{j-1}}{R(t\xi^{0'}+\eta')P_+(t\xi^{0'}+\eta', t\mu^0+\lambda)P(t\xi^0+\eta)}\right.\right.\tag{4.2}$$

$$\left.\left.-\sum_{j=0}^{N-1}\tilde{F}_j(\eta, \lambda)t^{-j/L}\right\}\right|\leqq a(1+|\eta|+|\lambda|)^c$$

for $(\eta, \lambda)\in(\Xi^n - is\vartheta)\times\Xi$ *and* $t\geqq 1$, *where s is large enough.*

Remark. We can also prove the above lemma, making use of (2.21) without Seidenberg's lemma.

Put

$$\tilde{\Gamma}_{(\xi^{0'},\mu^0)}=\bigcap_{k=1}^{r_0}\{(\eta, \lambda)\in\Xi^{n+1};\ (\eta', \lambda)\in\Gamma(p_{s_k(\xi^{0'},\mu^0)}, \vartheta)\},\tag{4.3}$$

$$\Gamma_{\xi^0,\mu^0}=(\Gamma(P_{\xi^0}, \vartheta)\times\Xi)\cap\tilde{\Gamma}_{(\xi^{0'},\mu^0)}\cap(\dot{\Sigma}_{\xi^{0'}}\times\Xi^2).\tag{4.4}$$

Lemma 4.3. *Let* $\tilde{K}$ *be a compact set in* $\gamma_1(\vartheta, 0)+\Gamma_{\xi^0,\mu^0}$. *Then there exist positive constants* $a_{j\tilde{K}}$ *and* $c_{j\tilde{K}}$ *such that*

$$|\tilde{F}_j(\eta, \lambda)|\leqq a_{j\tilde{K}}(1+|\eta|+|\lambda|)^{c_{j\tilde{K}}}\qquad for\quad (\eta, \lambda)\in\Xi^{n+1}-i\Lambda_{\tilde{K}},\tag{4.5}$$

where $\Lambda_{\tilde{K}}=\{t(\eta, \lambda);\ (\eta, \lambda)\in\tilde{K},\ t\geqq 1\}$.

Now we can prove Theorem 1.1. In fact, we have

$$\exp[-it\{(x'-y')\cdot\xi^{0'}+x_n\mu^0-y_n\xi_n^0\}]F(x, y)\tag{4.6}$$

$$=(2\pi)^{-(n+1)}\int_{\Xi^{n+1}}i^{-1}\sum_{j,k=1}^{l}\exp[i\{(x'-y')\cdot(\zeta'+i\eta')+\lambda x_n$$

$$-y_n(\zeta_n+i\eta_n)\}]$$

$$\times\frac{R_{jk}(t\xi^{0'}+\zeta'+i\eta')B_k(t\xi^0+\zeta+i\eta)(t\mu^0+\lambda)^{j-1}}{R(t\xi^{0'}+\zeta'+i\eta')P_+(t\xi^{0'}+\zeta'+i\eta', t\mu^0+\lambda)P(t\xi^0+\zeta+i\eta)}d\zeta d\lambda,$$

where $\eta=-s\vartheta$ and s is sufficiently large. From Lemmas 4.1 and 4.2 and (4.6) (1.13) easily follows, putting

$$(4.7)\qquad F_j(x, y)=(2\pi)^{-(n+1)}\int_{\Xi^{n+1}} i^{-1}\exp[i\{(x'-y')\cdot(\zeta'+i\eta') + \lambda x_n - y_n(\zeta_n+i\eta_n)\}]\tilde{F}_j(\zeta+i\eta, \lambda)d\zeta d\lambda.$$

(1.15) follows from the Riemann-Lebesgue theorem. From Lemma 4.3 we have (1.16), applying the Paley-Wiener theorem to (4.7). This completes the proof of Theorem 1.1.

The following theorem shows that the localization of $\tilde{R}_0(\xi')$ at $\xi^{0\prime}$ coincides with $Q_0^0(\eta')$ under some assumptions.

Theorem 4.1. *Assume that each $p_j^0(\xi^{0\prime}, \mu)=0$ has no real multiple roots. Then the localization $Q_0(\eta')$ of $R(\xi')$ at $\xi^{0\prime}$ is a hyperbolic polynomial and $\dot{\Sigma}_{\xi^{0\prime}}=\Gamma(Q_0(\eta'), \vartheta')$. Moreover $Q_0^0(\eta')$ is equal to the localization of $\tilde{R}_0(\xi')$ at $\xi^{0\prime}$, if at least one of the following conditions is satisfied*: (i) *The system $\{P(-D), B_j(-D)\}$ satisfies the Lopatinski condition.* (ii) $\xi^{0\prime}\in\partial\dot{\Sigma}$.

Proof. The first assertion is obvious. Since the real roots of $p_j^0(\xi^{0\prime}, \mu)=0$, $1\leqq j\leqq q$, are simple, $R(t\xi')$ can be continued analytically to a neighborhood of $\xi^{0\prime}$ when $t>T$. Moreover the $\tilde{R}_j(\xi')$ are holomorphic in the above neighborhood. If $\tilde{R}_0(\xi^{0\prime})$ does not vanish, then it follows that the localization $Q_0(\eta')$ of $R(\xi')$ at $\xi^{0\prime}$ is equal to $\tilde{R}_0(\xi^{0\prime})$. Thus it suffices to show that $Q_0^0(\eta')$ is equal to the localization of $\tilde{R}_0(\xi')$ at $\xi^{0\prime}$ when $\tilde{R}_0(\xi^{0\prime})=0$. Since $\tilde{R}_0(\xi^{0\prime}+\nu\eta'+s\vartheta')\not\equiv 0$, applying Weierstrass, preparation theorem, we have

$$\tilde{R}_0(\xi^{0\prime}+\nu\eta'+s\vartheta')=\{s^l+b_1^0(\xi^{0\prime}+\nu\eta')s^{l-1}+\cdots+b_l^0(\xi^{0\prime}+\nu\eta')\}$$
$$\times S_0(\xi^{0\prime}+\nu\eta'; s)\qquad \text{for}\quad |\nu|<\nu_0,\quad |s|<s_0,$$

where $S_0(\xi^{0\prime}+\nu\eta'; s)\neq 0$ for $|\nu|<\nu_0$, $|s|<s_0$ and $b_j^0(\xi^{0\prime})=0$, $1\leqq j\leqq l$, and η' is fixed in Ξ^{n-1}. Put

$$s^l+b_1^0(\xi^{0\prime}+\nu\eta')s^{l-1}+\cdots+b_l^0(\xi^{0\prime}+\nu\eta')=\prod_{k=1}^l(s+r_k^0(\xi^{0\prime}+\nu\eta')).$$

Then the $r_k^0(\xi^{0\prime}+\nu\eta')$ can be expanded in Puiseux series of the form

$$r_k^0(\xi^{0\prime}+\nu\eta')=\sum_{j=1}^{\infty}d_{kj}^0(\eta')\nu^{j/n_k}, \qquad 1\leqq k\leqq l.$$

Denote by $j(k)$ a number such that $d_{k1}^0=\cdots=d_{kj(k)-1}^0=0$ and $d_{kj(k)}^0\neq 0$. Then $j(k)/n_k$ is an integer. In fact, if $j(k)/n_k$ is not an integer, a branch of $r_k^0(\xi^{0\prime}+\nu\eta')$ has positive imaginary part for some real ν. This contradicts the fact that

$$\tilde{R}_0(\xi^{0\prime}+\nu\eta'+s\vartheta')\neq 0 \qquad \text{for} \quad \operatorname{Im}\nu=0 \text{ and } \operatorname{Im}s<0.$$

Moreover it follows that $j(k)=n_k$ and $d_{kn_k}^0(\eta')>0$ for $\eta'\in\dot{\Sigma}$. In fact,

$$\tilde{R}_0(\xi^{0\prime}+\nu\eta'+s\vartheta')\neq 0 \qquad \text{for} \quad \operatorname{Im}\nu<0 \text{ and } \operatorname{Im}s\leqq 0,$$

and, therefore, $\operatorname{Im}r_k^0(\xi^{0\prime}+\nu\eta')<0$ for $\operatorname{Im}\nu<0$. Since

$$\text{(4.8)} \qquad \nu^{h_0}R(\nu^{-1}\xi^{0\prime}+\eta'+\nu^{-1}s\vartheta')=\tilde{R}_0(\xi^{0\prime}+\nu\eta'+s\vartheta')$$
$$+\nu\tilde{R}_1(\xi^{0\prime}+\nu\eta'+s\vartheta')+\cdots \qquad \text{for} \quad |s|<s_0, \quad 0<\nu<\nu_0,$$

we have

$$\nu^{h_0}R(\nu^{-1}\xi^{0\prime}+\eta'+\nu^{-1}s\vartheta')=\{s^l+b_1(\xi^{0\prime}+\nu\eta';\nu)s^{l-1}+$$
$$\cdots+b_l(\xi^{0\prime}+\nu\eta';\nu)\}S(\xi^{0\prime}+\nu\eta';\nu,s) \qquad \text{for} \quad |s|<s_0, \quad 0<\nu<\nu_0,$$

where $S(\xi^{0\prime}+\nu\eta';\nu,s)\neq 0$ for $|s|<s_0$, $|\nu|<\nu_0$ and $b_j(\xi^{0\prime};0)=0$, $1\leqq j\leqq l$. Here we note that the right hand side of (4.8) can be defined for $|s|<s_0$ and $|\nu|<\nu_0$. So we can define $\nu^{h_0}R(\nu^{-1}\xi^{0\prime}+\eta'+\nu^{-1}s\vartheta')$ for $|s|<s_0$, $|\nu|<\nu_0$. Then $R(\nu^{-1}\xi')$ does not always coincide with the analytic continuation to a neighborhood of $\nu^{-1}\xi^{0\prime}$ of the Lopatinski determinant for the system $\{P(D), B_j(D)\}$, but with the Lopatinski determinant $R^-(|\nu|^{-1}\xi')$ for the system $\{P(-D), B_j(-D)\}$ apart from a constant factor when $-\nu_0<\nu<0$, $|\xi'-\xi^{0\prime}|<\varepsilon$ and $\xi'\in\Xi^{n-1}-i\gamma_0\vartheta'-i\dot{\Gamma}$. Now put

$$s^l+b_1(\xi^{0\prime}+\nu\eta';\nu)s^{l-1}+\cdots+b_l(\xi^{0\prime}+\nu\eta';\nu)=\prod_{k=1}^{l}(s+r_k(\xi^{0\prime}+\nu\eta';\nu)).$$

Then the $r_k(\xi^{0\prime}+\nu\eta';\nu)$ can be also expanded in Puiseux series of the form

$$r_k(\xi^{0\prime}+\nu\eta';\nu)=\sum_{j=1}^{\infty}d_{kj}(\eta')\nu^{j/\tilde{n}_k}, \qquad 1\leqq k\leqq l.$$

Denote by $\tilde{j}(k)$ a number such that $d_{k1}=\cdots=d_{k\tilde{j}(k)-1}=0$ and $d_{k\tilde{j}(k)}\neq 0$.

Then $\tilde{j}(k)/\tilde{n}_k=1/2$ or $\geqq 1$. This follows from the fact that

$$v^{-1}\operatorname{Im} r_k(\xi^{0\prime}+v\eta'; v)\leqq \gamma_1 \qquad \text{for} \quad 0<v<v_0.$$

Moreover if $\tilde{j}(k)/\tilde{n}_k=1/2$, it follows that $\operatorname{Im} d_{k\tilde{j}(k)}=0$.

(i) When the system $\{P(-D), B_j(-D)\}$ satisfies the Lopatinski condition, i.e.

$$R^-(\xi')\neq 0 \qquad \text{for} \quad \operatorname{Im}\xi_1<\gamma_2, \quad \xi'''\in\Xi^{n-2},$$

it follows that $\tilde{j}(k)/\tilde{n}_k\geqq 1$. In fact,

$$|v|^{-1}\operatorname{Im} r_k(\xi^{0\prime}+v\eta'; v)\leqq \gamma_2 \qquad \text{for} \quad -v_0<v<0.$$

(ii) When $\xi^{0\prime}\in\partial\dot{\Sigma}$, it follows that $\tilde{j}(k)/\tilde{n}_k\geqq 1$. In fact,

$$R(v^{-1}\xi^{0\prime}+\eta'+v^{-1}s\vartheta')\neq 0 \qquad \text{for} \quad 0<\operatorname{Re} v<v_0,$$

$$|\operatorname{Im} v|\leqq\delta_0\operatorname{Re} v \quad \text{and} \quad \operatorname{Im}(v^{-1}\xi^{0\prime}+v^{-1}s\vartheta')\in -\gamma_1\vartheta'-\dot{\Sigma},$$

where δ_0 is sufficiently small. Putting $v=\tilde{v}(1+i\delta)$, where $0<\tilde{v}<v_0$, we have

$$\tilde{v}^{-1}\operatorname{Im}[(1+i\delta)^{-1}\xi^{0\prime}+(1+i\delta)^{-1}r_k(\xi^{0\prime}+\tilde{v}(1+i\delta)\eta'; \tilde{v}(1+i\delta))\vartheta']$$

$$\notin -\gamma_1\vartheta'-\dot{\Sigma}.$$

If $\tilde{j}(k)/\tilde{n}_k=1/2$, then it follows that

$$(1+\delta^2)^{-1}\xi^{0\prime}\pm 1/2\,\tilde{v}^{1/2}d_{k\tilde{j}(k)}\vartheta'+o(\tilde{v}^{1/2})\notin\dot{\Sigma} \qquad \text{for } \tilde{v} \text{ small}$$

$$\text{enough and } |\delta|\geqq\tilde{v}^{1/N},$$

where N is large enough. This implies that $d_{k\tilde{j}(k)}=0$, which is a contradiction to $d_{k\tilde{j}(k)}\neq 0$. Thus we have $\tilde{j}(k)/\tilde{n}_k\geqq 1$. When $\tilde{j}(k)/\tilde{n}_k\geqq 1$, it is easy to see that the localization of $\tilde{R}_0(\xi')$ at $\xi^{0\prime}$ is the principal part $Q_0^0(\eta')$ of the localization $Q_0(\eta')$ of $R(\xi')$ at $\xi^{0\prime}$. Q.E.D.

§5. Some examples

Let us visualize Theorem 1.1 for some simple examples. $\operatorname{supp}_x E_{\xi^0}(x-y)$ is an incident bicharacteristic line emanated from a point $y=(0,$

$y_2, \dots, y_n) \in \mathbf{R}^n_+$. $\cup_{j=0}^{\infty} \mathrm{supp}_x F_j(x, y)$ is an reflected bicharacteristic line corresponding to the above incident bicharacteristic line.

Example 5.1. Consider the following hyperbolic polynomial with respect to $\vartheta = (1, 0, 0)$:

$$P(\xi) = (\xi_1^2 - \xi_2^2 - \xi_3^2)(\xi_1^2 - \xi_2^2/4 - \xi_3^2/4). \tag{5.1}$$

The roots of the equation $P(\xi', \lambda) = 0$ in λ are

$$\lambda_1^{\pm}(\xi') = \mu_1^{\pm}(\xi') = \mathrm{sgn}(\mp \xi_1)\sqrt{\xi_1^2 - \xi_2^2}, \qquad \text{when } |\xi_1| \geqq |\xi_2|, \tag{5.2}$$

$$\lambda_2^{\pm}(\xi') = \mu_2^{\pm}(\xi') = \mathrm{sgn}(\mp \xi_1)\sqrt{4\xi_1^2 - \xi_2^2}, \qquad \text{when } |\xi_1| \geqq |\xi_2|/2.$$

Assume that the Lopatinski determinant $R(\xi')$ for the system $\{P(D), B_1(D), B_2(D)\}$ satisfies the uniform Lopatinski condition, i.e.

$$R(\xi') = \tilde{R}_0(\xi') \neq 0 \qquad \text{for } \xi' \in (\Xi^2 \backslash \{0\}) - i0\vartheta'. \tag{5.3}$$

For example we may put

$$B_1(\xi) = 1, \quad B_2(\xi) = \xi_1 + \xi_2 + \xi_3. \tag{5.4}$$

Then

$$R(\xi') = \tilde{R}_0(\xi') = -1. \tag{5.5}$$

It suffices to consider the localizations of P at the points ξ^0 such that $\xi_1^0 = 1$, $\xi_3^0 > 0$ and $\xi^0 \in \mathrm{Re}\, A$. When $\xi_3^0 \leqq 0$, $\mathrm{supp}_x E_{\xi^0}(x-y)$ does not intersect the boundary plane $x_3 = 0$ and, therefore, it is independent of reflection. In fact, when $\xi_3^0 \leqq 0$, we have

$$(\cup_{j=0}^{\infty} \mathrm{supp}_x F_j(x, y)) \cap \mathbf{R}^3_+ = \emptyset. \tag{5.6}$$

For $\xi^0 = (1, \xi_2^0, \mu_1^-(1, \xi_2^0))$, $|\xi_2^0| < 1$, and $\mu^0 = \mu_1^+(\xi^{0\prime})$ we obtain

$$\Gamma_{\xi^0, \mu^0} = (\Gamma(P_{\xi^0}, \vartheta) \times \Xi) \cap \tilde{\Gamma}_{(\xi^{0\prime}, \mu^0)}, \tag{5.7}$$

where

$$\Gamma(P_{\xi^0}, \vartheta) = \{\eta \in \Xi^3;\ \eta_1 - \xi_2^0 \eta_2 - \mu_1^-(\xi^{0\prime})\eta_3 > 0\}, \tag{5.8}$$

$$\tilde{\Gamma}_{(\xi^{0\prime}, \mu^0)} = \{(\eta, \lambda) \in \Xi^4;\ \eta_1 - \xi_2^0 \eta_2 - \mu_1^+(\xi^{0\prime})\lambda > 0\}. \tag{5.9}$$

Thus $\cup_{j=0}^{\infty} \operatorname{supp}_x F_j(x, y)$ is included in the half-line defined by the equation

$$(5.10) \qquad x_1 - y_3/\mu_1^-(\xi^{0\prime}) = -x_2/\xi_2^0 - y_3/\mu_1^-(\xi^{0\prime}) = -x_3/\mu_1^+(\xi^{0\prime}),$$

$$x_1 > 0, \quad x_3 > 0.$$

Here we have assumed that $y=(0, 0, y_3)$. This line intersects the hyperplane $x_3=0$ at $(x_1, x_2)=(y_3/\mu_1^-(\xi^{0\prime}), -\xi_2^0 y_3/\mu_1^-(\xi^{0\prime}))$. $\operatorname{supp}_x E_{\xi^0}(x-y)$ is included in the half-line defined by the equation

$$(5.11) \quad x_1 - y_3/\mu_1^-(\xi^{0\prime}) = -x_2/\xi_2^0 - y_3/\mu_1^-(\xi^{0\prime}) = -x_3/\mu_1^-(\xi^{0\prime}), \qquad x_1 > 0.$$

This line also intersects the hyperplane $x_3=0$ at $(x_1, x_2)=(y_3/\mu_1^-(\xi^{0\prime}), -\xi_2^0 y_3/\mu_1^-(\xi^{0\prime}))$. For $(\xi^0, \mu^0)=(1, \xi_2^0, \mu_1^-(1, \xi_2^0), \mu_2^+(\xi^{0\prime}))$, $|\xi_2^0|<1$, and $(\xi^0, \mu^0)=(1, \xi_2^0, \mu_2^-(1, \xi_2^0), \mu_j^+(\xi^{0\prime}))$, $|\xi_2^0|<1, j=1, 2$, and $(\xi^0, \mu^0)=(1, \xi_2^0, \mu_2^-(1, \xi_2^0), \mu_2^+(\xi^{0\prime}))$, $1<|\xi_2^0|<2$, we can calculate in the same way. Next we consider the case where $(\xi^0, \mu^0)=(1, \pm 1, \mu_2^-(1, \pm 1), \mu_2^+(1, \pm 1))$. Then

$$(5.12) \qquad \Gamma_{\xi^0,\mu^0} = (\Gamma(P_{\xi^0}, \vartheta) \times \Xi) \cap \tilde{\Gamma}_{(\xi^{0\prime},\mu^0)} \cap (\dot{\Sigma}_{\xi^{0\prime}} \times \Xi^2),$$

where

$$(5.13) \qquad \Gamma(P_{\xi^0}, \vartheta) = \{\eta \in \Xi^3;\ \eta_1 \mp \eta_2/4 - \sqrt{3}\eta_3/4 > 0\},$$

$$(5.14) \qquad \tilde{\Gamma}_{(\xi^{0\prime},\mu^0)} = \{(\eta, \lambda) \in \Xi^4;\ \eta_1 \mp \eta_2/4 + \sqrt{3}\lambda/4 > 0\},$$

$$(5.15) \qquad \dot{\Sigma}_{\xi^{0\prime}} = \{\eta' \in \Xi^2;\ \eta_1 \mp \eta_2 > 0\}.$$

Thus we obtain

$$(5.16) \quad \cup_{j=0}^{\infty} \operatorname{supp}_x F_j(x, y) \subset \{x \in \mathbf{R}_+^3;\ x_1 = 4(x_3+y_3)/\sqrt{3} + u,$$

$$x_2 = \mp(x_3+y_3)/\sqrt{3} \mp u,\ u \geqq 0\}.$$

This is related to the lateral waves.

Example 5.2. Next consider the following hyperbolic polynomial with respect to $\vartheta=(1, 0, 0)$:

$$(5.17) \qquad P(\xi) = (\xi_1^2 - \xi_2^2/4 - \xi_3^2)(\xi_1^2 - \xi_2^2 - \xi_3^2/4).$$

We assume that the uniform Lopatinski condition is satisfied. The complicated cases are that $(\xi^0, \mu^0)=(1, \pm 2/\sqrt{5}, 2/\sqrt{5}, -2/\sqrt{5})$. Then we have

$$(5.18)\quad \Gamma(P_{\xi^0}, \vartheta)=\{\eta\in\Xi^3;\ \xi_1\mp\xi_2/(2\sqrt{5})-2\xi_3/\sqrt{5}>0 \text{ and } \xi_1\mp 2\xi_2/\sqrt{5}-\xi_3/(2\sqrt{5})>0\},$$

$$(5.19)\quad \tilde{\Gamma}_{(\xi^{0\prime},\mu^0)}=\{(\eta, \lambda)\in\Xi^4;\ \xi_1\mp\xi_2/(2\sqrt{5})+2\lambda/\sqrt{5}>0 \text{ and } \xi_1\mp 2\xi_2/\sqrt{5}+\lambda/(2\sqrt{5})>0\},$$

$$(5.20)\quad \dot{\Sigma}_{\xi^{0\prime}}=\Xi^2.$$

Thus we obtain

$$(5.21)\quad \cup_{j=0}^{\infty}\mathrm{supp}_x F_j(x, y)\subset\{x\in\mathbf{R}_+^3;\ x_1=\sqrt{5}x_3/2+\sqrt{5}y_3/2+2u,$$
$$x_2=\mp(x_3/4+y_3/4+\sqrt{5}u,\ 0\leqq u\leqq 3\sqrt{5}(x_3+y_3)/4\}$$
$$=\{x\in\mathbf{R}_+^3;\ x_3=(x_1+6\sqrt{5}t-2\sqrt{5}y_3)/(2\sqrt{5}-3\sqrt{5}\theta/2)$$
$$=(\mp x_2+15t-4y_3)/(4-15\theta/4),\qquad 0\leqq\theta\leqq 1,\quad 0\leqq t\leqq y_3/4\}.$$

Example 5.3. Put

$$(5.22)\quad P(\xi)=\xi_1^2-\xi_2^2-\xi_3^2,$$

$$(5.23)\quad B(\xi)=a\xi_2+\xi_3,\qquad a>0.$$

Then

$$(5.24)\quad \lambda^{\pm}(\xi')=\mu^{\pm}(\xi')=\mathrm{sgn}(\mp\xi_1)\sqrt{\xi_1^2-\xi_2^2},\quad \text{when}\quad |\xi_1|\geqq|\xi_2|,$$

$$(5.25)\quad R(\xi')=\tilde{R}_0(\xi')=\mu^+(\xi')+a\xi_2.$$

Thus $\{P(D), B(D)\}$ is $\mathscr{E}$-well posed. We have

$$(5.26)\quad \tilde{R}_0(\pm 1, \pm 1/\sqrt{1+a^2})=0.$$

The interesting cases are that $(\xi^0, \mu^0)=(\pm 1, \pm 1/\sqrt{1+a^2}, \pm a/\sqrt{1+a^2}, \mp a/\sqrt{1+a^2})$. Then

$$Q_0(\xi')=Q_0^0(\xi')=-\sqrt{1+a^2}\eta_1/a+(1+a^2)\eta_2/a. \tag{5.27}$$

Thus we have

$$\Gamma(P_{\xi^0}, \vartheta)=\{\eta\in\Xi^3;\ \eta_1-\eta_2/\sqrt{1+a^2}-a\eta_3/\sqrt{1+a^2}>0\}, \tag{5.28}$$

$$\tilde{\Gamma}_{(\xi^{0\prime},\mu^0)}=\{(\eta, \lambda)\in\Xi^4;\ \eta_1-\eta_2/\sqrt{1+a^2}+a\lambda/\sqrt{1+a^2}>0\}, \tag{5.29}$$

$$\dot{\Sigma}_{\xi^{0\prime}}=\{\eta'\in\Xi^2;\ \eta_1-\sqrt{1+a^2}\eta_2>0\}. \tag{5.30}$$

From Theorem 1.1 we obtain

$$\cup_{j=0}^{\infty}\mathrm{supp}_x F_j(x, y)\subset\{x\in\mathbf{R}_+^3;\ x_1=\sqrt{1+a^2}(x_3+y_3)/a+u,\ x_2=-(x_3+y_3)/a-\sqrt{1+a^2}u,\ u\geqq 0\}. \tag{5.31}$$

This is related to the boundary waves.

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