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On Linear Exceptional Sets of Solutions of Linear Partial Differential Equations with Constant Coefficients

By

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Introduction

Thus far we have studied in [1], [2] and [4] the possibility of extension of solutions of linear partial differential equations $p(D)u=0$ with constant coefficients to various exceptional sets. There, the exceptional sets were of those types to which the convex analysis, or the technique of the Fourier transform based on the growth order estimation, was applicable. Here we treat a new kind of exceptional set. Let $K=\{(0,\dots,0,x_n);\ -1<x_n<1\}$ be the line segment on the x_n-axis of the n-dimensional Euclidean space $\mathbf{R}^n$. Let U be an open neighborhood of K. (This means that U contains K as a closed subset.) Let $\mathscr{B}_p$ and $\mathscr{A}_p$ be the hyperfunction and the real analytic solutions of $p(D)u=0$ respectively. We give a necessary and sufficient condition on $p(D)$ for $\mathscr{B}_p(U\backslash K)/\widehat{\mathscr{B}_p(U)}=0$, where $\widehat{\mathscr{B}_p(U)}=\mathscr{B}_p(U)/\mathscr{B}_p[K]$, and a sufficient condition for $\mathscr{A}_p(U\backslash K)/\mathscr{A}_p(U)=0$. (See Theorem 1.4 and Corollary 2.4.)

Let L be the closure of K. In the course of proof we must reduce the support of a hyperfunction from L to $L\backslash K$ under some additional condition. Since the convex hull of $L\backslash K$ agrees with L, the routine technique in the Fourier transform based on the growth condition has no use. Instead, we rely on a new tool for non-convex Fourier analysis developed in [6]. (See Lemma 2.2 below.)

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§1. Continuation of Hyperfunction Solutions to Linear Exceptional Sets

We fix a system of coordinates $(x_1,\dots, x_{n-1}, x_n)$ in $\mathbb{R}^n$. As in the introduction, we put

$$K=\{(0,\dots, 0, x_n);\ -1<x_n<1\},$$

without loss of generality. Let U be an open neighborhood of K. This means that U is an open set in $\mathbb{R}^n$ containing K as a closed subset. Let L be the closure of K in $\mathbb{R}^n$. The set $L\backslash K$ consists of the two points $P^{\pm}=(0,\dots, 0, \pm 1)$. Let $p(D)$ be a linear partial differential operator with constant coefficients corresponding to the polynomial $p(\zeta)$, where $D=(D_1,\dots, D_{n-1}, D_n)$ and $D_j=\sqrt{-1}\partial/\partial x_j$, $j=1,\dots, n$. Let $\mathscr{B}_p$ be the sheaf of hyperfunction solutions of $p(D)u=0$. By the Harvey-Komatsu theorem (see [9]) we have the flabby resolution of $\mathscr{B}_p$ by the sheaf of hyperfunctions:

$$0 \longrightarrow \mathscr{B}_p \longrightarrow \mathscr{B} \xrightarrow{p(D)} \mathscr{B} \longrightarrow 0,$$

and the isomorphisms:

$$H^i(U, \mathscr{B}_p)=0, \qquad i\geqq 1.$$

Thus, from the fundamental exact sequence

$$\begin{aligned} 0 \longrightarrow H^0_K(U, \mathscr{B}_p) &\longrightarrow H^0(U, \mathscr{B}_p) \longrightarrow H^0(U\backslash K, \mathscr{B}_p) \\ &\longrightarrow H^1_K(U, \mathscr{B}_p) \longrightarrow H^1(U, \mathscr{B}_p), \end{aligned}$$

we obtain

$$H^1_K(U, \mathscr{B}_p)=\mathscr{B}_p(U\backslash K)/\widehat{\mathscr{B}_p(U)},$$

where we have put

$$\mathscr{B}_p(U)=\mathscr{B}_p(U)/\mathscr{B}_p[K],$$

$$\mathscr{B}_p(U)=H^0(U,\ \mathscr{B}_p),\quad \mathscr{B}_p(U\backslash K)=H^0(U\backslash K,\ \mathscr{B}_p),$$

$$\mathscr{B}_p[K]=H^0_K(U,\ \mathscr{B}_p).$$

Since $H^0_K(U, \mathscr{B}_p)$ does not depend on U, the last notation is reasonable. From the cochain complex of section modules

$$0 \longrightarrow H^0_K(U,\ \mathscr{B}) \xrightarrow{p(D)} H^0_K(U,\ \mathscr{B}) \longrightarrow 0,$$

we obtain

(1.1) $$H^1_K(U,\ \mathscr{B}_p)=\mathscr{B}[K]/p(D)\mathscr{B}[K],$$

where $\mathscr{B}[K]=H^0_K(U, \mathscr{B})$ also does not depend on U. Further, to the triple of the open sets $X=\mathbf{R}^n$, $Y=\mathbf{R}^n\backslash(L\backslash K)$, $Z=\mathbf{R}^n\backslash L$, we apply the fundamental exact sequence with the sheaf $\mathscr{B}_p$. By way of the excision theorem we thus obtain

$$\begin{aligned} 0 &\longrightarrow H^0_{L\backslash K}(\mathbf{R}^n,\ \mathscr{B}_p) \longrightarrow H^0_L(\mathbf{R}^n,\ \mathscr{B}_p) \longrightarrow H^0_K(U,\ \mathscr{B}_p) \\ &\longrightarrow H^1_{L\backslash K}(\mathbf{R}^n,\ \mathscr{B}_p) \xrightarrow{\lambda} H^1_L(\mathbf{R}^n,\ \mathscr{B}_p) \longrightarrow H^1_K(U,\ \mathscr{B}_p) \\ &\longrightarrow H^2_{L\backslash K}(\mathbf{R}^n,\ \mathscr{B}_p). \end{aligned}$$

The last term vanishes because $\mathscr{B}_p$ is of flabby dimension $\leqq 1$. Thus, with the above definition of the mapping λ, we have

$$H^1_K(U,\ \mathscr{B}_p)=H^1_L(\mathbf{R}^n,\ \mathscr{B}_p)/\lambda H^1_{L\backslash K}(\mathbf{R}^n,\ \mathscr{B}_p).$$

Note that

$$H^1_{L\backslash K}(\mathbf{R}^n,\ \mathscr{B}_p)=H^1_{P^+}(\mathbf{R}^n,\ \mathscr{B}_p)\oplus H^1_{P^-}(\mathbf{R}^n,\ \mathscr{B}_p).$$

Since L and $P^{\pm}$ are convex compact sets in $\mathbf{R}^n$, we can apply the Fundamental Principle ([2], Theorem 3.8). Namely, we have the following exact sequences of homomorphisms:

$$\widetilde{\mathscr{B}[L]} \xrightarrow{p(\zeta)} \widetilde{\mathscr{B}[L]} \xrightarrow{d} \widetilde{\mathscr{B}[L]}\{p,\ d\} \longrightarrow 0,$$

$$\widetilde{\mathscr{B}[P^{\pm}]} \xrightarrow{p(\zeta)} \widetilde{\mathscr{B}[P^{\pm}]} \xrightarrow{d} \widetilde{\mathscr{B}[P^{\pm}]}\{p,\ d\} \longrightarrow 0.$$

Here $\widetilde{\mathscr{B}[L]}$ denotes the Fourier image of $\mathscr{B}[L]$. The symbol d denotes a normal noetherian operator; we can employ the restriction to each irreducible component of the associated algebraic variety $N(p)=\{p(\zeta)=0\}$ composed with the derivatives to a transversal direction of the order up to the corresponding multiplicity minus one. $\widetilde{\mathscr{B}[L]}\{p, d\}$ denotes the space of vectors of holomorphic functions on the irreducible components of $N(p)$ such that they are locally in the image of d and globally satisfy the growth condition of the type $\widetilde{\mathscr{B}[L]}$. Written explicitly, this growth condition is: given $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$|F(\zeta)| \leqq C_\varepsilon \exp(\varepsilon|\zeta| + |\operatorname{Im} \zeta_n|).$$

There are similar meanings for $\widetilde{\mathscr{B}[P^\pm]}\{p, d\}$. By the isomorphisms similar to (1.1) and by the Fundamental Principle we have

$$H^1_L(\mathbf{R}^n, \mathscr{B}_p) = \mathscr{B}[L]/p(D)\mathscr{B}[L] = \widetilde{\mathscr{B}[L]}/p(\zeta)\widetilde{\mathscr{B}[L]} = \widetilde{\mathscr{B}[L]}\{p, d\},$$

$$H^1_{P^\pm}(\mathbf{R}^n, \mathscr{B}_p) = \widetilde{\mathscr{B}[P^\pm]}\{p, d\}.$$

Combining all these isomorphisms we finally obtain

Theorem 1.1.
$$\begin{aligned}\mathscr{B}_p(U\backslash K)/\widehat{\mathscr{B}_p(U)} &= H^1_K(U, \mathscr{B}_p)\\ &= \mathscr{B}[K]/p(D)\mathscr{B}[K]\\ &= H^1_L(\mathbf{R}^n, \mathscr{B}_p)/\lambda H^1_{L\backslash K}(\mathbf{R}^n, \mathscr{B}_p)\\ &= \widetilde{\mathscr{B}[L]}\{p, d\}/\tilde{\lambda}[\widetilde{\mathscr{B}[P^+]}\{p, d\} \oplus \widetilde{\mathscr{B}[P^-]}\{p, d\}].\end{aligned}$$

By chasing the above isomorphisms we can give the definite correspondence: Take $u \in \mathscr{B}_p(U\backslash K)$. Let $[u] \in \mathscr{B}(U)$ be one of the extensions, which exists because $\mathscr{B}$ is flabby. Then $p(D)[u] \in \mathscr{B}[K]$. Take an extension of $p(D)[u]$ to $\mathbf{R}^n$ with the smallest support, say $[[p(D)[u]]] \in \mathscr{B}[L]$. Applying the Fourier transform and the operator d, we obtain an element

$$d\widetilde{[[p(D)[u]]]} \in \widetilde{\mathscr{B}[L]}\{p, d\}.$$

Obviously this element is determined with the ambiguity modulo $\tilde{\lambda}[\widetilde{\mathscr{B}[P^+]}\{p, d\} \oplus \widetilde{\mathscr{B}[P^-]}\{p, d\}]$, where $\tilde{\lambda}$ is the mapping naturally induced from λ.

In our present case $\mathscr{B}_p[K]$ does not necessarily vanish even though we treat a non-trivial single equation.

Proposition 1.2. *In order that $\mathscr{B}_p[K] \neq 0$, it is necessary and sufficient that $p(\zeta)$ has an irreducible component whose principal part consists only of ζ_n.*

Remark. This condition depends on the lower order terms. See, e.g., $\zeta_1\zeta_n + 1$.

Proof. Assume that there exists a non-trivial element $u \in \mathscr{B}_p[K]$. By applying the irreducible factors of $p(D)$ one by one, the result reduces to zero first at some step. Considering exactly this factor, we can show that its principal part contains only ζ_n. Thus we can assume that $p(\zeta)$ itself is irreducible. Let $[[u]] \in \mathscr{B}[L]$ be one of the extensions of u. Then we can write

$$p(D)\,[[u]] = v^+ + v^-, \tag{1.2}$$

where $v^\pm \in \mathscr{B}[P^\pm]$. Thus applying Fourier transform we can write

$$p(\zeta)\,[[u]] = e^{\sqrt{-1}\zeta_n} J^+(\zeta) + e^{-\sqrt{-1}\zeta_n} J^-(\zeta),$$

where $J^\pm(\zeta)$ are entire functions of infra-exponential growth, namely the Fourier images of some hyperfunctions with support at the origin. Taking the restriction to the algebraic variety $N(p)$, we have

$$e^{\sqrt{-1}\zeta_n} J^+(\zeta)|_{N(p)} = -e^{-\sqrt{-1}\zeta_n} J^-(\zeta)|_{N(p)}. \tag{1.3}$$

Now assume that the principal part of $p(\zeta)$ contains a variable other than ζ_n. Then the both sides of (1.3) must vanish identically. In fact, let p_m be the principal part of the operator p of m-th order. The assumption implies that we can choose a suitable system of coordinates in $(x_1,\dots, x_{n-1})$-space such that $p_m(\zeta) = \zeta_n^k q_{m-k}(\zeta)$, where $q_{m-k}(\zeta)$ is an $(m-k)$-th order homogeneous polynomial satisfying $q_{m-k}(1, 0,\dots, 0) \neq 0$. Thus for each fixed $\zeta^* = (\zeta_2,\dots, \zeta_{n-1})$ the equation $p(\zeta) = 0$ in ζ_1 has a root holomorphic in ζ_n and with the asymptotic of the form

$$(1.4) \qquad \zeta_1 = a(\zeta^*)\zeta_n + o(\zeta_n),$$

on $\mathrm{Re}\,\zeta_n \geqq R(\zeta^*)$, where $a(\zeta^*) \in \mathbf{C}$ and $R(\zeta^*) > 0$ are constants depending on ζ^*. This fact is proved in Lemma 1.3 below. Taking this root into (1.3) we obtain

$$(1.5) \qquad e^{\sqrt{-1}\zeta_n} J^+(a(\zeta^*)\zeta_n + o(\zeta_n), \zeta^*, \zeta_n)$$
$$= -e^{-\sqrt{-1}\zeta_n} J^-(a(\zeta^*)\zeta_n + o(\zeta_n), \zeta^*, \zeta_n).$$

Let $F(\zeta_n)$ be the function holomorphic on $\mathrm{Re}\,\zeta_n \geqq R(\zeta^*)$ defined by the both sides of (1.5). Then, considering the left respectively right hand side of (1.5) for $\pm\mathrm{Im}\,\zeta_n \geqq 0$, we see that $F(\zeta_n)$ satisfies for every $\varepsilon > 0$,

$$(1.6) \qquad |F(\zeta_n)| \leqq C_\varepsilon \exp(\varepsilon|\zeta_n| - |\mathrm{Im}\,\zeta_n|).$$

By Carlson's theorem (see [11]) applied on $\mathrm{Re}\,\zeta_n \geqq R(\zeta^*)$, $F(\zeta_n)$ must vanish identically because of this estimate. Since ζ^* is arbitrary, we see that the both sides of (1.3) vanish on an open subset of the variety $N(p)$. Since $N(p)$ is irreducible, the unique continuation holds and we conclude that the both sides of (1.3) vanish identically.

Now by the Fundamental Principle we conclude that there exist $w^\pm \in \mathscr{B}[P^\pm]$ such that

$$e^{\pm\sqrt{-1}\zeta_n} J^\pm(\zeta) = \widetilde{v^\pm}(\zeta) = p(\zeta)\widetilde{w^\pm}(\zeta).$$

Thus (1.2) can be rewirtten as

$$p(D)([[u]] - w^+ - w^-) = 0,$$

hence

$$[[u]] = w^+ + w^-.$$

This implies that u is trivial as an element of $\mathscr{B}_p[K]$. This is a contradiction, and the necessity is proved.

Conversely, assuming the condition on p we can esaily show a non-trivial element of $\mathscr{B}_p[K]$ based on the result of [7] concerning the fundamental solutions of hyperbolic equations. But we give here a direct proof. We can assume that p itself is irreducible and its principal part is equal to ζ_n^m. It suffices to give infra-exponential entire functions

$J^{\pm}(\zeta)$ such that they cannot be divided by $p(\zeta)$ though $e^{\sqrt{-1}\zeta_n}J^+(\zeta)+e^{-\sqrt{-1}\zeta_n}J^-(\zeta)$ can. Put $J^+(\zeta)=1$ and

$$J^-(\zeta)=\sum_{k=0}^{\infty}\left(\frac{1}{(mk)!}+\cdots+\frac{\zeta_n^{m-1}}{(mk+m-1)!}\right)(p(\zeta)-\zeta_n^m)^k .$$

The latter is the remainder of $e^{2\sqrt{-1}\zeta_n}$ divided by $p(\zeta)$. Since $p(\zeta)-\zeta_n^m$ is a polynomial of order less than m, we can easily show employing the method of majorant that this series defines an infra-exponential entire function. Q.E.D.

Lemma 1.3. *Let $p(\zeta_1, \zeta_n)$ be a polynomial of two variables. Assume that the principal part (the collection of the highest order terms) has a factor $\zeta_1-a\zeta_n$, where $a\in\mathbf{C}$. Then the equation $p(\zeta_1, \zeta_n)=0$ in ζ_1 has a root $\zeta_1=\tau(\zeta_n)$ multivalued holomorphic in ζ_n on $|\zeta_n|>R$ and with the asymptotic of the form $\tau(\zeta_n)=a\zeta_n+o(\zeta_n)$.*

Proof (suggested by Professor K. Saito). Since $(\zeta_1-a\zeta_n)$ is a factor of the principal part of some irreducible component of p, we can assume that p itself is irreducible. Assume that the principal part of p has the form $\zeta_n^k\prod_{j=1}^{m-k}(\zeta_1-a_j\zeta_n)$, where $a_1=a$ and some of the other a_j may also agree with a. Put $\lambda=\zeta_1/\zeta_n$ and $\mu=1/\zeta_n$. Then we obtain an irreducible polynomial

$$q(\lambda, \mu)=\frac{1}{\zeta_n^m}p(\zeta_1, \zeta_n)=\prod_{j=1}^{m-k}(\lambda-a_j)+\mu q_1(\lambda, \mu) .$$

Thus $(a, 0)$ is a regular or at most an isolated singular point of the algebraic curve $q(\lambda, \mu)=0$. As is well known, we have a parametrization of the form

$$\lambda=a+\sum_{r=1}^{\infty}b_r t^r ,$$

$$\mu=t^{r_0},$$

with $r_0>0$ (see, e.g., [12]). Thus we obtain a multi-valued holomorphic root $\lambda=a+\sum_{r=1}^{\infty}b_r\mu^{r/r_0}$ on $|\mu|<1/R$. Back to the initial notation, we have proved the assertion. Q.E.D.

Theorem 1.4. *$\mathscr{B}_p(U\backslash K)/\widehat{\mathscr{B}_p(U)}=0$ if and only if the principal part of $p(\zeta)$ contains only ζ_n.*

Proof. This condition on p is clearly equivalent to the same condition posed on every irreducible component of p. If the principal part of p consists only of ζ_n, then on the variety $N(p)$ the three growth conditions corresponding to $\widetilde{\mathscr{B}[L]}$, $\widetilde{\mathscr{B}[P^\pm]}$ reduce to the same. Therefore from the last isomorphism in Theorem 1.1 we obtain $\mathscr{B}_p(U\backslash K)/\widehat{\mathscr{B}_p(U)}=0$.

Conversely, assuming $\mathscr{B}_p(U\backslash K)/\widehat{\mathscr{B}_p(U)}=0$, we see from the same isomorphism that $\mathscr{B}_q(U\backslash K)/\widehat{\mathscr{B}_q(U)}=0$ for every irreducible component q of p. Thus we can treat each irreducible component independently. Therefore we assume that p itself is irreducible. Let E be a solution of $p(D)E=\delta$ in $\mathscr{B}(\mathbb{R}^n)$. Then $E\in\mathscr{B}_p(U\backslash K)$, hence by the assumption, with a suitable element $u\in\mathscr{B}[K]$ we have $p(D)(E-u)=0$. Then, taking an extension $[[u]]\in\mathscr{B}[L]$ we have

$$p(D)[[u]]=\delta+v^+ +v^-, \tag{1.7}$$

with some $v^\pm\in\mathscr{B}[P^\pm]$. Applying the Fourier transform and taking the restriction to $N(p)$, we have

$$e^{\sqrt{-1}\zeta_n}J^+(\zeta)|_{N(p)}=(-1-e^{-\sqrt{-1}\zeta_n}J^-(\zeta)|_{N(p)}\,. \tag{1.8}$$

Now assume that the principal part of p contains a variable other than ζ_n. By the same reason as in the proof of Proposition 1.2, we can assume the existence of a root (1.4). This time we can apply Carlson's theorem to the function $G(\zeta_n)$ which is obtained from (1.8) after substituting (1.4) and multiplying by $e^{-\sqrt{-1}\zeta_n/2}$. Thus by the same argument we conclude that both sides of (1.8) vanish identically. By the Fundamental Principle there exists $w^+\in\mathscr{B}[P^+]$ such that $e^{\sqrt{-1}\zeta_n}J^+(\zeta)=p(\zeta)\widetilde{w^+}(\zeta)$, hence $v^+=p(D)w^+$. Similarly there exists $w^-\in\mathscr{B}[P^-]$ satisfying $v^-=p(D)w^-$. Thus from (1.7) we have

$$p(D)([[u]]-w^+-w^-)=\delta.$$

This gives a contradiction, by applying the Fourier transform and substituting $p(\zeta)=0$. Q.E.D.

Remark. Referring to Proposition 1.2, we see that there are no case where the hyperfunction solutions on $U \backslash K$ can be extended uniquely to U.

§2. Continuation of Real Analytic Solutions

Now we consider the real analytic solutions $\mathscr{A}_p$ of $p(D)u=0$. For the sake of simplicity we give the result first for a fixed system of coordinates. We somewhat extend the situation.

Theorem 2.1. *Let L be a compact convex subset of $\mathbf{R}^n$ with $n \geqq 3$, contained in the hyperplane $\{x_1=0\}$. Put $K=L \cap \{-1<x_n<1\}$. Let U be an open neighborhood of K. Let $p(D)$ be an m-th order linear partial differential operator with constant coefficients, which is irreducible and Kowalevskian with respect to D_1. Put $\zeta'=(\zeta_2,\dots,\zeta_n)$ and $\zeta''=(\zeta_3,\dots,\zeta_{n-1})$. Assume that the roots $\tau_j(\zeta')$, $j=1,\dots,m$ of the equation $p(\zeta)=0$ with respect to ζ_1 satisfy*

$$|\operatorname{Im}\tau_j(\zeta')| \leqq a(|\operatorname{Re}\zeta_2|^q+|\operatorname{Re}\zeta_n|^q)+b(|\operatorname{Im}\zeta_2|+|\operatorname{Im}\zeta_n|)+c|\zeta''|, \tag{2.1}$$

for $\zeta' \in \mathbf{C}^{n-1}$, where $q<1$ and a, b, c are real constants. Then the image of the natural mapping

$$\mathscr{A}_p(U\backslash K)/\mathscr{A}_p(U) \longrightarrow \mathscr{B}_p(U\backslash K)/\mathscr{B}_p(U) \tag{2.2}$$

is zero.

Remark. We assume that L agrees with the closure of K in $\mathbf{R}^n$, because the part in $|x_n|>1$ is of no use for us. Then $L\backslash K$ consists of (at most) two convex connected components contained in the hyperplanes $\{x_n=\pm 1\}$. They may be denoted by $P^{\pm}$ without any confusion. The isomorphisms in Theorem 1.1 hold for our present situation, as long as we employ the growth conditions corresponding to the sets under consideration. Since our conditions on p imply $\mathscr{B}_p[K]=0$ due to Theorem 2 in [10], we have $\widehat{\mathscr{B}_p(U)}=\mathscr{B}_p(U)$ and the mapping corresponding to λ in §1 is injective.

Proof. Take $u \in \mathscr{A}_p(U\backslash K)$ arbitrarily. We consider it as an element of $\mathscr{B}_p(U\backslash K)$. Then, in the same way as in the explanation after Theorem 1.1, we can take a representative $F(\zeta)=\widetilde{[[p(D)[u]]]}|_{N(p)}$ of the image of u in $\widetilde{\mathscr{B}[L]}\{p, d\}/[\widetilde{\mathscr{B}[P^+]}\{p, d\}\oplus\widetilde{\mathscr{B}[P^-]}\{p, d\}]$, where $[u]\in\mathscr{B}(U)$ and $[[p(D)[u]]]\in\mathscr{B}[L]$ are some of the extensions. Let $\chi(x)$ be a function of Gevrey class on U such that $\operatorname{supp}\chi$ is contained in the ε-neighborhood of K, χ is identically equal to one on a smaller neighborhood of K, and $\overline{\operatorname{supp}\chi}\cap\partial U\subset L\backslash K$. Here supp denotes the support of a hyperfunction, the upper bar denotes the closure operator and ∂U denotes the boundary of U in $\mathbb{R}^n$. We specialize the regularity of χ in later step. Let $[(1-\chi(x))u]_0$ denote an extended function of Gevrey class on U, which is defined to be identically equal to zero on a neighborhood of K. Then we have obviously

$$
\begin{aligned}
&p(D)[[[(1-\chi(x))u]_0-[u]]]\\
&\qquad\equiv[[p(D)[(1-\chi(x))u]_0]]-[[p(D)[u]]]\qquad \operatorname{mod}\mathscr{B}[L\backslash K],
\end{aligned}
$$

where [[ ]] denotes an extension from $\mathscr{B}(U)$ to $\mathscr{B}_*(\mathbb{R}^n)$ with the smallest support. Thus we have

$$
F(\zeta)\equiv\widetilde{[[p(D)[(1-\chi(x))u]_0]]}|_{N(p)}\qquad \operatorname{mod}\widetilde{\mathscr{B}[P^+]}\{p, d\}\oplus\widetilde{\mathscr{B}[P^-]}\{p, d\}.
$$

Now let $J(D)$ be a local operator with constant coefficients. (For this concept of differential operator of infinite order we refer to a survey in §1 of [3].) Then we have $J(D)u\in\mathscr{A}_p(U\backslash K)$, hence the above representation is applicable to $J(D)u$. Let $J(\zeta)$ be the total symbol of $J(D)$, namely, the Fourier image of $J(D)\delta$. Since $J(\zeta)F(\zeta)$ is obviously one of the representatives of $J(D)u$, we have

$$
\begin{aligned}
J(\zeta)F(\zeta)\equiv&\widetilde{[[p(D)[(1-\chi(x))J(D)u]_0]]}|_{N(p)}\\
&\qquad \operatorname{mod}\widetilde{\mathscr{B}[P^+]}\{p, d\}\oplus\widetilde{\mathscr{B}[P^-]}\{p, d\}.
\end{aligned}
$$

Next, let $\varphi(x)$ be a function of Gevrey class such that $\operatorname{supp}\varphi$ is contained in the ε-neighborhood of $L\backslash K$ and φ is identically equal to one on the $(\varepsilon/2)$-neighborhood of $L\backslash K$. Put

$$
\begin{aligned}
&w(x)=(1-\varphi(x))p(D)[(1-\chi(x))J(D)u]_0,\\
&v^+(x)+v^-(x)=[[p(D)[(1-\chi(x))J(D)u]_0]]-w(x),
\end{aligned}
$$

where $\operatorname{supp} v^{\pm}$ is contained in the ε-neighborhood of $P^{\pm}$, and $w(x)$ is considered to be zero on the $(\varepsilon/2)$-neighborhood of $L \backslash K$. Note that $w(x)$ is in Gevrey class corresponding to χ, φ. Thus we have

$$
(2.3) \qquad J(\zeta)F(\zeta) \equiv \tilde{w}(\zeta)|_{N(p)} + \widetilde{v^{+}}(\zeta)|_{N(p)} + \widetilde{v^{-}}(\zeta)|_{N(p)}
$$
$$
\mod \widetilde{\mathscr{B}[P^{+}]}\{p, d\} \oplus \widetilde{\mathscr{B}[P^{-}]}\{p, d\}.
$$

By the Fundamental Principle we can adjust the elements of $\widetilde{\mathscr{B}[P^{+}]}\{p, d\} \oplus \widetilde{\mathscr{B}[P^{-}]}\{p, d\}$ modifying $v^{\pm}$. Therefore we can assume that (2.3) is a true identity if we will. We must deduce from this relation the necessary information. Since the convex hull of $\operatorname{supp}(v^{+} + v^{-})$ may cover the whole set L, the usual analysis based on the growth conditions fails to work. Instead, we employ the following tool. (For the concept of Fourier hyperfunctions we refer to a survey in §1 of [3].)

Lemma 2.2. *In order that a Fourier hyperfunction $u(x)$ is real analytic in a neighborhood of the origin, it is necessary and sufficient that every derivative $J(D)u(x)$ has the finite value at the origin. Here $J(D)$ runs over the local operators. The value is defined by*

$$
\lim_{\varepsilon \downarrow 0} (2\pi)^{-n} \int_{\boldsymbol{R}^n} J(\xi)\tilde{u}(\xi) \exp(-\varepsilon\sqrt{|\xi|^2+1})d\xi .
$$

For the proof see [6], Theorem 3.8.

In order to apply this lemma, we need to introduce entire functions instead of holomorphic functions on $N(p)$. The method employing symmetric polynomials developed in [4] badly twists the information. Therefore we introduce here a technique from the boundary value theory. To a holomorphic function $F(\zeta)$ on $N(p)$, we make correspond the entire function of the form

$$
(2.4) \qquad \tilde{f}(\zeta) = \tilde{f}_0(\zeta') + \zeta_1 \tilde{f}_1(\zeta') + \cdots + \zeta_1^{m-1} \tilde{f}_{m-1}(\zeta')
$$

on $\mathbf{C}^n$. The correspondence $\tilde{f}(\zeta) \to F(\zeta)$ is given by the restriction $F(\zeta) = \tilde{f}(\zeta)|_{N(p)}$. The inverse correspondence B: $F(\zeta) \to \tilde{f}(\zeta)$ is given by defining the coefficients $\tilde{f}_j(\zeta')$ by Cramér's formula:

$$\tilde{f}_j(\zeta') = \frac{\begin{array}{c} \hphantom{1\ \tau_1(\zeta')\cdots F(\tau_1(} j \hphantom{), \zeta')\cdots\tau_1(\zeta')^{m-1}} \\ \left|\begin{matrix} 1 & \tau_1(\zeta') \cdots F(\tau_1(\zeta'), \zeta') & \cdots & \tau_1(\zeta')^{m-1} \\ \vdots & \vdots & & \vdots \\ 1 & \tau_m(\zeta') \cdots F(\tau_m(\zeta'), \zeta') & \cdots & \tau_m(\zeta')^{m-1} \end{matrix}\right| \end{array}}{\left|\begin{matrix} 1 & \tau_1(\zeta') \cdots\cdots\cdots\cdots\cdots\cdots & \tau_1(\zeta')^{m-1} \\ \vdots & \vdots & \vdots \\ 1 & \tau_m(\zeta') \cdots\cdots\cdots\cdots\cdots\cdots & \tau_m(\zeta')^{m-1} \end{matrix}\right|}. \tag{2.5}$$

Since the correspondence is linear, with the coefficients of rational growth, the growth conditions are kept. In fact, assume that $F(\zeta)$ satisfies for any $\eta > 0$,

$$|F(\zeta)| \leqq C_\eta \exp(\eta|\zeta| + H_L(\operatorname{Im}\zeta)),$$

where $H_L(\operatorname{Im}\zeta) = \sup_{x \in L} \operatorname{Re} x \cdot \zeta$ is the supporting function of L. Since L is contained in $\{x_1 = 0\}$, $H_L(\operatorname{Im}\zeta)$ in fact contains only $\operatorname{Im}\zeta'$. Since p is Kowalevskian with respect to D_1, we have $|\tau_j(\zeta')| \leqq M|\zeta'|$, $j = 1, \dots, m$. Thus for the coefficients of $\tilde{f} = B[F]$ we obtain from (2.5)

$$|\tilde{f}_j(\zeta')| \leqq C_\eta \exp(\eta|\zeta'| + H_L(\operatorname{Im}\zeta')).$$

(Precisely speaking, the direct estimate is possible only outside the zeros of the discriminant $\Delta(\zeta')$ of $p(\zeta) = 0$ as an equation of ζ_1. Since $p(\zeta)$ is irreducible, $\Delta(\zeta') \not\equiv 0$. Hence the maximum principle can be applied and we have the estimate on the whole $\mathbf{C}^{n-1}$.) Conversely the latter growth condition clearly implies the former for $F = B^{-1}[f]$. Similar argument holds with L replaced by $P^\pm$. Thus the composed correspondence $u(x) \to F(\zeta) \to (\tilde{f}_0(\zeta'), \dots, \tilde{f}_{m-1}(\zeta'))$ gives the isomorphism

$$\mathscr{B}_p(U \backslash K)/\mathscr{B}_p(U) \longrightarrow \widetilde{\mathscr{B}[L]}^m / [{}'\widetilde{\mathscr{B}[P^+]}^m \oplus {}'\widetilde{\mathscr{B}[P^-]}^m], \tag{2.6}$$

where ${}'\mathscr{B}[L]$ denotes the space of hyperfunctions of $(n-1)$-variables $x' = (x_2, \dots, x_n)$ with support in L, and similarly for ${}'\mathscr{B}[P^\pm]$. We apply the mapping B to (2.3). Put

$$B[v^\pm(\zeta)|_{N(p)}] = \tilde{g}^\pm(\zeta) = \tilde{g}_0^\pm(\zeta') + \zeta_1 \tilde{g}_1^\pm(\zeta') + \cdots + \zeta_1^{m-1} \tilde{g}_{m-1}^\pm(\zeta'),$$

$$B[\widetilde{w(\zeta)}|_{N(p)}] = \tilde{h}(\zeta) = \tilde{h}_0(\zeta') + \zeta_1 \tilde{h}_1(\zeta') + \cdots + \zeta_1^{m-1} \tilde{h}_{m-1}(\zeta').$$

We have for every $\eta > 0$,

$$|\tilde{v}^\pm(\zeta)| \leqq C_\eta \exp(\eta|\zeta| + \varepsilon|\operatorname{Im}\zeta| + H_{P^\pm}(\operatorname{Im}\zeta)). \tag{2.7}$$

Thus, from (2.5) and the condition on $\tau_j(\zeta')$ we have

$$(2.8)\qquad |\tilde{g}_j^{\pm}(\zeta')|\leqq C'_\eta \exp(c\varepsilon|\mathrm{Re}\,\zeta''|+\eta|\zeta'|+(1+b+c)\varepsilon|\mathrm{Im}\,\zeta'|+H_{P^{\pm}}(\mathrm{Im}\,\zeta')).$$

Now we specialize the regularity of $\chi(x)$, $\varphi(x)$ so that the following estimate holds:

$$(2.9)\qquad |\tilde{w}(\zeta)|\leqq C\exp(\varepsilon|\mathrm{Im}\,\zeta|+H_L(\mathrm{Im}\,\zeta)-A|\mathrm{Re}\,\zeta|^q),$$

with $A>a$ and the given $q<1$. Thus from (2.5) and the condition on $\tau_j(\zeta')$ we have

$$(2.10)\qquad |\tilde{h}_j(\zeta')|$$
$$\leqq C|\zeta'|^{m(m-1)/2}\exp(c\varepsilon|\mathrm{Re}\,\zeta''|+(1+b+c)\varepsilon|\mathrm{Im}\,\zeta'|+H_L(\mathrm{Im}\,\zeta')-a'|\mathrm{Re}\,\zeta'|^q),$$

where we have assumed $a'=A-\varepsilon a>0$. Now in (2.3) we employ those $J(\zeta')$ containing only ζ'. Then applying the linear mapping B to both sides, we obtain

$$(2.11)\qquad J(\zeta')\tilde{f}_j(\zeta')=\tilde{h}_j(\zeta')+\tilde{g}_j^{+}(\zeta')+\tilde{g}_j^{-}(\zeta'),\qquad j=0, 1,\dots, m-1.$$

Here, as remarked above, $\tilde{f}_j(\zeta')$ are the Fourier images of hyperfunctions with support in L. But $\tilde{g}_j^{\pm}(\zeta')$, $\tilde{h}_j(\zeta')$ have the estimates where $\mathrm{Re}\,\zeta''$ appears with a definite modulus. Hence they cannot be considered as the Fourier images of hyperfunctions. To overcome the difficulty we restrict the variables ζ' to the real ξ' and multiply by $\exp(-2c\varepsilon\sqrt{|\xi''|^2+1})$. Then we have

$$(2.12)\qquad J(\xi')\tilde{f}_j(\xi')\exp(-2c\varepsilon\sqrt{|\xi''|^2+1})$$
$$=\tilde{h}_j(\xi')\exp(-2c\varepsilon\sqrt{|\xi'|^2+1})+\tilde{g}_j^{+}(\xi')\exp(-2c\varepsilon\sqrt{|\xi''|^2+1})$$
$$+\tilde{g}_j^{-}(\xi')\exp(-2c\varepsilon\sqrt{|\xi''|^2+1}).$$

From the estimate (2.10) we see that the integral

$$\int_{\mathbf{R}^{n-1}}\tilde{h}_j(\xi'')\exp(-2c\varepsilon\sqrt{|\xi''|^2+1}-\sqrt{-1}x'\xi')d\xi'$$

converges absolutely for any $x'\in\mathbf{R}^{n-1}$. On the other hand, from the estimate (2.8) we see that $F_{n-1}^{-1}[\tilde{g}_j^{\pm}(\xi')\exp(-2c\varepsilon\sqrt{|\xi''|^2+1}])$ are Fourier hyperfunctions whose (analytic) singular supports are contained, respectively, in the $(1+b+c)\varepsilon$-neighborhood of $P^{\pm}+\{x_2=x_n=0\}$, where

$$F_{n-1}^{-1}[\tilde{f}(\xi')]=(2\pi)^{-(n-1)}\int_{\mathbf{R}^{n-1}}\tilde{f}(\xi')\exp(-\sqrt{-1}x'\cdot\xi')d\xi'$$

denotes the $(n-1)$-dimensional inverse Fourier transform. We check this assertion in the following lemma.

Lemma 2.3. *Let L be a compact convex subset of $\mathbf{R}^n$. Let $x=(x', x'')$ be a partition of the variables, and let $\zeta=(\zeta', \zeta'')$ be the corresponding partition of the dual variables. Let $f(x)$ be a Fourier hyperfunction. Assume that its Fourier image $\tilde{f}(\xi)$ can be extended analytically to*

$$|\mathrm{Im}\,\zeta'|\leqq\varepsilon(|\mathrm{Re}\,\zeta'|+1),\qquad |\mathrm{Im}\,\zeta''|\leqq\varepsilon,$$

and there satisfies the following estimate: Given $\eta>0$, there exists C_η such that

$$|\tilde{f}(\zeta)|\leqq C_\eta\exp(\eta|\zeta|-\varepsilon|\mathrm{Re}\,\zeta''|+H_L(\mathrm{Im}\,\zeta)).$$

Then $f(x)$ is real analytic outside $L+\{x'=0\}$.

Proof. Assume that x_1 is contained in the part x'. We show that $f(x)$ is real analytic at a point x^0 on $x_1\geqq\sup\{x_1;\, x\in L\}+\delta$ for any $\delta>0$. Then by the linear coordinate transformations we can prove the general assertion. Let $\sigma=(\sigma_1,\dots,\sigma_n)$, $\sigma_j=\pm1$ be a multi-signature. Consider the defining function of $f(x)$ on the σ-th orthant $\{\sigma_j\,\mathrm{Im}\,z_j>0,\ j=1,\dots,n\}$

$$F_\sigma(z)=(2\pi)^{-n}\int_{\{\sigma_j\xi_j\leqq0,\,j=1,\dots,n\}}\tilde{f}(\xi)e^{-\sqrt{-1}z\cdot\xi}d\xi.$$

We are going to deform the path of integration to the complex region so that the above integral may converge locally uniformly in z even when $\sigma\,\mathrm{Im}\,z>-\varepsilon'$ for some $\varepsilon'>0$. Put

$$\Gamma=\{\zeta=\xi+\sqrt{-1}(\varepsilon|\xi'|,0,\dots,0);\ \sigma_j\xi_j\leqq0,\ j=1,\dots,n\}.$$

By the assumption we can deform the above integral to that on Γ.

When $\zeta \in \Gamma$ and $\operatorname{Re} z = x^0$, we have

$$|\tilde{f}(\zeta)| \leqq C_\eta \exp(\eta|\zeta| - \varepsilon|\xi''| + H_L(\operatorname{Im}\zeta)),$$

$$|e^{-\sqrt{-1}z\cdot\zeta}| \leqq \exp(\operatorname{Im} z \cdot \xi - \varepsilon x_1^0|\xi'|),$$

and

$$H_L(\operatorname{Im}\zeta) = \sup_{x\in L} \varepsilon x_1|\xi'| = \varepsilon|\xi'|\sup\{x_1;\ x \in L\} \leqq \varepsilon x_1^0|\xi'| - \varepsilon\delta|\xi'|.$$

Therefore if we choose η so small that $\eta < \varepsilon' = \min(\varepsilon\delta, \varepsilon)/2$, we have for $\zeta \in \Gamma$,

$$|\tilde{f}(\zeta)e^{-\sqrt{-1}z\zeta}| \leqq C\exp(\operatorname{Im} z \cdot \xi - \varepsilon'(|\xi'| + |\xi''|)).$$

Thus we conclude that the integral converges locally uniformly in z on $|\operatorname{Im} z| < \varepsilon'$, hence it is analytic in z there. Q.E.D.

Remark. The above lemma is a modification of Lemma 5.1.2 in [7]. There a beautiful criterion for $\operatorname{sing.supp} f(x) \subset K$ is given when K is convex and compact. Lemma 2.3 above is for temporary use and may be refined to give a necessary and sufficient condition as the latter. (It seems to the author, however, that the condition $|F(\zeta)| \leqq A_\varepsilon \exp(\varepsilon|\zeta| + H_K(\operatorname{Im}\zeta))$ in the latter is the misprint of $|F(\zeta)| \leqq A_{\varepsilon,\eta} \exp(\eta|\zeta| + \varepsilon|\operatorname{Im}\zeta| + H_K(\operatorname{Im}\zeta))$, where $\eta > 0$ runs independently of ε.)

End of Proof of Theorem 2.1. Thus by Lemma 2.2 the following limit exists when $x' = (x_2, \dots, x_n)$ does not belong to the $(1+b+c)\varepsilon$-neighborhood of $P^\pm + \{x_2 = x_n = 0\}$ in $\mathbb{R}^{n-1}$.

$$\lim_{\delta\downarrow 0}(2\pi)^{-(n-1)}\int_{\mathbb{R}^{n-1}} \tilde{g}_j^\pm(\xi')\exp(-2c\varepsilon\sqrt{|\xi''|^2+1} - \sqrt{-1}x'\xi' - \delta\sqrt{|\xi'|^2+1})d\xi'.$$

Summing up, we conclude that when x' does not belong to the $(1+b+c)\varepsilon$-neighborhood of $(L\backslash K) + \{x_2 = x_n = 0\}$, the following limit exists

$$\lim_{\delta\downarrow 0}(2\pi)^{-(n-1)}\int_{\mathbb{R}^{n-1}} J(\xi')\tilde{f}_j(\xi')\exp(-2c\varepsilon\sqrt{|\xi''|^2+1} - \sqrt{-1}x'\xi' - \delta\sqrt{|\xi'|^2+1})d\xi'.$$

Thus, employing the inverse implication in Lemma 2.2, we conclude that the Fourier hyperfunction $u_{j,\varepsilon}(x') = F_{n-1}^{-1}[\tilde{f}_j(\xi')\exp(-2c\varepsilon\sqrt{|\xi''|^2+1})]$

is real analytic outside the $(1+b+c)\varepsilon$-neighborhood of $(L\backslash K)+\{x_2=x_n=0\}$. Since

$$u_{j\,\varepsilon}(x')=f_j(x')\ast\{F_{n-3}^{-1}[\exp(-2c\varepsilon\sqrt{|\xi''|^2+1})]\delta(x_2)\delta(x_n)\}$$

where $f_j(x')=F_{n-1}^{-1}[\tilde{f}_j(\xi')]$, and

$$\operatorname{supp} f_j(x')\subset L,$$

$$\operatorname{supp}\{F_{n-3}^{-1}[\exp(-2c\varepsilon\sqrt{|\xi''|^2+1})]\delta(x_2)\delta(x_n)\}\subset\{x_2=x_n=0\},$$

we have

$$\operatorname{supp} u_{j,\varepsilon}(x')\subset L+\{x_2=x_n=0\}.$$

Therefore, by the uniqueness of analytic continuation we conclude that $u_{j,\varepsilon}(x')$ is identically equal to zero outside the $(1+b+c)\varepsilon$-neighborhood of $(L\backslash K)+\{x_2=x_n=0\}$.

Now we let ε tend to zero. We assert that $u_{j,\varepsilon}(x')$ tends to $f_j(x')$ and the estimate of support is kept through this limit process, thus obtaining $\operatorname{supp} f_j(x')\subset(L\backslash K)+\{x_2=x_n=0\}$. Since the usual topology in hyperfunction theory cannot be localizable, we must be careful in these arguments. We proceed as in the proof of Theorem 3.8 in [6], employing the boundary value theory of Komatsu-Kawai: $u_{j,\varepsilon}(x')$ is naturally considered as a hyperfunction of (ε, x') on the real analytic manifold $\{\varepsilon>0\}\times\mathbf{R}_{x_2}\times\mathbf{S}_{x''}^{n-3}\times\mathbf{R}_{x_n}$, where $\mathbf{S}_{x''}^{n-3}$ is the one point compactification of the $(n-3)$-dimensional Euclidean space of $x''=(x_3,\dots, x_{n-1})$, and it satisfies the following differential equation there

$$\left[\frac{\partial^2}{\partial\varepsilon^2}+4c^2\left(\frac{\partial^2}{\partial x_3^2}+\cdots+\frac{\partial^2}{\partial x_{n-1}^2}-1\right)\right]u_{j,\varepsilon}(x')=0. \tag{2.13}$$

In fact, on the finite subdomain $\{\varepsilon>0\}\times\mathbf{R}^{n-1}$ we define $u_{j,\varepsilon}(x')$ by the defining function given by the inverse Fourier transform. Then $u_{j,\varepsilon}(x')$ contains ε as a complex holomorphic parameter on $\operatorname{Re}\varepsilon>0$ and we can easily check (2.13) by the defining function. At the infinity of $\mathbf{S}_{x''}^{n-3}$, we employ the system of coordinates $y''=x''/|x''|^2$, $x''=y''/|y''|^2$ and define

$$u_{j,\varepsilon}(x_2, y'', x_n)$$
$$= \int_{\mathbf{R}^{n-3}} f_j(x_2, t'', x_n) F_{n-3}^{-1}[\exp(-2c\varepsilon\sqrt{|\xi''|^2+1})](y''/|y''|^2 - t'')dt''.$$

Note that the integration is in fact performed on a compact fiber. As is well known (see [6], the proof of Theorem 3.8), $F_{n-3}^{-1}[\exp(-2c\varepsilon\sqrt{|\xi''|^2+1})]$ $(y''/|y''|^2)$ is uniquely extended as an infinitely differentiable function of y'' to $y''=0$, and holomorphic in ε on $\operatorname{Re}\varepsilon>0$. Thus the above integral defines a natural extension of $u_{j,\varepsilon}(x_2, y'', x_n)$ to $y''=0$. In this system of coordinates (2.13) becomes

$$(2.14) \quad \left[\frac{\partial^2}{\partial\varepsilon^2} + 4c^2\left\{|y''|^4 \sum_{k=3}^{n-1} \frac{\partial^2}{\partial y_k^2} + (10-2n)|y''|^2 \sum_{k=3}^{n-1} y_k \frac{\partial}{\partial y_k}\right\}\right]$$
$$u_{j,\varepsilon}(x_2, y'', x_n) = 0,$$

and it is also obviously satisfied. Since the hypersurface $\{\varepsilon=0\}$ is everywhere non-characteristic with respect to the equation (2.13)-(2.14), we can take the boundary value in the sense of Komatsu-Kawai [10].

On the other hand, for each fixed $\varepsilon>0$, we have $u_{j,\varepsilon}(x') \in {}'\mathscr{B}[\{0\} \times \mathbf{S}_{x''}^{n-3} \times \{0\}]$, and when we let $\varepsilon \downarrow 0$, it converges to $f_j(x')$ in this space. In fact, we can check the weak convergence. Let $\varphi(x')$ be a real analytic function on a neighborhood of $\{0\} \times \mathbf{S}_{x''}^{n-3} \times \{0\}$. Then we have obviously

$$< u_{j,\varepsilon}(x'), \varphi(x') >_{x'} = < f_j(x'), \varphi \underset{x''}{*} F_{n-3}^{-1}[\exp(-2c\varepsilon\sqrt{|\xi''|^2+1})] >_{x'},$$

and the right hand member in the last inner product converges to $\varphi(x')$ in the space of real analytic functions.

Thus by Corollary 2.6 in [6], we conclude that $f_j(x')$ is the boundary value of $u_{j,\varepsilon}(x')$ with respect to (2.13). Since $\operatorname{supp} u_{j,\varepsilon}(x')|_{\{\varepsilon>0\}\times\mathbf{R}^{n-1}} \subset \{(\varepsilon, x'); \operatorname{dis}[x', (L\backslash K)+\{x_2=x_n=0\}] \leqq \varepsilon\}$ and the process of taking the boundary value to a non-characteristic surface is local, we conclude that $\operatorname{supp} f_j(x') \subset (L\backslash K)+\{x_2=x_n=0\}$, hence $\operatorname{supp} f_j(x') \subset L\backslash K$. This means $\tilde{f}_j(\zeta') \in {}'\widetilde{\mathscr{B}[P^+]} \oplus {}'\widetilde{\mathscr{B}[P^-]}$. Returning to $F(\zeta)$, we have shown that $F(\zeta) \in \widetilde{\mathscr{B}[P^+]}\{p, d\} \oplus \widetilde{\mathscr{B}[P^-]}\{p, d\}$. This implies the triviality of the image of (2.2). Q.E.D.

Remark. The conclusion of Theorem 2.1 implies that $u \in \mathscr{A}_p(U \backslash K)$ can be uniquely extended as a hyperfunction solution to the whole U. In order that $\mathscr{A}_p(U \backslash K)/\mathscr{A}_p(U)$ itself is trivial, the propagation of regularity $\mathscr{A}(U \backslash K) \cap \mathscr{B}_p(U) \subset \mathscr{A}(U)$ is further needed. The latter holds, e.g., if $p(D)$ is simply characteristic and every bicharacteristic line is transversal to K (Kawai [8]).

Summing up, we have

Corollary 2.4. *Let* $K = \{(0, \dots, 0, x_n);\ -1 < x_n < 1\}$. *Assume that each irreducible component* q *of* p *satisfies the following two conditions.*

1) q *satisfies the assumption of Theorem* 2.1 *for a suitable system of coordinates in* $(x_1, \dots, x_{n-1})$ *space.*

2) *The propagation of regularity holds, namely,*

$$\mathscr{A}_q(U \backslash K)/\mathscr{A}_q(U) \longrightarrow \mathscr{B}_q(U \backslash K)/\widehat{\mathscr{B}_q(U)}$$

is injective.

Then we have $\mathscr{A}_q(U \backslash K)/\mathscr{A}_p(U) = 0$.

Proof. Let $p = p_1 p_2 \cdots p_k$ be the irreducible decomposition of p, where the multiple factors are repeated. Take $u \in \mathscr{A}_p(U \backslash K)$. We apply Theorem 2.1 to $f_1 = p_2(D) \cdots p_k(D) u \in \mathscr{A}_{p_1}(U \backslash K)$ and conclude that it can be extended to an element of $\mathscr{B}_{p_1}(U)$. Since the propagation of regularity is assumed, we conclude that $f_1 \in \mathscr{A}(U)$ as remarked above. Next we solve the equation

$$p_2(D) \cdots p_k(D) v_1 = f_1$$

and obtain a solution $v_1 \in \mathscr{A}(U_\varepsilon)$, where U_ε is a convex open neighborhood of $K_\varepsilon = \{(0, \dots, 0, x_n);\ -1 + \varepsilon < x_n < 1 - \varepsilon\}$. (For this existence theorem we refer to [9], Theorem 3.1.) Put $u_2 = u - v_1$. Then $u_2 \in \mathscr{A}(U_\varepsilon \backslash K_\varepsilon)$ and

$$p_2(D) \cdots p_k(D) u_2 = 0,$$

Thus repeating this argument we obtain $v_j \in \mathscr{A}(U_{j\varepsilon})$, $u_{j+1} = u - (v_1 + \cdots + v_j)$, $j = 1, \dots, k-1$, and finally conclude that $u_k \in \mathscr{A}(U_{(k-1)\varepsilon})$. Thus we have

$$u = (v_1 + \cdots + v_{k-1}) + u_k \in \mathscr{A}(U_{(k-1)\varepsilon}).$$

Since ε is arbitrary, we conclude that $u \in \mathscr{A}(U)$. Q.E.D.

Example. The ultra-hyperbolic operators

$$p(D)=D_1^2+\cdots+D_k^2-D_{k+1}^2-\cdots-D_n^2 \qquad (1\leqq k\leqq n-2)$$

are covered by our result. We can apply Theorem 2.1 taking ζ_{n-1} instead of ζ_2. In fact, put $\zeta=\xi+\sqrt{-1}\eta$. For $\xi_{n-1}^2+\xi_n^2\leqq 3(\eta_{n-1}^2+\eta_n^2)$ we have

$$|\operatorname{Im}\zeta_1|\leqq\sqrt{4(\eta_{n-1}^2+\eta_n^2)+|\zeta''|^2}\leqq 2(|\eta_{n-1}|+|\eta_n|)+|\zeta''|,$$

where $\zeta''=(\zeta_2,\dots,\zeta_{n-2})$ in the present notation. For $3|\zeta''|^2\geqq\xi_{n-1}^2+\xi_n^2\geqq 3(\eta_{n-1}^2+\eta_n^2)$ we have

$$|\operatorname{Im}\zeta_1|\leqq\sqrt{4|\zeta''|^2}=2|\zeta''|.$$

Finally if $\xi_{n-1}^2+\xi_n^2\geqq\max\{3|\zeta''|^2, 3(\eta_{n-1}^2+\eta_n^2)\}$, we have

$$|\operatorname{Im}\zeta_1|\leqq\frac{|\xi_2\eta_2+\cdots+\xi_n\eta_n|}{\sqrt{(\xi_{n-1}^2+\xi_n^2)/3}}\leqq\sqrt{3}\,(|\eta_{n-1}|+|\eta_n|)+\frac{1}{2}|\zeta''|.$$

Since the propagation of regularity holds due to the above remark, Corollary 2.4 can also be applied. Thus we have strengthened the result of [4], which corresponds to the case $P^-=\emptyset$. Note that for $k=0$ or n the operator is elliptic and has a trivial counter-example. Finally for $k=n-1$ the fundamental solution $E(x_1,\dots,x_{n-1})$ of the Laplacian $D_1^2+\cdots+D_{n-1}^2$ gives a non-trivial element of $\mathscr{A}_p(U\backslash K)/\mathscr{A}_p(U)$. Since the propagation of regularity holds, E cannot be extended to the x_n-axis even as a hyperfunction solution.

Though Theorem 2.1 itself is fairly sharp, we expect that there are milder conditions when we consider only those K contained in the x_n-axis.

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