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On Smooth Semifree S^1 Actions on Cohomology Complex Projective Spaces

By

Tomoyoshi YOSHIDA*

§0. Introduction and Statement of Results

Let X^{2n} be an oriented closed differentiable $2n$-manifold. Let CP^n be the complex projective n-space.

Definition. X^{2n} is a cohomology CP^n (X is a coh CP^n) if there is an element $\alpha \in H^2(X; \mathbf{Z})$ such that $H^{**}(X; \mathbf{Z})$ is isomorphic to the truncated polynomial ring of α, $\mathbf{Z}[\alpha]/(\alpha^{n+1})$ (here $\mathbf{Z}$ denotes the ring of the rational integers).

We may assume that the Kronecker pairing $\langle \alpha^n, [X] \rangle$ equals 1 where $[X]$ is the fundamental class. We call α a cohomology generator of X. Let $\hat{\mathscr{A}}(X)$ be the total $\hat{\mathscr{A}}$-class of X defined by

$$\hat{\mathscr{A}}(X) = \Pi(x_j/2)(\sinh x_j/2)^{-1} \in H^{**}(X; \mathbf{Q}),$$

where the elementary symmetric functions of the $(x_j)^2$ give the Pontrjagin classes of X, and $\mathbf{Q}$ denotes the field of the rational numbers.

A circle group action is called semifree if it is free outside the fixed point set. Now our result is as follows.

Theorem 0.1. *Let X be a* coh CP^n *with a cohomology generator α. If X admits a non-trivial smooth semifree circle group action, then*

$$\hat{\mathscr{A}}(X) = (\alpha/e^{\alpha/2} - e^{-\alpha/2})^{n+1}.$$

Corollary 0.2. *Let f be an orientation preserving homotopy equiva-*

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* Department of Mathematics, Tsuda College, Tokyo.

lence from an oriented closed differentiable manifold X to CP^n. If X admits a non-trivial smooth semifree circle group action, then $\hat{\mathscr{A}}(X) = f^*\hat{\mathscr{A}}(CP^n)$.

In the case with $n=3$, Corollary 0.2 implies that X is diffeomorphic to CP^3 by the result of D. Montgomery and C. T. Yang [3]. The motivation of this paper is the conjecture of T. Petrie which asserts that the conclusion of Corollary 0.2 holds even if we miss the condition 'semifree'.

Theorem 0.1 will be proved by the use of the Atiyah-Singer-Segal index formula which is formulated by T. Petrie for spin^c-manifold in [4]. In §1, we will state some results of G. E. Bredon and J. C. Su on circle group actions on cohomology complex projective spaces. In §2, some properties of the fixed point set will be given in the semifree case. In §3, the index formula will be given in a special form for our purpose. In §4, Theorem 0.1 and Corollary 0.2 will be proved.

Notation. 1) S^1 denotes the circle group which is identified with the group of the complex number with determinant 1. 2) For a Lie group G, if X is a right G-space and Y is a left G-space, $X \times_G Y$ denotes the space obtained from $X \times Y$ by identifying $(xg, g^{-1}y)$ with (x, y) for $x \in X$, $y \in Y$, $g \in G$.

§1. Preliminaries

Let X be a cohCP^n with a cohomology generator α (see §0). Let $\phi: S^1 \times X \to X$ be a smooth S^1 action on X. Let $F = \cup F_j$ be the fixed point set of ϕ, where $\{F_j\}$ are its connected components. Each F_j is an orientable smooth submanifold of X.

Proposition 1.1 (G. E. Bredon [2], J. C. Su [5]). *Each F_j is a* cohCP^{h_j} *for some h_j and $\Sigma(h_j+1)=n+1$. If α_j is the restriction of α to F_j, then α_j is a generator of $H^2(F_j: \mathbf{Z})$.*

Let η be the complex line bundle over X whose first Chern class $c_1(\eta)$ is α. We call η the line bundle associated to α. As $H^1(X; \mathbf{Z})=0$, there is an S^1 action $\tilde{\phi}$ on η which is a lifting of ϕ (Su [5]). This means that there is a smooth S^1 action on $E(\eta)$, the total space of η, $\tilde{\phi}: S^1 \times E(\eta) \to E(\eta)$, such that $\tilde{\phi}(s, \ \ )$ is a bundle map for each fixed $s \in S^1$ and the diagram

$$\begin{array}{ccc} S^1 \times E(\eta) & \xrightarrow{\tilde{\phi}} & E(\eta) \\ \downarrow{\scriptstyle 1\times q} & & \downarrow{\scriptstyle q} \\ S^1 \times X & \xrightarrow{\phi} & X \end{array}$$

commutes where q is the projection. If p_j is a point of F_j, $\tilde{\phi}$ induces a 1-dimensional complex representation of S^1 in the fibre $q^{-1}(p_j)$. Let t denote the canonical representation $t: S^1 = U(1)$. Then the above representation may be written as t^{a_j} for some integer a_j. We will write this situation as $\tilde{\phi}|p_j = t^{a_j}$. Thus we have a set of integers $\{a_j\}$. Let $\mathbf{Z}_k \subset S^1$ be the subgroup of the k-th roots of unity. $F(\mathbf{Z}_k)$ denotes the set of the points of X fixed by the whole group $\mathbf{Z}_k$. Clearly $F(\mathbf{Z}_k) \supset F$.

Proposition 1.2 (T. Petrie [4]). *$\{a_j\}$ has the following properties*
1) *for each $i \neq j$, the difference $a_i - a_j$ is not zero and it depends only on ϕ and η, and does not depend on the choice of $\tilde{\phi}$, and*
2) *for a prime power p^r, F_i and F_j are contained in a same connected component of $F(\mathbf{Z}_{p^r})$ if the difference $a_i - a_j \equiv 0 \pmod{p^r}$.*

§ 2. Properties of Fixed Point Set

Let ϕ be a non-trivial semifree smooth S^1 action on X which is a coh CP^n with a cohomology generator α. Let F be the fixed point set of ϕ. The main purpose of this section is to prove Proposition 2.5.

Proposition 2.1. *F has just two connected components.*

Proof. By Proposition 1.1, F has necessarily at least two connected components. Assume that F has at least three connected components. Let F_1, F_2 and F_3 be three ones different from each other. Let η be the line bundle associated to α. Choose a lifting $\tilde{\phi}$ of ϕ in η. Then we have a set of integers $\{a_1, a_2, a_3\}$ (see § 1). By Proposition 1.2 $a_i \neq a_j$

if $i\neq j$. Hence for some $1\leqq i<j\leqq 3$, a_i-a_j is divisible by some prime number $p\geqq 2$. By Proposition 1.2, 2), F_i and F_j are contained in a same connected component of $F(\mathbf{Z}_p)$. Therefore $F(\mathbf{Z}_p)\neq F$ and ϕ is not semifree. This is a contradiction. Q.E.D.

Now let F_0 and F_1 be the two connected components of F. By Proposition 1.1, F_0 is a coh CP^p and F_1 is a coh CP^q for some non-negative integers p, q such that $p+q=n-1$. The restriction of α to F_j, α_j, is a generator of $H^2(F_j; \mathbf{Z})$ $(j=0, 1)$. Let η be the line bundle associated to α. Let $\tilde{\phi}$ be a lifting of ϕ in η. If $\tilde{\phi}|p_j=t^{a_j}$ for $j=0, 1$, then $a_1-a_0=\pm 1$ by Proposition 1.2 and the semifreeness. We may make the following assumption with no loss of generality.

Assumption (*) $a_1-a_0=+1$.

Let $\tilde{X}$ be the sphere bundle of η. Let $q:\tilde{X}\rightarrow X$ be the projection. Then $q:\tilde{X}\rightarrow X$ is a principal S^1 bundle over X. For $A\subset X$, denote $q^{-1}(A)$ by $\tilde{A}$.

Now throughout this section, the cohomology groups will be understood with integer coefficients.

Lemma 2.2. *$\tilde{X}, \tilde{F}_0$ and $\tilde{F}_1$ are $\mathbf{Z}$-cohomology spheres, that is* $H^*(\tilde{X})=H^*(S^{2n+1})$, $H^*(\tilde{F}_0)=H^*(S^{2p+1})$ *and* $H^*(\tilde{F}_1)=H^*(S^{2q+1})$.

Proof. This follows from the Gysin cohomology exact sequences associated to the S^1 bundles

$$\tilde{X}\longrightarrow X, \quad \tilde{F}_j\longrightarrow F_j \qquad (j=0, 1).$$ Q.E.D.

The following lemma is a preparation for the next Lemma 2.4.

Lemma 2.3. *Let K be a finite dimensional locally finite CW-complex on which S^1 acts semifreely with fixed point set L. We assume that L is a subcomplex of K. If both K and L have the same integral cohomology rings as the m-sphere S^m, then $j^*: H^m(K)\rightarrow H^m(L)$ is an isomorphism, where $j: L\subsetneq K$ is the inclusion map.*

Proof. Let (ES^1, p, BS^1) be the universal S^1 principal bundle. As S^1 acts on $K-L$ freely, $(K-L)\times_{S^1}ES^1$ is homotopically equivalent to the orbit space $K-L/S^1$. Let

$$\cdots \to H^i(K\times_{S^1}ES^1) \longrightarrow H^i(L\times BS^1) \longrightarrow H^i(K-L/S^1)\to\cdots$$

be the exact sequence of the Čech cohomology rings. This is an exact sequence of $H^*(BS^1)$-modules. As $K-L/S^1$ is a finite dimensional space, $H^*(K-L/S^1)$ is a $H^*(BS^1)$-torsion module. Let s be a generator of $H^2(BS^1)$ and let s^{-1} be the formal inverse. Tensoring the above exact sequence with $\mathbf{Z}[s^{-1}]$, we have an isomorphism

$$(j\times 1)^*\colon H^*(K\times_{S^1}ES^1)[s^{-1}]\cong H^*(L\times BS^1)[s^{-1}].$$

Now the right hand side is $(H^m(L)\oplus H^0(L))\otimes H^*(BS^1)[s^{-1}]$, hence the left hand side must be isomorphic to $(H^m(K)\oplus H^0(K))\otimes H^*(BS^1)[s^{-1}]$ and $j^*\colon H^m(K)\to H^m(L)$ must be an isomorphism. Q.E.D.

Lemma 2.4. *The linking number of $\tilde{F}_0$ and $\tilde{F}_1$ in $\tilde{X}$ equals ± 1.*

Proof. By assumption (*), there exists a lifting of ϕ, $\tilde{\phi}_0$, in η such that $\tilde{\phi}_0|p_0=1$ and $\tilde{\phi}_0|p_1=t$ for $p_j\in F_j$ $(j=0, 1)$. $\tilde{\phi}_0$ induces a semifree S^1 action on $\tilde{X}$. $\tilde{F}_1$ is invariant under this action and the restricted action on $\tilde{X}-\tilde{F}_1$ is semifree with the fixed point set $\tilde{F}_0$. By the Alexander duality and Lemma 2.3, $H^*(\tilde{X}-\tilde{F}_1)\cong H^*(S^{2p+1})$. Hence by Lemma 2.2 and Lemma 2.3, $J^*\colon H^{2p+1}(\tilde{X}-\tilde{F}_1)\to H^{2p+1}(\tilde{F}_0)$ is an isomorphism where $J\colon \tilde{F}_0\subsetneq \tilde{X}-\tilde{F}_1$ is the inclusion map. Q.E.D.

Let N_j be the normal bundle of F_j in X. The dimension of the fibre of N_0 is $2(q+1)$ and that of N_1 is $2(p+1)$. S^1 acts on N_j by bundle automorphisms. This action is free in each fibres outside the zero-section. Hence N_0 and N_1 have complex structures such that the S^1 action in each fibres are the complex representations $\underbrace{t+\cdots+t}_{q+1}$ and $\underbrace{t+\cdots+t}_{p+1}$ respectively. *From now on, we consider N_j as a complex vector bundle with this complex structure.*

Let $D(N_j)$ and $S(N_j)$ be the disk and the sphere bundles of N_j respectively. By introducing some invariant Riemannian metric on X, we may consider $D(N_j)$ as an equivariant tubular neighborhood of F_j in X such that $D(N_0)\cap D(N_1)=\phi$. Put $Y=$the closure of $(X-(D(N_0)\cup D(N_1)))$ and $\tilde{Y}=q^{-1}(Y)$. The boundary of $\tilde{Y}$ is $\widetilde{S(N_0)}\cup\widetilde{S(N_1)}$. Let

$\tilde{i}_j\colon \widetilde{S(N_j)} \subsetneq \tilde{Y}$ be the inclusion. $\tilde{Y}$ is homotopically equivalent to $\tilde{X}-(\tilde{F}_0 \cup \tilde{F}_1)$. Hence from the Alexander duality and Lemma 2.4, it follows that $\tilde{i}_j^*\colon H^*(\tilde{Y})=H^*(\widetilde{S(N_j)})=H^*(S^{2p+1}\times S^{2q+1})$ as rings.

Now the restriction of ϕ to Y is free and we denote the orbit space Y/ϕ by $\bar{Y}$. The following diagram commutes

$$\begin{array}{ccccc} \widetilde{S(N_j)} & \longrightarrow & S(N_j) & \longrightarrow & P(N_j) \\ {\scriptstyle \tilde{i}_j}\downarrow & & {\scriptstyle i_j}\downarrow & & {\scriptstyle \bar{i}_j}\downarrow \\ \tilde{Y} & \longrightarrow & Y & \longrightarrow & \bar{Y} \end{array},$$

where the horizontal maps are the orbit maps and the vertical ones are the inclusions. Since $\tilde{i}_j$ induces an isomorphism of the cohomology rings, the Gysin exact sequence associated to the orbit maps show that $i_j^*\colon H^*(Y)\to H^*(S(N_j))$ and $\bar{i}_j^*\colon H^*(\bar{Y})\to H^*(P(N_j))$ are both isomorphisms. Let $\pi_j\colon P(N_j)\to F_j$ be the projection. We denote $(\bar{i}_j^*)^{-1}(\pi_j^*\alpha_j)$ also by $\pi_j^*\alpha_j$. Then $(\pi_0^*\alpha_0)^{p+1}=0$ and $(\pi_1^*\alpha_1)^{q+1}=0$ in $H^*(\bar{Y})$.

Proposition 2.5. *Let $c(N_j)$ be the total Chern class of the normal bundle N_j of F_j. Then*

$$c(N_0)=(1-\alpha_0)^{q+1} \qquad \textit{and}$$

$$c(N_1)=(1+\alpha_1)^{p+1},$$

where α_j is the restriction of α to F_j.

Proof. Let γ be the first Chern class of the S^1 principal bundle $Y\to\bar{Y}$. First we show that $\gamma=-\pi_0^*\alpha_0+\pi_1^*\alpha_1$ in $H^2(\bar{Y})$.

Let $\tilde{\phi}_0$ and $\tilde{\phi}_1$ be two liftings of ϕ in η such that $\tilde{\phi}_0|p_0=1$, $\tilde{\phi}_0|p_1=t$ and $\tilde{\phi}_1|p_0=t^{-1}$, $\tilde{\phi}_1|p_1=1$ for $p_j\in F_j$. Then $\tilde{\phi}_j$ induces a semifree S^1 action on $\tilde{X}$ with the fixed point set $\tilde{F}_j$. The diagram

$$\begin{array}{ccccccc} Y & \longleftarrow & \tilde{Y} & \subsetneq & \tilde{X}-\tilde{F}_0 & \supsetneq & \tilde{F}_1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \bar{Y} & \overset{q_1}{\longleftarrow} & \dfrac{\tilde{Y}}{\tilde{\phi}_0} & \overset{k_1}{\subsetneq} & \dfrac{\tilde{X}-\tilde{F}_0}{\tilde{\phi}_0} & \overset{k_2}{\supsetneq} & F_1=\dfrac{\tilde{F}_1}{\tilde{\phi}_0} \end{array}$$

commutes, where $\supsetneq$ and $\subsetneq$ denote the inclusions and the other maps are the orbit maps and $/\tilde{\phi}_0$ denotes the orbit spaces. As was shown before, $H^*(\tilde{X}-\tilde{F}_0)=H^*(\tilde{F}_1)$ via the inclusion. Hence the Gysin sequence shows that $k_2^*\colon H^*(\tilde{X}-\tilde{F}_0/\tilde{\phi}_0)\cong H^*(F_1)$. The restriction of $\tilde{\phi}_0$ to $\tilde{F}_1$ coincides with the bundle S^1 action, so that the right hand square of the above diagram shows that the first Chern class of the S^1 bundle $\tilde{X}-\tilde{F}_0\to\tilde{X}-\tilde{F}_0/\tilde{\phi}_0$ is $(k_2^*)^{-1}\alpha_1$. Now $k_1^*(k_2^*)^{-1}\alpha_1=q_1^*\pi_1^*(\alpha_1)$ and it is the first Chern class of the S^1 bundle $\tilde{Y}\to\tilde{Y}/\tilde{\phi}_0$. But $q_1^*\gamma$ is also the first Chern class of the same bundle. Therefore $q_1^*\gamma=q_1^*\pi_1^*(\alpha_1)$. Now $\widetilde{S(N_0)}$ is a sphere bundle over $\tilde{F}_0$. Let $\pi_0'\colon\widetilde{S(N_0)}\to\tilde{F}_0$ be the projection. Each fibre of π_0' may be assumed to be invariant under $\tilde{\phi}_0$. Let $\tilde{\pi}_0\colon\frac{\widetilde{S(N_0)}}{\tilde{\phi}_0}\to\tilde{F}_0$ be the map induced by π_0'. The bundle S^1 action on $\tilde{X}$ induces free S^1 actions on $\frac{\tilde{Y}}{\tilde{\phi}_0}$, $\frac{\widetilde{S(N_0)}}{\tilde{\phi}_0}$ and $\tilde{F}_0$, and $\tilde{\pi}_0$ is equivariant with respect to these actions. There is a commutative diagram

$$\begin{array}{ccccc} \frac{\tilde{Y}}{\tilde{\phi}_0} & \supset & \frac{\widetilde{S(N_0)}}{\tilde{\phi}_0} & \xrightarrow{\tilde{\pi}_0} & \tilde{F}_0 \\ \Big\downarrow{\scriptstyle q_1} & & \Big\downarrow & & \Big\downarrow \\ \bar{Y} & \supset & P(N_0) & \xrightarrow{\pi_0} & F_0\ . \end{array}$$

Hence the first Chern class of the S^1 bundle $q_1\colon\tilde{Y}/\tilde{\phi}_0\to\bar{Y}$ is $\pi_0^*\alpha_0$, and $\pi_0^*\alpha_0$ generates the kernel of $q_1^*|H^2(\bar{Y})$. Therefore we see that $\gamma=\pi_1^*\alpha_1 \bmod \pi_0^*\alpha_0$. By replacing $\tilde{\phi}_0$ by $\tilde{\phi}_1$ and carrying a similar argument, we see that $\gamma=-\pi_0^*\alpha_0 \bmod \pi_1^*\alpha_1$ (we note that the restriction of $\tilde{\phi}_1$ on $\tilde{F}_0$ is the conjugation of the bundle S^1 action). Consequently we have $\gamma=-\pi_0^*\alpha_0+\pi_1^*\alpha_1$.

Now $((-\gamma)-\pi_0^*\alpha_0)^{q+1}=0$ and $((-\gamma)+\pi_1^*\alpha_1)^{p+1}=0$. By the Grothendieck's definition of the Chern classes, we obtain the result. Q.E.D.

§3. Index Formula for Semifree S^1 Action on coh CP^n

Let $\mathrm{Spin}(m)$ be the spinor-group (the simply connected double fold covering of $SO(m)$). Let $\lambda\colon\mathrm{Spin}(m)\to SO(m)$ be the covering map. The inverse image of the unit of $SO(m)$, $\lambda^{-1}(1)$, is a cyclic group of order 2. Thus $\mathbf{Z}_2$ $(\subset S^1)$ acts on $\mathrm{Spin}(m)$ by the right multiplication of $\lambda^{-1}(1)$. The complex spinor group $\mathrm{Spin}^c(m)$ is defined by $\mathrm{Spin}^c(m)=\mathrm{Spin}(m)$

$\times_{\mathbf{Z}_2} S^1$. Let $[g, t]$ denote an equivalence class in $\mathrm{Spin}^c(m)$ determined by $g \in \mathrm{Spin}(m)$ and $t \in S^1$. There are two fibre maps λ_1, λ_2,

$$\lambda_1 : \mathrm{Spin}^c(m) \longrightarrow SO(m)$$

$$\lambda_2 : \mathrm{Spin}^c(m) \longrightarrow S^1$$

such that $\lambda_1([g, t]) = \lambda(g)$ and $\lambda_2([g, t]) = t^2$. The fibre of λ_1 is S^1 and that of λ_2 is $\mathrm{Spin}(m)$. Thus $\mathrm{Spin}^c(m)$ acts on $\mathbf{R}^m$ and $\mathbf{C}$ from the left by λ_1 and λ_2 respectively.

Let X^m be an oriented C^∞-m-manifold. A Spin^c-structure on X is a principal $\mathrm{Spin}^c(m)$-bundle P such that $P \times_{\mathrm{Spin}^c(m)} \mathbf{R}^m$ is equivalent to TX, the tangent bundle over X. If P is a Spin^c-structure on X we have a complex line bundle over X, $\omega = P \times_{\mathrm{Spin}^c(m)} \mathbf{C}$. Let $c_1(\omega)$ be the first Chern class of ω. The mod 2 reduction of $c_1(\omega)$ is $w_2(X)$, the second Stiefel-Whitney class of X. It is well known that under the map, $P \to c_1(\omega)$, the set of the Spin^c-structures on X is in one-one correspondence with the set of those elements of $H^2(X; \mathbf{Z})$ whose mod 2 reduction is $w_2(X)$.

Let X^m be a Spin^c-manifold with a Spin^c-structure P. We assume that $H^1(X; \mathbf{Z}) = 0$. Let ϕ be an S^1 action on X. Then P has a left S^1 action ψ which is compatible with the right $\mathrm{Spin}^c(m)$ action on P and the diagram

$$\begin{array}{ccc} S^1 \times P & \xrightarrow{\psi} & P \\ \downarrow{\scriptstyle 1\times q} & & \downarrow{\scriptstyle q} \\ S^1 \times X & \xrightarrow{\phi} & X \end{array}$$

commutes, where q is the projection (T. Petrie [4]). Let $K^*_{S^1}(?)$ be the equivariant K-theory. According to T. Petrie ([4]), if $H^1(X; \mathbf{Z}) = 0$, there is an orientation class δ_{S^1} in $K^*_{S^1}(TX)$ and the Thom-isomorphism $K^*_{S^1}(X) \to K^*_{S^1}(TX)$ defined by $u \to u\delta_{S^1}$ for $u \in K^*_{S^1}(X)$. The index homomorphism $Id: K^*_{S^1}(TX) \to R(S^1)$ is defined, where $R(S^1)$ denotes the complex representation ring of S^1 ([1]). In [4], Part I, T. Petrie has given the explicit formula of $Id(u\delta_{S^1})$ by the terms of the normal bundles of the fixed point set and the representations of S^1 in its fibres. This formula is essential for our purpose, but the general formula is needless to us. Hence we will write down the formula only for our special case in the bellow.

We begin with the following lemma.

Lemma 3.1. *Let X^{2n} be a* coh CP^n *and let α be a generator of $H^2(X;\mathbf{Z})$. Then $w_2(X)=0$ if n is odd, and $w_2(X)=\bar{\alpha}$ if n is even where $\bar{\alpha}$ is the* mod 2 *reduction of α.*

Proof. Let $v_2(X)$ be the second Wu class of X. As X is orientable $w_1(X)=0$ and $w_2(X)=v_2(X)$. Now $v_2(X)\cup\bar{\alpha}^{n-1}=Sq^2\bar{\alpha}^{n-1}=(n-1)\bar{\alpha}^n$. Hence if n is odd $w_2(X)=0$ and if n is even $w_2(X)=\bar{\alpha}$. Q. E. D.

Now let X be a coh CP^n with a cohomology generator α. Let ϕ be a smooth semifree S^1 action on X. As was shown in §2, the fixed point set F has two connected components F_0^{2p} and F_1^{2q} with $p+q=n-1$. Let η be the line bundle associated to α. Under Assumption (*) in §2, there is a lifting action $\tilde{\phi}_0$ of ϕ in η such that $\tilde{\phi}_0|p_0=1$ and $\tilde{\phi}_0|p_1=t$ for $p_j\in F_j$. In §3 and §4, we consider η as an S^1 vector bundle by $\tilde{\phi}_0$. Let N_j be the normal bundle of F_j in X. Then by Proposition 2.5, we have $c(N_0)=(1-\alpha_0)^{q+1}$ and $c(N_1)=(1+\alpha_1)^{p+1}$.

Now we separate our consideration into two cases.

i) The case where n is odd.

As $w_2(X)=0$ by Lemma 3.1, we may take a Spin^c-structure on X, P, such that $c_1(\omega)$ is zero. Thus $\omega=P\times_{\mathrm{Spin}^c(2n)}\mathbf{C}$ is a trivial complex line bundle. As was mentioned before, there is a left S^1 action on P, ψ which is a lifting of ϕ and compatible with the right Spin$^c(2n)$ action on P. ψ induces an S^1 action $\bar{\psi}$ on ω which is a lifting of ϕ. Since ω is a trivial bundle we have $\bar{\psi}|p_0=\bar{\psi}|p_1=t^w$ for $p_j\in F_j$ and some integer w. By considering the representations of S^1 in the fibres of N_j, we see that w is odd (even) if p and q are even (odd respectively).

N_j is an oriented bundle by the complex structure given in §2 and we give an orientation to F_j by the orientations of X and N_j.

Proposition 3.2. *Let X^{2n}, F_0^{2p}, F_1^{2q} and η be as above. If n is odd, then $Id(\eta^k\delta_{S^1})\in R(S^1)=\mathbf{Z}[t,t^{-1}]$ is given by*

$$Id(\eta^k\delta_{S^1})=\langle e^{k\alpha_0}\hat{\mathscr{A}}(F_0)(t^{-\frac{1}{2}}e^{\frac{\alpha_0}{2}}-t^{\frac{1}{2}}e^{-\frac{\alpha_0}{2}})^{-(q+1)}t^{\frac{w}{2}}, [F_0]\rangle$$

$$+\langle e^{k\alpha_1}\hat{\mathscr{A}}(F_1)(t^{-\frac{1}{2}}e^{-\frac{\alpha_1}{2}}-t^{\frac{1}{2}}e^{\frac{\alpha_1}{2}})^{-(p+1)}t^{k+\frac{w}{2}}, [F_1]\rangle$$

where $\hat{\mathscr{A}}(F_j)$ *is the total* $\hat{\mathscr{A}}$*-class of* F_j *and* $[F_j]$ *is the homology fundamental class of* F_j *and* $\langle\ ,\ \rangle$ *denotes the evaluation. As was mentioned above,* $w\equiv 0 \pmod 2$ *if* $p, q\not\equiv 0 \pmod 2$ *and* $w\not\equiv 0 \pmod 2$ *if* $p, q\equiv 0 \pmod 2$. *In particular, for* $1\in S^1$

$$Id(\eta^k\delta_{S^1})(1)=\langle ch\,\eta^k\hat{\mathscr{A}}(X), [X]\rangle,$$

where ch denotes the Chern character.

ii) The case where n is even.

By Lemma 3.1, we may take a Spin^c-structure on X, P, such that $c_1(\omega)=\alpha$. Let ψ be an S^1 action on P which is a lifting of ϕ and is compatible with the right Spin$^c(2n)$ action on P. Let $\bar{\psi}$ be the S^1 action on ω induced by ψ. Since ω is equivalent to η as a complex line bundle, we have $\bar{\psi}|p_0=t^w$ and $\bar{\psi}|p_1=t^{w+1}$ for $p_j\in F_j$ and some integer w. By considering the S^1 representations in the fibres of N_j, we see that w is odd (even) if p is odd and q is even (p is even and q is odd respectively).

Proposition 3.3. *Let* X^{2n}, F_0^{2p}, F_1^{2q} *and* η *be as above. If* n *is even, then*

$$Id(\eta^k\delta_{S^1})=\langle e^{(k\alpha_0+\frac{\alpha_0}{2})}\hat{\mathscr{A}}(F_0)(t^{-\frac{1}{2}}e^{\frac{\alpha_0}{2}}-t^{\frac{1}{2}}e^{-\frac{\alpha_0}{2}})^{-(q+1)}t^{\frac{w}{2}}, [F_0]\rangle$$

$$+\langle e^{(k\alpha_1+\frac{\alpha_1}{2})}\hat{\mathscr{A}}(F_1)(t^{-\frac{1}{2}}e^{-\frac{\alpha_1}{2}}-t^{\frac{1}{2}}e^{\frac{\alpha_1}{2}})^{-(p+1)}t^{k+\frac{w+1}{2}}, [F_1]\rangle$$

where $w\equiv 0 \pmod 2$ *if* $p\equiv 0$ *and* $q\not\equiv 0 \pmod 2$ *and* $w\not\equiv 0 \pmod 2$ *if* $p\not\equiv 0$ *and* $q\equiv 0 \pmod 2$. *In particular for* $1\in S^1$,

$$Id(\eta^k\delta_{S^1})(1)=\langle ch\,\eta^k e^{\alpha/2}\hat{\mathscr{A}}(X), [X]\rangle.$$

Proposition 3.2 and Proposition 3.3 are obtained by applying Proposition 5.2 and 5.3 in [4], Part I, to our case, but details will be omitted.

§4. Proofs

In this section, we will prove Theorem 0.1 and Corollary 0.2 stated in §0.

Put $A(x)=(x/2)(\sinh x/2)^{-1}$.

Corollary 0.2 is an consequence of Theorem 0.1 and the fact that $\hat{\mathscr{A}}(CP^n)=A(c)^{n+1}$ and $f^*(c)$ is a cohomology generator of X, where c is the first Chern class of the canonical Hopf bundle H.

Now we proceed to the proof of Theorem 0.1. Our task is to show that $\hat{\mathscr{A}}(X)=A(\alpha)^{n+1}$. First we consider the linear semifree S^1 action Φ on CP^n defined by the equation

$$\Phi(t, [z_0,\dots, z_n])=[z_0,\dots, z_p, tz_{p+1},\dots, tz_n]$$

for $t\in S^1$ and $[z_0\dots z_n]\in CP^n$. The fixed point set is $F_0=CP^p\cup F_1=CP^q$. If we give a complex structure to N_j, the normal bundle of F_j, as in §2, then $N_0=\underbrace{\bar{H}+\cdots+\bar{H}}_{q+1}$ and $N_1=\underbrace{H+\cdots+H}_{p+1}$, where $\bar{H}$ is the conjugate bundle of H. Let c_j be the restriction of c to F_j. Then $\hat{\mathscr{A}}(F_0)=A(c_0)^{p+1}$ and $\hat{\mathscr{A}}(F_1)=A(c_1)^{q+1}$.

Lemma 4.1. *$\hat{\mathscr{A}}(X)=A(\alpha)^{n+1}$ if and only if $\hat{\mathscr{A}}(F_0)=A(\alpha_0)^{p+1}$ and $\hat{\mathscr{A}}(F_1)=A(\alpha_1)^{q+1}$.*

Proof. Assume that $\hat{\mathscr{A}}(X)=A(\alpha)^{n+1}$. By Proposition 2.5, we have $\hat{\mathscr{A}}(N_0)=A(\alpha_0)^{q+1}$ and $\hat{\mathscr{A}}(N_1)=A(\alpha_1)^{p+1}$. Let $i_j\colon F_j\subsetneq X$ be the inclusion. Then $i_j^*\hat{\mathscr{A}}(X)=\hat{\mathscr{A}}(F_j)\hat{\mathscr{A}}(N_j)$. Hence we have $\hat{\mathscr{A}}(F_0)=A(\alpha_0)^{p+1}$ and $\hat{\mathscr{A}}(F_1)=A(\alpha_1)^{q+1}$.

Conversely assume that $\hat{\mathscr{A}}(F_0)=A(\alpha_0)^{p+1}$ and $\hat{\mathscr{A}}(F_1)=A(\alpha_1)^{q+1}$. Let $f\colon X\to CP^n$ be a map such that $f^*(c)=\alpha$. Then $f^*H=\eta$. Now $Id(\eta^k\delta_{S^1})(1)=\lim_{t\to 1}Id(\eta^k\delta_{S^1})$, and the right hand side may be calculated by the formula of Proposition 3.2 or 3.3. Since F_0 is a coh CP^p and F_1 is a coh CP^q and α_j is a generator of $H^2(F_j;\mathbf{Z})$, the right hand side of the formula of Proposition 3.2 (or 3.3) formally coincides with that of the corresponding formula for the linear S^1 action Φ on CP^n and $Id(H^k\delta_{S^1})$. Therefore we have $Id(\eta^k\delta_{S^1})(1)=Id(H^k\delta_{S^1})(1)$ for each k.

But $Id(\eta^k\delta_{S^1})(1)=\langle ch\,\eta^k\hat{\mathscr{A}}(X), [X]\rangle$ and $Id(H^k\delta_{S^1})(1)=\langle ch\, H^k\hat{\mathscr{A}}(CP^n), [CP^n]\rangle$ $=f^*\langle ch\, H^k\hat{\mathscr{A}}(CP^n), [CP^n]\rangle=\langle ch\,\eta^k A(\alpha)^{n+1}, [X]\rangle$. Hence $\langle ch\,\eta^k\hat{\mathscr{A}}(X), [X]\rangle$ $=\langle ch\,\eta^k A(\alpha)^{n+1}, [X]\rangle$ for each k. Since $\{ch\,\eta^k, (k=0, 1,\dots, n)\}$ gives an additive base of $H^{**}(X; \mathbf{Q})$, $\hat{\mathscr{A}}(X)$ equals $A(\alpha)^{n+1}$. Q.E.D.

Now we will give the proof of Theorem 0.1 only when n is odd. The proof in the case with n even is almost parallel, and we will omit it. Thus from now on, we assume that n is odd.

Put $S(\alpha)=\hat{\mathscr{A}}(X)A(\alpha)^{-(n+1)}$. $S(\alpha)$ is a power series of α^2 with rational coefficients and its leading term is 1, that is $S(\alpha)=1+b_1\alpha^2+b_2\alpha^4+\cdots$, $b_i\in Q$. Our purpose is to show that $S(\alpha)$ is equal to 1. Now $\hat{\mathscr{A}}(X)$ $=S(\alpha)A(\alpha)^{n+1}$ and $\hat{\mathscr{A}}(F_j)=i_j^*\hat{\mathscr{A}}(X)\hat{\mathscr{A}}(N_j)^{-1}$, so that we have $\hat{\mathscr{A}}(F_0)$ $=S(\alpha_0)A(\alpha_0)^{p+1}$ and $\hat{\mathscr{A}}(F_1)=S(\alpha_1)A(\alpha_1)^{q+1}$, where $S(\alpha_j)=1+b_1\alpha_j^2+b_2\alpha_j^4$ $+\cdots$.

Lemma 4.2. *$S(\alpha)$ equals 1 if and only if $S(\alpha_0)$ and $S(\alpha_1)$ both equal 1.*

Proof. This is a restatement of Lemma 4.1. Q.E.D.

If $\hat{\mathscr{A}}(F_0)$ and $\hat{\mathscr{A}}(F_1)$ are replaced by $A(\alpha_0)^{p+1}$ and $A(\alpha_1)^{q+1}$ respectively in the formula of Proposition 3.2, the resulting formula (denoted by $B(k)(t)$) formally coincides with the formula for the linear S^1 action on CP^n and $Id(H^k\delta_{S^1})$, possibly up to a factor t^N for some N. Therefore this resulting formula is a finite Laurent series of t. On the other hand $Id(\eta^k\delta_{S^1})$ is a finite Laurent series, hence $K(k)(t)=Id(\eta^k\delta_{S^1})-B(k)(t)$ is a finite Laurent series. Put $S'(\alpha)=S(\alpha)-1=b_1\alpha^2+b_2\alpha^4+\cdots$ and $S'(\alpha_j)=S(\alpha_j)-1=b_1\alpha_j^2+b_2\alpha_j^4+\cdots$. Then $\hat{\mathscr{A}}(F_0)-A(\alpha_0)^{p+1}=S'(\alpha_0)A(\alpha_0)^{p+1}$ and $\hat{\mathscr{A}}(F_1)-A(\alpha_1)^{q+1}=S'(\alpha_1)A(\alpha_1)^{q+1}$, and $K(k)(t)$ is obtained by replacing $\hat{\mathscr{A}}(F_0)$ and $\hat{\mathscr{A}}(F_1)$ by $S'(\alpha_0)A(\alpha_0)^{p+1}$ and $S'(\alpha_1)A(\alpha_1)^{q+1}$ respectively in the formula of Proposition 3.2. Now we slightly deform the formula of $K(k)(t)$ as follows;

$$(3.1)' \qquad K(k)(t)=\langle S'(\alpha_0)G_0(\alpha_0)(1-te^{-\alpha_0})^{-(q+1)}t^{\frac{w+q+1}{2}}, [F_0^{2p}]\rangle$$
$$+\langle S'(\alpha_1)G_1(\alpha_1)(1-te^{\alpha_1})^{-(p+1)}t^{k+\frac{w+p+1}{2}}, [F_1^{2q}]\rangle$$

where $G_0(\alpha_0)=e^{\left(k-\frac{q+1}{2}\right)\alpha_0}A(\alpha_0)^{p+1}$, and

$$G_1(\alpha_1)=e^{\left(k+\frac{p+1}{2}\right)\alpha_1}A(\alpha_1)^{q+1}.$$

We separate the proof into the two cases whether $p=q$ or not.

Case 1. $p\neq q$.

Let ${}_mC_r=m!/(m-r)!r!$ be the binomial coefficient. Using the expansion $(1-x)^{-m}=\sum_{r=0}^{\infty}{}_{m+r-1}C_{m-1}x^r$, we have

$$(1-te^{-\alpha_0})^{-(q+1)}=\sum_{r=0}^{\infty}{}_{q+r}C_q t^r(1-t)^{-(q+r+1)}(e^{-\alpha_0}-1)^r$$

$$(1-te^{\alpha_1})^{-(p+1)}=\sum_{r=0}^{\infty}{}_{p+r}C_p t^r(1-t)^{-(p+r+1)}(e^{\alpha_1}-1)^r$$

If $p\leqq 1$ and $q\leqq 1$, then $S'(\alpha_0)=S'(\alpha_1)=0$ and there is nothing to be proved. Hence we assume that $p\geqq 2$ or $q\geqq 2$. Now we consider $K(0)(t)$. The possible maximal power of $(1-t)^{-1}$ occurring in $K(0)(t)$ is $(1-t)^{-(p+q-1)}$. The term containing it in $K(0)(t)$ is $b_1(1-t)^{-(p+q-1)}((-1)^{p-2}{}_{p+q-2}C_q t^u+{}_{p+q-2}C_p t^v)$, where $u=p-2+\frac{w+q+1}{2}$ and $v=q-2+\frac{w+p+1}{2}$. Since $K(0)(t)$ is a finite Laurent series, if $b_1\neq 0$, then $((-1)^{p-2}{}_{p+q-2}C_q t^u+{}_{p+q-2}C_p t^v)$ must be divisible by $(1-t)$. But this is impossible if $p\neq q$. Therefore $b_1=0$. Hence the possible maximal power of $(1-t)^{-1}$ occurring in $K(0)(t)$ is $(1-t)^{-(p+q-3)}$. If $p\geqq 3$ or $q\geqq 3$, then a similar procedure shows that $b_2=0$ and so on. Consequently we have $S'(\alpha_0)=S'(\alpha_1)=0$, hence $S(\alpha)=1$ by Lemma 4.2.

Case 2. $p=q$.

If $p\leqq 1$, nothing is to be proved. Assume that $p\geqq 2$. We consider $K(1)(t)$. The possible maximal power of $(1-t)^{-1}$ occurring in $K(1)(t)$ is $(1-t)^{-(2p-1)}$. The term containing it in $K(1)(t)$ is $b_1(1-t)^{-(2p-1)}{}_{2p-2}C_p((-1)^{p-2}t^u+t^{u+1})$, where $u=p-2+\frac{w+p+1}{2}$.

First assume that p is even. $((-1)^{p-2}t^u+t^{u+1})$ is not divisible by $(1-t)$. Hence if b_1 is not zero, then $K(1)(t)$ cannot be a finite Laurent series. Therefore $b_1=0$ and $(1-t)^{-(2p-3)}$ is the possible maximal power of $(1-t)^{-1}$. If $p\geqq 4$, a similar procedure shows that $b_2=0$, and so on.

Next assume that p is odd. Then $((-1)^{p-2}t^u+t^{u+1})=-(1-t)t^u$. The possible maximal power of $(1-t)^{-1}$ occurring in $K(1)(t)$ is $(1-$

$t)^{-(2p-2)}$. The term containing it in $K(1)(t)$ is

$$b_1(1-t)^{-(2p-2)}\left(-{}_{2p-2}C_p t^u-\frac{p-1}{2}{}_{2p-3}C_p t^{u-1}+\frac{p+3}{2}{}_{2p-3}C_p t^u\right).$$

The factor in the bracket is not divisible by $(1-t)$. Hence if b_1 is not zero, then $K(1)(t)$ is not a finite Laurent series. Therefore $b_1=0$. If $p\geqq 5$, a similar procedure shows that $b_2=0$, and so on.

This completes the proof of Theorem 0.1 in the case when n is odd. When n is even, necessarily $p\neq q$. Using the formula of Proposition 3.3, a similar argument as Case 1 in the above may be carried out.

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