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Cohomology mod 2 of the Classifying Space of $PSp(4n+2)$

By

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§0. Introduction

As is well known, the symplectic group $Sp(m)$ of m variables has the center isomorphic to Z_2. The quotient of $Sp(m)$ by the center is also a compact, connected Lie group, denoted by $PSp(m)$, and called the projective symplectic group.

Since $H^*(PSp(2m); Z_2)$ is not primitively generated (cf. [1]), it seems to be difficult to determite $H^*(BPSp(2m); Z_2)$. In this paper we will determine the module structure of the cohomology mod 2 of the classifying space $BPSp(4n+2)$ of $PSp(4n+2)$ by making use of the Eilenberg-Moore spectral sequence $\{E_r(PSp(4n+2)), d_r\}$, which has the following properties;

(1) $$E_2=\mathrm{Cotor}^A(Z_2, Z_2) \quad \text{for} \quad A=H^*(PSp(4n+2); Z_2),$$

(2) $$E_\infty=\mathcal{G}_r H^*(BPSp(4n+2); Z_2).$$

Our result is

Theorem 4.8. *As a module*

$$H^*(BPSp(4n+2); Z_2)\cong Z_2[y_2, y_3, y_5, v_{16l+16}, a_4, a(I)]/R,$$

where $1\leqq l\leqq 2n$ *and* I *runs over all sequences of integers satisfying* (2.3) *and* R *is the ideal generated by* $y_5a(I)$, $a(I)^2+\sum_{j=1}^{r} v_{16i_1+16}\cdots a^2_{8i_j+4}\cdots v_{16i_r+16}$ *and* $a(I)a(J)+\Sigma f_i a(I_i)$. (For details see Theorem 2.4).

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The paper is organized as follows. In §1 we define Lie groups $G(m)$ and determine the Hopf algebra structure of their cohomology mod 2. In §2 we calculate $\mathrm{Cotor}^A(Z_2, Z_2)$ for $A=H^*(PSp(4n+2); Z_2)$ by making use of the twisted tensor product ([4], [5]). Various subgroups of $Sp(2m)$ and $PSp(2m)$ are considered in §3. We use these groups to determine the Poincaré series of some subalgebras of $H^*(BPSp(4n+2); Z_2)$. The main purpose of the paper, namely a proof of collapsing of the Eilenberg-Moore spectral sequence for $PSp(4n+2)$ with Z_2-coefficient, is shown in §4. Some algebra relations in $H^*(BPSp(4n+2); Z_2)$ are given in §5. The next section, §6, is a sort of appendix, in which $H^*(BPSp(2n+1); Z_2)$ is determined. This is one of the results in [6]. A key proposition used in §4 to prove the collapsing of the Eilenberg-Moore spectral sequence is proved in the last section, §7.

Throughout the paper X^n stands for the product $X\times\cdots\times X$ of n objects X in the category whenever the product is defined. For a homomorphism $f: H\to G$ between two topological groups we use the same symbol $f: BH\to BG$ for the induced map. $\bar{\phi}$ denotes the reduced form of the coalgebra structure of $H^*(G; Z_2)$, $\bar{\phi}: \tilde{H}^*(G; Z_2)\to\tilde{H}^*(G; Z_2)\otimes\tilde{H}^*(G; Z_2)$, induced from the multiplication on the group G. Further, $H^*(X)$ denotes $H^*(X; Z_2)$ unless otherwise stated. The symbol Z_2 denotes not only the cyclic group of order 2 but also the prime field of characteristic 2 by abuse of notation. Let $\sum_{i=0}^{\infty} a_i t^i$ and $\sum_{i=0}^{\infty} b_i t^i \in \mathbb{Z}[[t]]$ then $\Sigma a_i t^i \gg \Sigma b_i t^i$ means $a_i \geqq b_i$ for any $i\geqq 0$.

§1. Hopf Algebra Structures of Certain Semi-simple Lie Groups

Notation. For simplicity we denote by $(a_1,\dots, a_n)$ the diagonal matrix $\begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \in Sp(n)$ and so $(1,\dots, 1)=I_n$ is the unit matrix. We also denote

$$\Delta(n)=\{\pm I_n\}\subset Sp(n).$$

Note that $\Delta(n)$ is the center of $Sp(n)$.

The following propositions are well known ([2]).

Proposition 1.1. (1) $H^*(Sp(n); Z)\cong\Lambda(\tilde{e}_3, \tilde{e}_7,\dots, \tilde{e}_{4n-1})$, *where* $\deg\tilde{e}_i=i$

and $\tilde{e}_{4j-1}$ *is universally transgressive.*

(2) $H^*(BSp(n); Z) \cong Z[q_1, \dots, q_n]$, *where* q_i *is the i-th universal symplectic Pontrjagin class which is the universal transgression image of* $\tilde{e}_{4i-1}$.

Proposition 1.2. *The Serre spectral sequence for the fibering*

$$Sp(nm)/(Sp(n))^m \longrightarrow B(Sp(n))^m \xrightarrow{i} BSp(nm)$$

with Z-coefficient collapses for $n, m > 0$.

Proof. $H_*(Sp(nm); Z)$ and $H_*((Sp(n))^m; Z)$ are torsion free and the rank of $Sp(nm)$ and $(Sp(n))^m$ are same. So $H^{odd}(Sp(nm)/(Sp(n))^m; Z) = 0$ (cf. § 13 of [3]) and by Proposition 1.1 $H^{odd}(BSp(nm); Z) = 0$. So we can easily get the result. Q. E. D.

Note that

$$\mathrm{Im}\, i^* = H^*(B(Sp(1))^m; Z)^{\mathfrak{S}_m} = Z[t_1, \dots, t_m]^{\mathfrak{S}_m},$$

where $\deg t_i = 4$, $\mathfrak{S}_m$ is the symmetric group operating on $H^*(B(Sp(1))^m; Z) = Z[t_1, \dots, t_m]$ as permutation of t_i's and $Z[t_1, \dots, t_m]^{\mathfrak{S}_m}$ is the invariant subalgebra under $\mathfrak{S}_m$. Note that $Z[t_1, \dots, t_m]^{\mathfrak{S}_m} \cong Z[\sigma_1, \dots, \sigma_m]$, where σ_i is the i-th elementary symmetric function of t_i's.

Notation. $G(m) = (Sp(1))^m / \Delta(m)$.

Remark that this is a compact, connected Lie group, where we have

(1.3) $\Delta(j) \cong Z_2$, *and hence*

$$H^*(B\Delta(j)) \cong Z_2[\mu] \quad \text{with } \deg \mu = 1.$$

Recall that

(1.4) $$H^*((Sp(1))^m) \cong \Lambda(\tilde{\alpha}_1, \dots, \tilde{\alpha}_m),$$

(1.5) $$H^*((BSp(1))^m) \cong Z_2[t_1, \dots, t_m],$$

where $\deg \tilde{\alpha}_i = 3$ *and* $\deg t_i = 4$.

The natural inclusion $i: \Delta(m) \to (Sp(1))^m$ induces the homomorphism $i^*: H^*((BSp(1))^m) \to H^*(B\Delta(m))$, where we have $i^*(t_r) = \mu^4$ for $1 \leqq r \leqq m$.

Therefore the Serre spectral sequence for the fibering $(Sp(1))^m \xrightarrow{\pi} G(m) \to B\Delta(m)$ yields

Proposition 1.6. *$H^*(G(m))=Z_2[\mu]/(\mu^4)\otimes\Delta(\alpha_1,\dots,\alpha_{m-1})$, where $\deg\mu=1$, $\deg\alpha_i=3$, and there holds*

$$\pi^*(\alpha_i)=\tilde{\alpha}_i+\tilde{\alpha}_m \qquad \text{for} \quad 1\leqq i\leqq m-1.$$

Notation. Let

$$p_i\colon G(m) \longrightarrow G(2) \qquad (1\leqq i\leqq m-1)$$

be the homomorphism induced by the correspondence

$$(\alpha_1,\dots,\alpha_m)\longrightarrow(\alpha_i,\alpha_m), \quad \alpha_i\in Sp(1)$$

and put

$$p=\prod_1^{m-1} p_i\colon G(m) \longrightarrow (G(2))^{m-1}.$$

For simplicity we express for the case $m=2$:

$$H^*(G(2))=Z_2[\mu]/(\mu^4)\otimes\Delta(\alpha).$$

Then we may suppose

$$\alpha_i=p_i^*(\alpha) \qquad \text{for} \quad 1\leqq i\leqq m-1.$$

Lemma 1.7. *In* Proposition 1.6 *the elements α_i may be chosen to be universally transgressive. Similarly for above α.*

Proof. This is equivalent to $\tau(\mu)\cdot\tau(\mu^2)\neq 0$. Consider the diagram

$$\begin{array}{ccc} Sp(1) & \xrightarrow{\Delta_m} & (Sp(1))^m \\ \downarrow & & \downarrow \\ SO(3)=PSp(1) & \xrightarrow{\bar{\Delta}_m} & G(m) \end{array}$$

where Δ_m is the diagonal map, $\bar{\Delta}_m$ is the induced one and the vertical arrows are the natural projections. This diagram induces the commutative one

$$\begin{array}{ccc} H^*(BSp(1)) & \xleftarrow{\Delta_m^*} & H^*((BSp(1))^m) \\ \uparrow & & \uparrow \\ H^*(BSO(3)) & \xleftarrow[\bar{\Delta}_m^*]{} & H^*(BG(m)) \end{array}$$

where there hold $\bar{\Delta}_m^*(\tau(\mu^{i-1}))=w_i$, the i-th Stiefel-Whitney class, for $i=2, 3$. Therefore $\tau(\mu)\cdot\tau(\mu^2)\neq 0$. Q.E.D.

Remark 1.8. Note that $\alpha^2=0$, since α^2 is primitive and since there are no non-trivial primitive elements at this degree.

Theorem 1.9. *$H^*(G(m))\cong Z_2[\mu]/(\mu^4)\otimes\Lambda(\alpha_1,\dots,\alpha_{m-1})$, where $\deg\mu=1$ and $\deg\alpha_i=3$. Further there hold*

$$\bar{\phi}(\mu)=\bar{\phi}(\alpha_i)=0, \qquad \textit{for} \quad 1\leqq i\leqq m-1.$$

The Borel's theorem ([2] or §9 (B) of [3]) and Lemma 1.7 immediately give rise to

Corollary 1.10.

$$H^*(BG(m))=Z_2[y_2, y_3, x_1,\dots, x_{m-1}],$$

where $\deg y_i=i$, $\deg x_j=4$ *for* $1\leqq j\leqq m-1$ *and* $Sq^1y_2=y_3$.

§2. Determination of $\mathrm{Cotor}^A(Z_2, Z_2)$

Recall from [1]

Proposition 2.1 (Baum-Browder).

$$H^*(PSp(4n+2))\cong Z_2[t]/(t^8)\otimes\Lambda(e_3, e_{11}, e_{15},\dots, e_{16n+7}),$$

where $\bar{\phi}(e_{8j-5})=0$ $(1\leqq j\leqq 2n+1)$,

$$\bar{\phi}(e_{8j+7})=e_{8j+3}\otimes t^4 \qquad (1\leqq j\leqq 2n).$$

Notation. $A=H^*(PSp(4n+2))$.

(See [8] for the details of the following.)

Regard A as a coalgebra over Z_2. Let L be a Z_2-submodule of

$A^+=\sum_{i>0} H^i(PSp(4n+2))$ generated by $\{t, t^2, t^4, e_{8i-5}, e_{8j+7}\}$, $1\leqq i\leqq 2n+1$, $1\leqq j\leqq 2n$. Let $s: L\to sL$ be the suspension and denote by $sL=\{y_2', y_3', y_5', a_{8i-4}', b_{8j+8}'\}$ the corresponding elements. Let $\iota: L\to A$ be the inclusion and $\theta: A\to L$ be the projection such that $\theta\circ\iota=1_L$. Define $\bar{\theta}$: $A\to sL$ by $\bar{\theta}=s\circ\theta$ and $\bar{\iota}: sL\to A$ by $\bar{\iota}=\iota\circ s^{-1}$. Let I be the two sided ideal of the free tensor algebra $T(sL)$ generated by $\mathrm{Im}\,(\psi\circ(\bar{\theta}\otimes\bar{\theta})\circ\bar{\phi})\circ\mathrm{Ker}\,\bar{\theta}$, where ψ is the product of $T(sL)$. Then $\bar{X}=T(sL)/I$ is isomorphic to $Z_2[y_2', y_3', y_5', a_{8i-4}', b_{8j+8}']$, $1\leqq i\leqq 2n+1$, $1\leqq j\leqq 2n$. The map $\bar{d}=\cdot\circ(\bar{\theta}\otimes\bar{\theta})\circ\bar{\phi}\circ\bar{\iota}$ on sL can be extended over $\bar{X}$ satisfying $\bar{d}\circ\bar{d}=0$. Thus $\bar{X}$ is a differential algebra.

Remark 2.2. By definition

$$\bar{d}y_k'=0 \quad \text{for} \quad k=2, 3, 5,$$

$$\bar{d}a_{8i-4}'=0 \quad \text{for} \quad 1\leqq i\leqq 2n+1,$$

$$\bar{d}b_{8j+8}'=y_5'a_{8j+4}' \quad \text{for} \quad 1\leqq j\leqq 2n.$$

Then we construct the twisted tensor product $X=A\otimes\bar{X}$ with respect to $\bar{\theta}$, that is, $X=A\otimes\bar{X}$ is a differential A-comodule with the differential operator d such that $d|1\otimes\bar{X}=\bar{d}$ and

$$d(t^i\otimes 1)=1\otimes y_{i+1}' \quad \text{for} \quad i=1, 2, 4,$$

$$d(e_{8i-5}\otimes 1)=1\otimes a_{8i-4}' \quad \text{for} \quad 1\leqq i\leqq 2n+1,$$

$$d(e_{8i+7}\otimes 1)=1\otimes b_{8j+8}'+e_{8j+3}\otimes y_5' \quad \text{for} \quad 1\leqq j\leqq 2n.$$

Then it is easy to see that X is acyclic and hence $X=A\otimes\bar{X}$ is an injective resolution of Z_2 over A. By definition

$$H^*(\bar{X}: \bar{d})=\mathrm{Cotor}^A(Z_2, Z_2).$$

Let $I=(i_1,\dots, i_r)$ be a sequence of integers satisfying

(2.3) $$1\leqq r\leqq 2n \quad \text{and} \quad 1\leqq i_1<\cdots<i_r\leqq 2n.$$

Put $a'(I)=\frac{1}{y_5'}\bar{d}(b_{8i_1+8}'\dots b_{8i_r+8}')$. Clearly $d(a'(I))=0$.

Theorem 2.4. *Let* $A=H^*(PSp(4n+2))$. *Then as an algebra*

$$\mathrm{Cotor}^A(Z_2, Z_2) \cong Z_2[\bar{y}_2, \bar{y}_3, \bar{y}_5, \bar{v}_{16l+16}, \bar{a}_4, \bar{a}(I)]/R,$$

where $1 \leqq l \leqq 2n$ *and* I *runs over all sequences satisfying* (2.3). *Further,* R *is the ideal generated by* $\bar{y}_5\bar{a}(I)$, $\bar{a}(I)^2 + \sum_{j=1}^{r} \bar{v}_{16i_1+16} \cdots \bar{a}^2_{8i_j+4} \cdots \bar{v}_{16i_r+16}$ *and* $\bar{a}(I)\bar{a}(J) + \sum_i f_i \bar{a}(I_i)$, *where* f_i *is a polynomial of* $\bar{y}_2, \bar{y}_3, \bar{y}_5$ *and* $\bar{v}_{16l+16}$.

Remark 2.5. (1) $\bar{y}_i$, $\bar{v}_{16l+16}$, $\bar{a}_4$ and $\bar{a}(I)$ are represented by y'_i, b'^2_{8l+8}, a'_4 and $a'(I)$ respectively.

(2) $a'(i) = a_{8i+4}$.

We call $\bar{y}_5\bar{a}(I) = 0$ the relation of type I and $\bar{a}(I)\bar{a}(J) + \cdots = 0$ the relation of type II.

§3. Subgroups of $Sp(2m)$ and $PSp(2m)$

In this section we consider various subgroups of $Sp(2m)$ and $PSp(2m)$.

Notation. For simplicity we denote by $(A_1, \ldots, A_k)$ the matrix

$$\begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_k \end{bmatrix}$$

for $A_i \in Sp(2)$.

Definition 3.1. $\varepsilon_i = \pm I_2$,

$$\begin{aligned} \tilde{H}(m) &= \{(\varepsilon_1 A, \ldots, \varepsilon_m A);\ A \in Sp(2)\}, \\ \tilde{I}(m) &= \{(\varepsilon_1, \ldots, \varepsilon_m)\}, \\ \tilde{J}(m) &= \{(\varepsilon_1, \ldots, \varepsilon_{m-1}, I_2)\}, \\ \tilde{K}(m) &= \{(A, \ldots, A);\ A \in Sp(2)\}. \end{aligned}$$

Lemma 3.2.

(1) $\tilde{H}(m) \supset \tilde{I}(m) \supset \tilde{J}(m) \supset \Delta(2m)$,

(2) $\tilde{J}(m) \cap \tilde{K}(m) = I_{2m}$,

(3) $\tilde{I}(m) \cong (Z_2)^m$ *and* $\tilde{J}(m) \cong (Z_2)^{m-1}$,

(4) $\tilde{I}(m) \subset \mathrm{Center}\, \tilde{H}(m)$.

Notation. $M(m)=\tilde{M}(m)/\Delta(2m)$ for $M=H$, I or K.

Lemma 3.3. (1) *$\tilde{K}(m)$ is a closed, normal subgroup of $\tilde{H}(m)$ and isomorphic to $Sp(2)$,*

(2) $\tilde{H}(m)\cong\tilde{K}(m)\times\tilde{J}(m)$ *as Lie groups,*

(3) $H(m)\cong K(m)\times\tilde{J}(m)$ *as Lie groups,*

(4) $K(m)\cong PSp(2)\cong SO(5)$.

The proofs of these two lemmas are easy.

Let $i_1:\tilde{I}(2m)\to Sp(2m)$ be the natural inclusion.

Lemma 3.4. $\mathrm{Ker}\, i_1^*=(q_1, q_3,\dots, q_{2m-1})$, *where* $i_1^*: H^*(BSp(2m))\to H^*(B\tilde{I}(m))$ *and* q_i's *are generators in* Proposition 1.1.

Proof. Let $s_i\in H^1(B\tilde{I}(m))$ be the generator corresponding to the dual element of

$Z_2\cong\{(I_2,\dots, I_2, \varepsilon_i, I_2,\dots, I_2)\}$. Then

$$H^*(B\tilde{I}(m))\cong Z_2[s_1,\dots, s_m].$$

Consider the sequence

$$i_1: B\tilde{I}(m) \xrightarrow{i_2} BSp(1)^{2m} \longrightarrow BSp(2m),$$

where i_2 is the map induced by the natural inclusion. Recall (cf. § 1) that

$$H^*(B(Sp(1)^{2m}))\cong Z_2[t_1,\dots, t_{2m}].$$

Clearly

$$i_2^*(t_{2j-1})=i_2^*(t_{2j})=s_j^4,$$

from which follows the lemma. Q.E.D.

Lemma 3.5. (1) $H^*(B\tilde{H}(m))=H^*(B\tilde{K}(m))\otimes H^*(B\tilde{J}(m))=Z_2[\bar{q}_1, \bar{q}_2, \alpha_1, \dots, \alpha_{m-1}]$, *where* $\mathrm{Ker}\, j_1^*=(\bar{q}_1)$ *for the natural map* $j_1: B\tilde{I}(m)\to B\tilde{H}(m)$.

The proof follows from the observation $j_1=id\times\lambda_m: \tilde{J}(m)\times\Delta(2m)\to\tilde{J}(m)\times\tilde{K}(m)$, where $\lambda_m: \Delta(2m)\to\tilde{K}(m)$ is the natural map

Lemma 3.6. $H^*(BH(m))=H^*(BK(m))\otimes H^*(B\tilde{J}(m))=Z_2[\bar{w}_2, \bar{w}_3, \bar{w}_4, \bar{w}_5, \alpha_1,\dots, \alpha_{m-1}]$.

Remark 3.7. For the projection $\pi_2: \tilde{H}(m)\to H(m)$, we have

$$\mathrm{Ker}\,\pi_2^* = (\bar{w}_2, \bar{w}_3, \bar{w}_5),$$
$$\pi_2^*(\bar{w}_4)=\bar{q}_1.$$

§4. The Eilenberg-Moore Spectral Sequence

Consider the following commutative diagram

$$\begin{array}{ccccc} H^*((BSp(1))^{4n+2}) & \xleftarrow{k^*} & H^*(BSp(4n+2)) & \xrightarrow{k'^*} & H^*((B\tilde{H}(2n+1)) \\ \uparrow{\scriptstyle \pi_1^*} & & \uparrow{\scriptstyle \pi_0^*} & & \uparrow{\scriptstyle \pi_2^*} \\ H^*(BG(4n+2)) & \xleftarrow{\bar{k}^*} & H^*(BPSp(4n+2)) & \xrightarrow{\bar{k}'^*} & H^*(BH(2n+1)) \end{array}$$

where π_i is the natural projection for $i=0, 1, 2$ ($\pi_0=\pi$) and k and k' (resp. $\bar{k}$ and $\bar{k}'$) are the natural inclusions (resp. the induced ones).

Lemma 4.1. *$H^*(BPSp(4n+2))\cong Z_2[y_2, y_3, y_5, a_4]$ for $*\leq 5$, where $y_5=Sq^2y_3+y_2y_3$, $y_3=Sq^1y_2$,* $\deg a_4=4$, $\deg y_i=i$.

Proof. Recall that

$$H^*(PSp(4n+2))\cong Z_2[t]/(t^8)\otimes\Lambda(e_3) \qquad \text{for } *\leqq 4.$$

Then $y_{i+1}=\tau(t^i)$ for $i=1, 2, 4$. Further, e_3 is universally transgressive and $\tau(e_3)=a_4$, since $\bar{k}^*(y_2y_3)=\bar{k}^*(y_2)\bar{k}^*(y_3)=y_2y_3\neq 0$ (cf. the proof of Lemma 3.4). Q.E.D.

Lemma 4.2. (1) *$\bar{k}^*(y_i)=y_i$ for $i=2, 3$,*

(2) $\bar{k}^*(y_5)=0$,

(3) *$\pi_i^*(y_j)=0$ for $i=0, 1, 2$ and any j.*

The proof is clear.

Consider the following statement:

(4.3.h) *the Eilenberg-Moore spectral sequence for $PSp(4n+2)$ with Z_2-coefficient collapses for degrees $\leqq h$.*

Notation. Under the assumption (4. 3. h) we denote by y_i, a_4, $a(I)$ and v_{16i} of $H^*(BPSp(4n+2))$ the elements expressed by $\bar{y}_i$, $\bar{a}_4$, $\bar{a}(I)$ and $\bar{v}_{16i}$ of $\mathrm{Cotor}^A(Z_2, Z_2)$ respectively for degrees $\leqq h$.

Definition. Let $P_2(h)$ be the subalgebra of $H^*(BPSp(4n+2))$ generated by $\{y_j\ (j=2, 3, 5), a_4, v_{16i}\ (16i\leqq h)\}$ and $P_1(h)$ the subalgebra generated by $\{y_j\ (j=2, 3), a_4, a(I)\ (\deg a(I)\leqq h), v_{16i}\ (16i\leqq h)\}$. Denote by $\bar{P}_i(h)$ the corresponding subalgebra of $\mathrm{Cotor}^A(Z_2, Z_2)$.

Remark 4.4. (1) In general,

$$PS(\bar{P}_i(h))\gg PS(P_i(h)) \qquad \text{for} \quad h\geqq 6.$$

(2) If $PS(\bar{P}_i(h))=PS(P_i(h))$ for $i=1, 2$, then $(4, 3, h+1)$ is true. (Of course $(4, 3, h+1)$ implies $(4, 3, h)$.)

(3) $(4, 3, h)$ is true for $h=5$ by Lemma 4.1.

Let $m=2n+1$.

Proposition 4.5. *Under the assumption* $(4, 3, h)$

(1) $\bar{k}^*|P_1(h)$ *is injective,*

(2) $\bar{k}'^*|P_2(h)$ *is injective.*

The proof will be given in the last section, § 7.

Corollary 4.6. *Under the assumption* $(4, 3, h)$

$$PS(P_i(h))\gg PS(\bar{P}_i(h)) \qquad \textit{for} \quad i=1, 2.$$

The proof is clear from Proposition 4.5.

Thus we have proved

Theorem 4.7. *The Eilenberg-Moore spectral sequence for* $PSp(4n+2)$ *with* Z_2*-coefficient collapses.*

As an immediate corollary we have

Theorem 4.8. *As a module*

$$H^*(BPSp(4n+2))\cong \mathrm{Cotor}^A(Z_2, Z_2)$$
$$\cong Z_2[y_2, y_3, y_5, v_{16i+16}, a_4, a(I)]/R,$$

where $1 \leqq l \leqq 2n$ *and* I *runs over all sequences satisfying* (2.3) *and* R *is the ideal generated by* $y_5 a(I)$, $a(I)^2 + \sum_{j=1}^{r} v_{16i_1+16} \cdots a_{8i_j+4}^2 \cdots v_{16i_r+16}$ *and* $a(I)a(J) + \Sigma f_i a(I_i)$. (See Theorem 2.4 for the notations).

§5. Remark on Some Algebra Relations

The following is Theorem of [6]

Lemma 5.1. *The homomorphism*

$$\tilde{\Delta}_{2n+1}^{*}: H^*(BPSp(4n+2)) \longrightarrow H^*(BPSp(2))$$

is an isomorphism for $i \leqq 10$ *and a monomorphism for* $i \leqq 11$.

So we have the isomorphism as algebras over A_2:

$$H^*(BPSp(4n+2)) \cong H^*(BPSp(2)) = H^*(BSO(5)) \qquad \text{for } * \leqq 10. \tag{5.2}$$

Notation. Denote by y_2, y_3, a_4, y_5 the image of w_2, w_3, w_4, w_5 under this isomorphism respectively, where we have $w_5 = Sq^2 w_3 + w_2 w_3$, $w_3 = Sq^1 w_2$ by the Wu-formula. (This assures us that choosing the generators of $H^*(BPSp(4n+2))$ in this way does not contradict to those in the previous section.)

By a similar argument to that in §5 of [8] we can show

Proposition 5.3. (1) *In* $H^*(PSp(4n+2)) \cong Z_2[t]/(t^8) \otimes \Lambda(e_3, e_{11}, e_{15}, \ldots, e_{16n+7})$ *the elements* e_{8j-5} *may be chosen to be universally transgressive for* $1 \leqq j \leqq 2n+1$.

(2) *With suitably chosen* $a_{8j-4} = \tau(e_{8j-5})$ *there holds*

$$y_5 a_{8j-4} = 0 \qquad \text{for } 1 \leqq j \leqq 2n.$$

Proof. See Proposition 5.6 of [8] for the method to choose $a_{8j-4} = \tau(e_{8j-5})$.

We will prove Proposition 5.3 for the case $j = 2s+1$. Clearly $\tau(e_{16s+4}) = Sq^8 e_{16s-4} + \text{decomp.}$, since $Sq^8 e_{16s-5} = e_{16s+3}$. The Wu-formula $Sq^i w_5 = w_i w_5$ $(0 \leqq i \leqq 5)$ gives $Sq^i y_5 = y_i y_5$ $(0 \leqq i \leqq 5)$. ($w_1 = 0$ and hence $y_1 = 0$). Put

$$a_{16s+4}=Sq^8a_{16s-4}+y_2Sq^6a_{16s-4}+y_3Sq^5a_{16s-4}+a_4Sq^4a_{16s-4}$$
$$+y_5Sq^3a_{16s-4}$$

and then $y_5a_{16s+4}=Sq^8(y_5a_{16s-4})=0$. Q.E.D.

§6. $H^*(BPSp(2n+1))$

In this section we give an alternative proof of the result in §4 of [6].

Notation. $F(k)=Sq(k)/(Sq(1))^k$.

Remark 6.1. $F(k)=PSp(k)/G(k)$.

As is well known, the Serre spectral sequence with Z-coefficient for the fibering

$$F(k)\longrightarrow(BSp(1))^k\longrightarrow BSp(k)$$

collapses, since $(Sp(1))^k$ is of maximal rank and since $H_*((Sp(1))^k; Z)$ is torsion free (cf. [2]). In particular,

Proposition 6.2. $PS(H^*(F(k); Z))=(1-t^8)\dots(1-t^{4k})/(1-t^4)^{k-1}$.

Recall from [1]

Proposition 6.3 (Baum-Browder).

$$H^*(PSp(2n+1))\cong Z_2[t]/(t^4)\otimes\Lambda(e_7,\dots, e_{8n+3}),$$

where $\deg t=1$ *and* $\deg e_i=i$ *and* $\bar\phi(t)=\bar\phi(e_i)=0$.

Notation. $B_{2n+1}=H^*(PSp(2n+1))$.

By an easy calculation

Lemma 6.4. *As an algebra*

$$\mathrm{Cotor}^{B_{2n+1}}(Z_2, Z_2)\cong Z_2[\bar y_2, \bar y_3, \bar y_8,\dots, \bar y_{8n+4}].$$

We shall prove

Theorem 6.5. *The Eilenberg-Moore spectral sequence for* $PSp(2n+1)$ *with* Z_2*-coefficient collapses.*

Proof. The Serre spectral sequence with Z_2-coefficient for the fibering

$$F(2n+1) \longrightarrow BG(2n+1) \longrightarrow BPSp(2n+1)$$

gives

(6.6) $PS(H^*(BPSp(2n+1))) \gg f(t) = \{(1-t^2)(1-t^3)(1-t^8)\dots(1-t^{8n+4})\}^{-1}$.

On the other hand we have

$$PS(\mathrm{Cotor}^{B_{2n+1}}(Z_2, Z_2)) = f(t).$$

Thus the Eilenberg-Moore spectral sequence for $PSp(2n+1)$ with Z_2-coefficient collapses. Q.E.D.

Corollary 6.6. *There exist elements* $y_i \in H^i(BPSp(2n+1))$ *such that*

$$H^*(BPSp(2n+1)) \cong Z_2[y_2, y_3, y_8, y_{12}, \dots, y_{8n+4}].$$

Since the equality holds in (6.6), we obtain

Corollary 6.7. *The Serre spectral sequence with* Z_2*-coefficient for the fibering*

$$F(2n+1) \longrightarrow BG(2n+1) \xrightarrow{i} BPSp(2n+1)$$

collapses. In particular, $i^*: H^*(BPSp(2n+1)) \to H^*(BG(2n+1))$ *is injective.*

(cf. [7])

§7. A proof of Proposition 4.5

We prepare a lemma which will be used in the proof of the proposition below.

Let k be a commutative field. Let X_i $(1 \leqq i \leqq n)$ and Y_j $(1 \leqq j \leqq m)$ be indeterminates with suitable positive degrees and $R = k[X_1, \dots, X_n, Y_1, \dots, Y_m]$ and $\bar{R} = k[Y_1, \dots, Y_m]$ be graded polynomial algebras over k.

Let R' be a graded commutative algebra generated by homogeneous elements x_i $(1 \leqq i \leqq n)$ and f_j $(1 \leqq j \leqq s)$ over k, where $\deg x_i > 0$ and $\deg f_j > 0$. Let R'' be the subalgebra of R' generated by $\{f_1, \dots, f_s\}$. Let ψ: $R' \to R$ be a homomorphism of graded algebras such that $\psi(e) = e$ (e; the unit) and ψ preserves the degree. Let $p: R \to \bar{R}$ be the projection.

Lemma 7.1. *If ψ satisfies*

(1) $(p \circ \psi)|R''$ *is injective,*

(2) $\psi(x_i) = X_i$ *for all i,*

then ψ is injective.

Proof. Define the weight w as follows:

$$w(X_i) = 0, \quad w(Y_i) = \deg Y_i, \quad w(x_i) = 0, \quad w(f_i) = \deg f_i.$$

Introduce a filtration F_i in $R^{(\prime)}$ by

$$F_i(R^{(\prime)}) = \{x \in R^{(\prime)};\ w(x) \geqq i\}.$$

Put $E_0(R^{(\prime)}) = \sum_{i=0}^{\infty} F_i/F_{i+1}$.

Then the induced homomorphism $\psi_0 = E_0(\psi): E_0(R') \to E_0(R)$ satisfies $\psi_0(x_i) = X_i$. Further, for a homogeneous element $g \in R''$ $(g \neq 0, \deg g > 0)$, $\psi_0(g)$ is a non-zero polynomial of $Y_1, \dots, Y_m$. For a sequence of non-negative integers, $I = (i_1, \dots, i_n)$ put $x^I = x_1^{i_1} \dots x_n^{i_n}$ and $X^I = X_1^{i_1} \dots X_n^{i_n}$. Consider the homogeneous element $\sum_{f_I \in R''} f_I x^I$. Then $\psi_0(\Sigma f_I x^I) = \Sigma \psi_0(f_I) X^I = 0$ implies $\psi_0(f_I) = 0$ implies $f_I = 0$ implies the injectivity of ψ_0. Thus ψ is injective. Q.E.D.

A proof of Proposition 4.5.

(1) By the commutativity of the diagram

$$\begin{array}{ccc} H^*((BSp(1))^{4n+2}) & \xleftarrow{k^*} & H^*(BSp(4n+2)) \\ \uparrow{\scriptstyle \pi_1^*} & & \uparrow{\scriptstyle \pi_0^*} \\ H^*(BG(4n+2)) & \xleftarrow{\bar{k}^*} & H^*(BPSp(4n+2)) \end{array}$$

we have $\pi_1^* \circ \bar{k}^* | P_1(h) = k^* \circ \pi_0^* | P_1(h)$, where k^* is injective. Observe that the relations of type II in $H^*(BPSp(4n+2))$ are mapped by π_0^* to the trivial identity in $H^*(BSp(4n+2))$ and that $\mathrm{Ker}(\pi_0^* | P_1(h)) = (y_2, y_3)$.

Let $\overline{P_1(h)}$ be the subalgebra of $P_1(h)$ generated by $\{a_4, a(I), v_j; \deg h\}$. Since $\operatorname{Ker}\pi_1^*=(y_2, y_3)$ and $k^*(y_i)=y_i$ $(i=2, 3)$ and since $k^*\circ\pi_0^*|P_1(h)$ is injective, we have $k^*|P_1(h)$ is injective by Lemma 7.1.

(2) Consider the commutative diagram

$$\begin{array}{ccc} & BΔ(4n+2) & \\ \swarrow & & \searrow \\ B\tilde{H}(2n+1) & \xrightarrow{k'} & BSp(4n+2) \\ \downarrow{\scriptstyle \pi_2} & & \downarrow{\scriptstyle \pi_0} \\ BH(2n+1) & \xrightarrow[\bar{k}']{} & BPSp(4n+2) \end{array}$$

Then by the naturality of the transgression,

$$\bar{k}'^*(y_j)=\bar{w}_j \qquad \text{for} \quad j=2, 3, 5.$$

Consider the commutative diagram

$$\begin{array}{ccccc} H^*(B\tilde{I}(2n+1)) & \xleftarrow{j_1^*} & H^*(B\tilde{H}(2n+1)) & \xleftarrow{k'^*} & H^*(BSp(4n+2)) \\ & & \uparrow{\scriptstyle \pi_2^*} & & \uparrow{\scriptstyle \pi_0^*} \\ & & H^*(BH(2n+1)) & \xleftarrow[\bar{k}'^*]{} & H^*(BPSp(4n+2))\,. \end{array}$$

the homomorphism $j_1^*\circ\pi_2^*\circ\bar{k}'^*$ maps the subalgebra generated by v_{16i} monomorphically, since $j_1^*\circ k'^*=i_1^*$ and since $\pi_0^*(v_{16i})=q_{2i}^2+v'_{16i}$, where v'_{16i} is the term consisting of the elements of lower index. The relation $\pi_0^*(y_4)=q_1$ implies $k'^*\circ\pi_0^*(y_4)=\bar{q}_1$, and hence $\bar{k}'^*(y_4)=\bar{w}_4$. Since $\operatorname{Ker} j_1^* =(\bar{q}_1)$ by Lemma 3.5, the homomorphism $\pi_2^*\circ\bar{k}'^*$ is injective on the subalgebra generated by a_4 and v_{16i} by Lemma 7.1. Now the result follows from the fact $\operatorname{Ker}\pi_2^*=(\bar{w}_2, \bar{w}_3, \bar{w}_5)$ and Lemma 7.1. Q.E.D.

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