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Spectral Representation for Schrödinger Operators with Long-Range Potentials, II
—Perturbation by Short-Range Potentials—

By

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§0. Introduction

In our previous work (Ikebe [1]) we have obtained a spectral representation for the Schrödinger operator $-\Delta+V(x)$ with a *purely* long-range, real-valued potential $V(x)$ acting in the Euclidean three-space $\boldsymbol{R}^3$. That is, we have assumed that $V(x)=O(|x|^{-1/2-\delta})$, grad $V(x)=O(|x|^{-3/2-\delta})$ and $\Lambda V(r\omega)=O(r^{-\delta})$ for $|x|=r\to\infty, \delta>0$, where Λ is the negative Laplace-Beltrami operator acting on the aungular variable $\omega\in\Omega=\{x\in\boldsymbol{R}^3|\ |x|=1\}$. In [1] we have pointed out that the introduction of a short-range perturbation $V_S(x)=O(|x|^{-1-\delta})$ is not trivial, and cursorily indicated how to handle the matter. The purpose of the present paper is to show that this is actually possible.

What we have done in [1] is roughly as follows. Let H be the self-adjoint realization in the Hilbert space $\boldsymbol{H}=L_2(\boldsymbol{R}^n)$ (see below) of the Schrödinger operator $T=-\Delta+V(x)$ with a long-range potential $V(x)$ in $\boldsymbol{R}^n$, and let E be the associated spectral measure. (Although we have treated in [1] the case $n=3$, the dimension of the underlying space is no essential restriction.) For $\gamma\in\boldsymbol{R}=\boldsymbol{R}^1$ and $G\subset\boldsymbol{R}^n$ let

$$L_{2,\gamma}(G)=\{u|\|u\|^2_{\gamma,G}=\int_G|(1+|x|)^\gamma u(x)|^2dx<\infty\}.$$

We shall omit γ or G if $\gamma=0$ or $G=\boldsymbol{R}^n$. Let $\boldsymbol{h}=L_2(\Omega)$, square integrable functions over Ω with respect to the ordinary surface measure,

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and $\hat{\boldsymbol{H}}=L_2((0, \infty); \boldsymbol{h})$, $\boldsymbol{h}$-valued square integrable functions over $(0, \infty)$. If we choose $\gamma>1/2$ but close to $1/2$, then there exists for each $\lambda>0$ an operator $\mathscr{F}(\lambda)\in\mathscr{B}(L_{2,\gamma}; \boldsymbol{h})$ such that for any Borel set $B\subset(0, \infty)$ and for $f, g\in L_{2,\gamma}$

$$(0.1)\qquad (E(B)f, g)=(\chi_B(\cdot)\mathscr{F}(\cdot)f, \mathscr{F}(\cdot)g)_{\boldsymbol{H}}\equiv\int_B(\mathscr{F}(\lambda)f, \mathscr{F}(\lambda)g)_{\boldsymbol{h}}d\lambda,$$

where χ_B denotes the characteristic function of B. ($\mathscr{B}(X; Y)$ denotes the Banach space of all bounded linear operators: $X\to Y$, X and Y being Banach spaces. $\mathscr{B}(X)=\mathscr{B}(X; X)$. $\|\ \|_X$ denotes the norm of X. $(\ ,\ )_K$ denotes the inner product of a Hilbert space K. $\|\ \|_{\boldsymbol{H}}$ is simply denoted by $\|\ \|$, and $(\ ,\ )_{\boldsymbol{H}}$ by $(\ ,\ )$.) A spectral representation for H is thus obtainable by means of $\{\mathscr{F}(\lambda)\}$, $\lambda\in(0, \infty)$.

Now let us consider the *perturbed* Schrödinger operator $T_S=-\Delta+V(x)+V_S(x)$, where $V_S(x)$ is short-range. The corresponding self-adjoint realization and spectral measure will be denoted by H_S and E_S. The resolvents at z of H and H_S will be designated by $R(z)$ and $R_S(z)$. In this paper we shall show that if we define $\mathscr{F}_S(\lambda)\in\mathscr{B}(L_{2,\gamma}; \boldsymbol{h})$ by

$$\mathscr{F}_S(\lambda)=\mathscr{F}(\lambda)(1-V_SR_S(\lambda-i0)),$$

where $R_S(\lambda-i0)f=\lim_{\varepsilon\downarrow 0}R_S(\lambda-i\varepsilon)f$, $f\in L_{2,\gamma}$, which is known to exist by Ikebe-Saitō [2], then formula (0.1) obtains, where E and $\mathscr{F}(\lambda)$ are replaced by E_S and $\mathscr{F}_S(\lambda)$, respectively. This gives a spectral representation for the perturbed Schrödinger operator H_S.

Our method is perturbation-theoretic and largely abstract, and is based essentially upon the facts that $R_S(\lambda\pm i0)$ and $R(\lambda\pm i0)$ exist and that we have a complete knowledge of a spectral representation for the *unperturbed* operator H.

§1. Assumption and Some Known Results

The Schrödinger operator we shall study is of the form

$$T_S=-\Delta+V(x)+V_S(x)\qquad(T=-\Delta+V(x)),$$

where Δ is the n-dimensional Laplacian, and $V(x)$ and $V_S(x)$ are a long-range and a short-range potential, respectively. We assume:

(1.1) $V(x)$ *is a real-valued* C^2 (*twice continuously differentiable*) *function such that for some* $\delta>0$

$$V(x)=O(|x|^{-1/2-\delta}) \qquad (|x|\to\infty),$$

$$\text{grad } V(x)=O(|x|^{-3/2-\delta}) \qquad (|x|\to\infty),$$

$$\Lambda V(r\omega)=O(r^{-\delta}) \qquad (r\to\infty,\ \omega\in\Omega),$$

where Λ *is the negative Laplace-Beltrami operator on the* $(n-1)$-*sphere* $\Omega=\{x|\ |x|=1\}$.

(1.2) $V_S(x)$ *is a real-valued* C^0 (*continuous*) *function such that*

$$V_{\mathscr{S}}(x)=O(|x|^{-1-\delta}) \qquad (|x|\to\infty).$$

Although in Ikebe [1] it has been assumed that $V(x)$ is a smooth function (over $\boldsymbol{R}^3$), as we can easily check, it is enough as to the regularity of $V(x)$ to assume that $V(x)$ is a C^2 function (over $\boldsymbol{R}^n$). The short-range potential $V_S(x)$ may have singularities, if not too strong. But just for simplicity's sake we have assumed as above.

The following notation will be used. H_S (H) is the self-adjoint realization in $\boldsymbol{H}=L_2$ of T_S (T). $R_S(z)$ $(R(z))$ is the resolvent of H_S (H) at z. E_S (E) is the spectral measure associated with H_S (H). $\boldsymbol{H}_{S,ac}$ $(\boldsymbol{H}_{ac})$ is the absolutely continuous subspace for H_S (H). $P_{S,ac}$ (P_{ac}) is the orthogonal projection onto $\boldsymbol{H}_{S,ac}$ $(\boldsymbol{H}_{ac})$. It is known that $P_{S,ac}=E_S((0,\infty))$ and $P_{ac}=E((0,\infty))$.

Now we shall collect some known results from Ikebe [1] and Ikebe-Saitō [2] which will be needed for the development of our subsequent discussion.

Theorem 1.1. *Let* $f\in L_{2,\gamma}$, *where* γ *is greater than but sufficiently near* $1/2$ $(0<2\gamma-1<(1+2\delta)/4)$. *Then* $R_S(z)f$ *and* $R(z)f$ *as* $L_{2,-\gamma}$-*valued functions of* z *can be continuously extended to the closures of the upper and lower half complex planes exclusive of* $(-\infty, 0]$ (*but for* $\lambda>0$ $R_S(\lambda+i0)f$ $[R(\lambda+i0)f]$ *is generally different from* $R_S(\lambda-i0)f$ $[R(\lambda-i0)f]$).

Theorem 1.2. *Associated with the operator H and $\lambda>0$ there exists a bounded linear operator $\mathscr{F}(\lambda)\in\mathscr{B}(L_{2,\gamma};\mathbb{h})$ such that $\mathscr{F}(\lambda)f, f\in L_{2,\gamma}$, is strongly continuous in $\lambda>0$, and the following relations hold*:

$$(1.3)\qquad (\mathscr{F}(\lambda)f, \mathscr{F}(\lambda)g)_{\mathbb{h}}=(2\pi i)^{-1}(R(\lambda+i0)f-R(\lambda-i0)f, g),$$

$$(1.4)\qquad (E(B)f, g)=\int_B(\mathscr{F}(\lambda)f, \mathscr{F}(\lambda)g)_{\mathbb{h}}d\lambda,$$

where $f, g\in L_{2,\gamma}$ and B is a Borel set of $(0, \infty)$. *The operator $\mathscr{F}$: $L_{2,\gamma}\to\hat{\mathbb{H}}$ defined by $(\mathscr{F}f)(\lambda)=\mathscr{F}(\lambda)f$ can be uniquely extended by continuity to a partially isometric operator $\in\mathscr{B}(\mathbb{H};\hat{\mathbb{H}})$ which is unitary on $\mathbb{H}_{ac}=P_{ac}\mathbb{H}$ onto $\hat{\mathbb{H}}$, and will also be denoted by $\mathscr{F}$. $\mathscr{F}^*$, the adjoint of $\mathscr{F}$, admits of the following representation*

$$\mathscr{F}^*\hat{f}=\text{s-lim}_{N\to\infty}\int_{N^{-1}}^{N}\mathscr{F}(\lambda)^*\hat{f}(\lambda)d\lambda\qquad(\hat{f}\in\hat{\mathbb{H}}),$$

where s-lim *means the strong limit (in $\mathbb{H}$), and where $\mathscr{F}(\lambda)^*$, the adjoint of $\mathscr{F}(\lambda)$, is a linear operator $\in\mathscr{B}(\mathbb{h};L_{2,-\gamma})$ defined by $(\mathscr{F}(\lambda)^*\phi, g)=(\phi, \mathscr{F}(\lambda)g)_{\mathbb{h}}$ for $\phi\in\mathbb{h}$, $g\in L_{2,\gamma}$.*

As to Theorem 1.2 let us remark the following. In [1] we have constructed $\mathscr{F}(\lambda)$ as an operator $\in\mathscr{B}(L_{2,1};\mathbb{h})$. But we have shown there that this $\mathscr{F}(\lambda)$ can be extended to an operator $\in\mathscr{B}(L_{2,\gamma};\mathbb{h})$ (by using Theorem 1.1 and relation (1.3) valid with $f, g\in L_{2,1}$). The strong continuity in $\lambda>0$ of $\mathscr{F}(\lambda)f, f\in L_{2,1}$, in the topology of $\mathbb{h}$ has been shown also in [1]. That this is the case with $f\in L_{2,\gamma}$ may follow from the following argument. First, by using (1.3), it may be noted that for this purpose it suffices to show that $(\mathscr{F}(\lambda)f, \phi)_{\mathbb{h}}$ is continuous in λ for smooth $\phi\in\mathbb{h}$. Then one can show this by using the definition of $\mathscr{F}(\lambda)$ and Green's formula. (This sort of argument has been made actually in [1], though the continuity in λ of $\mathscr{F}(\lambda)f, f\in L_{2,\gamma}$ has not been explicitly stated. As a matter of fact, in our discussion that follows, the continuity in λ of $\mathscr{F}(\lambda)$ will not be absolutely necessary. Instead, in most cases, the measurability and local boundedness of $\mathscr{F}(\lambda)$ alone will suffice.)

Assumption (1.1) will never be used explicitly, because almost everything necessary is included in the above two theorems. We shall make an explicit use of (1.2) solely for recalling that V_S is a bounded linear operator $\in \mathscr{B}(L_{2,-\gamma}; L_{2,\gamma})$.

§2. Spectral Representation for H_S

To start with let us consider the resolvent equation

$$R_{\mathscr{S}}(z)-R(z)=-R(z)V_S R_S(z)=-R_S(z)V_S R(z) \tag{2.1}$$

for non-real z. Although (2.1) is valid usually as an equation in $\mathscr{B}(\mathbb{H})$, it can also be interpreted as an equation for $R_S(z)$, $R(z)\in\mathscr{B}(L_{2,\gamma}; L_{2,-\gamma})$, because, as is easily checked by (1.2), $V_S\in\mathscr{B}(L_{2,-\gamma}; L_{2,\gamma})$. Then by Theorem 1.1 we see that (2.1) is also valid for the boundary values of $R_S(z)$ and $R(z)$. Namely, we have the following

Lemma 2.1. *For* $\lambda>0$ $R_S(\lambda\pm i0)-R(\lambda\pm i0)=-R(\lambda\pm i0)V_S R_S(\lambda\pm i0)$ $=-R_S(\lambda\pm i0)V_S R(\lambda\pm i0)$.

Definition 2.2. $\mathscr{F}_S(\lambda)=\mathscr{F}(\lambda)(1-V_S R_S(\lambda-i0))$ *for* $\lambda>0$.

$\mathscr{F}_S(\lambda)$ is well-defined as an operator $\in\mathscr{B}(L_{2,\gamma}; \mathbb{h})$, for we have $V_S R_S(\lambda-i0)\in\mathscr{B}(L_{2,\gamma})$.

The adjoints V_S^* and $R_S(\lambda\pm i0)^*$ of $V_S\in\mathscr{B}(L_{2,-\gamma}; L_{2,\gamma})$ and $R_S(\lambda\pm i0)\in\mathscr{B}(L_{2,\gamma}; L_{2,-\gamma})$ are defined by the following relations:

$$(V_S^* u, v)=(u, V_S v), \quad (R_S(\lambda\pm i0)^* f, g)=(f, R_S(\lambda\pm i0)g)$$

for $u, v\in L_{2,-\gamma}$ and $f, g\in L_{2,\gamma}$. Similarly for $R(\lambda\pm i0)^*$. Clearly, $V_S^*\in\mathscr{B}(L_{2,-\gamma}; L_{2,\gamma})$ and $R_S(\lambda\pm i0)^*\in\mathscr{B}(L_{2,\gamma}; L_{2,-\gamma})$. The following lemma is an obvious consequence of these definitions, (1.2) and Theorem 1.1.

Lemm 2.3. i) $V_S^*=V_S$. ii) $R_S(\lambda\pm i0)^*=R_S(\lambda\mp i0)$, $R(\lambda\pm i0)^*=R(\lambda\mp i0)$ *for* $\lambda>0$.

Lemma 2.4. *For each* $f\in L_{2,\gamma}$ $\mathscr{F}_S(\lambda)f$ *is a continuous function of* $\lambda>0$ *in the topology of* $\mathbb{h}$, *and the following relation holds*:

$$(\mathscr{F}_S(\lambda)f, \mathscr{F}_S(\lambda)g)_{\mathbb{h}}=(2\pi i)^{-1}(R_S(\lambda+i0)f-R_S(\lambda-i0)f, g).$$

Proof. The continuity of $\mathscr{F}_S(\lambda)f$ is obvious from that of $\mathscr{F}(\lambda)f$

(Theorem 1.2), since $V_S R_S(\lambda - i0)f$ is continuous in λ in $\boldsymbol{h}$ by Theorem 1.1.

To prove the last half we compute as follows:

$$(\mathscr{F}_S(\lambda)f, \mathscr{F}_S(\lambda)g)_{\boldsymbol{h}} = (\mathscr{F}(\lambda)(1 - V_S R_S(\lambda - i0))f, \mathscr{F}(\lambda)(1 - V_S R_S(\lambda - i0))g)_{\boldsymbol{h}}$$

(by Definition 2.2)

$$= (2\pi i)^{-1}([R(\lambda + i0) - R(\lambda - i0)][1 - V_S R_S(\lambda - i0)]f, [1 - V_S R_S(\lambda - i0)]g)$$

(by Theorem 1.2)

$$= (2\pi i)^{-1}\{([1 - V_S R_S(\lambda - i0)]f, R(\lambda - i0)[1 - V_S R_S(\lambda - i0)]g)$$
$$- (R(\lambda - i0)[1 - V_S R_S(\lambda - i0)]f, [1 - V_S R_S(\lambda - i0)]g)\}$$

(by Lemma 2.3 ii))

$$= (2\pi i)^{-1}\{([1 - V_S R_S(\lambda - i0)]f, R_S(\lambda - i0)g)$$
$$- (R_S(\lambda - i0)f, [1 - V_S R_S(\lambda - i0)]g)\} \qquad \text{(by Lemma 2.1)}$$

$$= (2\pi i)^{-1}\{(R_S(\lambda + i0)f, g) - (R_S(\lambda - i0)f, g)\} \qquad \text{(by Lemma 2.3 i), ii))}$$

which was to be shown. The proof of the lemma is thus complete.

Now it is easy to derive with the aid of Lemma 2.4 the relation (1.3) of Theorem 1.2, where E and $\mathscr{F}(\lambda)$ are replaced with E_S and $\mathscr{F}_S(\lambda)$. Consequently, the proof of the following theorem is standard (see, e.g., [1]) except for the proof of the onto-ness of the operator $\mathscr{F}_S$ (to be defined in the theorem). This we shall do after stating the theorem.

Theorem 2.5. a) *For any* $f, g \in L_{2,\gamma} \subset \boldsymbol{H}$ *and for any Borel set* $B \subset (0, \infty)$ *we have*

$$(E_S(B)f, g) = \int_B (\mathscr{F}_S(\lambda)f, \mathscr{F}_S(\lambda)g)_{\boldsymbol{h}} d\lambda .$$

b) *Define* $\mathscr{F}_S$ *by*

$$(\mathscr{F}_S f)(\lambda) = \mathscr{F}_S(\lambda)f \qquad \text{for } f \in L_{2,\gamma}.$$

Then $\mathscr{F}_S: L_{2,\gamma} \to \hat{\boldsymbol{H}} = L_2((0, \infty); \boldsymbol{h})$ *can be extended by continuity to a partial isometry on* $\boldsymbol{H}$ *which is unitary from* $\boldsymbol{H}_{S,ac} = P_{S,ac}\boldsymbol{H}$ *onto* $\hat{\boldsymbol{H}}$, *and the following relation holds: For any bounded Borel function* $\alpha(\lambda)$

$$(\mathscr{F}_S \alpha(H_S) P_{S,ac} f)(\lambda) = \alpha(\lambda)(\mathscr{F}_S f)(\lambda) \qquad \text{for a.e. } \lambda > 0,$$

where we have denoted the extended operator also by $\mathscr{F}_S$.

c) *For any bounded Borel set* B *whose closure is contained in* $(0, \infty)$, *define* $\mathscr{F}_B^*$ *by*

$$\mathscr{F}_B^* \hat{f} = \int_B \mathscr{F}_S(\lambda)^* \hat{f}(\lambda) d\lambda \qquad (\hat{f} \in \hat{\boldsymbol{H}}),$$

where $\mathscr{F}_S(\lambda)^*$ *is defined in the same manner as* $\mathscr{F}(\lambda)^*$ *is defined in Theorem* 1.2. *Then* $\mathscr{F}_B^*$ *is not only in* $\mathscr{B}(\hat{\boldsymbol{H}}; L_{2,-\gamma})$ *but also in* $\mathscr{B}(\hat{\boldsymbol{H}}; \boldsymbol{H})$, *and we have* $\mathscr{F}_B^* = E_S(B)\mathscr{F}_S^*$. *For any Borel set* B *let* $B_N = B \cap [N^{-1}, N]$ $(N > 1)$. *Then the strong limit as* $N \to \infty$ *of* $\mathscr{F}_{B_N}^*$ *exists and* $\text{s-lim}_{N\to\infty} \mathscr{F}_{B_N}^* = E_S(B)\mathscr{F}_S^*$. *In particular, the following inversion formula is valid*:

$$f = \text{s-lim}_{N\to\infty} \int_{N^{-1}}^{N} \mathscr{F}_S(\lambda)^* (\mathscr{F}_S f)(\lambda) d\lambda .$$

d) $\mathscr{F}_S(\lambda)^*$ *is an eigenoperator for* H_S *or* T_S *with eigenvalue* $\lambda > 0$ *in the sense that for any* $\phi \in \boldsymbol{h}$ *and any smooth function* f *with compact support*

$$(T_S \mathscr{F}_S(\lambda)^* \phi, f) = \lambda (\mathscr{F}_S(\lambda)^* \phi, f) .$$

Proof that $\mathscr{F}_S$ *maps onto* $\hat{\boldsymbol{H}}$. The proof is completely the same as the one given in Ikebe [1] (§ 3) except for the last step.

That $\mathscr{F}_S$ maps onto $\hat{\boldsymbol{H}}$ is equivalent to that the null space of $\mathscr{F}_S^*$ consists only of 0. Thus we need to show that $\mathscr{F}_S^* \hat{f} = 0$ implies $\hat{f} = 0$. Arguing as in [1], from $\mathscr{F}_S^* \hat{f} = 0$ we can get to $\mathscr{F}_S(\lambda)^* \hat{f}(\lambda) = 0$ for a.e. $\lambda > 0$. We have by Lemma 2.1 and Definition 2.2 $\mathscr{F}(\lambda) = \mathscr{F}_S(\lambda)(1 + V_S R(\lambda - i0))$, and hence $\mathscr{F}(\lambda)^* = (1 + R(\lambda + i0)V_S)\mathscr{F}_S(\lambda)^*$. Therefore, $\mathscr{F}_S(\lambda)^* \hat{f}(\lambda) = 0$ for a.e. $\lambda > 0$ implies $\mathscr{F}(\lambda)^* \hat{f}(\lambda) = 0$ for a.e. $\lambda > 0$, i.e., $\mathscr{F}^* \hat{f} = 0$ by Theorem 1.2. But since $\mathscr{F}$ maps onto $\hat{\boldsymbol{H}}$ by Theorem 1.2, we have $\hat{f} = 0$, which completes the proof.

In concluding the present paper we remark that Kato and Kuroda's general abstract stationary method (see, e.g., [3]) may be applied to our situation. Here, we have chosen a rather direct way of constructing a spectral representation or eigenfunction expansion.

References

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