Publ. RIMS, Kyoto Univ.
11 (1976), 497-521

On Stable Homotopy Types of Some Stunted Spaces

By

Hideaki ŌSHIMA*

1. Introduction

In this note we shall study the stable homotopy types (S-types) of the stunted spaces $N_k^{n+k}(G)=N^{n+k}(G)/N^{k-1}(G)$, where $N^n(G)=S^{4n+3}$ mod G are quotients of S^{4n+3} by free orthogonal actions of a closed subgroup G of S^3. In §2, we show that $N_k^{n+k}(G)$ are homeomorphic to the Thom spaces $N^n(G)^{k\xi}$. If G is not finite, then G is S^1, S^3 or the normalizer $N(S^1)$ of S^1 in S^3. The case with $G=S^1$ or S^3 has been treated by Feder and Gitler [8], [9]. We consider the case with $G=N(S^1)$ in §3. The case with $G=Z_m$ (cyclic group of order m) has been treated in [12], [15]. On and after §4, we consider the remaining cases, i.e. the cases with G the binary dihedral or binary polyhedral groups (see §2 for definitions). We examine the representation groups of the generalized quaternion groups $D^*(2^{m+1})$ in §4 and evaluate the orders of some elements of $K_F(N^n(D^*(2^{m+1})))$ in §5 or $J(N^n(D^*(2^{m+1})))$ in §§6–7 and study the S-types of $N_k^{n+k}(G)$ in the final section §8.

2. Thom Spaces and Subgroups of S^3

In this note G-space means a left G-space and F-vector space (or bundle) implies a right F-vector space (or bundle) for a field F. For a G-space X its orbit space is denoted by X mod G and if G acts on Y also, $X\times Y$ mod G denotes the orbit space by diagonal action. For a vector bundle α over a finite CW-complex X, X^α denotes the associated

Communicated by N. Shimada, February 5, 1975.
* Department of Mathematics, Osaka City University, Osaka.

Thom space, that is, the one point compactification of the total space of α.

Let $O_F(m)$ denote the orthogonal group $O(m)$ for $F=R$ (the real numbers), the unitary group $U(m)$ for $F=C$ (the complex numbers) and the symplectic group $Sp(m)$ for $F=H$ (the quaternions) in m dimensions respectively. We say that a representation $d: G\to O_F(m)$ of a topological group G is free if the action of G restricted to the unit sphere $S(V)$ is free, where V is a representation space of d with an inner product $(\ |\ )$. Let kV denote the sum $V\oplus\cdots\oplus V$ (k factors) with the inner product $(a|b)=\Sigma(a_i|b_i)$ for $a=(a_1,\dots, a_k)$, $b=(b_1,\dots, b_k)\in kV$. For $k<k'$, we regard kV as a subspace of $k'V$ by the identification $(a_1,\dots, a_k)=(a_1, \dots, a_k, 0,\dots, 0)$.

For a given free representation $d: G\to O_F(m)$, we introduce the following notations:

$$N^n(G, d)=S((n+1)V)\bmod G\,,$$

$$N_k^{n+k}(G, d)=N^{n+k}(G, d)/N^{k-1}(G, d)\,,$$

and $\xi_n(G, d)$ means the canonical bundle

$$S((n+1)V)\times V\bmod G \longrightarrow N^n(G, d)\,.$$

Then we have the following theorem.

Theorem 2.1. *There exists a homeomorphism*

$$N_k^{n+k}(G, d)\approx N^n(G, d)^{k\xi_n(G,d)}\,.$$

Proof. Consider the map $f: S((n+1)V)\times D(kV)\to S((n+k+1)V)$ defined by

$$f(x, y)=(y, x\lambda)$$

where $D(kV)$ denotes the unit disk of kV and $x=(x_0,\dots, x_n)\in S((n+1)V)$, $y=(y_0,\dots, y_{k-1})\in D(kV)$, $\lambda=\sqrt{1-(y|y)}$ and $x\lambda=(x_0\lambda,\dots, x_n\lambda)$. It is easy to show that f defines a G-equivariant homeomorphism

$$S((n+1)V)\times(D(kV)-S(kV)) \longrightarrow S((n+k+1)V)-S(kV)$$

and then we have a homeomorphism

$$S((n+1)V)\times(D(kV)-S(kV)) \bmod G \longrightarrow (S((n+k+1)V)-S(kV)) \bmod G.$$

$S((n+1)V)\times(D(kV)-S(kV)) \bmod G$ may be identified with the total space of $k\xi_n(G, d)$. Compactifying the both spaces by adding one point, we have

$$N^n(G, d)^{k\xi_n(G,d)} \approx N_k^{n+k}(G, d)$$

as desired.

For example, we have

$$S_k^{n+k} = S^{n+k}/S^{k-1} \approx S^k \vee S^{n+k}$$

for the trivial representation $1 \to O(1)$ and

$$FP_k^{n+k} = FP^{n+k}/FP^{k-1} \approx (FP^n)^{k\xi}$$

for the identity $O_F(1) \to O_F(1)$ and

$$L_k^{n+k}(m) = L^{n+k}(m)/L^{k-1}(m) \approx L^n(m)^{k\xi}$$

for $Z_m \subset U(1)$, where FP^n indicates the F-projective space and $L^n(m)$ the standard mod m lens space. These are well-known.

We say that two spaces X and Y are stably homotopy equivalent (S-equivalent) if the suspensions $S^u \wedge X$ and $S^v \wedge Y$ are homotopy equivalent for some u and v.

The classifications of S-types of S_k^{n+k}, CP_k^{n+k} and HP_k^{n+k} have been completed. The sphere case is trivial and the complex or quaternion projective space case has been done by Feder and Gitler [8], [9].

It is known that which compact group admit a free representation. Finite groups admitting a free representation are listed in [23, Chapter 6]. If a compact group G including infinite elements has a free representation, then G is a Lie group ([17, V Th. 2]) and must be S^1, S^3 or the normalizer $N(S^1)$ of S^1 in S^3 [5, III 8.5].

From now on, we will treat the case with G a closed subgroup of S^3 and d the inclusion $d_1: G \subset S^3 = Sp(1)$. And we will use the notations:

$N^n(G)=N^n(G, d_1)$,

$N_k^{n+k}(G)=N_k^{n+k}(G, d_1)$,

$\xi_n(G)=\xi_n(G, d_1)$,

$\xi_n(G)_F$, the underlying F-vector bundle of $\xi_n(G)$.

$\pi(G, H)\colon N^n(G)\to N^n(H)$, the natural projection for $G\subset H\subset S^3$. Closed subgroups of S^3 are maximal tori S^1, $N(S^1)$'s (any two of them are conjugate each other respectively), S^3 itself or finite subgroups. Concerning finite subgroups of S^3, we have

2.2 (Wolf [23, 2.6.7]). *Every finite subgroup of S^3 is a cyclic, binary dihedral or binary polyhedral group. If two finite subgroups of S^3 are isomorphic, they are conjugate in S^3.*

We remark that if two subgroups G, H of S^3 are conjugate, then $N^n(G)$ and $N^n(H)$ are naturally homeomorphic, and this homeomorphism induces the isomorphism between $\xi_n(G)$ and $\xi_n(H)$. Thus we may assume that $N^n(G)$ and $\xi_n(G)$ are defined for the conjugate classes of subgroups of S^3. So we describe the subgroups of S^3 in terms of generators and relations as follows: the binary dihedral group $D^*(4m)$ of order $4m$ ($m\geqq 2$), the binary tetrahedral group T^* of order 24, the binary octahedral group O^* of order 48 and the binary icosahedral group I^* of order 120 are given by

$$D^*(4m)\colon x^m=(yx)^2=y^2,$$

$$T^* \quad \colon x^3=(yx)^3=y^2,\ y^4=1,$$

$$O^* \quad \colon x^4=(yx)^3=y^2,\ y^4=1,$$

$$I^* \quad \colon x^5=(yx)^3=y^2,\ y^4=1$$

(see [23] or [18, 6.2]). T^*, O^* and I^* are called the binary polyhedral groups. $D^*(2^{m+1})$ is called the generalized quaternion group.

3. $N^n(N(S^1))$

In this section we examine the S-types of $N_k^{n+k}(N(S^1))$.

For simplicity we use the notations

$$N^n = N^n(N(S^1)),\ N_k^{n+k} = N_k^{n+k}(N(S^1)).$$

For $0 \leqq k \leqq n$, we define the cells in $S((n+1)H) = S^{4n+3}$ as follows:

$$\underline{e}^{4k} = \{(z_1, \dots, z_{2k+1}, 0, \dots, 0);\ z_{2k+1} \neq 0,\ \arg(z_{2k+1}) = 0\},$$

$$\underline{e}^{4k+1} = \{(z_1, \dots, z_{2k+2}, 0, \dots, 0);\ z_{2k+1} \neq 0,\ z_{2k+2} \neq 0,$$
$$\arg(z_{2k+1}) = \arg(z_{2k+2}) = 0\},$$

$$\underline{e}^{4k+2} = \{(z_1, \dots, z_{2k+2}, 0, \dots, 0);\ z_{2k+1} \neq 0,\ 0 < \arg(z_{2k+1}) < \pi,$$
$$z_{2k+2} \neq 0,\ \arg(z_{2k+2}) = 0\},$$

and their images in N^n by the natural projection $S^{4n+3} \to N^n$ are denoted by e^{4k}, e^{4k+1} and e^{4k+2} respectively, here we regard H as the complex 2-space by the replacement $q = z + z'j$. Then it is easy to check the following proposition.

Proposition 3.1. *$\{e^{4k}, e^{4k+1}, e^{4k+2}; 0 \leqq k \leqq n\}$ gives a CW-decomposition of N^n.*

Remark that the above CW-decomposition satisfies the condition that the $4m+2$-skeleton of N^n is N^m for $0 \leqq m \leqq n$.

It is easy to show that the Serre spectral sequence of the fibration

$$N^0 = RP^2 \longrightarrow N^n \longrightarrow HP^n$$

is trivial and therefore we have the following proposition.

Proposition 3.2. *For any coefficients A, we have*

$$H^*(N^n; A) \cong H^*(HP^n; \mathbf{Z}) \otimes H^*(RP^2; A).$$

Let K_F be real ($F = \mathbf{R}$), complex ($F = \mathbf{C}$) or symplectic ($F = \mathbf{H}$) K-theory and θ_F be the representation

$$N(S^1) \xrightarrow[\text{quotient}]{} N(S^1)/S^1 = O(1) \subset O_F(1)$$

and $\hat{\theta}_F$ be the associated F-line bundle

$$S((n+1)H) \times F \bmod N(S^1) \longrightarrow N^n.$$

Proposition 3.3. *There exists a split exact sequence*

$$0 \longrightarrow K_F(HP^n) \longrightarrow K_F(N^n) \longrightarrow Z_{2^{f(n;F)}} \longrightarrow 0,$$

where $f(n; \mathbf{R}) = 2[n/2]+2, f(n; \mathbf{C}) = n+1, f(n; \mathbf{H}) = 2[(n+1)/2]$ *and the reduced element* $\hat{\theta}_F - 1 \in \tilde{K}_F(N^n)$ *generates the direct summand* $Z_{2^{f(n;F)}}$*. Here* $[a]$ *denotes the greatest integer which does not exceed* a.

Proof. (i) $F=\mathbf{C}$-case. Consider the commutative triangle

$$\begin{array}{ccc} & K_{\mathbf{C}}(CP^{2n+1}) & \\ {}^{\pi_1^!}\nearrow & & \nwarrow^{\pi_3^!} \\ K_{\mathbf{C}}(N^n) & \xleftarrow{\ \pi_2^!\ } & K_{\mathbf{C}}(HP^n) \end{array}$$

where $\pi_1 = \pi(S^1, N(S^1))$, $\pi_2 = \pi(N(S^1), S^3)$ and $\pi_3 = \pi(S^1, S^3)$. Let η_{2n+1} be the canonical complex line bundle over CP^{2n+1}. Put $\mu = \eta_{2n+1} - 1 \in \tilde{K}_{\mathbf{C}}(CP^{2n+1})$ and $\upsilon_n = \xi_n(S^3)_{\mathbf{C}} - 2 \in \tilde{K}_{\mathbf{C}}(HP^n)$. Then it is well-known that

$$K_{\mathbf{C}}(CP^{2n+1}) = \mathbf{Z}[\mu]/\mu^{2n+2},$$

$$K_{\mathbf{C}}(HP^n) = \mathbf{Z}[\upsilon_n]/\upsilon_n^{n+1},$$

$$\pi_3^*(\xi_n(S^3)_{\mathbf{C}}) = \eta_{2n+1} \oplus \bar{\eta}_{2n+1}$$

where $^-$ denotes the complex conjugation. Since

$$\pi_3^!(\upsilon_n) = \mu + \bar{\mu} = \mu^2 - \mu^3 + \cdots - \mu^{2n+1},$$

the image $Im\pi_3^!$ of $\pi_3^!$ is a direct summand of $K_{\mathbf{C}}(CP^{2n+1})$. In the commutative diagram

$$\begin{array}{ccccc} K_{\mathbf{C}}(HP^n) & \longrightarrow & K_{\mathbf{C}}(HP^n)\otimes Q & \xrightarrow{ch} & H^*(HP^n; Q) \\ \downarrow{\scriptstyle \pi_2^!} & & \downarrow{\scriptstyle \pi_2^!\otimes Q} & & \downarrow{\scriptstyle \pi_2^*} \\ K_{\mathbf{C}}(N^n) & \longrightarrow & K_{\mathbf{C}}(N^n)\otimes Q & \xrightarrow{ch} & H^*(N^n; Q) \end{array}$$

π_2^* is an isomorphism by (3.2) and then $\pi_2^!$ is monomorphic and the cokernel of $\pi_2^!$ is finite, where ch denotes the Chern character. And therefore $Im\pi_3^!$ and $Im\pi_1^!$ have the same rank. Then, since $Im\pi_3^! \subset Im\pi_1^!$ and $Im\pi_3^!$ is a direct summand of the free module $K_{\mathbf{C}}(CP^{2n+1})$, we know that $Im\pi_3^! = Im\pi_1^!$ and therefore $\pi_2^!$ is an isomorphism onto a direct summand of $K_{\mathbf{C}}(N^n)$.

By definition, we have

$$\pi_2^!(\hat{\theta}_{\mathbf{C}}-1)=0,$$

and then the aboves imply that $\hat{\theta}_{\mathbf{C}}-1$ has a finite order. Put $\sigma=\pi(\mathbf{Z}_4, S^1)^!\mu \in K_{\mathbf{C}}(L^{2n+1}(4))$. It is easy to see that

$$\pi(\mathbf{Z}_4, N(S^1))^!(\hat{\theta}_{\mathbf{C}}-1)=\sigma^2+2\sigma.$$

Since the order $\#(\sigma^2+2\sigma)$ of $\sigma^2+2\sigma$ is 2^{n+1} [14, Th. A], $\#(\hat{\theta}_{\mathbf{C}}-1)$ is a multiple of 2^{n+1}.

Let $\{E_r\}$ be the Atiyah-Hirzebruch spectral sequence for $K_{\mathbf{C}}^*(N^n)$. Then $E_2^{p,q}=H^p(N^n; K_{\mathbf{C}}^q)$ and

$$\#\,\mathrm{Tor}(K_{\mathbf{C}}(N^n)) \leqq \#\,\mathrm{Tor}\,(\sum_p E_2^{p,-p}) = \#\sum_{k=0}^{n} E_2^{4k+2,-4k-2} = 2^{n+1}$$

by (3.2), where $\mathrm{Tor}(A)$ denotes the torsion submodule of a module A. Hence $\#(\hat{\theta}_{\mathbf{C}}-1)$ is a divisor of 2^{n+1}. Therefore $\#(\hat{\theta}_{\mathbf{C}}-1)=2^{n+1}$ and $\hat{\theta}_{\mathbf{C}}-1$ generates $\mathrm{Tor}(K_{\mathbf{C}}(N^n))$. This completes the proof of the proposition for $F=\mathbf{C}$.

Remark. (3.2) implies that the above spectral sequence collapses.

(ii) $F=\mathbf{R}$-case. Let $c\colon K_{\mathbf{R}}\to K_{\mathbf{C}}$ be the complexification and $r\colon K_{\mathbf{C}}\to K_{\mathbf{R}}$ be the real restriction. Since $r\circ c=2$ and $c(\hat{\theta}_{\mathbf{R}}-1)=\hat{\theta}_{\mathbf{C}}-1$, we have

$$\#(\hat{\theta}_{\mathbf{R}}-1)=2^{n+1} \quad \text{or} \quad 2^{n+2}$$

by (i).

Consider the Atiyah-Hirzebruch spectral sequence $\{{}_{\mathbf{R}}E_r\}$ for $K_{\mathbf{R}}^*(N^n)$.

Then ${}_{\mathbf{R}}E_2^{p,q}=H^p(N^n;\ K_{\mathbf{R}}^q)$ and

$$\# \mathrm{Tor}(K_{\mathbf{R}}(N^n)) \leqq \# \mathrm{Tor}(\sum_p {}_{\mathbf{R}}E_\infty^{p,-p}).$$

Since the rank of $K_{\mathbf{R}}(N^n)$ equals the rank of $K_{\mathbf{C}}(N^n)$, $n+1$, we have

$${}_{\mathbf{R}}E_\infty^{4k,-4k} \cong Z \qquad \text{for} \quad 0 \leqq k \leqq n,$$

and then

$$\# \mathrm{Tor}(\sum_p {}_{\mathbf{R}}E_\infty^{p,-p}) = \# \mathrm{Tor}(\sum_{p\equiv 1,2(8)} {}_{\mathbf{R}}E_\infty^{p,-p}) \leqq \# \sum_{p\equiv 1,2(8)} {}_{\mathbf{R}}E_2^{p,-p} = 2^{2\left[\frac{n+2}{2}\right]}.$$

Then we have

$$\# \mathrm{Tor}(K_{\mathbf{R}}(N^n)) \leqq 2^{2[(n+2)/2]}.$$

Since

$$f(n;\mathbf{R}) = 2[(n+2)/2] = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n+2 & \text{if } n \text{ is even,} \end{cases}$$

we know that

$$\#(\hat{\theta}_{\mathbf{R}}-1) = 2^{n+1} \qquad \text{for} \quad n \text{ odd.}$$

An easy computation shows that

$$H^*(N^{2m+1}, N^{2m};\ A) \cong H^*(HP^{2m+1}, HP^{2m};\ \mathbf{Z}) \otimes H^*(RP^2;\ A)$$

and then by the Atiyah-Hirzebruch spectral sequence, we have

$$K_{\mathbf{R}}^{\pm 1}(N^{2m+1}, N^{2m}) = 0, \quad K_{\mathbf{R}}(N^{2m+1}, N^{2m}) \cong Z.$$

Then the long exact sequence of the pair (N^{2m+1}, N^{2m}) induces the following short exact sequence

$$0 \longrightarrow K_{\mathbf{R}}(N^{2m+1}, N^{2m}) \longrightarrow K_{\mathbf{R}}(N^{2m+1}) \xrightarrow{\iota^!} K_{\mathbf{R}}(N^{2m}) \longrightarrow 0.$$

Then $\iota^!$ induces an isomorphism between the torsion submodules of $K_{\mathbf{R}}(N^{2m+1})$ and $K_{\mathbf{R}}(N^{2m})$. And since $\iota^!(\hat{\theta}_{\mathbf{R}}-1) = \hat{\theta}_{\mathbf{R}}-1$, we have

$$\#(\hat{\theta}_{\mathbf{R}}-1) = 2^{n+2} \qquad \text{for } n \text{ even.}$$

Therefore

$$\#(\hat{\theta}_{\mathbf{R}}-1)=2^{f(n;\mathbf{R})}$$

and $\hat{\theta}_{\mathbf{R}}-1$ generates $\mathrm{Tor}(K_{\mathbf{R}}(N^n))$.

Let $N^n_{(s)}$ be the s-skeleton of N^n with respect to the CW-decomposition (3.1) and $K_{\mathbf{R}}(N^n)_s$ be the kernel of the restriction $K_{\mathbf{R}}(N^n)\to K_{\mathbf{R}}(N^n_{(s-1)})$. Then ${}_{\mathbf{R}}E_\infty^{s,-s}=K_{\mathbf{R}}(N^n)_s/K_{\mathbf{R}}(N^n)_{s+1}$. The facts $\mathrm{Tor}(K_{\mathbf{R}}(N^n))=Z_{2^{f(n;\mathbf{R})}}$ and $\#\mathrm{Tor}(\sum_p {}_{\mathbf{R}}E_2^{p,-p})=2^{f(n;\mathbf{R})}$ imply that we may think that ${}_{\mathbf{R}}E_\infty^{4k,-4k}\cong Z(0\leqq k\leqq n)$ is a direct summand of $K_{\mathbf{R}}(N^n)$, that is, an element of $K_{\mathbf{R}}(N^n)_{4k}$ which represents a generator of ${}_{\mathbf{R}}E_\infty^{4k,-4k}=Z$ generates a direct summand of $K_{\mathbf{R}}(N^n)$.

Put $\underline{v}=\pi_2^!(v_n)\in K_{\mathbf{C}}(N^n)$. Comparing the spectral sequences for $K_{\mathbf{C}}(HP^n)$ and $K_{\mathbf{C}}(N^n)$, we know that $\underline{v}^s$ represents a generator of $E_\infty^{4s,-4s}=E_2^{4s,-4s}$.

Let $c\colon {}_{\mathbf{R}}E_r\to E_r$ be the homomorphism induced by the complexification $c\colon K_{\mathbf{R}}(N^n)\to K_{\mathbf{C}}(N^n)$. Since $c\colon {}_{\mathbf{R}}E_2^{p,q}=H^p(N^n;K^q_{\mathbf{R}})\to E_2^{p,q}=H^p(N^n;K^q_{\mathbf{C}})$ is induced by the coefficients homomorphism $c\colon K^q_{\mathbf{R}}\to K^q_{\mathbf{C}}$, $c\colon {}_{\mathbf{R}}E_2^{8k+4,-8k-4}\to E_2^{8k+4,-8k-4}$ coincides with the multiplication $H^{8k+4}(N^n;\mathbf{Z})\to H^{8k+4}(N^n;\mathbf{Z})$ by 2. Then the aboves imply that $\underline{v}^s$ is not in the image of $c\colon K_{\mathbf{R}}(N^n)\to K_{\mathbf{C}}(N^n)$ for s odd and $0\leqq s\leqq n$. On the other hand, the image of $c\colon K_{\mathbf{R}}(HP^n)\to K_{\mathbf{C}}(HP^n)$ is generated by $\varepsilon_k v^k$ for $0\leqq k\leqq n$, where $\varepsilon_k=1$ for k odd or 2 for k even [20, 3.11]. Then (i) and the commutative diagram

$$\begin{array}{ccc} K_{\mathbf{R}}(HP^n) & \xrightarrow{c} & K_{\mathbf{C}}(HP^n) \\ \downarrow{\scriptstyle \pi_2^!} & & \downarrow{\scriptstyle \pi_2^!} \\ K_{\mathbf{R}}(N^n) & \xrightarrow{c} & K_{\mathbf{C}}(N^n) \end{array}$$

imply that the composition

$$K_{\mathbf{R}}(HP^n)\xrightarrow{\pi_2^!}K_{\mathbf{R}}(N^n)\longrightarrow K_{\mathbf{R}}(N^n)/\mathrm{Tor}$$

is an isomorphism. Therefore we have the split exact sequence

$$0\longrightarrow K_{\mathbf{R}}(HP^n)\longrightarrow K_{\mathbf{R}}(N^n)\longrightarrow Z_{2^{f(n;\mathbf{R})}}\longrightarrow 0$$

as desired.

(iii) $F=\mathbf{H}$-case. Identifying KSp with $K_{\mathbf{R}}^{-4}$, we can prove the proposition for $F=\mathbf{H}$ by the same methods with (ii). And we complete the proof of Proposition.

Corollary 3.4. (i) *We have the exact sequence*

$$0 \longrightarrow \tilde{K}_F(N_k^{n+k}) \longrightarrow K_F(N^{n+k}) \longrightarrow K_F(N^{k-1}) \longrightarrow 0.$$

(ii) *The complex conjugation* $t: K_{\mathbf{C}}(N^n) \longrightarrow K_{\mathbf{C}}(N^n)$ *is the identity.*

Proof. (3.3) and the exact sequence

$$0 \longrightarrow \tilde{K}_F(HP_k^{n+k}) \longrightarrow K_F(HP^{n+k}) \longrightarrow K_F(HP^{k-1}) \longrightarrow 0$$

imply (i). Since

$$t(\hat{\theta}_{\mathbf{C}}) = t \circ c(\hat{\theta}_{\mathbf{R}}) = c(\hat{\theta}_{\mathbf{R}}) = \hat{\theta}_{\mathbf{C}}$$

and $t: K_{\mathbf{C}}(HP^n) \to K_{\mathbf{C}}(HP^n)$ is the identity function, we have (ii) by (3.3).

We shall evaluate the J-groups $J(N^n)$ [3]. Let $\Psi_F^k: K_F(X) \to K_F(X)$ be the Adams operation for $F=\mathbf{R}$ or $\mathbf{C}$. By now proved Adams conjecture [2] we may identify $J(X)$ with $\tilde{K}_{\mathbf{R}}(X)/\bigcap_e Y_e$, where $e: \mathbf{Z} \to \{0, 1, 2, \ldots\}$ and $Y_e = \sum_{k \in \mathbf{Z}} k^{e(k)}(\Psi_{\mathbf{R}}^k - 1)K_{\mathbf{R}}(X)$. We have

$$\Psi_{\mathbf{R}}^k(\hat{\theta}_{\mathbf{R}}) = \hat{\theta}_{\mathbf{R}}^k = \begin{cases} \hat{\theta}_{\mathbf{R}} & \text{if } k \text{ is odd} \\ 1 & \text{if } k \text{ is even.} \end{cases}$$

Then, since $\Psi_{\mathbf{R}}^k$ commutes with $\pi_2^!$, we have the following proposition by (3.3).

Proposition 3.5. *There exists a split exact sequence*

$$0 \longrightarrow J(HP^n) \longrightarrow J(N^n) \longrightarrow Z_{2^{2[n/2]+2}} \longrightarrow 0$$

and then the J-orders of the canonical symplectic line bundles over HP^n *and* N^n *are equal.*

Let B_n be the J-order of the canonical symplectic line bundle $\xi_n(S^3)$ over HP^n. (B_n has been computed by Sigrist and Suter [21].) Then by Atiyah [3, 2.6] we have

Theorem 3.6. *If* $k-l\equiv 0(B_n)$, *then* N_k^{n+k} *and* N_l^{n+l} *are of the same stable homotopy type.*

Using above propositions and corollary, we may prove the following theorem by following faithfully the proof of [9, §4] which has treated HP_k^{n+k} instead of N_k^{n+k}.

Theorem 3.7. *If* N_k^{n+k} *and* N_l^{m+l} *are of the same stable homotopy type, then* $m=n$ *and one of the following conditions hold*:

(i) $k-l\equiv 0(B_n)$

(ii) $k-l\equiv 0(B_{n-1})$ *and* $k+l\equiv 0(B_n)$.

4. Representations of the Generalized Quaternion Groups

In this section we examine the representation groups of the generalized quaternion groups $D^*(2^{m+1})$ according to Pitt [19].

Let $R_F(G)$ denote real$(F=\mathbf{R})$, complex$(F=\mathbf{C})$ or symplectic$(F=\mathbf{H})$ representation group of a group G. There exist the natural homomorphisms

$$R_{\mathbf{R}}(G) \underset{r}{\overset{c_{\mathbf{R}}}{\rightleftarrows}} R_{\mathbf{C}}(G) \underset{c'}{\overset{h}{\rightleftarrows}} R_{\mathbf{H}}(G)$$

satisfying the relations

$$r\circ c_{\mathbf{R}}=2, \qquad c_{\mathbf{R}}\circ r=1+t$$

$$h\circ c'=2, \qquad c'\circ h=1+t,$$

where $t\colon R_{\mathbf{C}}(G)\to R_{\mathbf{C}}(G)$ is complex conjugation. Being $R_F(G)$ free, $c_{\mathbf{R}}$ and c' are monomorphisms and in what follows we shall identify $R_{\mathbf{R}}(G)$ and $R_H(G)$ with their images in $R_{\mathbf{C}}(G)$ under $c_{\mathbf{R}}$ and c'.

Recall that $D^*(2^{m+1})=\{x, y; x^{2^{m-1}}=(yx)^2=y^2\}$. We consider the following complex representations of $D^*(2^{m+1})$:

$$1\begin{cases} x \longrightarrow 1 \\ y \longrightarrow 1 \end{cases} \qquad b\begin{cases} x \longrightarrow -1 \\ y \longrightarrow 1 \end{cases}$$

$$a\begin{cases} x \longrightarrow 1 \\ y \longrightarrow -1 \end{cases} \qquad c\begin{cases} x \longrightarrow -1 \\ y \longrightarrow -1 \end{cases}$$

$$d_k\begin{cases} x \longrightarrow \begin{bmatrix} \omega^k & 0 \\ 0 & \omega^{-k} \end{bmatrix} \\ y \longrightarrow \begin{bmatrix} 0 & (-1)^k \\ 1 & 0 \end{bmatrix} \end{cases}, \quad k \in Z,$$

where ω is a primitive 2^m-th root of unity. The characters of these representations are

$$\chi_1(x^u y^v) = 1, \qquad \chi_b(x^u y^v) = (-1)^u,$$

$$\chi_a(x^u y^v) = (-1)^v, \qquad \chi_c(x^u y^v) = (-1)^{u+v},$$

$$\chi_{d_k}(x^u y^v) = (\omega^{uk} + \omega^{-uk})(1-v),$$

where $u = 1, 2, \dots, 2^m$, $v = 0, 1$. Evaluating the characters, we have the relations

4.1.
$$a^2 = b^2 = c^2 = 1, \qquad ab = c, \qquad bc = a, \qquad ca = b,$$
$$d_0 = 1 + a, \qquad d_{2^{m-1}} = b + c, \qquad d_{-k} = d_k,$$
$$d_{2^{m-1}+k} = d_{2^{m-1}-k}, \qquad d_k d_j = d_{k+j} + d_{k-j}, \qquad a d_k = d_k,$$
$$b d_k = c d_k = d_{2^{m-1}-k}.$$

Then we have

4.2. *$R_{\mathbf{C}}(D^*(2^{m+1}))$ is free abelian on* $1, b$ *and* d_k $(0 \leqq k \leqq 2^{m-1})$ *and generated multiplicatively by* $1, a, b$ *and* d_1. *Therefore* $t =$ *identity on* $R_{\mathbf{C}}(D^*(2^{m+1}))$.

4.3. *$R_{\mathbf{R}}(D^*(2^{m+1}))$ is free abelian on* $1, b, d_{2k}$ $(0 \leqq k \leqq 2^{m-2})$ *and* $2d_{2k+1}$ $(0 \leqq k < 2^{m-2})$ *and generated multiplicatively by* $1, a, b, 2d_1$ *and* d_1^2.

4.4. *$R_{\mathbf{H}}(D^*(2^{m+1}))$ is free abelian on $2, 2b, 2d_{2k}$ $(0\leqq k\leqq 2^{m-2})$ and d_{2k+1} $(0\leqq k<2^{m-2})$.*

Let $\lambda^k(\ )$ be the exterior k-th power operation and put

$$\lambda_t(x)=\sum_{k\geqq 0}\lambda^k(x)t^k\in R_{\mathbf{C}}(G)[[t]] \qquad \text{for} \quad x\in R_{\mathbf{C}}(G).$$

Then it is well known that

$$\lambda_t(x+y)=\lambda_t(x)\lambda_t(y).$$

Hence

$$\lambda_t(nd_1)=(\lambda_t(d_1))^n=(1+d_1t+t^2)^n.$$

Therefore we have

Lemma 4.5. $\lambda_{-1}(nd_1)=(2-d_1)^n.$

For the proof of Proposition 5.7, we prepare the following lemma.

Lemma 4.6. *In $R_{\mathbf{C}}(D^*(2^{m+1}))$ we have the relations*

$$d_1^{2k}=\frac{1}{2}\sum_{-\infty<t<\infty}\binom{2k}{k+2^{m-1}t}d_0+\sum_{j=1}^{2^{m-2}-1}\sum_{-\infty<t<\infty}\binom{2k}{k+2^{m-1}t-j}d_{2j}$$
$$+\frac{1}{2}\sum_{-\infty<t<\infty}\binom{2k}{k+2^{m-1}t-2^{m-2}}d_{2^{m-1}},$$

$$d_1^{2k+1}=\sum_{j=0}^{2^{m-2}-1}\sum_{-\infty<t<\infty}\binom{2k+1}{k+2^{m-1}t-j}d_{2j+1}.$$

Proof. Using (4.1), we may prove this by induction on k. The proof is elementary and easy, so we omit it.

5. $K_F(N^n(G))$

Hereafter G denotes a finite subgroup of S^3.

Let $V(=H)$ be the representation space of $d_1: G\subset S^3=Sp(1)=SU(2)$. Put $E=(n+1)V$ and consider the following exact sequence of equivariant K-theory.

$$\cdots \longrightarrow KF_G(DE, SE) \longrightarrow KF_G(DE) \longrightarrow$$

$$KF_G(SE) \longrightarrow KF_G^1(DE, SE) \longrightarrow \cdots,$$

where F denotes $\mathbf{R}$ or $\mathbf{C}$. By Thom isomorphism, this induces the exact sequence

$$\cdots \longrightarrow KF_G^{-4n-4}(pt.) \xrightarrow{\psi_F} R_F(G) \xrightarrow{\phi_F} K_F(N^n(G))$$

$$\longrightarrow KF_G^{1-4n-4}(pt.) \longrightarrow \cdots .$$

Recalling that

$$KC_G^{-4n-4}(pt.) = R_C(G),$$

$$KR_G^{-4n-4}(pt.) = \begin{cases} R_R(G) & \text{if } n \text{ is odd} \\ R_H(G) & \text{if } n \text{ is even,} \end{cases}$$

then ψ_F is the multiplication by $\lambda_{-1}((n+1)d_1) = (2-d_1)^{n+1}$ which is contained in $R_{\mathbf{R}}(G)$ (if n is odd) or $R_{\mathbf{H}}(G)$ (if n is even). When $F=\mathbf{C}$, these are as usual. In case $F=R$, see [19]. ϕ_F maps a representation of G to its associated vector bundle induced from the principal G bundle $S((n+1)V) \to N^n(G)$. Hence $\phi_{\mathbf{C}}(d_1) = \xi_n(G)_{\mathbf{C}}$ and $\phi_{\mathbf{R}}(r(d_1)) = \xi_n(G)_{\mathbf{R}}$. Since $KC_G^{odd}(pt.) = 0$ and $KR_G^{-k}(pt.) = 0$ for $k \equiv 3, 7(8)$ ([4]), we obtain the exact sequence

$$KF_G^{-4n-4}(pt.) \xrightarrow{\psi_F} R_F(G) \xrightarrow{\phi_F} K_F(N^n(G)) \longrightarrow 0,$$

and then we have

Proposition 5.1. (cf. [10], [19]) $\phi_F\colon R_F(G) \to K_F(N^n(G))$ *induces the isomorphisms*

$$K_{\mathbf{C}}(N^n(G)) \cong R_{\mathbf{C}}(G)/(2-d_1)^{n+1} R_{\mathbf{C}}(G)$$

$$K_{\mathbf{R}}(N^n(G)) \cong \begin{cases} R_{\mathbf{R}}(G)/(2-d_1)^{n+1} R_{\mathbf{R}}(G) & \textit{if } n \textit{ is odd} \\ R_{\mathbf{R}}(G)/(2-d_1)^{n+1} R_{\mathbf{H}}(G) & \textit{if } n \textit{ is even.} \end{cases}$$

In the rest of this section, we consider the case with G a generalized quaternion group $D^*(2^{m+1})$. For simplicity we will use the notations

$$N^n(m)=N^n(D^*(2^{m+1})),\quad N_k^{n+k}(m)=N^{n+k}(m)/N^{k-1}(m),$$

$$\xi_n(m)=\xi_n(D^*(2^{m+1})),$$

$$\delta'_n(m)=\xi_n(m)_{\mathbf{C}}-2\in\tilde{K}_{\mathbf{C}}(N^n(m))\quad\text{and}$$

$$\delta_n(m)=\xi_n(m)_{\mathbf{R}}-4\in\tilde{K}_{\mathbf{R}}(N^n(m)).$$

The remaining part of this section is devoted to evaluate the orders of $\delta'_n(m)=\phi_{\mathbf{C}}(d_1-2)$ and $\delta_n(m)=\phi_{\mathbf{R}}(r(d_1)-4)$.

Proposition 5.2.

$$\#\delta'_n(m)^k=\begin{cases}2^{m+2(n-k)+1} & \text{if } 1\leqq k\leqq n\\ 1 & \text{if } k>n \text{ or } n=0.\end{cases}$$

Proof. By (5.1), we have that $\delta'_n(m)^k=\phi_{\mathbf{C}}((d_1-2))^k=0$ for $k>n$ or $n=0$. Let η be the canonical complex line bundle over CP^{2n+1}. Put $\sigma=\pi(Z_{2^m}, S^1)^*\eta-1\in\tilde{K}_{\mathbf{C}}(L^{2n+1}(2^m))$. Then we have

$$\begin{aligned}\pi(Z_{2^m}, D^*(2^{m+1}))^!\delta'_n(m)^k&=\pi(Z_{2^m}, S^1)^!\pi(S^1, S^3)^!(\xi_n(S^3)_{\mathbf{C}}-2)^k\\&=(\sigma+\bar{\sigma})^k\\&=\sigma^{2k}+\text{higher terms.}\end{aligned}$$

By [13, 1.1], we have

$$\#\sigma^k=\begin{cases}2^{m+2n+1-k} & \text{if } 1\leqq k\leqq 2n+1\\ 1 & \text{if } k>2n+1.\end{cases}$$

Then we know that $\#\delta'_n(m)^k$ is a multiple of $2^{m+2(n-k)+1}$ if $1\leqq k\leqq n$.

To obtain an upper bound of $\#\delta'_n(m)^k$, we use the complex cobordism theory U^*.

5.3. (Conner-Floyd [7]). *There exists a monomorphism $\tilde{K}_{\mathbf{C}}(X)\to U^2(X)$ for any finite connected CW-complex X.*

Since the tangent bundle $\tau N^n(m)$ of $N^n(m)$ satisfies the condition

$$\tau N^n(m)\oplus 1\cong(n+1)\xi_n(D^*(2^{m+1}))_{\mathbf{R}}$$

[22, 3.3], $N^n(m)$ is a U-manifold. Then there is a duality isomorphism

$$U^k(N^n(m))\cong U_{4n+3-k}(N^n(m)),$$

and in particular we have

$$U^2(N^n(m))\cong U_{4n+1}(N^n(m)).$$

Since $N^n(m)$ is the $4n+3$-skeleton of $BD^*(2^{m+1})=\bigcup_n N^n(m)$ [10], we have

$$U_{4n+1}(N^n(m))\cong U_{4n+1}(N^{n+1}(m))\cong\cdots\cong U_{4n+1}(BD^*(2^{m+1})).$$

Hence we have a monomorphism

$$\tilde{K}_{\mathbf{C}}(N^n(m))\longrightarrow U_{4n+1}(BD^*(2^{m+1})).$$

Since $H_*(BD^*(2^{m+1});\mathbf{Z})$ is periodic ([6, XII]), the Atiyah-Hirzebruch spectral sequence for $U_*(BD^*(2^{m+1}))$ collapses ([16]) and then the Thom map $\mu\colon U_*(BD^*(2^{m+1}))\to H_*(BD^*(2^{m+1}))$ is epimorphic. Recall $D^*(2^{m+1})=\{x, y; x^{2^{m-1}}=(yx)^2=y^2\}$. We will identify $\mathbf{Z}_{2^m}$ and $\mathbf{Z}_4$ with the subgroups of $D^*(2^{m+1})$ generated by x and y respectively. Let $i'_1\colon\mathbf{Z}_{2^m}\to D^*(2^{m+1})$ and $i'_2\colon\mathbf{Z}_4\to D^*(2^{m+1})$ be those inclusions. And let $i_1\colon BZ_{2^m}=L^\infty(2^m)\to N^\infty(m)=BD^*(2^{m+1})$ and $i_2\colon BZ_4=L^\infty(4)\to BD^*(2^{m+1})$ be the induced maps (see §2). And we will write the following inclusions by the same letter ι:

$$N^k(m)\subset BD^*(2^{m+1}),\ L^{2k}(2^m)\subset L^\infty(2^m)=BZ_{2^m}\quad\text{and}$$

$$L^{2k}(4)\subset L^\infty(4)=BZ_4.$$

Then $\{\mu[N^k(m),\iota],\ \mu i_{1*}[L^{2k}(2^m),\iota],\ \mu i_{2*}[L^{2k}(4),\iota];\ 0\leqq k\}$ generates $\tilde{H}_*(BD^*(2^{m+1});\mathbf{Z})$ and then $\{[N^k(m),\iota],\ i_{1*}[L^{2k}(2^m),\iota],\ i_{2*}[L^{2k}(4),\iota];\ 0\leqq k\}$ generates the U_*-module $\tilde{U}_*(BD^*(2^{m+1}))$. The orders of these U_*-module generators have been computed by K. Shibata and Y. Katsube (unpublished) as follows:

5.4. $\quad\#[N^k(m),\iota]=2^{m+2k+1},$

$$\sharp i_{1*}[L^{2k}(2^m), \iota] = \begin{cases} 2 & \text{if } k=0 \\ 2^{m+2k-1} & \text{if } k>0 \end{cases}, \text{ and}$$

$$\sharp i_{2*}[L^{2k}(4), \iota] = 2^{2k+1}.$$

This implies

$$2^{m+2n-1} U_{4n+1}(BD^*(2^{m+1})) = 0$$

and then

$$2^{m+2n-1} \tilde{K}_{\mathbf{C}}(N^n(m)) = 0.$$

Comparing this upper bound with the above lower bound of $\sharp\delta'_n(m)$, we have

5.5.
$$\sharp\delta'_n(m) = \begin{cases} 2^{m+2n-1} & \textit{if } n>0 \\ 1 & \textit{if } n=0. \end{cases}$$

To compute $\sharp\delta'_n(m)^k$, we prepare the following lemma.

Lemma 5.6. (cf. [19, 5.2]) *For* $\lambda \in \mathbf{Z}$, $\alpha \in R_{\mathbf{C}}(D^*(2^{m+1}))$ *and* $k \geqq 2$, $\lambda(d_1-2) = \alpha(d_1-2)^{n+1}$ *holds if and only if* $\lambda(d_1-2)^k = \alpha(d_1-2)^{n+k}$.

Proof. Only if part is trivial. (5.1) and (5.5) imply

$$2^{m+2n-1}(d_1-2) = \beta(d_1-2)^{n+1} \quad \text{for some} \quad \beta \in R_{\mathbf{C}}(D^*(2^{m+1})),$$

and in particular

$$2^{m+1}(d_1-2) = \beta(d_1-2)^2 \quad \text{for some} \quad \beta \in R_{\mathbf{C}}(D^*(2^{m+1})).$$

Then

$$\beta^{k-1}(d_1-2)^k = 2^{k(m+1)}(d_1-2)$$

and hence

$$\beta^{k-1}(d_1-2)^{n+k} = 2^{k(m+1)}(d_1-2)^{n+1}.$$

Let $\lambda(d_1-2)^k = \alpha(d_1-2)^{n+k}$. Then

$$2^{k(m+1)}\lambda(d_1-2)=\lambda\beta^{k-1}(d_1-2)^k=\beta^{k-1}\alpha(d_1-2)^{n+k}$$
$$=2^{k(m+1)}\alpha(d_1-2)^{n+1}.$$

But $R_{\mathbf{C}}(D^*(2^{m+1}))$ is free, so we have

$$\lambda(d_1-2)=\alpha(d_1-2)^{n+1}.$$

Thus the lemma (5.6) follows.

(5.6) implies

$$\#\delta'_n(m)^k=\#\delta'_{n-k+1}(m) \qquad \text{for} \quad 1\leqq k\leqq n$$

and hence

$$\#\delta'_n(m)^k=2^{m+2(n-k)+1} \qquad \text{for} \quad 1\leqq k\leqq n.$$

This completes the proof of the proposition.

Since d_1 is self conjugate (i.e. $t(d_1)=d_1$), we have

$$t(\delta'_n(m))=\delta'_n(m) \qquad \text{and} \qquad c(\delta_n(m))=2\delta_n(m).$$

Then we have

$$\#\delta_n(m)=2^{m+2n-2} \qquad \text{or} \qquad 2^{m+2n-1}.$$

Pitt [19, 5.5] has proved

$$\#\delta_1(m)=2^{m+1}.$$

Using (4.7) and the method of Pitt, the author has checked the following proposition.

Proposition 5.7.

$$\#\delta_n(2)=\begin{cases} 2^{2n+1} & \textit{if } n \textit{ is odd} \\ 2^{2n} & \textit{if } n \textit{ is even} \end{cases}, \quad \#\delta_n(3)=\begin{cases} 2^{2n+2} & \textit{if } n \textit{ is odd} \\ 2^{2n+1} & \textit{if } n \textit{ is positive and even} \\ 1 & \textit{if } n=0 \end{cases}$$

and

$$\#\delta_2(m)=2^{m+2}.$$

In case $m=2$, this has been proved by Fujii [11], more generally he has determined the additive structure of $K_{\mathbf{R}}(N^n(2))$. The proof of (5.7) is long and routine and so we omit it.

Remark. By (5.7), we may conjecture that for $n>0$

$$\#\delta_n(m)=\begin{cases} 2^{m+2n-1} & \text{if } n \text{ is odd} \\ 2^{m+2n-2} & \text{if } n \text{ is even.} \end{cases}$$

6. $J(N^n(2))$

The purpose of this section is to prove the following theorem.

Theorem 6.1. *J-homomorphism* $J\colon \tilde{K}_{\mathbf{R}}(N^n(2))\to J(N^n(2))$ *is an isomorphism.*

Since $\Psi_{\mathbf{C}}^{-1}=t$ is identity on $K_{\mathbf{C}}(N^n(m))$ by (4.2) and (5.1), we have $\Psi_F^k=\Psi_F^{-k}$ on $K_F(N^n(m))$. So we will consider Ψ_F^k for k non negative only.

Consider Adams operation $\Psi_F^k\colon R_F(G)\to R_F(G)$. Concerning the characters it is well known that

$$\chi_{\Psi_F^k(\theta)}(g)=\chi_\theta(g^k) \quad \text{for} \quad \theta\in R_F(G) \quad \text{and} \quad g\in G$$

(see [1, 4.4]). Then (4.2) and a short character computation show that

$$\Psi_F^k=\text{identity on } R_F(D^*(8)) \text{ for } k \text{ odd.}$$

Then (5.1) and the following commutative diagram

$$\begin{array}{ccc} K_F(N^n(2)) & \xleftarrow{\ \phi_F\ } & R_F(D^*(8)) \\ \downarrow{\scriptstyle \Psi_F^k} & & \downarrow{\scriptstyle \Psi_F^k} \\ K_F(N^n(2)) & \xleftarrow[\ \phi_F\ ]{} & R_F(D^*(8)) \end{array}$$

imply

Lemma 6.2. $\Psi_F^k=$ *identity on* $K_F(N^n(2))$ *for* k *odd.*

Now we prove Theorem 6.1. Since $\tilde{K}_F(N_n(m))$ is a 2-primary group (see e.g. [6], [10], [11]), we have $2^N\tilde{K}_{\mathbf{R}}(N^n(2))=0$ for some N. Let us choose $e: \mathbf{Z}\to\{0, 1, 2,\dots\}$ so that $e(k)\geqq N$ for k even. Then $k^{e(k)}(\Psi_{\mathbf{R}}^k-1)K_{\mathbf{R}}(N^n(2))=0$ for k even. But for k odd $\Psi_{\mathbf{R}}^k$ is identity on $K_{\mathbf{R}}(N^n(2))$ by (6.2), so that $k^{e(k)}(\Psi_{\mathbf{R}}^k-1)K_{\mathbf{R}}(N^n(2))=0$. Thus we have $Y_e=0$ for this function e, and hence $\bigcap_e Y_e=0$ (see § 3 for the definition of Y_e). This completes the proof of Theorem 6.1.

As a corollary of this theorem and (5.7), we have the following.

Corollary 6.3.

$$\# J(\delta_n(2))=\begin{cases} 2^{2n+1} & \textit{if } n \textit{ is odd} \\ 2^{2n} & \textit{if } n \textit{ is even.}\end{cases}$$

7. $J(N^n(G))$

In this section we evaluate the J-order of $\xi_n(G)$.

For simplicity we will use the notation $J(\tau)$ instead of $J(\tau\text{-}\dim_{\mathbf{R}}\tau)$ for a vector bundle τ.

Consider the induced homomorphism $\pi(G, S^3)^*: J(HP^n)\to J(N^n(G))$. Then, since $\pi(G, S^3)^*J(\xi_n(S^3))=J(\xi_n(G))$, we have

Proposition 7.1. *$\# J(\xi_n(G))$ is a factor of B_n.*

By (5.2) and (5.7) we have

Proposition 7.2. (i) *$\# J(\xi_n(D^*(2^{m+1})))$ is a factor of 2^{m+2n-1}.*

(ii) *$\# J(\xi_n(D^*(16)))$ is a factor of 2^{2n+2} (if n is odd) or 2^{2n+1} (if n is even).*

(iii) *$\# J(\xi_2(D^*(2^{m+1})))$ is a factor of 2^{m+2}.*

Let $\mathbf{Z}_k$ be a cyclic subgroup of G and $\eta_{2n+1}(k)$ be the canonical complex line bundle over $L^{2n+1}(k)$. Since $\pi(\mathbf{Z}_k, G)^*\xi_n(G)_{\mathbf{C}}=\eta_{2n+1}(k)+\bar{\eta}_{2n+1}(k)$, we have

$$\pi(\mathbf{Z}_k, G)^*J(\xi_n(G))=2J(\eta_{2n+1}(k)).$$

Then we have

Proposition 7.3. *If* $Z_k \subset G$, *then* $\# J(\xi_n(G))$ *is a multiple of* $\# 2J(\eta_{2n+1}(k))$.

Remark. $\# J(\eta_n(k))$ has been determined by Kambe-Matsunaga-Toda [12] and Kobayashi-Sugawara [15] when $k=p$ or p^2 for p prime.

When $D^*(8)\subset G$ i.e. $G=D^*(8m)$, T^*, O^* or I^*, we obtain the following proposition by (6.3), since $\pi(D^*(8), G)^* J(\xi_n(G))=J(\xi_n(D^*(8)))$.

Proposition 7.4. *If* $D^*(8)\subset G$, *then* $\# J(\xi_n(G))$ *is a multiple of* 2^{2n+1} (*if* n *is odd*) *or* 2^{2n} (*if* n *is even*).

As a corollary of this we have

Corollary 7.5. *If* $n=2^u+2v+1$ *for* $\frac{u-3}{4}\leqq v\leqq 2^{u-1}-1$ *and* $u\geqq 1$, *then* $\# J(\xi_n(D^*(2^{m+1})))=2^{2n+1}$.

Proof. Recall that

$$\upsilon_2(B_n)=\max\{2n+1,\ 2j+\upsilon_2(j);\ 1\leqq j\leqq n\}$$

(see [21]), where $\upsilon_2(w)$ denotes the largest integer for which $2^{\upsilon_2(w)}$ divides w. If n satisfies the above condition, then $\upsilon_2(B_n)=2n+1$ and then (7.1) and (7.4) imply (7.5).

8. S-types of $N_k^{n+k}(G)$

Evaluating the (co)homology groups of $N_k^{n+k}(G)$ (see [6, XII §§7, 8, 9]), we have

Theorem 8.1. *If* $N_j^{m+j}(G)$ *and* $N_k^{n+k}(H)$ *are of the same stable homotopy type, then* G *is conjugate with* H *and* $m=n$.

By Atiyah [3, 2.6] and (2.1) we have

Proposition 8.2. *If* $j\equiv k\ (\# J(\xi_n(G)))$, *then* $N_j^{n+j}(G)$ *and* $N_k^{n+k}(G)$ *are of the same stable homotopy type.*

Put $B_n(m)=\min\{m+2n-1, \upsilon_2(B_n)\}$. Then (8.2) implies the following theorem by (5.1), (7.1) and (7.2).

Theorem 8.3. (i) *If* $j\equiv k(2^{B_n(m)})$, *then* $N_j^{n+j}(m)$ *is S-equivalent to* $N_k^{n+k}(m)$.

(ii) *For a fixed G, all* $N_n^n(G)$ *are of the same stable homotopy type.*

(iii) *If* $m=2$ *or* 3 *and* $j\equiv k \bmod \begin{cases} 2^{2n+m-1} & \text{if } n \text{ is odd} \\ 2^{2n+m-2} & \text{if } n \text{ is even} \end{cases}$, *then* $N_j^{n+j}(m)$ *is S-equivalent to* $N_k^{n+k}(m)$.

For the converse of this, we have the following theorem by methods of Kobayashi-Sugawara [15, 1.1].

Theorem 8.4. *If* $N_j^{n+j}(2)$ *and* $N_k^{n+k}(2)$ *are of the same stable homotopy type for* $n\geqq 1$, *then* $j\equiv k(2^{2n-2})$.

Proof. Consider the Puppe exact sequence

$$\tilde{K}_{\mathbf{C}}(S^1\wedge N^{n+j}(m))\xrightarrow{(1\wedge\iota)^!}\tilde{K}_{\mathbf{C}}(S^1\wedge N^{j-1}(m))\longrightarrow\tilde{K}_{\mathbf{C}}(N_j^{n+j}(m))$$
$$\xrightarrow{p^!}\tilde{K}_{\mathbf{C}}(N^{n+j}(m)).$$

Since Atiyah-Hirzebruch spectral sequences for $\tilde{K}_{\mathbf{C}}(S^1\wedge N^u(m))$ and $\tilde{K}_{\mathbf{C}}(N_j^{n+j}(m))$ collapse, we have $\tilde{K}_{\mathbf{C}}(S^1\wedge N^u(m))=\mathbf{Z}$ and $(1\wedge\iota)^!=0$. Hence above sequence induces the following exact one

$$0\longrightarrow\mathbf{Z}\longrightarrow\tilde{K}_{\mathbf{C}}(N_j^{n+j}(m))\xrightarrow{p^!}\tilde{K}_{\mathbf{C}}(N^{n+j}(m))$$

and then $p^!$ is monomorphic on $\mathrm{Tor}(\tilde{K}_{\mathbf{C}}(N_j^{n+j}(m)))$. Then by (6.2), we know that $\Psi_{\mathbf{C}}^{v}$ is identity on $\mathrm{Tor}(\tilde{K}_{\mathbf{C}}(N_j^{n+j}(2)))$ for v odd. Consider the following diagram

$$\begin{array}{ccc}\tilde{K}_{\mathbf{C}}(N_j^{n+j}(2)) & \xrightarrow{I^u} & \tilde{K}_{\mathbf{C}}(S^{2u}\wedge N_j^{n+j}(2))\\ \Psi_{\mathbf{C}}^{v}\downarrow & & \downarrow\Psi_{\mathbf{C}}^{v}\\ \tilde{K}_{\mathbf{C}}(N_j^{n+j}(2)) & \xrightarrow[I^u]{} & \tilde{K}_{\mathbf{C}}(S^{2u}\wedge N_j^{n+j}(2))\end{array}$$

where I indicates the Bott isomorphism. Then we have $\Psi_{\mathbf{C}}^{v}I^u=v^uI^u\Psi_{\mathbf{C}}^{v}$.

Therefore we have

8.5. $\Psi_{\mathbf{C}}^{2v+1}=(2v+1)^u$ *on* $\mathrm{Tor}(\tilde{K}_{\mathbf{C}}(S^{2u}\wedge N_j^{n+j}(2))$.

If $S^u\wedge N_j^{n+j}(m)$ is homotopy equivalent to $S^v\wedge N_k^{n+k}(m)$, then $v=u+4(j-k)$ by their cohomology groups.

Now suppose that there exists a homotopy equivalence

$$g: s^{2u+4(j-k)}\wedge N_k^{n+k}(2)\longrightarrow S^{2u}\wedge N_j^{n+j}(2)$$

and consider the following commutative diagram

$$\begin{array}{ccc} K_{\mathbf{C}}(S^{2u}\wedge N_j^{n+j}(2)) & \xrightarrow{g^!} & K_{\mathbf{C}}(S^{2u+4(j-k)}\wedge N_k^{n+k}(2)) \\ \Psi_{\mathbf{C}}^{2v+1}\big\downarrow & & \big\downarrow\Psi_{\mathbf{C}}^{2v+1} \\ K_{\mathbf{C}}(S^{2u}\wedge N_j^{n+j}(2)) & \xrightarrow[g^!]{} & K_{\mathbf{C}}(S^{2u+4(j-k)}\wedge N_k^{n+k}(2)) \end{array}$$

Then (8.5) implies that

8.6. $(2v+1)^{u+2(j-k)}g^!=(2v+1)^u g^!$ *on* $\mathrm{Tor}(\tilde{K}_{\mathbf{C}}(S^{2u}\wedge N_j^{n+j}(2)))$.

Since

$$\begin{aligned}\tilde{K}_{\mathbf{C}}(S^{2u}\wedge N_j^{n+j}(2))&\cong\tilde{K}_{\mathbf{C}}(N_j^{n+j}(2))\\ &\cong K_{\mathbf{C}}^{-4j}(N^n(2)) \quad \text{(Thom isomorphism)}\\ &\cong K_{\mathbf{C}}(N^n(2)),\end{aligned}$$

there is an element of order 2^{2n+1} in $\tilde{K}_{\mathbf{C}}(S^{2u}\wedge N_j^{n+j}(2))$ by (5.2). Then (8.6) implies that

$$(2v+1)^{u+2(j-k)}-(2v+1)^u\equiv 0(2^{2n+1}),$$

that is

$$(2v+1)^{2(j-k)}-1\equiv 0(2^{2n+1}).$$

It was proved by Adams [1, 8.1] that

$$\text{if } b=(2a+1)2^f, \text{ then } 3^b-1\equiv 2^{f+2}(2^{f+3}).$$

This implies

$$3^{2(j-k)}-1\equiv 2^{\upsilon_2(2(j-k))+2}(2^{\upsilon_2(2(j-k))+3}).$$

Then

$$2n+1\leqq \upsilon_2(2(j-k))+2=\upsilon_2(j-k)+3,$$

and therefore

$$j-k\equiv 0(2^{2n-2}).$$

This completes the proof of Theorem 8.4.

References

[1] Adams, J. F., Vector fields on spheres, *Ann. of Math.* **75** (1962), 603–632.

[2] ———, On the groups $J(X)$–I, *Topology* **2** (1963), 181–195.

[3] Atiyah, M. F., Thom complexes, *Proc. London Math. Soc.* **11** (1961), 291–310.

[4] ——— and Segal, G., Equivariant K-theory and completions, *J. Differential Geometry* **3** (1968), 1–18.

[5] Bredon, G. E., *Introduction to compact transformation groups*, Academic Press, New York, 1972.

[6] Cartan, H. and Eilenberg, S., *Homological Algebra,* Princeton Univ. Press, Princeton, New Jersey, 1956.

[7] Conner, P. E., and Floyd, E. E., The relation of cobordism ot K-theoreis, *Lecture Notes in Math.* **28**, Springer-Verlag, 1966.

[8] Feder, S. and Gitler, S., Stunted projective spaces and the J-order of the Hopf bundle, *Bull. Amer. Math. Soc.* **80** (1974), 748–749.

[9] ———, The classification of stunted projective spaces by stable homotopy type, preprint.

[10] Fujii, K., On the K-ring of S^{4n+3}/H_m, *Hiroshima Math. J.* **3** (1973), 251–265.

[11] ———, On the KO-rings of S^{4n+3}/H_m, *Hiroshima Math. J.* **4** (1974), 459–475.

[12] Kambe, T., Matsunaga, H. and Toda, H., A note on stunted space, *J. Math. Kyoto Univ.* **5** (1966), 143–149.

[13] Kawaguchi, T. and Sugawara, M., K- and KO-rings of the lens space $L^n(p^2)$ for odd prime p, *Hiroshima Math. J.* **1** (1971), 273–286.

[14] Kobayashi, T. and Sugawara, M., K_Λ-rings of lens spaces $L^n(4)$, *Hiroshima Math. J.* **1** (1971), 253–271.

[15] Kobayashi, T. and Sugawara, M., On stable homotopy types of stunted lens spaces, *Hiroshima Math. J.* **1** (1971), 281–304.

[16] Landweber, P. S., Complex bordism of Classifying spaces, *Proc. Amer. Math. Soc.* **27** (1971), 175–179.

[17] Montgomery, D. and Zippin, L., *Topological transformation group,* Interscience Publishers, Inc., New York, 1955.

[18] Orlik, P., Seifert manifolds, *Lecture Notes in Math.* **291**, Springer (1972).

[19] Pitt, D., Free actions of generalized quaternion groups on spheres, *Proc. London Math. Soc.* **26** (1973), 1–18.
[20] Sanderson, B. J., Immersions and embeddings of projective spaces, *Proc. London Math. Soc.* **14** (1964), 137–153.
[21] Sigrist, F. and Suter, U., Cross-sections of symplectic Stiefel manifolds, *Trans. Amer. Math. Soc.* **184** (1973), 247–259.
[22] Szczarba, R. H., On tangent bundles of fibre spaces and quotient spaces, *Amer. J. Math.* **86** (1964), 685–697.
[23] Wolf, J. A., *Spaces of constant curvature,* McGraw Hill, New York, 1967.