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A Remark on a Necessary Condition of the Cauchy-Kowalevski Theorem

By

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1. Introduction

We are concerned with a necessary condition of the Cauchy-Kowalevski theorem for the differential operator L with the analytic coefficients in the neighbourhood of the origin:

$$L=\partial_t^m-\sum_{j=1}^{m} a_j(t, x; \partial)\partial_t^{m-j}.\ ^{1)}$$

Let

$$a_j(t, x; \partial)\equiv\sum_{\alpha} a_{j,\alpha}(t, x)\partial^\alpha\equiv\sum_{\alpha,\mu} a_{j,\alpha}^{\mu}(x)t^\mu\partial^\alpha\equiv\sum_{\mu} t^\mu a_j^\mu(x; \partial)\,.$$

Professor S. Mizohata [3] defined the weight q of L by

$$q\equiv\mathrm{Min}\{q;\ \underset{\partial}{\mathrm{order}}\, a_j(t, x; \partial)\leqq q\cdot j,\qquad j=1, 2,\dots, m\}$$

and, denoting by $h_j(t, x; \partial)$ the homogeneous part of $a_j(t, x; \partial)$ with order $q\cdot j$, he showed that, in order that the Cauchy-Kowalevski theorem for L hold at the origin, it is necessary that

$$h_j(0, x; \partial)\equiv 0\quad (j=1, 2,\dots, m),\qquad \text{if}\quad q>1.$$

Mr. M. Miyake [2] investigates the first order operator L^1

$$L^1=\partial_t-a(t, x; \partial)$$

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1) We use the following abbreviations: $\partial^\alpha=\partial_x^\alpha$, ∂_t^j stand for $\left(\frac{\partial}{\partial x}\right)^\alpha$, $\left(\frac{\partial}{\partial t}\right)^j$ respectively.

and proved that, in order that the Cauchy-Kowalevski theorem for L^1 hold at the origin, it is necessary and sufficient that

$$\operatorname*{order}_{\partial} a(t, x; \partial) \leqq 1.$$

Professor S. Mizohata [4] investigates again a necessary condition of the Cauchy-Kowalevski theorem for L and showed that it is necessary that

$$p \leqq 1,$$

where p is the modified weight of L which is defined by

$$p = \operatorname{Min}\{p;\ |\alpha| + p(m-j-\mu) \leqq p \cdot m, \qquad a^{\mu}_{j,\alpha}(x) \not\equiv 0\},$$

and that, in particular,

$$\operatorname*{order}_{\partial} a_j(0, x; \partial) \leqq j \qquad (j=1, 2,\dots, m).$$

Mrs. Y. Hasegawa [1] remarked that it is moreover necessary that

$$\operatorname*{order}_{\partial} a^1_j(x; \partial) \leqq j \qquad (j=1, 2,\dots, m),$$

and sent us her manuscript.

In a series of these researches, they proved their results by means of the formal solution. Using their techniques, especially that of Mrs. Y. Hasegawa, we obtained the following result.

Let

$$n(j, \alpha) = \operatorname{Min}\{\mu;\ a^{\mu}_{j,\alpha}(x) \not\equiv 0\}.$$

Then the modified weight p of L is given by

$$p = \operatorname*{Max}_{j,\alpha}\left\{\frac{|\alpha|}{n(j, \alpha)+j}\right\}.$$

We define the modified principal part of L by

$$\sum_{j=1}^{m} \sum_{|\alpha| = p(n(j,\alpha)+j)} t^{n(j,\alpha)} a^{n(j,\alpha)}_{j,\alpha}(x) \partial^{\alpha} \partial_t^{m-j}$$

and we say that the terms

$$a^{\mu}_{j,\alpha}(x)t^{\mu}\partial^{\alpha}\partial_t^{m-j}; \quad \text{for which} \quad |\alpha|\leqq j$$

belong to the kowalevskian part of L.

Theorem. *In order that the Cauchy-Kowalevski theorem for L hold at the origin, it is necessary that the modified principal part of L is composed uniquely of the terms belonging to the kowalevskian part of L.*

Let

$$p_k=\mathrm{Max}\left\{\frac{|\alpha|}{n(j,\alpha)+j},\ |\alpha|\leqq j\right\},\ p_v=\mathrm{Max}\left\{\frac{|\alpha|}{n(j,\alpha)+j},\ |\alpha|>j\right\}.$$

Then our theorem is also represented as follows.

In order that the Cauchy-Kowalevski theorem for L hold at the origin, it is necessary that

$$p_v<p_k\,.$$

Remarks. Suppose that the Cauchy-Kowalevski theorem for L holds at the origin.

1) The theorem says that the (j, α) for which $p=\dfrac{|\alpha|}{n(j,\alpha)+j}$ satisfy $|\alpha|\leqq j$. This implies $p\leqq 1$.

2) In $a_j(t, x; \partial)$ we observe the terms $a_j^1(x;\partial)=\sum_{\alpha} a_{j,\alpha}^1(x)\partial^{\alpha}$. The theorem says that there does not exist the terms for which $|\alpha|=p(1+j)$ unless $p<1$. We have therefore $|\alpha|<1+j$, i.e. $|\alpha|\leqq j$. That is $\operatorname{order}_{\partial} a_j^1(x;\partial)\leqq j$.

2. Formal Solution and Recurrence Formula

We treat the following Cauchy problem;

$$\text{(2.1)} \qquad \begin{cases} L(u)=n_0t^{n_0-1}f(x), \\ u|_{t=0}=\cdots=\partial_t^{m-1}u|_{t=0}=0\,. \end{cases}$$

Setting $\vec{u}(t, x)={}^t(u^1(t, x), .., u^m(t, x))={}^t(u, \partial_t u,\dots, \partial_t^{m-1}u)$, we have

$$(2.2)\quad \begin{cases} \partial_t \vec{u}(t,x) = \begin{pmatrix} 0 & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ \mathbf{0} & & 0 & 1 \\ a_m(t,x;\partial), & a_{m-1}(t,x;\partial), & \dots, & a_1(t,x;\partial) \end{pmatrix} \vec{u}(t,x) + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ n_0 t^{n_0-1} f(x) \end{pmatrix}, \\ \vec{u}(0,x) = {}^t(0,\dots,0). \end{cases}$$

We denote the formal solution of (2.2) by $\vec{u}(t,x) \sim \sum_n t^n \vec{u}_n(x) = \sum_n t^n\, {}^t(u_n^1(x), \dots, u_n^m(x))$, then $\{\vec{u}_n(x)\}$ are determined by the recurrence formula:

$$(2.3)\quad \begin{cases} \vec{u}_n(x) = \vec{0} \quad \text{for} \quad n < n_0, \\ \vec{u}_{n_0}(x) = {}^t(0,\dots,0,f(x)), \\ \vec{u}_n(x) = \dfrac{1}{n} \begin{pmatrix} 0, & 1 & & \mathbf{0} \\ & \ddots & \ddots & \\ \mathbf{0} & & 0, & 1 \\ 0, & \dots\dots & \dots, & 0 \end{pmatrix} \vec{u}_{n-1}(x) \\ \qquad + \dfrac{1}{n} \displaystyle\sum_{\mu=0}^{n-1} \begin{pmatrix} & \mathbf{0} & \\ a_m^\mu(x;\partial), & \dots, & a_1^\mu(x;\partial) \end{pmatrix} \vec{u}_{n-\mu-1}(x) \quad \text{for} \quad n \geqq n_0+1. \end{cases}$$

This also gives the recurrence formula for $\{u_n^i(x)\}$:

$$(2.4)\quad \begin{cases} u_{n_0}^i(x) = 0 \quad (i=1,2,\dots,m-1), \qquad u_{n_0}^m(x) = f(x) \\ u_n^i(x) = \dfrac{1}{n} u_{n-1}^{i+1}(x) \qquad (i=1,2,\dots,m-1) \\ u_n^m(x) = \dfrac{1}{n} \displaystyle\sum_{\mu=0}^{n-1} \sum_{j=1}^{m} a_j^\mu(x;\partial) u_{n-\mu-1}^{m-j+1}(x) \qquad \text{for} \quad n \geqq n_0+1. \end{cases}$$

Let

$$\sigma(j,n) = \begin{cases} 1, & (j=1) \\ \dfrac{1}{n(n-1)\dots(n-j+2)}, & (j=2,3,\dots,m) \end{cases}$$

then we have from (2.4)

$$u_n^{m-j+1}(x)=\sigma(j,n)u_{n-j+1}^m(x) \tag{2.5}$$

and

$$u_n^m(x)=\frac{1}{n}\sum_{\mu=0}^{n-1}\sum_{j=1}^m \sigma(j,n-\mu-1)a_j^\mu(x;\partial)u_{n-\mu-j}^m(x)\,. \tag{2.6}$$

Denote briefly $u_n^m(x)$ by $v_n(x)$. Then (2.6) becomes

$$\begin{cases} v_n(x)=0 \qquad \text{for} \quad n\leqq n_0-1, \\ v_{n_0}(x)=f(x), \\ v_n(x)=\dfrac{1}{n}\displaystyle\sum_{\mu=0}^{n-1}\sum_{j=1}^m \sigma(j,n-\mu-1)a_j^\mu(x;\partial)v_{n-\mu-j}(x) \qquad \text{for } n\geqq n_0+1 \\ \qquad =\dfrac{1}{n}\displaystyle\sum_{j=1}^m\sum_\alpha\sum_{\mu=n(j,\alpha)}^{n-1}\sigma(j,n-\mu-1)a_{j,\alpha}^\mu(x)\partial^\alpha v_{n-\mu-j}(x)\,, \end{cases} \tag{2.7}$$

where we choose the number n_0 in such a way that $n_0\geqq \operatorname{Max}_{j,\alpha}\{n(j,\alpha)+1\}$ and that $p\cdot n_0$ is an integer.

Lemma 1.

$$v_n(x)=Q_n(x;\partial)f(x)+R_n(x;\partial)f(x) \qquad \text{for} \quad n\geqq n_0,$$

where (1) $Q_n(x;\partial)\equiv 0$ *for* n *such that* $p\cdot n$ *is non integer,*

(2) *if* $Q_n(x;\partial)\not\equiv 0$, *then* $\operatorname{order}_\partial Q_n(x;\partial)=p(n-n_0)$ *for* n *such that* $p\cdot n$ *is integer,*

(3) $\operatorname{order}_\partial R_n(x;\partial)<p(n-n_0)$.

Moreover $\{Q_n(x;\partial)\}$ *satisfy the following recurrence formula*:

$$\begin{cases} Q_{n_0}(x;\partial)=1 \\ Q_n(x;\partial)=\dfrac{1}{n}\displaystyle\sum_{j=1}^m\sum_{|\alpha|=p(n(j,\alpha)+j)}\delta_n(j,\alpha)a_{j,\alpha}^{n(j,\alpha)}(x)Q_{n-\frac{|\alpha|}{p}}(x;\partial)\partial^\alpha \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } n\geqq n_0+1 \end{cases}$$

where $\delta_n(j,\alpha)=\sigma(j,n-n(j,\alpha)-1)$

$$=\begin{cases} 1 & (j=1), \\ \dfrac{1}{\left(n-\frac{|\alpha|}{p}+j-1\right)\left(n-\frac{|\alpha|}{p}+j-2\right)\cdots\left(n-\frac{|\alpha|}{p}+1\right)} & (j=2,\dots,m)\,, \end{cases}$$

We prove Lemma 1 by the induction. When $n=n_0$, it is clear. Suppose that Lemma 1 is valid for any h; $n_0\leqq h\leqq n-1$. Then we have

$$
(2.8)\quad \begin{cases}
v_n(x)=\dfrac{1}{n}\displaystyle\sum_{j=1}^{m}\sum_{\alpha}\sum_{\mu=n(j,\alpha)}^{n-1}\sigma(j,\, n-\mu-1)\, a^{\mu}_{j,\alpha}(x)\partial^{\alpha} \\
\qquad\qquad [Q_{n-\mu-j}(x;\partial)f(x)+R_{n-\mu-j}(x;\partial)f(x)] \\
=\dfrac{1}{n}\Big[\displaystyle\sum_{j=1}^{m}\sum_{|\alpha|=p(n(j,\alpha)+j)}\sigma(j,\, n-n(j,\alpha)-1)a^{n(j,\alpha)}_{j,\alpha}(x)Q_{n-n(j,\alpha)-j} \\
\quad (x;\partial)\partial^{\alpha}f(x)\Big]+\dfrac{1}{n}\Big[\displaystyle\sum_{j=1}^{m}\sum_{|\alpha|<p(n(j,\alpha)+j)}\sigma(j,\, n-n(j,\alpha)-1) \\
\quad a^{n(j,\alpha)}_{j,\alpha}(x)Q_{n-n(j,\alpha)-j}(x;\partial)\partial^{\alpha}f(x) \\
\quad +\displaystyle\sum_{j=1}^{m}\sum_{\alpha}\sum_{\mu>n(j,\alpha)}\sigma(j,\, n-\mu-1)a^{\mu}_{j,\alpha}(x)Q_{n-\mu-j}(x;\partial)\partial^{\alpha}f(x) \\
\quad +\displaystyle\sum_{j=1}^{m}\sum_{\alpha}\sum_{\mu\geqq n(j,\alpha)}\sigma(j,\, n-\mu-1)a^{\mu}_{j,\alpha}(x) \\
\qquad (\partial^{\alpha}Q_{n-\mu-j}(x;\partial)-Q_{n-\mu-j}(x;\partial)\partial^{\alpha}+\partial^{\alpha}R_{n-\mu-j}(x;\partial))f(x)\Big] \\
=Q_n(x;\partial)f(x)+R_n(x;\partial)f(x),
\end{cases}
$$

where

$$
(2.9)\quad Q_n(x;\partial)=\frac{1}{n}\sum_{j=1}^{m}\sum_{|\alpha|=p(n(j,\alpha)+j)}\sigma(j,\, n-n(j,\alpha)-1)\, a^{n(j,\alpha)}_{j,\alpha}(x)Q_{n-n(j,\alpha)-j}(x;\partial)\partial^{\alpha}.
$$

It is easy to verify that $Q_n(x;\partial)$ and $R_n(x;\partial)$ satisfy the requirements of Lemma 1.

3. Proof of Theorem

Let $N_j=\mathrm{Max}\{|\alpha|-j;\ |\alpha|=p(n(j,\alpha)+j)\}$ and $N=\mathrm{Max}_j N_j$. It is represented by $N\geqq 1$ that the modified principal part of L has a term which does not belong to the kowalevskian part of L.

To prove the theorem, it suffices that if $N\geqq 1$, we can construct a right-hand side $f(x)$ such that the formal solution $\sum_n t^n v_n(x)$ for $f(x)$ does not converge in any neighbourhood of the origin. Let us show this fact.

We assume $N\geqq 1$ and let j_0 be the minimum number satisfying $N_j=N$. We treat in the first place the case of one variable, and we see that the general case is reducible, in some sense, to that of one variable.

Case of one variable. We can put by Lemma 1,

$$v_n(x)=\alpha_n(x)\partial^{p(n-n_0)}f(x)+R_n(x;\partial)f(x) \qquad \text{for} \quad n\geqq n_0, \tag{3.1}$$

where $\{\alpha_n(x)\}$ satisfy the following

$$\begin{cases} \alpha_{n_0}(x)=1, \\ \alpha_n(x)=0 \qquad \text{for} \quad n: p\cdot n \text{ non integer}, \\ \alpha_n(x)=\dfrac{1}{n}\sum_{j=1}^{m}\sum_{\alpha=p(n(j,\alpha)+j)}\delta_n(j,\alpha)a_{j,\alpha}^{n(j,\alpha)}(x)\alpha_{n-\frac{\alpha}{p}}(x) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } n: p\cdot n \text{ integer}. \end{cases} \tag{3.2}$$

From the assumption

$$a_{j_0,j_0+N}^{n(j_0,j_0+N)}(x)\neq 0 \tag{3.3}$$

and at this time we may assume (see S. Mizohata [3] p. 225, [4])

$$a_{j_0,j_0+N}^{n(j_0,j_0+N)}(0)\neq 0\,. \tag{3.4}$$

Hereafter we denote briefly $\alpha_n=\alpha_n(0)$, $a_{j,\alpha}=a_{j,\alpha}^{n(j,\alpha)}(0)$. (3.2) can be written

$$\begin{cases} \alpha_{n_0}=1, \\ \alpha_n=0 \qquad \text{for} \quad n: p\cdot n \text{ non integer}, \\ \alpha_n=\dfrac{1}{n}\sum_{j=1}^{m}\sum_{\alpha=p(n(j,\alpha)+j)}\delta_n(j,\alpha)a_{j,\alpha}\alpha_{n-\frac{\alpha}{p}} \qquad \text{for } n: p\cdot n \text{ integer}. \end{cases} \tag{3.5}$$

Lemma 2. *There exists an infinite sequence* $\{\alpha_n\}$ *satisfying*

$$|\alpha_n|>\frac{K^n}{(p(n-n_0))!}\left[\frac{p\cdot n}{2(j_0+N)}\right]!$$

where $[\ \ ]$ *denotes the Gauss' sign, and* K *some positive constant.*

Proof. Let $\nu_k=k/p$, and

$$\beta_k = \begin{cases} \nu_k \nu_{k-1} \cdots \nu_{k_1} \alpha_n & \text{for} \quad k = p \cdot n \\ 0 & \text{for} \quad k < k_1 \quad \text{or} \quad k \neq p \cdot n \quad (k_1 = p \cdot n_0). \end{cases}$$

Then we have from (3.5) the recurrence formula for $\{\beta_k\}$:

$$(3.6) \quad \begin{cases} \beta_{k_1} = n_0, & \\ \beta_k = 0 & \text{for} \quad k \neq p \cdot n, \\ \beta_k = \sum_{j=1}^{m} \sum_{\alpha = p(n(j,\alpha)+j)} \varepsilon_k(j, \alpha) a_{j,\alpha} \beta_{k-\alpha} & \text{for} \quad k = p \cdot n, \end{cases}$$

where

$$\varepsilon_k(j, \alpha) = \begin{cases} \nu_{k-1}\nu_{k-2} \cdots \nu_{k-\alpha+1} & (j=1), \\ \dfrac{\nu_{k-1}\nu_{k-2} \cdots \nu_{k-\alpha+1}}{(\nu_{k-\alpha}+j-1)(\nu_{k-\alpha}+j-2) \cdots (\nu_{k-\alpha}+1)} & (j = 2, 3, \dots, m). \end{cases}$$

Lemma. *If $N \geqq 1$ and that n_0 is chosen sufficiently large, then there exists an infinite subsequence $\{k_r\}$ of $\{k\}$ satisfyng*

$$|\beta_{k_{r+1}}| \geqq \sqrt{k_r} |\beta_{k_r}|$$

and

$$k_{r+1} - k_r \leqq j_0 + N.$$

In fact, if k_r is defined, then k_{r+1} is defined as the minimum number h $(h > k_r)$ satisfying

$$|\beta_h| \geqq \sqrt{k_r} |\beta_{k_r}|.$$

Accordingly, it suffices to prove the following fact:

If, posing briefly $k_r = k$,

$$|\beta_{k+j}| < \sqrt{k} |\beta_k| \qquad \text{for all} \quad j: j = 1, 2, \dots, j_0 + N - 1,$$

then

$$|\beta_{k+j_0+N}| \geqq \sqrt{k} |\beta_k|.$$

We show that this fact is valid as follows: We have from (3.6)

$$(3.7)\qquad \beta_{k+j_0+N}=[\varepsilon_{k+j_0+N}(j_0, j_0+N)a_{j_0,j_0+N}\beta_k]$$

$$+\Big[\sum_{\substack{j>j_0\\N_j=N}}\varepsilon_{k+j_0+N}(j, j+N)a_{j,j+N}\beta_{k+j_0-j}\Big]$$

$$+\Big[\sum_{j<j_0}\sum_{\substack{\alpha\leqq N_j+j\\ \alpha=p(n(j,\alpha)+j)}}\varepsilon_{k+j_0+N}(j, \alpha)a_{j,\alpha}\beta_{k+j_0+N-\alpha}$$

$$+\sum_{\substack{\alpha<j_0+N\\ \alpha=p(n(j_0,\alpha)+j_0)}}\varepsilon_{k+j_0+N}(j_0, \alpha)a_{j_0,\alpha}\beta_{k+j_0+N-\alpha}$$

$$+\sum_{\substack{j>j_0\\N_j\leqq N}}\sum_{\substack{\alpha<j_0+N\\ \alpha=p(n(j,\alpha)+j)}}\varepsilon_{k+j_0+N}(j, \alpha)a_{j,\alpha}\beta_{k+j_0+N-\alpha}\Big]$$

$$+\sum_{\substack{j>j_0\\N_j=N}}\sum_{\substack{j_0+N\leqq\alpha<j+N\\ \alpha=p(n(j,\alpha)+j)}}\varepsilon_{k+j_0+N}(j, \alpha)a_{j,\alpha}\beta_{k+j_0+N-\alpha}$$

$$+\sum_{\substack{j>j_0\\N_j<N}}\sum_{\substack{j_0+N\leqq\alpha\leqq j+N_j\\ \alpha=p(n(j,\alpha)+j)}}\varepsilon_{k+j_0+N}(j, \alpha)a_{j,\alpha}\beta_{k+j_0+N-\alpha}\Big]$$

$$=A+B+C+D.$$

Let us evaluate A, B, C and D. Firstly, note the next facts:

(1) if $h\leqq k=k_r$, then $|\beta_h|\leqq|\beta_k|$

(2) for any p ($p>0$), there exist constants c ($c>0$), d ($d>1$)

$$c\leqq\prod_{i=1}^{s}\frac{\nu_{k+i+i_0}}{\nu_k+i+i_0}\leqq d\qquad (s=1, 2,\dots, m,\ i_0=0, 1, 2,\dots)$$

(3) let $\Gamma=\nu_{k+j_0+N-1}\nu_{k+j_0+N-2}\cdots\nu_{k+j_0}$, then from (2)

$$c\cdot\Gamma\leqq\varepsilon_{k+j_0+N}(j, j+N)\leqq d\cdot\Gamma$$

and

$$\varepsilon_{k+j_0+N}(j, \alpha)\leqq\frac{d}{\nu_{k+j_0}}\cdot\Gamma\qquad(\alpha\leqq j+N-1)\,.$$

We put

$$L=\text{Max}\,\{|a_{j,\alpha}|;\, j>j_0,\ N_j=N,\ \alpha=j+N=p(n(j, \alpha)+j)\}$$

$$M=\text{Max}\,\{|a_{j,\alpha}|;\, \alpha=p(n(j, \alpha)+j)\}\,.$$

Then we have

$$|A| \geqq c\Gamma |a_{j_0,j_0+N}| \, |\beta_k|,$$

$$|B| \leqq dmL\Gamma |\beta_k|,$$

$$|C| \leqq dmp(m+N)M\frac{\Gamma}{\sqrt{k}}|\beta_k|$$

and

$$|D| \leqq dm^2pM\frac{\Gamma}{k}|\beta_k| .$$

Therefore we have from (3.7)

$$(3.8) \qquad |\beta_{k+j_0+N}| \geqq \Gamma\left[c|a_{j_0,j_0+N}| - dmL - \frac{dmp(2m+N)}{\sqrt{k}}M\right]|\beta_k|$$

and, if necessary, choose λ suitably and make the substitution

$$t=\lambda\cdot\tau, \qquad x=\lambda^{-\frac{1}{p}}\cdot y$$

for L, we can suppose that

$$c|a_{j_0,j_0+N}| - dmL > 0.$$

Since $N \geqq 1$, we have $\Gamma \geqq \nu_{k+j_0} \geqq k/p$. Now we choose k_1 (i.e. n_0) so large that

$$(3.9) \qquad \frac{\Gamma}{\sqrt{k}}\left[c|a_{j_0,j_0+N}| - dmL - \frac{dmp(2m+N)}{\sqrt{k}}M\right] \geqq 1 \quad \text{for } k \geqq k_1 .$$

Hereafter we take n_0 $(n_0 = k_1/p)$ such a way. Consequently we obtain from (3.8) and (3.9),

$$|\beta_{k+j_0+N}| \geqq \sqrt{k}\,|\beta_k| \qquad \text{for} \quad k \geqq k_1.$$

The proof of Lemma is complete.

Let us return to the proof of Lemma 2. From the Lemma, there exist constants K_0 and K such that

$$|\beta_{k_r}| \geqq K_0^{k_r}\left[\frac{k_r}{2(j_0+N)}\right]!$$

that is, for infinitely many n,

$$|\alpha_n| \geqq \frac{K^n}{(p(n-n_0))!}\left[\frac{p\cdot n}{2(j_0+N)}\right]!$$

This completes the proof of Lemma 2.

Now we return to the proof of the theorem. Let us construct $f(x)$. Let

$$f(x) = \sum_{\substack{n \geqq n_0 \\ pn:\text{ integer}}} e^{i\theta_n} x^{p(n-n_0)} \equiv \sum_n{}' e^{i\theta_n} x^{p(n-n_0)}. \tag{3.10}$$

Then we have from (3.1)

$$\begin{aligned} v_n(0) &= (\alpha_n(x)\partial^{p(n-n_0)} + R_n(x;\partial))f(x)|_{x=0} \\ &= \psi_n(\theta_m;\ m<n) + e^{i\theta_n}(p(n-n_0))!\alpha_n, \end{aligned} \tag{3.11}$$

where $\psi_n(\theta_m;\ m<n)$ depends only on $\theta_m\ (m<n)$. We determine θ_n in such a way that

$$\theta_n = \arg\psi_n(\theta_m;\ m<n) - \arg\alpha_n \quad \text{and} \quad \theta_{n_0} = 0.$$

For this $f(x)$, we have

$$|v_n(0)| \geqq (p(n-n_0))!|\alpha_n| \tag{3.12}$$

and from Lemma 2

$$|v_n(0)| \geqq K^n\left[\frac{p\cdot n}{2(j_0+N)}\right]! \tag{3.13}$$

for infinitely many n. This shows that the formal solution $\sum_n t^n v_n(x)$ for $f(x)$ can not converge in any neighbourhood of the origin.

General case. By the assumption and the definition of N and j_0,

$$\left[\sum_{|\alpha|=j_0+N} a_{j_0,\alpha}^{n(j_0,\alpha)}(x)t^{n(j_0,\alpha)}\partial^\alpha\right]\partial_t^{j_0} = t^{\frac{j_0+N}{p}-j_0}\left[\sum_{|\alpha|=j_0+N} a_{j_0,\alpha}^{n(j_0,\alpha)}(x)\partial^\alpha\right]\partial_t^{j_0} \not\equiv 0.$$

Hence, there exists $\xi=(\xi_1,\dots,\xi_l)\neq 0$ such that

$$\sum_{|\alpha|=j_0+N} a_{j_0,\alpha}^{n(j_0,\alpha)}(x)\xi^\alpha \not\equiv 0$$

and we proceed our argument under the assumption (see, S. Mizohata [3], [4])

$$\sum_{|\alpha|=j_0+N} a_{j_0,\alpha}^{n(j_0,\alpha)}(0)\xi^\alpha \neq 0 .$$

For the sake of simplicity, let $\xi_1 \neq 0$ and make the substitution in L

$$X_1 = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_l x_l$$

$$X_i = x_i \qquad (i=2, 3,\dots, l).$$

Then the modified weight p, N and j_0 are invariant with respect to this change of variables and the modified principal part of L is transformed into the modified principal part of the new operator and the term

$$\sum_{|\alpha|=j_0+N} a_{j_0,\alpha}^{n(j_0,\alpha)}(x)\partial^\alpha$$

is transformed into

$$\Big(\sum_{|\alpha|=j_0+N} a_{j_0,\alpha}^{n(j_0,\alpha)}(x)\xi^\alpha\Big)\partial_{X_1}^{j_0+N} + R_0(X;\partial_X)\,,$$

where $\underset{\partial_X}{\text{order}}\, R_0(X;\, \partial_X) = j_0+N$ and $\underset{\partial_{X_1}}{\text{order}}\, R_0(X;\, \partial_X) < j_0+N$.

If necessary, we make the above substitution and observing one of the variables x_i, for example x_1, we follow our reasoning in the case of one variable: If we adopt a function of x: $f(x)$ as

$$f(x) = \sum_n{}' e^{i\theta_n} x_1^{p(n-n_0)}$$

then the formal solution $\sum_n t^n v_n(x)$ for $f(x)$ can not converge in any neighbourhood of the origin. This completes the proof of the theorem.

References

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