

Hypoelliptic Degenerate Evolution Equations of the Second Order

By

Masatake MIYAKE*

§0. Introduction

We shall study in this paper the hypoellipticity of the following partial differential operator,

$$(0.1) \quad P(x, t; D_x, \partial_t) = \partial_t^2 + A(x, t; D_x)\partial_t + B(x, t; D_x),$$

$$A(x, t; D_x) = \sum_{j=0}^p a_j(x, t)t^{l_j}D_x^{p-j}, \quad (l_j \geq 0, \text{ integer}),$$

$$B(x, t; D_x) = \sum_{k=0}^{2p} b_k(x, t)t^{m_k}D_x^{2p-k}, \quad (m_k \geq 0, \text{ integer}),$$

where $\partial_t = \partial/\partial t$, $D_x = -i\partial/\partial x$, $(x, t) \in R_x \times I_t$, $I_t = (-1, 1)$,

$$a_j(x, t), b_k(x, t) \in \mathcal{C}^\infty(\bar{I}_t; \mathcal{B}_x^\infty)^1, \bar{I}_t = [-1, 1].$$

For degenerate parabolic differential operators, the study of hypoellipticity has been made by many authors (see [1]~[9]). But for degenerate p -parabolic differential operators, its study has not been made so detailed (see F. Trèves [10]). So we shall give a sufficient condition for the operator given by (0.1) to be hypoelliptic by constructing very regular parametrices of the operator (see Y. Kato [2], T. Matsuzawa [8]). The operator considered here is a special one, but we want to give an elementary relation between the order of degeneracy of the co-

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* Department of Mathematics, University of Tsukuba, Ibaraki, Japan.

- 1) $a(x, t) \in \mathcal{C}^\infty(\bar{I}_t; \mathcal{B}_x^\infty)$ means that $a(x, t) \in \mathcal{B}_x^\infty$ for any fixed $t \in \bar{I}_t$ and infinitely continuously differentiable with respect to t in the usual topology of \mathcal{B}_x^∞ , where $\mathcal{B}_x^\infty = \{u(x) \in C^\infty(R_x); |\partial_x^\alpha u(x)| \leq M_\alpha \text{ for any } \alpha\}$.

efficient and that of derivative, based on the notion of “*modified order*” defined in M. Miyake [12]. And for general degenerate p -parabolic differential operators of type (0.1), our result in this paper will become a fundamental one (see K. Shinkai [15]).

In Appendix A we shall study the hypoellipticity for the degenerate parabolic differential operator given by

$$(0.2) \quad P(x, t; D_x, \partial_t) = \partial_t + \sum_{j=0}^{2m} t^{l_j} a_j(x, t) D_x^{2m-j}.$$

In Appendix B we shall see that for the study of hypoellipticity of degenerate parabolic differential operators the notion of modified order is a useful one, and we shall be concerned with the operator,

$$(0.3) \quad P(t; D_x, \partial_t) = \partial_t + at^{l_0} D_x^{2m} + bt^{l_1} D_x^{2n},$$

where $m > n$ and, a and b are non-zero real numbers.

Finally, we note that our result in section 1 can be extended to the operator defined in $R_x^n \times I_t$, and it is given in section 8.

§ 1. Result

Our purpose in this paper is to give a sufficient condition for the operator $P(x, t; D_x, \partial_t)$ given by (0.1) to be hypoelliptic. And for this purpose let us assume:

- (H) $\left\{ \begin{array}{l} \text{(i) } p \text{ is a positive even integer,} \\ \text{(ii) } p/(l_0 + 1) \geq (p - j)/(l_j + 1), \quad j = 1, 2, \dots, p, \\ \text{(iii) } 2p/(m_0 + 2) \geq (2p - k)/(m_k + 2), \quad k = 1, 2, \dots, 2p, \\ \text{(iv) } p/(l_0 + 1) = 2p/(m_0 + 2), \text{ that is, } m_0 = 2l_0, \\ \text{(v) } \lambda^2 + a_0(x, t)\lambda + b_0(x, t) = 0 \text{ has simple roots } \lambda = \lambda_i(x, t) \text{ (} i = 1, 2 \text{)} \\ \text{or a double root } (\lambda_1 \equiv \lambda_2) \text{ in } R_x \times I_t, \text{ and they satisfy that} \\ \text{Re } \lambda_i(x, t) \leq -\delta \text{ for some positive constant } \delta. \end{array} \right.$

Then we have

Theorem. *Let $P(x, t; D_x, \partial_t)$ be the operator given by (0.1) and let P satisfy the conditions in (H).*

- (i) *If l_0 is a non-negative even integer, then P and its adjoint tP*

are hypoelliptic in $R_x \times I_t$.

(ii) If l_0 is a positive odd integer, then tP is hypoelliptic.

We note that under the conditions in (H) we can easily show that it suffices to consider the following operator instead of P given by (0.1),

$$(1.1) \quad P(x, t; D_x, \partial_t) = (\partial_t + A(x, t; D_x))(\partial_t + B(x, t; D_x)) + C(x, t; D_x),$$

$$A(x, t; D_x) = \sum_{j=0}^p t^{l_j} a_j(x, t) D_x^{p-j},$$

$$B(x, t; D_x) = \sum_{j=0}^p t^{l_j} b_j(x, t) D_x^{p-j},$$

$$C(x, t; D_x) = \sum_{k=1}^{2p} t^{m_k} c_k(x, t) D_x^{2p-k},$$

where we may assume without loss of generality that $a_j, b_j, c_k \in \mathcal{E}^\infty(\bar{I}_t; \mathcal{B}_x^\infty)$.

Then the conditions in (H) are transformed into

$$(\tilde{H}) \quad \left\{ \begin{array}{l} \text{(i)} \quad p \text{ is a positive even integer,} \\ \text{(ii)} \quad p/(l_0 + 1) \geq (p-j)/(l_j + 1), \quad j = 1, 2, \dots, p, \\ \text{(iii)} \quad p/(l_0 + 1) \geq (2p-k)/(m_k + 2), \quad k = 1, 2, \dots, 2p, \\ \text{(iv)} \quad \text{Re } a_0(x, t) \geq \delta, \text{ Re } b_0(x, t) \geq \delta \text{ in } R_x \times I_t \text{ for some positive} \\ \quad \quad \text{constant } \delta. \end{array} \right.$$

In fact, it suffices to check that $(p-j)/(l_j + 1) \leq p/(l_0 + 1)$ and $(p-k)/(l_k + 1) \leq p/(l_0 + 1)$ imply $(2p-j-k)/(l_j + l_k + 2) \leq p/(l_0 + 1)$ and that $(\partial_t + A)(\partial_t + B) + D(x, t; D_x)\partial_t + C(x, t; D_x) = (\partial_t + A + D)(\partial_t + B) - DB + C$.

Hence in this paper we shall prove our theorem for the operator P given by (1.1) under the conditions in (\tilde{H}) . And we shall only prove Theorem-(i), since the proof of Theorem-(ii) is obtained easily combining our argument and that of Y. Kannai [1] or Y. Kato [3]. Therefore throughout this paper we assume that l_0 is a non-negative even integer.

In section 2 we shall give an outline of the proof of our theorem and sections 3-7 are devoted to the proof of our theorem.

§2. An Outline of the Proof of Theorem

As mentioned in section 1, we shall prove only Theorem-(i), so we

assume that l_0 is a non-negative even integer.

Let $\mathcal{K}(x, y, t, s)$ be a kernel defined in $R_x \times R_y \times I_t \times I_s$ defined by

$$(2.1) \quad [\mathcal{K}v](x, t) = (2\pi)^{-1} \int_{-1}^t ds \int_{-\infty}^{\infty} e^{ix\xi} K(x, \xi, t, s) \hat{v}(\xi, s) d\xi,$$

where $v(y, s) \in C_0^\infty(R_y \times I_s)$ and $\hat{v}(\xi, s)$ denotes the partial Fourier transform of $v(y, s)$ with respect to y , that is,

$$\hat{v}(\xi, s) = \int_{-\infty}^{\infty} e^{-iy\xi} v(y, s) dy.$$

Then from a formal calculation we can obtain a condition on $K(x, \xi, t, s)$ that \mathcal{K} becomes a fundamental solution of P . Indeed,

$$\begin{aligned} P_{x,t}[\mathcal{K}v](x, t) &= (2\pi)^{-1} \int_{-1}^t ds \int_{-\infty}^{\infty} e^{ix\xi} P(x, t; \xi + D_x, \partial_t) K(x, \xi, t, s) \hat{v}(\xi, s) d\xi \\ &\quad + (2\pi)^{-1} \left[\int_{-\infty}^{\infty} e^{ix\xi} (\partial_t + B(x, t; \xi + D_x)) K(x, \xi, t, s) \hat{v}(\xi, s) d\xi \right]_{s=t} \\ &\quad + (2\pi)^{-1} (\partial_t + A(x, t; D_x)) \left[\int_{-\infty}^{\infty} e^{ix\xi} K(x, \xi, t, s) \hat{v}(\xi, s) d\xi \right]_{s=t} \end{aligned}$$

implies that in order that \mathcal{K} is a fundamental solution of P (i.e. $P[\mathcal{K}v] = v$), it must hold that

$$(2.2) \quad \begin{cases} P(x, t; \xi + D_x, \partial_t) K(x, \xi, t, s) = 0 & \text{in } -1 < s < t < 1, \\ K(x, \xi, t, s)|_{t=s} = 0, \quad \partial_t K(x, \xi, t, s)|_{t=s} = 1, \\ K(x, \xi, t, s) = 0 & \text{in } -1 < t < s < 1. \end{cases}$$

Now let $\{K_j(x, \xi, t, s)\}_{j=0}^\infty$ be a sequence of approximate solutions of (2.2) defined by

$$(2.3) \quad \begin{cases} P(x, t; \xi, \partial_t) K_0(x, \xi, t, s) = 0 & \text{in } -1 < s < t < 1, \\ K_0|_{t=s} = 0, \quad \partial_t K_0|_{t=s} = 1, \quad K_0 = 0 & \text{in } -1 < t < s < 1. \end{cases}$$

$$(2.4) \quad \begin{cases} P(x, t; \xi, \partial_t) K_j = -P_1(x, t, \xi; D_x, \partial_t) K_{j-1} & \text{in } -1 < s < t < 1, \\ K_j|_{t=s} = \partial_t K_j|_{t=s} = 0, \quad K_j = 0 & \text{in } -1 < t < s < 1, \quad j = 1, 2, \dots, \end{cases}$$

where $P_1(x, t, \xi; D_x, \partial_t) = P(x, t; \xi + D_x, \partial_t) - P(x, t; \xi, \partial_t)$.

Then obviously we have

$$(2.5) \quad P_{x,t} \left[\sum_{j=0}^k \mathcal{K}_j(x, y, t, s) v(y, s) \right] = v(x, t) + (2\pi)^{-1} \int_{-\infty}^t ds \int_{-\infty}^{\infty} e^{ix\xi} P_1(x, t, \xi; D_x, \partial_t) K_k(x, \xi, t, s) \hat{v}(\xi, s) d\xi.$$

for any $v(y, s) \in C_0^\infty(R_y \times I_s)$, where \mathcal{K}_j is a kernel defined by oscillatory integral,

$$(2.6) \quad \mathcal{K}_j(x, y, t, s) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i(x-y)\xi} K_j(x, \xi, t, s) d\xi.$$

Hence we have

$$(2.7) \quad P_{x,t} \left[\sum_{j=0}^k \mathcal{K}_j(x, y, t, s) \right] = \delta(x-y, t-s) + F_k(x, y, t, s),$$

where $(\sigma F_k)(x, \xi, t, s) = P_1(x, t, \xi; D_x, \partial_t) K_k(x, \xi, t, s)$.

Therefore in order to prove the hypoellipticity of tP , it suffices to show (see L. Schwartz [13]),

Proposition 2.1.

- (a) $\mathcal{K}_j(x, y, t, s)$ are regular in (x, t) and also in (y, s) .
- (b) $\mathcal{K}_j(x, y, t, s) \in C^\infty(W)$, $W = \{(x, y, t, s) \in R_x \times R_y \times I_t \times I_s; |x-y| + |t-s| > 0\}$,
- (c) For any fixed non-negative integer N , there exists k_0 such that $F_k(x, y, t, s) \in C^N(R_x \times R_y \times I_t \times I_s)$ for any $k \geq k_0$.

The proof of this proposition is given in section 3 assuming an elementary Proposition concerning $K_j(x, \xi, t, s)$, ($j=0, 1, 2, \dots$).

§3. Proof of Proposition 2.1

An elementary proposition for the proof of Proposition 2.1 is the following,

Proposition 3.1. *Under the assumptions in Proposition 2.1, there exists a positive constant d_0 satisfying*

$$(3.1)^2 \quad |\partial_x^v \partial_\xi^\mu K_j(x, \xi, t, s)| \leq C(t-s)(1+|\xi|)^{-jd_0-\mu} \\ \times \exp[-\varepsilon(t^{l_0+1}-s^{l_0+1})|\xi|^p],$$

$$(3.2)^3 \quad |\partial_{t,s}^{l+1} \partial_x^v \partial_\xi^\mu K_j| \leq C(1+|\xi|)^{lp-jd_0-\mu} \\ \times \exp[-\varepsilon(t^{l_0+1}-s^{l_0+1})|\xi|^p],$$

in $-1 < s \leq t < 1$, for some positive constants C and ε depending only on l, v, μ and j . We note that the constant d_0 does not depend on l, v, μ and j .

Remark 3.1. It also holds that

$$(3.3) \quad \partial_x^v \partial_\xi^\mu K_j \rightarrow 0 \text{ as } t \searrow s,$$

$$(3.4) \quad \partial_{t,s}^{l+1} \partial_x^v \partial_\xi^\mu K_j \rightarrow 0 \text{ as } t \searrow s \text{ if } \mu > lp,$$

$$(3.5) \quad \partial_{t,s}^l \partial_x^v \partial_\xi^\mu K_j \rightarrow 0 \text{ as } t \searrow s \text{ if } l \leq j,$$

where \rightarrow means uniform convergence in x and ξ . In (3.5) its convergence is uniform in x and ξ when ξ varies in a compact set.

The above proposition and remark imply Proposition 2.1 by the same way as T. Matsuzawa [8]. In fact, let us prove it assuming Proposition 3.1, and Proposition 3.1 will be proved in the following sections.

Proof of Proposition 2.1. At first let us show

Proposition 3.2. *The oscillatory integrals*

$$(3.6) \quad \mathcal{K}_j(x, y, t, s) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i(x-y)\xi} K_j(x, \xi, t, s) d\xi, \quad j=0, 1, \dots$$

2) (3.1) implies immediately

$$(3.1)' \quad |\partial_x^v \partial_\xi^\mu K_j| \leq C(1+|\xi|)^{-jd_0-\mu-p/(l_0+1)} \exp[-\varepsilon(t^{l_0+1}-s^{l_0+1})|\xi|^p].$$

In fact, it suffices to note that

$$t-s \leq C \cdot X^{p/(l_0+1)} \cdot |\xi|^{-p/(l_0+1)}, \quad (|\xi| > c > 0),$$

where X is defined by (4.11).

3) $\partial_{t,s}^l = \partial_{t_1}^{l_1} \partial_{t_2}^{l_2}$, ($l_1+l_2=l$).

belong to $C^\infty(W)$, where $W = \{(x, y, t, s) \in R_x \times R_y \times I_t \times I_s; |x - y| + |t - s| > 0\}$.

Proof. It is obvious that $\mathcal{H}_j(x, y, t, s)$ is infinitely differentiable when $t \neq s$. Let us consider

$$\begin{aligned} (x - y)^\mu \mathcal{H}_j(x, y, t, s) &= (2\pi)^{-1} \int_{-\infty}^{\infty} D_\xi^\mu e^{i(x-y)\xi} K_j(x, \xi, t, s) d\xi \\ &= (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i(x-y)\xi} (-D_\xi)^\mu K_j(x, \xi, t, s) d\xi. \end{aligned}$$

Then from Proposition 3.1 and its remark we see that for any fixed non-negative integer N , if we chose μ so that $\mu > Np - jd_0 + 1$, then for any l, v, κ such that $l + v + \kappa \leq N$ we have

$$\lim_{t \searrow s} \partial_{t,s}^l \partial_x^v \partial_y^\kappa (x - y)^\mu \mathcal{H}_j(x, y, t, s) = 0,$$

which implies that $(x - y)^\mu \mathcal{H}_j \in C^N(R_x \times R_y \times I_t \times I_s)$. Q. E. D.

Proposition 3.3. $\mathcal{H}_j(x, y, t, s)$ is regular in (x, t) and also in (y, s) .

Proof.

(i) The regularity in (x, t) follows immediately from that

$$\begin{aligned} \int_{R_y \times I_s} \mathcal{H}_j(x, y, t, s) v(y, s) dy ds &= (2\pi)^{-1} \int_{-1}^t ds \int_{-\infty}^{\infty} e^{ix\xi} \\ &\times K_j(x, \xi, t, s) \hat{v}(\xi, s) d\xi \quad \text{for } v(y, s) \in C_0^\infty(R_y \times I_s). \end{aligned}$$

(ii) For the proof of regularity in (y, s) , we use the following

Lemma 3.1. (Hörmander [6]) *Let $a(x, \xi) \in S^m(R_x \times R_\xi)$,⁴⁾ then for any $v(x) \in C_0^\infty(R_x)$ and any non-negative integer N we have*

$$(3.7) \quad \left| \int_{-\infty}^{\infty} e^{ix\xi} a(x, \xi) v(x) dx \right| \leq C_N (1 + |\xi|)^{m-N}$$

for some positive constant C_N .

4) $S^m(R_x \times R_\xi)$ ($= S_{1,0}^m(R_x \times R_\xi)$) denotes the class of symbols of pseudo-differential operator of order m . $S^{-\infty} = \bigcap_{m < \infty} S^m$.

Let us now consider

$$\begin{aligned} & \int_{-\infty}^{\infty} \mathcal{H}_j(x, y, t, s)u(x, t)dx \\ &= (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-iy\xi} \left(\int_{-\infty}^{\infty} e^{ix\xi} K_j(x, \xi, t, s)u(x, t)dx \right) d\xi \\ &= (2\pi)^{-1} \int_{-\infty}^{\infty} e^{-iy\xi} F(\xi, t, s) d\xi. \end{aligned}$$

Then from Lemma 3.1, $F(\xi, t, s) \in \mathcal{E}^\infty(\bar{A}_{t,s}; S^{-\infty})$, $(\bar{A}_{t,s} = \{(t, s); -1 < s \leq t < 1\})$ which implies that

$$\begin{aligned} & \int_{R_x \times I_t} \mathcal{H}_j(x, y, t, s)u(x, t)dxdt \\ &= (2\pi)^{-1} \int_s^1 dt \int_{-\infty}^{\infty} e^{-iy\xi} F(\xi, t, s) d\xi \in C^\infty(R_y \times I_s). \quad \text{Q. E. D.} \end{aligned}$$

Proposition 3.4. *For any fixed non-negative integer N , there exists k_0 such that for any $k \geq k_0$ it holds that*

$$\begin{aligned} & (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i(x-y)\xi} P_1(x, t, \xi; D_x, \partial_t) K_k(x, \xi, t, s) d\xi \\ & \in C^N(R_x \times R_y \times I_t \times I_s). \end{aligned}$$

Proof. It is obvious from Proposition 3.1 and its remark.

Summing up the above, the proof of Proposition 2.1 is completed. Consequently, the hypoellipticity of tP is obtained. The hypoellipticity of P follows if we repeat the same reasonings for the operator ${}^tP(x, -t; D_x, -\partial_t)$.

§4. Study of $K_0(x, \xi, t, s)$

Remember that $K_0(x, \xi, t, s)$ is the solution of the equation (2.3). We shall prove Proposition 3.1 for K_0 dividing many steps.

4.1. Preliminary. At first, let us consider the following ordinary differential equation,

$$(4.1) \quad \begin{cases} P_0(x, \xi, t; \partial_t) v = (\partial_t + A(x, t; \xi))(\partial_t + B(x, t; \xi)) v = 0 \\ \hspace{15em} \text{in } -1 < s < t < 1, \\ v|_{t=s} = 0, \quad \partial_t v|_{t=s} = 1. \end{cases}$$

Then the solution of (4.1) is given by

$$(4.2) \quad v(x, \xi, t, s) = \int_s^t \exp \left[\int_\tau^t B(x, \sigma; \xi) d\sigma + \int_s^\tau A(x, \sigma; \xi) d\sigma \right] d\tau,$$

in $-1 < s \leq t < 1$. Then for $v(x, \xi, t, s)$ we have

Proposition 4.1. *Under the assumptions in Theorem-(i) we have*

$$(4.3) \quad |\partial_x^\nu \partial_\xi^\mu v(x, \xi, t, s)| \leq C(t-s)(1+|\xi|)^{-\mu} \exp[-\varepsilon(t^{l_0+1} - s^{l_0+1})|\xi|^p],$$

$$(4.4) \quad |\partial_{t,s}^{l+1} \partial_x^\nu \partial_\xi^\mu v| \leq C(1+|\xi|)^{l-\mu} \exp[-\varepsilon(t^{l_0+1} - s^{l_0+1})|\xi|^p],$$

for some positive constants C and ε depending only on l, ν and μ . Moreover the facts stated in Remark 3.1 also hold for v .

For the proof of this proposition, we prepare the following two lemmas.

Lemma 4.1. (M. Miyake [12]) *The conditions that $p/(l_0+1) \geq (p-j)/(l_j+1)$, ($j=1, 2, \dots, p$) imply that*

$$(4.5) \quad \int_s^t |\tau|^{l_j} d\tau \leq C(t^{l_0+1} - s^{l_0+1})^{(p-j)/p}, \quad j=1, 2, \dots, p,$$

$$(4.6) \quad \max \{ (t-s)|t|^{l_j}, (t-s)|s|^{l_j} \} \leq C \int_s^t |\tau|^{l_j} d\tau,$$

for $-1 < s \leq t < 1$, where C is a positive constant independent of t and s .

Lemma 4.2. *Let us assume the conditions in Lemma 4.1 and let $A(x, t; \xi)$ be given in (4.1). Consider now*

$$(4.7) \quad E(x; \xi, t, s) = \exp \left[- \int_s^t A(x, \tau; \xi) d\tau \right], \quad -1 < s \leq t < 1,$$

then if $\text{Re } a_0(x, t) \geq \delta$ for some positive constant δ , we have

$$(4.8) \quad |\partial_{t,s}^l \partial_x^v \partial_\xi^\mu E| \leq C(1 + |\xi|)^{lp-\mu} \exp[-\varepsilon(t^{l_0+1} - s^{l_0+1}) |\xi|^p].$$

Moreover we have

$$(4.9) \quad |\partial_{t,s}^l \partial_x^v \partial_\xi^\mu E| \rightarrow 0 \text{ as } t \searrow s \text{ when } \mu > lp.$$

Proof. $\partial_{t,s}^l \partial_x^v \partial_\xi^\mu E$ is expressed by a linear combination of terms,

$$(4.10) \quad E(x, \xi, t, s) \times \prod_{i \in I} \partial_x^{v_i} \partial_\xi^{\mu_i} \int_s^t A(x, \tau; \xi) d\tau \times \prod_{j \in J} \partial_x^{l_j} \partial_x^{v_j} \partial_\xi^{\mu_j} A(x, t; \xi) \\ \times \prod_{k \in K} \partial_s^{l_k} \partial_x^{v_k} \partial_\xi^{\mu_k} A(x, s; \xi),$$

$\sum_i v_i + \sum_j v_j + \sum_k v_k = v$, $\sum_i \mu_i + \sum_j \mu_j + \sum_k \mu_k = \mu$, $|J| + |K| \leq l$, where $|I|$ denotes the number of elements in I . Without loss of generality, it suffices to assume that $|\xi| > c > 0$. Let

$$(4.11) \quad X = (t^{l_0+1} - s^{l_0+1})^{1/p} |\xi|.$$

then using the assumptions and Lemma 4.1, we obtain

$$(4.12) \quad |E(x, \xi, t, s)| \leq \text{const.} \exp[-cX^p + c' \sum_{j=1}^p X^{p-j}].$$

On the other hand we have easily that

$$(4.13) \quad \left| \partial_x^v \partial_\xi^\mu \int_s^t A(x, \tau; \xi) d\tau \right| \leq \text{const.} (1 + |\xi|)^{-\mu} \left\{ \sum_{j=0}^{p-\mu} \int_s^t |\tau|^{l_j} d\tau |\xi|^{p-j} \right\} \\ \leq \text{const.}' (1 + |\xi|)^{-\mu} \left\{ \sum_{j=0}^p X^{p-j} \right\},$$

$$(4.14) \quad \max \{ |\partial_t^l \partial_x^v \partial_\xi^\mu A(x, t; \xi)|, |\partial_s^l \partial_x^v \partial_\xi^\mu A(x, s; \xi)| \} \leq C(1 + |\xi|)^{p-\mu}.$$

Combining from (4.12) to (4.14), we have the desired inequality (4.8).

Finally, (4.9) is now obvious, since from the assumption that $\mu > lp$, in (4.10) there exists at least one of terms of $\partial_x^{v_i} \partial_\xi^{\mu_i} \int_s^t A(x, \tau; \xi) d\tau$.

Q. E. D.

Proof of Proposition 4.1. Let

$$(4.15) \quad G(x, \xi, t, \tau, s) = \exp \left[\int_\tau^t B(x, \sigma; \xi) d\sigma + \int_s^\tau A(x, \sigma; \xi) d\sigma \right].$$

Then from Lemma 4.2, we have easily that

$$|\partial_x^y \partial_\xi^\mu G(x, \xi, t, \tau, s)| \leq C(1 + |\xi|)^{-\mu} \exp[-\varepsilon(t^{l_0+1} - s^{l_0+1})|\xi|^p],$$

which implies immediately the inequality (4.3).

In order to prove (4.4), we use the following,

Lemma 4.3. (T. Matsuzawa [8]) *Let $f(t, \tau, s) \in C_{t,\tau,s}^\infty$, ($s \leq \tau \leq t$), then*

$$(4.16) \quad \begin{aligned} \partial_t^p \partial_s^q \left(\int_s^t f(t, \tau, s) d\tau \right) &= \int_s^t \partial_t^p \partial_s^q f(t, \tau, s) d\tau \\ &\quad - \partial_t^p \left[\sum_{j=1}^q C_j^q \partial_s^{q-j} \partial_\tau^{j-1} f(t, \tau, s) \right]_{\tau=s} \\ &\quad + \sum_{k=1}^q C_k^p \left[\partial_t^{p-k} \partial_\tau^{k-1} (\partial_s^q f(t, \tau, s)) \right]_{\tau=t}, \end{aligned}$$

where $C_s^p = p!/k!(p-k)!$.

Now we remark that we have by the same way as the proof of Lemma 4.2 that

$$(4.17) \quad \begin{aligned} &|\partial_{t,\tau,s}^l \partial_x^y \partial_\xi^\mu G(x, \xi, t, \tau, s)| \\ &\leq C(1 + |\xi|)^{l\mu - \mu} \exp[-\varepsilon(t^{l_0+1} - s^{l_0+1})|\xi|^p]. \end{aligned}$$

Hence, in order to evaluate $\partial_{t,s}^{l'+1} \partial_x^y \partial_\xi^\mu v$ it suffices only to consider

$$(4.18) \quad \int_s^t \partial_{t,s}^{l'+1} \partial_x^y \partial_\xi^\mu G(x, \xi, t, \tau, s) d\tau.$$

Now without loss of generality we may assume that $\partial_{t,s}^{l'+1} = \partial_{t,s}^l \partial_t$, then $\partial_{t,s}^l \partial_t \partial_x^y \partial_\xi^\mu G(x, \xi, t, \tau, s)$ is expressed by a linear combination of terms,

$$\partial_{t,s}^{l'} \partial_x^{y'} \partial_\xi^{\mu'} B(x, t; \xi) \times \partial_{t,s}^{l-l'} \partial_x^{y-y'} \partial_\xi^{\mu-\mu'} G, \quad (l' \leq l, y' \leq y, \mu' \leq \mu).$$

Considering (4.17), it suffices only to consider the term,

$$(4.19) \quad \int_s^t \partial_x^{y'} \partial_\xi^{\mu'} B(x, t; \xi) \partial_{t,s}^{l-l'} \partial_x^{y-y'} \partial_\xi^{\mu-\mu'} G(x, \xi, t, \tau, s) d\tau.$$

On the other hand, we can show from Lemma 4.1 that

$$\begin{aligned} (t-s) |\partial_x^{\nu'} \partial_{\xi}^{\mu'} B(x, t; \xi)| &\leq C(t-s) \left(\sum_{j=0}^p |t|^{l_j} |\xi|^{p-j} \right) (1+|\xi|)^{-\mu'} \\ &\leq C' \left(\sum_{j=0}^p X^{p-j} \right) (1+|\xi|)^{-\mu'}, \end{aligned}$$

which implies immediately the inequality (4.4). Q. E. D.

4.2. Integral equation. We note that the solution $K_0(x, \xi, t, s)$ is also the solution of the following integral equation,

$$\begin{aligned} (4.20) \quad K_0(x, \xi, t, s) &= v(x, \xi, t, s) - \int_s^t v(x, \xi, t, \tau) C(x, \tau; \xi) \\ &\quad \times K_0(x, \xi, \tau, s) d\tau. \end{aligned}$$

And the solution K_0 of (4.20) can be approximated as follows,

$$\begin{aligned} (4.21) \quad K_0^{(0)}(x, \xi, t, s) &= v(x, \xi, t, s), \\ K_0^{(n+1)}(x, \xi, t, s) &= v(x, \xi, t, s) \\ &\quad - \int_s^t v(x, \xi, t, \tau) C(x, \tau; \xi) K_0^{(n)}(x, \xi, \tau, s) d\tau, \quad n=0, 1, \dots \end{aligned}$$

Using (4.16) and (4.17) we shall prove Proposition 3.1.

4.3. Estimate of $\partial_x^{\nu} \partial_{\xi}^{\mu} K_0$. For this purpose let us prove

$$\begin{aligned} (4.22) \quad &|\partial_x^{\nu} \partial_{\xi}^{\mu} \{K_0^{(n+1)}(x, \xi, t, s) - K_0^{(n)}(x, \xi, t, s)\}| \\ &\leq (t-s) (1+|\xi|)^{-\mu} \exp[-\varepsilon(t^{l_0+1} - s^{l_0+1}) |\xi|^p] \\ &\quad \times \{MC(t, s, \xi)\}^{n+1} (t-s)^{2(n+1)} / 2(n+1)!, \end{aligned}$$

$n=0, 1, 2, \dots$, for some positive constants C, M and ε , where $C(t, s; \xi)$ is defined by

$$C(t, s; \xi) = \max \left\{ \sum_{k=1}^{2p} |t|^{m_k} |\xi|^{2p-k}, \sum_{k=1}^{2p} |s|^{m_k} |\xi|^{2p-k} \right\}.$$

In fact, let us show (4.22) by induction on n . Since

$$K_0^{(1)} - K_0^{(0)} = - \int_s^t v(x, \xi, t, \tau) C(x, \tau; \xi) v(x, \xi, \tau, s) d\tau,$$

$\partial_x^v \partial_\xi^\mu \{K_0^{(1)} - K_0^{(0)}\}$ is expressed by a linear combination of terms,

$$\int_s^t \partial_x^{v_1} \partial_\xi^{\mu_1} v \partial_x^{v_2} \partial_\xi^{\mu_2} C \partial_x^{v_3} \partial_\xi^{\mu_3} v d\tau, \quad v_1 + v_2 + v_3 = v, \mu_1 + \mu_2 + \mu_3 = \mu.$$

Hence using Proposition 4.1 we can obtain the desired inequality (4.22) for $n=0$. We note that c and M can be taken independently in v, μ if $|v|, |\mu| \leq N$ for fixed N . Considering that $K_0^{(n)}$ are determined by (4.21), we can easily obtain (4.22) by induction on n . (4.22) implies immediately that

$$(4.23) \quad |\partial_x^v \partial_\xi^\mu K_0(x, \xi, t, s)| \leq C(t-s)(1+|\xi|)^{-\mu} \times \exp[-\varepsilon(t^{l_0+1} - s^{l_0+1})|\xi|^p + (t-s)\sqrt{MC(t, s; \xi)}].$$

Lemma 4.4. *Under the assumptions in Theorem-(i), it holds that*

$$\max \{(t-s)^2 |t|^{mk}, (t-s)^2 |s|^{mk}\} \leq C(t^{l_0+1} - s^{l_0+1})^{(2p-k)/p},$$

$k=1, 2, \dots, 2p$, for some positive constant C .

This is a special case of Lemma 5.2.

Then from Lemma 4.4 we can show that

$$-\varepsilon(t^{l_0+1} - s^{l_0+1})|\xi|^p + C(t-s)\sqrt{C(t, s; \xi)} \leq \text{const.},$$

which implies

$$(4.24) \quad |\partial_x^v \partial_\xi^\mu K_0(x, \xi, t, s)| \leq C(t-s)(1+|\xi|)^{-\mu} \exp[-\varepsilon(t^{l_0+1} - s^{l_0+1})|\xi|^p],$$

for some positive constants C and ε .

4.4. Estimate of $\partial_t \partial_x^v \partial_\xi^\mu K_0$. Differentiating both sides of (4.20) with respect to t , we have

$$\partial_t K_0 = \partial_t v - \int_s^t \partial_t v \cdot C \cdot K_0 d\tau.$$

Now in view of (4.24) and Proposition 4.1, we have

$$|\partial_t \partial_x^v \partial_\xi^\mu K_0| \leq C(1+|\xi|)^{-\mu} \exp[-\varepsilon(t^{l_0+1} - s^{l_0+1})|\xi|^p] + C(1+|\xi|)^{-\mu} C(t, s; \xi)(t-s)^2 \exp[-\varepsilon(t^{l_0+1} - s^{l_0+1})|\xi|^p].$$

On the other hand, from Lemma 4.4 we have

$$(4.25) \quad (t-s)^2 C(t, s; \xi) \exp [-\varepsilon'(t^{l_0+1} - s^{l_0+1}) |\xi|^p] \leq \text{const.},$$

which implies

$$(4.26) \quad |\partial_t \partial_x^y \partial_\xi^\mu K_0| \leq \text{const.} (1 + |\xi|)^{-\mu} \exp [-\varepsilon(t^{l_0+1} - s^{l_0+1}) |\xi|^p],$$

for some positive constant ε .

4.5. Estimate of $\partial_s \partial_x^y \partial_\xi^\mu K_0$. Differentiating both sides of (4.20) with respect to s , we know that $\partial_s K_0$ should satisfy the following integral equation,

$$\begin{aligned} \partial_s K_0(x, \xi, t, s) &= \partial_s v(x, \xi, t, s) - \int_s^t v(x, \xi, t, \tau) C(x, \tau; \xi) \\ &\quad \times \partial_s K_0(x, \xi, \tau, s) d\tau. \end{aligned}$$

Then using Proposition 4.1, we can show by the same way as the estimate of $\partial_x^y \partial_\xi^\mu K_0$ that

$$(4.27) \quad |\partial_s \partial_x^y \partial_\xi^\mu K_0| \leq C(1 + |\xi|)^{-\mu} \exp [-\varepsilon(t^{l_0+1} - s^{l_0+1}) |\xi|^p].$$

4.6. Estimate of $\partial_{t,s}^l \partial_t \partial_x^y \partial_\xi^\mu K_0$. At first we note that

$$(4.28) \quad (t-s) C(t, s; \xi) \exp [-\varepsilon(t^{l_0+1} - s^{l_0+1}) |\xi|^p] \leq C(1 + |\xi|)^p.$$

Let operate $\partial_{t,s}^l \partial_t$ both sides of (4.20), and then using Lemma 4.3, the assumption of induction on l , (4.25) and (4.28) we have easily obtain the desired inequality,

$$(4.29) \quad |\partial_{t,s}^l \partial_t \partial_x^y \partial_\xi^\mu K_0| \leq C(1 + |\xi|)^{l p - \mu} \exp [-\varepsilon(t^{l_0+1} - s^{l_0+1}) |\xi|^p],$$

for some positive constants C and ε .

4.7. Estimate of $\partial_s^{l+1} \partial_x^y \partial_\xi^\mu K_0$. Let operate ∂_s^{l+1} to both sides of (4.20), then we can show that $\partial_s^{l+1} K_0$ should satisfy the following integral equation,

$$\begin{aligned} \partial_s^{l+1} K_0(x, \xi, t, s) &= G(x, \xi, t, s) - \int_s^t v(x, \xi, t, \tau) C(x, \tau; \xi) \\ &\quad \times \partial_s^{l+1} K_0(x, \xi, \tau, s) d\tau. \end{aligned}$$

Then from the assumption of induction we have

$$(4.31) \quad |\partial_x^\nu \partial_\xi^\mu G(x, \xi, t, s)| \leq C(1 + |\xi|)^{l_p - \mu} \exp[-\varepsilon(t^{l_0+1} - s^{l_0+1})|\xi|^p].$$

Hence using (4.31) we can get the desired inequality,

$$(4.32) \quad |\partial_s^{l_0+1} \partial_x^\nu \partial_\xi^\mu K_0| \leq C(1 + |\xi|)^{l_p - \mu} \exp[-\varepsilon(t^{l_0+1} - s^{l_0+1})|\xi|^p],$$

for some positive constants C and ε .

Summing up the aboves, Proposition 3.1 for $j=0$ is proved. On the other hand, the facts stated in Remark 3.1 are almost obvious considering (4.9) and the proof of our proposition.

§5. Preliminary for the Study of $K_j, j=1, 2, \dots$

Let us remember that $K_j, (j=1, 2, \dots)$ is the solution of the equation (2.4). On the other hand, we know that K_j is given by

$$(5.1) \quad K_j(x, \xi, t, s) = - \int_s^t K_0(x, \xi, t, \tau) P_1(x, \tau, \xi; D_x, \partial_t) K_{j-1}(x, \xi, \tau, s) d\tau,$$

in $-1 < s \leq t < 1$. Where P_1 is given by

$$(5.2) \quad P_1(x, t, \xi; D_x, \partial_t) = P(x, t; \xi + D_x, \partial_t) - P(x, t; \xi, \partial_t).$$

We shall prove Proposition 3.1 by induction on j . Then from the assumption of induction and conditions in (\tilde{H}) , it suffices to prove the inequalities in Proposition 3.1 when we put the following operators instead of P_1 ,

$$(5.3) \quad a(x, t) t^{l_j} \xi^{p-j} \partial_t, \quad p/(l_0+1) > (p-j)/(l_j+1),$$

$$(5.4) \quad a(x, t) t^{m_k} \xi^{2p-k}, \quad p/(l_0+1) > (2p-k)/(m_k+2),$$

where $a(x, t) \in \mathcal{E}^\infty(\bar{I}_t; \mathcal{B}_x^\infty)$.

Hence in the followings we shall study the following two integrals,

$$(5.5) \quad H_j^{(k)}(x, \xi, t, s) = \int_s^t K_0(x, \xi, t, \tau) \tau^{l_k} a(x, \tau) \xi^{p-k} K_{j-1}(x, \xi, \tau, s) d\tau,$$

where $p/(l_0 + 1) > (p - k)/(l_k + 1)$, $k = 1, 2, \dots, p$.

$$(5.6) \quad I_j^{(k)}(x, \xi, t, s) = \int_s^t K_0(x, \xi, t, \tau) \tau^{m_k} a(x, \tau) \xi^{2p-k} K_{j-1}(x, \xi, \tau, s) d\tau,$$

where $p/(l_0 + 1) > (2p - k)/(m_k + 2)$, $k = 1, 2, \dots, 2p$.

Let us now give two lemmas which are important to our purpose. Let $R (> 1)$ be defined by

$$(5.7) \quad R = \max \{r; (p - j)r/(l_j + 1) \leq p/(l_0 + 1), (p - j)r \leq p, \\ (2p - k)r/(m_k + 2) \leq p/(l_0 + 1), (2p - k)r \leq 2p\}.$$

Lemma 5.1. *Let R be the number defined by (5.7), then*

$$(5.8) \quad \int_s^t |\tau|^{l_j} d\tau \leq C(t^{l_0+1} - s^{l_0+1})^{(p-j)R/p}, \quad j = 1, 2, \dots, p,$$

for some positive constant C independent on t and s .

Its proof is essentially given in M. Miyake [12].

Lemma 5.2. *Let R be the number defined by (5.7), then*

$$(5.9) \quad \max \{(t - s)^2 |t|^{m_k}, (t - s)^2 |s|^{m_k}\} \leq C(t^{l_0+1} - s^{l_0+1})^{(2p-k)R/p},$$

$k = 1, 2, \dots, 2p$, for some positive constant C independent on t and s .

Proof. It suffices to prove our lemma when $0 \leq s \leq t < 1$.

(i) The case where $0 \leq s \leq 2s \leq t$. From the assumption, we have easily that

$$(t - s)^2 |t|^{m_k} \leq t^{m_k+2} \quad \text{and} \quad t^{l_0+1} - s^{l_0+1} \geq ct^{l_0+1}.$$

On the other hand, in view of the determination of R , it holds that $m_k + 2 \geq (l_0 + 1)(2p - k)R/p$ which implies immediately the inequality (5.9).

(ii) The case where $0 < s < t < 2s$. Since $t^{l_0+1} - s^{l_0+1} > (t - s)s^{l_0}$, it holds that

$$(t^{l_0+1} - s^{l_0+1})^{(2p-k)R/p} > \{(t - s)s^{l_0}\}^{(2p-k)R/p}.$$

Now because of that $t^{m_k}(t-s)^2 < c(t-s)^2s^{m_k}$ and that $(2p-k)R \leq 2p$ and $t-s < s$, it follows

$$t^{m_k}(t-s)^2 \leq c\{(t-s)s^{l_0}\}^{(2p-k)R/p} \cdot s^{\{p(m_k+2)-(l_0+1)(2p-k)R\}/p}.$$

From the determination of R we know that $p(m_k+2) \geq (l_0+1)(2p-k)R$, hence we get

$$t^{m_k}(t-s)^2 \leq c\{(t-s)s^{l_0}\}^{(2p-k)R/p},$$

which implies the inequality (5.15).

Q. E. D.

§6. Study of $H_j^{(k)}(x, \xi, t, s)$

We shall study $H_j^{(k)}$ dividing many steps.

6.1. Estimate of $\partial_x^\nu \partial_\xi^\mu H_j^{(k)}$. Recall that $H_j^{(k)}$ is given by

$$(6.1) \quad H_j^{(k)} = \int_s^t K_0(x, \xi, t, \tau) \tau^{l_k} a(x, \tau) \xi^{p-k} \partial_\tau K_{j-1}(x, \xi, \tau, s) d\tau,$$

where $p/(l_0+1) > (p-k)/(l_k+1)$, $k=1, 2, \dots, p$.

At first we note that Lemma 5.1 implies that

$$(6.2) \quad \max\{(t-s)|s|^{l_k}, (t-s)|t|^{l_k}\} \leq CX^{(p-k)R} |\xi|^{-(p-k)R},$$

where X is defined by (4.11) and R is the number defined by (5.7). Then from the assumption of induction and (6.2) we have

$$|\partial_x^\nu \partial_\xi^\mu H_j^{(k)}| \leq C(t-s)(1+|\xi|)^{-(p-k)(R-1)-(j-1)d_0-\mu} \times \exp[-\varepsilon(t^{l_0+1}-s^{l_0+1})|\xi|^p].$$

Hence, for any d_0 such that

$$(6.3) \quad 0 < d_0 < \min\{p/(l_0+1), (p-k)(R-1); k=1, \dots, p-1\},$$

it holds that

$$(6.4) \quad |\partial_x^\nu \partial_\xi^\mu H_j^{(k)}| \leq C(t-s)(1+|\xi|)^{-jd_0-\mu} \exp[-\varepsilon(t^{l_0+1}-s^{l_0+1})|\xi|^p].$$

6.2. Estimate of $\partial_t \partial_x^\nu \partial_\xi^\mu H_j^{(k)}$. Differentiating both sides of (6.1)

with respect to t we have

$$\partial_t H_j^{(k)} = \int_s^t \partial_t K_0 \cdot \tau^{l_k} a \xi^{p-k} \partial_\tau K_{j-1} d\tau.$$

Hence by the same way as the above we obtain

$$(6.5) \quad |\partial_t \partial_x^\nu \partial_\xi^\mu H_j^{(k)}| \leq C(1 + |\xi|)^{-j d_0 - \mu} \exp[-\varepsilon(t^{l_0+1} - s^{l_0+1})] |\xi|^p,$$

for d_0 defined by (6.3).

6.3. Estimate of $\partial_s \partial_x^\nu \partial_\xi^\mu H_j^{(k)}$. In this case, we can not obtain the desired inequality from the representation (6.1). So we rewrite it by

$$(6.6) \quad H_j^{(k)}(x, \xi, t, s) = - \int_s^t \partial_\tau \{K_0(x, \xi, t, \tau) \tau^{l_k} a(x, \tau)\} \xi^{p-k} \\ \times K_{j-1}(x, \xi, \tau, s) d\tau.$$

In order to study (6.6) we divide it into two cases:

- (A) The case where $(p-k)R \leq p/(l_0+1)$. In this case without loss of generality we may put $l_k=0$.
- (B) The case where $(p-k)R > p/(l_0+1)$.

At first we consider the case (A). In this case (6.6) can be rewritten by

$$(6.7) \quad -H_j^{(k)} = \int_s^t b(x, \tau) K_0(x, \xi, t, \tau) \xi^{p-k} K_{j-1}(x, \xi, \tau, s) d\tau \\ + \int_s^t a(x, \tau) \partial_\tau K_0(x, \xi, t, \tau) \xi^{p-k} K_{j-1}(x, \xi, \tau, s) d\tau \\ = I^{(k)}(x, \xi, t, s) + II^{(k)}(x, \xi, t, s).$$

Therefore it suffices only to study the part $II^{(k)}$. Then from the assumption of induction, we have

$$|\partial_s \partial_x^\nu \partial_\xi^\mu II^{(k)}| \leq C \cdot (t-s) |\xi|^{p-k} (1 + |\xi|)^{-(j-1)d_0 - \mu} \\ \times \exp[-\varepsilon(t^{l_0+1} - s^{l_0+1})] |\xi|^p.$$

Considering that $(t-s) \leq CX^{p/(l_0+1)} \cdot |\xi|^{-p/(l_0+1)}$, ($|\xi| > c > 0$), for such d_0 satisfying

$$(6.8) \quad 0 < d_0 < p/(l_0 + 1) - (p - k)$$

it follows that

$$(6.9) \quad |\partial_s \partial_x^\nu \partial_\xi^\mu I^{(k)}| \leq C(1 + |\xi|)^{-j d_0 - \mu} \exp[-\varepsilon(t^{l_0+1} - s^{l_0+1})|\xi|^p].$$

Next let us consider the case (B). Let

$$(6.10) \quad \begin{aligned} H_j^{(k)}(x, \xi, t, s) &= \int_s^t \tau^{l_k-1} b(x, \tau) \xi^{p-k} K_0(x, \xi, t, \tau) K_{j-1}(x, \xi, \tau, s) d\tau \\ &\quad + \int_s^t \partial_\tau K_0(x, \xi, t, \tau) \tau^{l_k} a(x, \tau) \xi^{p-k} K_{j-1}(x, \xi, \tau, s) d\tau \\ &= I^{(k)} + II^{(k)}. \end{aligned}$$

At first we consider $I^{(k)}$. The following inequality is obtained immediately,

$$\begin{aligned} |\partial_s \partial_x^\nu \partial_\xi^\mu I^{(k)}| &\leq C \max \{ (t-s) |s|^{l_k-1}, (t-s) |t|^{l_k-1} \} \\ &\quad \times |\xi|^{p-k} (1 + |\xi|)^{-p/(l_0+1) - (j-1)d_0 - \mu} \exp[-\varepsilon(t^{l_0+1} - s^{l_0+1})|\xi|^p]. \end{aligned}$$

Without loss of generality, we may assume that

$$(6.11) \quad p - k > p/(l_0 + 1).$$

In fact, in the case where $p - k \leq p/(l_0 + 1)$ we can easily see that our reasonings in the following are applicable for such d_0 satisfying $0 < d_0 < p/(l_0 + 1)$. From the assumption that $(p - k)/(l_k + 1) < p/(l_0 + 1)$ and (6.11), we have

$$(6.12) \quad \frac{(p - k) - p/(l_0 + 1)}{l_k} < \frac{p}{l_0 + 1}.$$

Now let

$$(6.13) \quad \begin{aligned} 1 < R_0 < \min \{ R, R'; \{(p - k) - p/(l_0 + 1)\} R' / l_k \leq p/(l_0 + 1), \\ \{(p - k) - p/(l_0 + 1)\} R' \leq p \}, \end{aligned}$$

then by the same way as the proof of Lemma 5.1 we have

$$\begin{aligned} \max \{ (t-s)|s|^{l_k-1}, (t-s)|t|^{l_k-1} \} &\leq C \int_s^t |\tau|^{l_k-1} d\tau \\ &\leq C' (t^{l_0+1} - s^{l_0+1})^{\{(p-k)-p/(l_0+1)\}R_0/p} \\ &= C' X^{\{(p-k)-p/(l_0+1)\}R_0} \cdot |\xi|^{-\{(p-k)-p/(l_0+1)\}R_0}. \end{aligned}$$

Hence for such d_0 satisfying

$$(6.14) \quad 0 < d_0 < \left\{ (p-k) - \frac{p}{l_0+1} \right\} (R_0 - 1),$$

it holds that

$$(6.15) \quad |\partial_s \partial_x^\nu \partial_\xi^\mu I^{(k)}| \leq C(1 + |\xi|)^{-j d_0 - \mu} \exp[-\varepsilon(t^{l_0+1} - s^{l_0+1}) |\xi|^p].$$

Next we consider $II^{(k)}$. In this case we can easily show that for such d_0 satisfying (6.3)

$$(6.16) \quad |\partial_s \partial_x^\nu \partial_\xi^\mu II^{(k)}| \leq C(1 + |\xi|)^{-j d_0 - \mu} \exp[-\varepsilon(t^{l_0+1} - s^{l_0+1}) |\xi|^p].$$

Summing up the aboves we have

$$(6.17) \quad |\partial_s \partial_x^\nu \partial_\xi^\mu H_j^{(k)}| \leq C(1 + |\xi|)^{-j d_0 - \mu} \exp[-\varepsilon(t^{l_0+1} - s^{l_0+1}) |\xi|^p],$$

$k = 1, 2, \dots, p.$

6.4. Estimate of $\partial_{t,s}^{l+1} \partial_x^\nu \partial_\xi^\mu H_j^{(k)}$ In this case, if we repeat the above reasonings we can obtain the desired inequality,

$$(6.18) \quad |\partial_{t,s}^{l+1} \partial_x^\nu \partial_\xi^\mu H_j^{(k)}| \leq C(1 + |\xi|)^{lp - j d_0 - \mu} \exp[-\varepsilon(t^{l_0+1} - s^{l_0+1}) |\xi|^p].$$

We only remark that for the proof of (6.18) in the case where $\partial_{t,s}^{l+1} = \partial_s^{l+1}$ we use (6.6) for the representation of $H_j^{(k)}$.

Finally we have to remark that it is easy to see that for $H_j^{(k)}$ the same facts stated in Remark 3.1 hold.

§7. Study of $I_j^{(k)}(x, \xi, t, s)$

Recall that $I_j^{(k)}$ is given by

$$(7.1) \quad I_j^{(k)} = \int_s^t K_0(x, \xi, t, \tau) \tau^{m_k} a(x, \tau) \xi^{2p-k} K_{j-1}(x, \xi, \tau, s) d\tau.$$

where $(2p - k)/(m_k + 2) < p/(l_0 + 1)$.

Since our desired inequalities are obtained by repeating the arguments in section 6, we only check different points. Lemma 5.2 implies

$$(7.2) \quad (t - s)^2 |\tau|^{m_k} \leq C X^{(2p-k)R} \cdot |\xi|^{-(2p-k)R}, \quad s \leq \tau \leq t,$$

where X is defined by (4.11). Hence for such d_0 satisfying

$$(7.3) \quad 0 < d_0 < \min \left\{ \frac{p}{l_0 + 1}, (2p - k)(R - 1); k = 1, 2, \dots, 2p - 1 \right\}$$

we can obtain the desired inequalities (3.1) and (3.2) for $\partial_x^y \partial_\xi^u I_j^{(k)}$, $\partial_t \partial_x^y \partial_\xi^u I_j^{(k)}$ and $\partial_s \partial_x^y \partial_\xi^u I_j^{(k)}$.

Let us consider now $\partial_{t,s}^{l_1+1} \partial_x^y \partial_\xi^u I_j^{(k)}$, and we may assume without loss of generality that $\partial_{t,s}^{l_1+1} = \partial_{t,s}^{l_1} \partial_t$. Let operate $\partial_{t,s}^{l_1} \partial_t = \partial_t^{l_1} \partial_s^{l_1-1} \partial_t$ to both sides of (7.1), then we have

$$(7.4) \quad \begin{aligned} \partial_{t,s}^{l_1} \partial_t I_j^{(k)} &= \int_s^t \partial_{t,s}^{l_1+1} K_0 \tau^{m_k} a \xi^{2p-k} \partial_s^{l_1-1} K_{j-1} d\tau \\ &\quad - \sum_{i=1}^{l_1-1} C_i^{l_1-1} \partial_t^{l_1} [\partial_s^{l_1-1-i} \partial_t^{i-1} \{\partial_t K_0 \tau^{m_k} a \xi^{2p-k} K_{j-1}\}]_{\tau=s} \\ &\quad + \sum_{i=1}^{l_1} C_i^{l_1} [\partial_t^{l_1-i} \partial_t^{i-1} \{\partial_t K_0 \tau^{m_k} a \xi^{2p-k} \partial_s^{l_1-1} K_{j-1}\}]_{\tau=t} \\ &= I^{(k)} + \sum_{i=1}^{l_1-1} II_i^{(k)} + \sum_{i=1}^{l_1} III_i^{(k)}. \end{aligned}$$

In order to study $I^{(k)}$, we divide it into two cases.

(A) The case where $\partial_{t,s}^{l_1} = \partial_t^{l_1}$. In this case, in view of (7.2) we have for such d_0 satisfying (7.3)

$$(7.5) \quad |\partial_x^y \partial_\xi^u I^{(k)}| \leq C(1 + |\xi|)^{lp-jd_0-\mu} \exp[-\varepsilon(t^{l_0+1} - s^{l_0+1}) |\xi|^p].$$

(B) The case where $\partial_{t,s}^{l_1} = \partial_t^{l_1} \partial_s^{l_1-1}$, $(l_1 - 1 \geq 1)$. Considering that $(t - s) |\tau|^{m_k} \leq C X^{(2p-k)R/2} |\xi|^{-(2p-k)R/2}$, we can also obtained the inequality (7.5) for such d_0 satisfying

$$(7.6) \quad 0 < d_0 < \min \left\{ \frac{p}{l_0 + 1}, \frac{(2p - k)(R - 1)}{2} + \frac{k}{2}; \right. \\ \left. k = 1, 2, \dots, 2p - 1 \right\}.$$

The study of $II_i^{(k)}$ is more easy, so we omit it. Finally let us consider $III_i^{(k)}$. In this case, using the following inequality we can get the desired inequality,

$$|t^{mk} \partial_x^\alpha \partial_\xi^\mu K_{j-1}| \leq C(1 + |\xi|)^{-(2p-k)R/2 - (j-1)d_0 - \mu} \times \exp[-\varepsilon(t^{l_0+1} - s^{l_0+1})|\xi|^p],$$

$k = 1, 2, \dots, 2p - 1$.

Summing up the aboves we have obtained

$$(7.7) \quad |\partial_{t,s}^{l_0+1} \partial_x^\alpha \partial_\xi^\mu I_j^{(k)}| \leq C(1 + |\xi|)^{l_0 p - j d_0 - \mu} \exp[-\varepsilon(t^{l_0+1} - s^{l_0+1})|\xi|^p].$$

Finally we have to note that the facts stated in Remark 3.1 also hold for $I_j^{(k)}$.

§8. Extension to the Operator Defined in $R_x^n \times I_t$

Our theorem stated in section 1 can be extended to the operator defined in $R_x^n \times I_t$,

$$(8.1) \quad P(x, t; D_x, \partial_t) = \partial_t^2 + A(x, t; D_x) \partial_t + B(x, t; D_x),$$

$$A(x, t; D_x) = \sum_{j=0}^p t^{l_j} a_{p-j}(x, t; D_x), \quad a_{p-j} = \sum_{|\alpha|=p-j} a_{\alpha,j}(x, t) D_x^\alpha,$$

$$B(x, t; D_x) = \sum_{k=0}^{2p} t^{m_k} b_{2p-k}(x, t; D_x), \quad b_{2p-k} = \sum_{|\beta|=2p-k} b_{\beta,k}(x, t) D_x^\beta,$$

where $\partial_t = \partial/\partial t$, $D_x = -i(\partial/\partial x_1, \dots, \partial/\partial x_n)$, $a_{\alpha,j}, b_{\beta,k} \in \mathcal{E}^\infty(\bar{I}_t; \mathcal{E}_x^\infty)$, and l_j and m_k are non-negative integers.

Then under the same conditions in section 1, we can obtain the results in our theorem. In fact, let us assume,

$$(8.2) \quad \left\{ \begin{array}{l} \text{(i)} \quad p \text{ is a positive even integer,} \\ \text{(ii)} \quad p/(l_0 + 1) \geq (p-j)/(l_j + 1), \quad j = 1, 2, \dots, p, \\ \text{(iii)} \quad 2p/(m_0 + 2) \geq (2p-k)/(m_k + 2), \quad k = 1, 2, \dots, 2p, \\ \text{(iv)} \quad p/(l_0 + 1) = 2p/(m_0 + 2), \text{ i.e., } m_0 = 2l_0, \end{array} \right.$$

- (v) $\lambda^2 + a_p(x, t; \xi)\lambda + b_{2p}(x, t; \xi) = 0$ has simple roots $\lambda = \lambda_i(x, t; \xi)$, ($i = 1, 2$) or a double root (i.e. $\lambda_1 \equiv \lambda_2$) in $R_x^n \times I_t$, and they satisfy that $\text{Re } \lambda_i(x, t; \xi) \leq -\delta|\xi|^p$ for some positive constant δ .

Then we have

Theorem 2. *Let the operator given by (8.1) satisfy the conditions in (8.2), then*

- (i) *If l_0 is a non-negative even integer, then P and its adjoint operator tP are hypoelliptic.*
- (ii) *If l_0 is a positive odd integer, then tP is hypoelliptic.*

The proof of the above theorem performed exactly by the same way as in section 2, except the proof of Proposition 3.1. For the proof of Proposition 3.1, we have to consider the following two cases,

- (a) the case where $|\xi|$ is bounded,
- (b) the case where $|\xi|$ is unbounded.

The later case (b) is not troublesome, since the reasonings in the previous sections are applicable. So we have to check the facts in Remark 3.1 for $K_0(x, \xi, t, s)$ when $|\xi|$ is bounded. Now for this purpose, let us transform K_0 to $v(x, \xi, t, s)$ by

$$(8.3) \quad K_0(x, \xi, t, s) = \exp\left[-\frac{1}{2} \int_s^t A(x, \tau; \xi) d\tau\right] v(x, \xi, t, s),$$

then we can easily see that v should satisfy the following ordinary differential equation,

$$(8.4) \quad \begin{cases} \partial_t^2 v = F(x, t; \xi) v & \text{in } -1 < s < t < 1, \\ v|_{t=s} = 0, \quad \partial_t v|_{t=s} = 1. \end{cases}$$

where

$$(8.5) \quad F(x, t; \xi) = \sum_{k=0}^{2p} t^{mk} f_{2p-k}(x, t; \xi), \quad f_{2p-k} = \sum_{|\beta|=2p-k} f_{\beta,k}(x, t) \xi^\beta,$$

$p/(l_0 + 1) \geq (2p - k)/(m_k + 2)$, $k = 0, 1, \dots, 2p$. We note that in view of the proof of Lemma 4.2, we have

Lemma 8.1. *If $|\mu| > lp$, then*

$$(8.6) \quad \partial_{t,s}^l \partial_x^y \partial_\xi^\mu \exp \left[-\frac{1}{2} \int_s^t A(x, \tau; \xi) d\tau \right] \Rightarrow 0 \quad \text{as } t \searrow s,$$

where \Rightarrow means uniformly convergence in x and ξ when ξ varies in a compact set.

On the other hand, for the solution v of (8.4) it holds

Lemma 8.2. *If $|\mu| > lp$, then*

$$(8.7) \quad |\partial_{t,s}^{l+1} \partial_x^y \partial_\xi^\mu v| \Rightarrow 0 \quad \text{as } t \searrow s,$$

$$(8.8) \quad |\partial_x^y \partial_\xi^\mu v| \Rightarrow 0 \quad \text{as } t \searrow s,$$

where \Rightarrow indicates the same meaning in Lemma 8.1.

Proof. We can show that v should satisfy the following integral equation,

$$v(x, \xi, t, s) = (t-s) + \int_s^t (t-\tau) F(x, \tau; \xi) v(x, \xi, \tau, s) d\tau.$$

Then in view of the arguments in section 4, we have

$$|\partial_x^y \partial_\xi^\mu v| \leq C(t-s)(1+|\xi|)^{-\mu} \exp[(t-s)\sqrt{M \cdot C(t, s; \xi)}],$$

$$|\partial_{t,s}^{l+1} \partial_x^y \partial_\xi^\mu v| \leq C(1+|\xi|)^{lp-\mu} \exp[(t-s)\sqrt{M \cdot C(t, s; \xi)}],$$

for some positive constants C and M , and $C(t, s; \xi)$ is defined by

$$C(t, s; \xi) = \max \left\{ \sum_{k=0}^{2p} |t|^{m_k} |\xi|^{2p-k}, \sum_{k=0}^{2p} |s|^{m_k} |\xi|^{2p-k} \right\}.$$

Therefore by induction on l , we can prove (8.7) and (8.8). Q.E.D.

Combining the above two lemmas, we can prove the facts stated in Remark 3.1 for K_0 when $|\xi|$ is bounded.

Appendix A.

In this appendix we shall give a sufficient condition for the follow-

ing operator to be hypoelliptic,

$$(A.1) \quad P(x, t; D_x, \partial_t) = \partial_t + \sum_{j=0}^{2m} t^{l_j} a_j(x, t) D_x^{2m-j},$$

where $(x, t) \in R_x \times I_t$, $I_t = (-1, 1)$, $a_j \in \mathcal{C}^\infty(I_t; \mathcal{B}_x^\infty)$.

For the operator given by (A.1), (or more general operator) the study has been made by many authors and detailed results are obtained. So we only give a result which is not covered in another paper.

Let us assume:

$$(A.2) \quad \left\{ \begin{array}{l} \text{There exists a sequence of integers } \{m_j\}_{j=0}^{k+1} \text{ satisfying,} \\ \text{(i) } 0 = m_0 < m_1 < \dots < m_k < m_{k+1} = m, \\ \text{(ii) } 2(m - m_j)/(l_{2m_j} + 1) = \max_{2m_j \leq i \leq 2m_{j+1} - 1} (2m - i)/(l_i + 1), \\ \text{(iii) } \operatorname{Re} a_{2(m - m_j)} \geq \delta > 0 \text{ in } R_x \times I_t, j = 0, \dots, k. \end{array} \right.$$

Then we have

Theorem A. *Let the operator P given by (A.1) satisfy the conditions in (A.2), then*

- (i) *If l_{2m_j} , ($j=0, 1, \dots, k$) are even integers, then P and its adjoint operator tP are hypoelliptic.*
- (ii) *If l_{2m_j} , ($j=0, 1, \dots, k$) are odd integers, then tP is hypoelliptic, but $P(t; D_x, \partial_t)$ is not hypoelliptic.*

Theorem A is proved by the same way as T. Matsuzawa [8] (see also section 2). Without loss of generality, we assume that l_{2m_j} are even integers. Let $\{K_j(x, \xi, t, s)\}_{j=0}^\infty$ be a sequence defined by

$$(A.3) \quad \left\{ \begin{array}{l} P(x, t; \xi, \partial_t) K_0(x, \xi, t, s) = 0 \quad \text{in } -1 < s < t < 1, \\ K_0|_{t=s} = 1, \quad K_0 = 0 \quad \text{in } -1 < t < s < 1, \end{array} \right.$$

$$(A.4) \quad \left\{ \begin{array}{l} P(x, t; \xi, \partial_t) K_j = -P_1(x, t, \xi; D_x) K_{j-1} \quad \text{in } -1 < s < t < 1, \\ K_j|_{t=s} = 0, \quad K_j = 0 \quad \text{in } -1 < t < s < 1, \quad (j=1, 2, \dots), \end{array} \right.$$

where $P(x, t; \xi + D_x, \partial_t) = P(x, t; \xi, \partial_t) + P_1(x, t, \xi; D_x)$. Then for $\{K_j\}_{j=0}^\infty$ we have

Lemma A.1. *There exists a positive constant d_0 such that*

$$(A.5) \quad |\partial_{t,s}^l \partial_x^v \partial_\xi^\mu K_j| \leq C(1 + |\xi|)^{2ml - jd_0 - \mu} \\ \times \exp \left[-\varepsilon \sum_{j=0}^k (t^{l_{2m_j} + 1} - s^{l_{2m_j} + 1}) |\xi|^{2(m - m_j)} \right],$$

in $-1 < s \leq t < 1$, for some positive constants C and ε depending only on l, v, μ and j . Moreover it holds that

$$(A.6) \quad \partial_{t,s}^l \partial_x^v \partial_\xi^\mu K_j \rightarrow 0 \text{ as } t \searrow s \text{ if } \mu > 2ml,$$

$$(A.7) \quad \partial_{t,s}^l \partial_x^v \partial_\xi^\mu K_j \rightarrow 0 \text{ as } t \searrow s \text{ if } 0 \leq l < j,$$

where the convergence in (A.7) is uniform in x and ξ when ξ varies in a compact set.

Its proof is done by the same way as that of Lemma 4.2. In fact, it suffices to define $X_j, (j=0, 1, \dots, k)$ by

$$(A.8) \quad X_j = (t^{l_{2m_j} + 1} - s^{l_{2m_j} + 1})^{1/2(m - m_j)} \cdot |\xi|, \quad j=0, 1, \dots, k,$$

instead of X defined by (4.11). So we omit the proof.

Appendix B.

B.1. Introduction

We shall study in this appendix the hypoellipticity for the following operator,

$$(B.1.1) \quad P(t; D_x, \partial_t) = \partial_t + at^{l_1} D_x^{2m} + bt^{l_2} D_x^{2n},$$

where $m > n, l_i (i=1, 2)$ are non-negative integers and a and b are non-zero real constants.

The study of hypoellipticity for degenerate parabolic operators are concentrated in the point that how the principal part control the lower order terms. That is, if

$$(B.1.2) \quad 2m/(l_1 + 1) \geq 2n/(l_2 + 1),$$

then P is hypoelliptic in a neighborhood of $t=0$ iff $\partial_t + at^{l_1} D_x^{2m}$ is hypo-

elliptic. (Y. Kato [3]). And the straight extensions of this result are given in Appendix A. But in the case where

$$(B.1.3) \quad 2m/(l_1+1) < 2n/(l_2+1),$$

the situation is quite different from the former case. So we shall study the hypoellipticity assuming (B.1.3). Now our results are stated as follows.

Theorem B.1. *If P is hypoelliptic in a neighborhood of $t=0$, then $\partial_t + bt^{l_2}D_x^{2n}$ is hypoelliptic.*

Theorem B.2. *If $ab > 0$ and $a < 0$, then P is hypoelliptic.*

B.2. Proof of Theorem B.1

We shall prove by contradiction, so we assume that $\partial_t + bt^{l_2}D_x^{2n}$ is not hypoelliptic, i.e., we assume that

$$(B.2.1) \quad l_2 \text{ is an odd integer and } b > 0.$$

In view of Appendix A, our theorem is obvious when l_1 is an odd integer and $a > 0$, therefore it suffices to consider the following two cases,

Case I. l_1 is an even integer.

Case II. l_1 is an odd integer and $a < 0$.

At first, let us consider the Case I. Without loss of generality we may assume that $a > 0$. In fact, if $a < 0$ it suffices only to change the coordinate t by $-t$. Now we remark that from Petrovsky's theorem forward Cauchy problem for P is uniformly well posed in \mathcal{S}' in the interval $[0, 1]$ and also in $[-1, -\varepsilon]$ for any fixed positive ε . But in the interval $[-1, 0]$ it is not uniformly well posed. This fact implies that there exists $f^-(x, t) \in \mathcal{E}'^\infty(\mathcal{S}_x)$, $t \in [-1, 0]$ such that the solution of the Cauchy problem,

$$(B.2.2) \quad \begin{cases} Pu^- = f^-(x, t) & \text{in } (-1, 0) \\ u^-|_{t=-1} = 0 \end{cases}$$

satisfies that $u^- \in \mathcal{E}'_t(\mathcal{S}_x)$, ($t \in (-1, 0)$) and $u^-(x, -0) = \lim_{t \nearrow 0} u^-(x, t) \notin C^\infty(\mathbb{R}_x)$. Let us now prove this fact. We note that for a given $f^-(x, t)$ the solution of (B.2.2) should be given by

$$(B.2.3) \quad u^-(x, t) = \mathcal{F}_\xi^{-1} \left(\int_{-1}^t \exp \left[-\frac{a}{l_1+1} (t^{l_1+1} - s^{l_1+1}) \xi^{2m} - \frac{b}{l_2+1} (t^{l_2+1} - s^{l_2+1}) \xi^{2n} \right] \hat{f}^-(\xi, s) ds \right) \\ = \mathcal{F}_\xi^{-1} [\hat{u}^-(\xi, t)],$$

where \mathcal{F}_ξ^{-1} denotes the Fourier inverse transform and $\hat{f}^-(\xi, t)$ denotes the partial Fourier transform of $f^-(x, t)$ with respect to x . Now let us define $\hat{f}^-(\xi, t)$ by

$$(B.2.4) \quad \hat{f}^-(\xi, t) = \begin{cases} 0 & \text{when } \xi \leq 0, t \leq 0 \\ \exp \left[-\frac{a}{l_1+1} t^{l_1+1} \xi^{2m} - \frac{b}{l_2+1} t^{l_2+1} \xi^{2n} \right] \\ \quad \times \varphi(t \xi^{2m/(l_1+1)}), & \text{when } t < 0, \xi > 0, \end{cases}$$

where $\varphi(\tau) \in C_0^\infty(\mathbb{R}_+)$, $\varphi(\tau) \geq 0$ and $\text{supp}[\varphi(\tau)] \subset \left[-\frac{1}{2}, -\frac{1}{4}\right]$. Considering that $a > 0$, $b > 0$ and the assumption (B.1.3), it is easy to show that

$$\lim_{t \nearrow 0} \partial_t^j \partial_\xi^k \hat{f}^-(\xi, t) = 0 \quad \text{in } \mathcal{S}'_\xi \quad \text{for any } j \text{ and } k.$$

On the other hand, it is easy to see that

$$\hat{u}^-(\xi, t) \in \mathcal{E}'_t(\mathcal{S}'_\xi), \quad t \in [-1, 0].$$

In fact, it suffices to see that

$$\hat{u}^-(\xi, t) = \xi^{-2m/(l_1+1)} \exp \left[-\frac{a}{l_1+1} t^{l_1+1} \xi^{2m} - \frac{b}{l_2+1} t^{l_2+1} \xi^{2n} \right] \\ \times \int_{-\xi^{2m/(l_1+1)}}^{t \xi^{2m/(l_1+1)}} \varphi(\tau) d\tau.$$

Hence for sufficiently large $\xi > 0$, $\lim_{t \nearrow 0} \hat{u}^-(\xi, t) = \int \varphi(\tau) d\tau \cdot \xi^{-2m/(l_1+1)}$, which

shows that $\lim_{t \nearrow 0} u^-(x, t) \notin C^\infty(R_x)$. Under the above preparation, let us define $f(x, t), t \in [-1, 1]$ by

$$(B.2.5) \quad f(x, t) = \begin{cases} 0, & t \geq 0, \\ f^-(x, t), & t < 0. \end{cases}$$

Then the equation $Pu=f$ has a solution $u(x, t)$ given by

$$u(x, t) = \begin{cases} u^-(x, t) & \text{in } t \leq 0 \text{ given by (B.2.3)} \\ u(x, t) & \text{in } t > 0, \text{ the solution of } Pu=0 \\ & \text{with Cauchy data } u|_{t=0} = \lim_{t \nearrow 0} u^-(x, t). \end{cases}$$

This proves the non-hypoellipticity of P .

Let us consider the Case II. In this case also we shall construct $f(x, t) \in \mathcal{E}'_i(\mathcal{S}_x)$ such that the equation $Pu=f$ has a non-smooth solution. Under the assumption of Case II we can easily see that the forward (or the backward) Cauchy problem for P is uniformly well posed in the interval $[-1, -\varepsilon]$ (or $[\varepsilon, 1]$) for any fixed positive ε , but not in $[-1, 0]$ (or $[0, 1]$). Hence from the arguments in the above, there exists $f^- \in \mathcal{E}'_i(\mathcal{S}_x), t \in [-1, 0]$ (or $f^+ \in \mathcal{E}'_i(\mathcal{S}_x), t \in [0, 1]$) such that the equation $Pu^- = f^-$ (or $Pu^+ = f^+$) has a solution $u^- \in \mathcal{E}'_i(\mathcal{S}_x), t \in (-1, 0)$ ($u^+ \in \mathcal{E}'_i(\mathcal{S}_x), t \in (0, 1)$) but $\lim_{t \nearrow 0} u^- \notin C^\infty$ ($\lim_{t \searrow 0} u^+ \notin C^\infty$). Let us now show that we can choose f^+ and f^- so that $\lim_{t \nearrow 0} u^- = \lim_{t \searrow 0} u^+$. Indeed, let $\hat{f}^-(\xi, t)$ be given by (B.2.4), then it suffices to give $\hat{f}^+(\xi, t)$ by $\hat{f}^+(\xi, t) = -\hat{f}^-(\xi, -t), t > 0$. This implies immediately the non-hypoellipticity of P . Q. E. D.

B.3. Proof of Theorem B.2

At first we note that under the assumption of Theorem B.2, it follows immediately that the solution of $Pu=f \in C^\infty(\Omega), (0 \in \Omega \subset R^2_{x,t})$ belongs to $C^\infty(\bar{\Omega}_+), (\bar{\Omega}_+ = \{(x, t) \in \Omega; t \geq 0\})$ by the usual method developed in [1] or [3]. But we can not prove that $u \in C^\infty(\bar{\Omega}_-), (\bar{\Omega}_- = \{(x, t) \in \Omega; t \leq 0\})$ by the method of fundamental solution. On the other hand, if we can show,

$$(B.3.1) \quad \lim_{t \nearrow 0} u(x, t) = u(x, -0) \in \mathcal{D}'(\Omega \cap \{t=0\}),$$

then we have $u \in C^\infty(\Omega)$. In fact if (B.3.1) is proved, let us decompose $u(x, t)$ as follows (see [13]),

$$(B.3.2) \quad u(x, t) = H(t)u(x, t) + H(-t)u(x, t) + \sum_{j=0}^N \delta^{(j)}(t) \otimes v_j(x)$$

for some $v_j(x) \in \mathcal{D}'(I_x)$, ($I_x \in \Omega \cap \{t=0\}$), where $H(t)$ denotes Heaviside's function. Then

$$\begin{aligned} Pu &= H(t)Pu + H(-t)Pu + \delta(t) \otimes \{u(x, +0) - u(x, -0)\} + \sum_{j=0}^N \delta^{(j+1)}(t) \otimes v_j(x) \\ &+ \sum_{j=0}^N at^{l_1} \delta^{(j)}(t) \otimes D_x^{2m} v_j(x) + \sum_{j=0}^N bt^{l_2} \delta^{(j)}(t) \otimes D_x^{2n} v_j(x). \end{aligned}$$

Considering that $Pu = f \in C^\infty(\Omega)$, it follows that $v_{N-l}(x) = 0$, ($l=0, 1, \dots, N$). Therefore, $\delta(t) \otimes \{u(x, +0) - u(x, -0)\} = 0$, that is, $u(x, +0) = u(x, -0)$. Thus we have $u(x, t) \in \mathcal{E}'_t(\mathcal{D}'(I_x))$, $t \in (-c, c)$ for some positive constant c . Consequently $u(x, t) \in \mathcal{E}'_t(\mathcal{D}'(I_x))$, which implies immediately that $u(x, t) \in C^\infty(I_x \times (-c, c))$.

Hence our purpose in the followings is to show (B.3.1). Without loss of generality we may assume that

$$(B.3.3) \quad u(x, t) = \partial_t^j \partial_x^k v(x, t), \text{ where } v(x, t) \in L^1(\Omega).$$

Since $u \in C^\infty(\bar{\Omega}_+)$, we can easily show that

$$(B.3.4) \quad \lim_{t \searrow 0} \partial_t^{j-l} \partial_x^k v(x, t) = v^{(j-l, k)}(x) \in \mathcal{D}'(I_x), \quad l=0, 1, \dots, j.$$

Indeed, $\partial_t^{j-l} \partial_x^k v(x, t)$ can be represented by

$$\begin{aligned} \partial_t^{j-l} \partial_x^k v(x, t) &= \int_0^t dt_1 \int_0^{t_1} dt_2 \dots \int_0^{t_{l-1}} u(x, \tau) d\tau \\ &+ \sum_{i=1}^l v_i(x) t^{l-i} / (l-i)!, \quad \text{for some } v_i \in \mathcal{D}'(I_x), (t > 0). \end{aligned}$$

Let us now rewrite $v(x, t)$ as follows,

$$v(x, t) = \lim_{\varepsilon \downarrow 0} \{H(t-\varepsilon)v + H(-t-\varepsilon)v\} \quad \text{in } \mathcal{D}'(\Omega).$$

Then we have

$$\begin{aligned}
 Pu = & \lim_{\varepsilon \searrow 0} \{H(t-\varepsilon)Pu + H(-t-\varepsilon)Pu + \delta_\varepsilon(t) \otimes u(x, \varepsilon) - \delta_{-\varepsilon}(t) \otimes u(x, -\varepsilon) \\
 & + \sum_{i=1}^j \delta_\varepsilon^{(i)}(t) \otimes \partial_t^{j-i} \partial_x^k v(x, \varepsilon) - \sum_{i=1}^j \delta_{-\varepsilon}^{(i)}(t) \otimes \partial_t^{j-i} \partial_x^k v(x, -\varepsilon) \\
 & + \sum_{i=1}^j at^{l_1} \delta_\varepsilon^{(i-1)}(t) \otimes \partial_t^{j-i} \partial_x^k D_x^{2m} v(x, \varepsilon) \\
 & - \sum_{i=1}^j at^{l_1} \delta_{-\varepsilon}^{(i-1)}(t) \otimes \partial_t^{j-i} \partial_x^k D_x^{2m} v(x, -\varepsilon) + \dots \}.
 \end{aligned}$$

On the other hand, we know from (B.3.4) that $\lim_{\varepsilon \searrow 0} \partial_t^{j-i} \partial_x^{k+k'} v(x, \varepsilon)$ exists in $\mathcal{D}'(I_x)$ for any $k' \geq 0$, therefore

$$\begin{aligned}
 \lim_{\varepsilon \searrow 0} \sum_{\mu=0}^j \delta_{-\varepsilon}^{(\mu)}(t) \otimes \{ \partial_t^{j-\mu} \partial_x^k v(x, -\varepsilon) + (\text{linear combination of} \\
 \partial_t^{j-\mu'} \partial_x^{k+k'} v(x, -\varepsilon), \mu' > \mu) \}
 \end{aligned}$$

has to exist in \mathcal{D}' . If we take as test function $\varphi(x, t)$ such a form as $\varphi(x, t) = f(x)g(t)$, where $g(t) = 1$ in a neighborhood of $t = 0$, we see that

$$\begin{aligned}
 \lim_{\varepsilon \searrow 0} \{ \partial_t^j \partial_x^k v(x, -\varepsilon) + (\text{linear combination of} \\
 \partial_t^{j-\mu'} \partial_x^{k+k'} v(x, -\varepsilon), \mu' > 0) \}
 \end{aligned}$$

exists in $\mathcal{D}'(I_x)$. Inductively we can prove that if we take as test function $\varphi(x, t)$ such a form as $\varphi(x, t) = t^\mu f(x)g(t)$, where $g(t) = 1$ in a neighborhood $t = 0$, then

$$\begin{aligned}
 \lim_{\varepsilon \searrow 0} \{ \partial_t^{j-\mu} \partial_x^k v(x, -\varepsilon) + (\text{linear combination of} \\
 \partial_t^{j-\mu'} \partial_x^{k+k'} v(x, -\varepsilon), \mu' > \mu) \}
 \end{aligned}$$

exists in $\mathcal{D}'(I_x)$, consequently, $\lim_{\varepsilon \searrow 0} \partial_x^k v(x, -\varepsilon) \in \mathcal{D}'(I_x)$. Conversely the above procedure, we can conclude that $\lim_{t \nearrow 0} u(x, t)$ exists in $\mathcal{D}'(I_x)$, which proves (B.3.1). Q. E. D.

Finally we note that recently R. Rubinstein [14] has studied an operator,

$$(B.3.5) \quad L = \partial_t + t^n D_x^2 - t^m D_x, \quad (n; \text{even}).$$

He proved that L is not solvable under the same assumption with (B.1.3), that is, if

$$(B.3.6) \quad \frac{2}{n+1} < \frac{1}{m+1},$$

then L is not solvable.

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