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On the Classification of Some $(n-3)$-Connected $(2n-1)$-Manifolds

Dedicated to Professor Ryoji Shizuma on his 60-th birthday

By

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Introduction

In the preceding paper [4], the author tried to classify $(n-2)$-connected $2n$-manifolds $(n\geqq 4)$ with torsion free homology groups up to diffeomorphism $\bmod\, \theta_{2n}$ by completely classifying the handlebodies of $\mathscr{H}(2n+1, k, n+1)$ $(n\geqq 4)$ up to diffeomorphism. As remarked there, the method is also applicable to the case of sufficiently connected odd dimensional manifolds.

In this paper, we try to classify the simply connected $(2n-1)$-manifolds $(n\geqq 6)$ with non-trivial homology groups only in dimensions 0, $n-2$, $n+1$, and $2n-1$, up to diffeomorphism $\bmod\, \theta_{2n-1}$ by completely classifying the handlebodies of $\mathscr{H}(2n, k, n+1)$ $(n\geqq 6)$ up to diffeomorphism. The results are listed up or given as theorems in the next section. Those contain the results of Tamura [10] as a special case, that is, as the case of type O. To classify the handlebodies of $\mathscr{H}(2n, k, n+1)$ $(n\geqq 6)$ up to diffeomorphism, we use Wall's classification theorem [11], similarly as in [4].

Throughout this paper, notations are due to those of [4], and manifolds are connected, closed, and differentiable.

Results

Let M be a simply connected $(2n-1)$-manifold $(n\geqq 6)$ satisfying the

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hypotheses

(H_1) $H_i(M)=0$ except dimensions $i=0,\ n-2,\ n+1$, and $2n-1$,

(H_2) M is $(n-2)$-parallelizable.[1] (This hypothesis is satisfied if $n=0,\ 1,\ 5$, and $7 \bmod 8$.)

Let $\Phi\colon H^{n-2}(M; Z_2)\to H^{n+1}(M; Z_2)$ be Adem's secondary cohomology operation associated to $S_q^3S_q^1+S_q^2S_q^2=0$. We note that there is no indeterminacy by the homological assumption of M. Let $\phi\colon H^{n-2}(M)\times H^{n-2}(M)\to Z_2$ be a bilinear form defined by $\phi(x, y)=<\Phi x_2 \cup y_2, [M]_2>$, where the suffixes 2 mean that those are considered in the Z_2-coefficient and $[M]$ denotes the fundamental class of $H_{2n-1}(M)$. It will be clear in §1 that ϕ is symmetric. So that the *type* of M is defined as in [4]. That is, M is of *type* O if $\operatorname{rank}\phi=0$, of *type* I if $\phi(x, x)\neq 0$ for some $x\in H^{n-2}(M)$ and $\operatorname{rank}\phi=k$ ($k=\operatorname{rank}H^{n-2}(M)$), and of *type* II if $\phi(x, x)=0$ for any $x\in H^{n-2}(M)$ and $\operatorname{rank}\phi=k$. M is of *type* (O+I) if $\phi(x, x)\neq 0$ for some $x\in H^{n-2}(M)$ and $0<\operatorname{rank}\phi<k$, and of *type* (O+II) if $\phi(x, x)=0$ for any $x\in H^{n-2}(M)$ and $0<\operatorname{rank}\phi<k$. M belongs to some type and the type is uniquely determined (See Lemma 1.1 of [4].)

Theorem 1. *Let M be a simply connected $(2n-1)$-manifold $(n\geqq 6)$ satisfying the hypotheses* (H_1), (H_2). *Then, M is represented* $\bmod \theta_{2n-1}$ *as shown in the following tables* 1, 2, *and* 3.

In these tables, A_α, B_β denote the $(n-2)$-sphere bundles over $(n+1)$-spheres with the characteristic elements $\alpha,\ \beta\in\pi_n(SO_{n-1})$ respectively such that $\pi(\alpha)=0$, $\pi(\beta)=1$ for $\pi\colon \pi_n(SO_{n-1})\to\pi_n(S^{n-2})\cong Z_2$ $(n\geqq 6)$, the homomorphism induced from the projection. $V\binom{\alpha_1}{\alpha_2}$ is the boundary of $W\binom{\alpha_1}{\alpha_2}$, where $W\binom{\alpha_1}{\alpha_2}$ is a handlebody of $\mathscr{H}(2n, 2, n+1)$ such that the link $f_1(\partial D_1^{n+1}\times o)\cup f_2(\partial D_2^{n+1}\times o)\subset\partial D^{2n}$ by the attaching maps f_1, f_2 has the non-zero linking element and the normal bundles of the spheres S_i^{n+1}, with hemispheres $D_i^{n+1}\times o$ and D_i^{n+1} in D^{2n}, $i=1, 2$, have the characteristic elements $\alpha_1, \alpha_2\in\pi_n(SO_{n-1})$ respectively such that $\pi(\alpha_1)=\pi(\alpha_2)=0$. $V\binom{\alpha_1}{\alpha_2}$ never has the homotopy type of the connected sum of the two $(n-2)$-sphere bundles over $(n+1)$-spheres (cf. [4], §8 and §1).

1) This means that M is parallelizable on its $(n-2)$-skeleton of a triangulation.

$W\binom{\alpha_1}{\alpha_2}$ is also constructed from $(n-1)$-disk bundles over $(n+1)$-spheres $\bar{A}_i$ with the characteristic elements α_i, $i=1, 2$, by plumbing along $S^1\times S^1$, where there are imbeddings f_i: $S^1\times S^1\to S_i^{n+1}$, $i=1, 2$, with the trivial normal bundles framed so that those Pontrjagin-Thom maps yield non-trivial elements of $\pi_{n+1}(S^{n-1})\cong Z_2$, and then by attaching two 2-cells with thickness D^{2n-2} and a 3-cell with thickness D^{2n-3} to the boundary. (See [3] p. 494, p. 506.)

For an integer $m\geqq 0$, mA_α, mB_β, $m(S^{n+1}\times S^{n-2})$, $mV\binom{\alpha_1}{\alpha_2}$ denote the connected sum of m-copies of A_α, B_β, $S^{n+1}\times S^{n-2}$, and $V\binom{\alpha_1}{\alpha_2}$ respectively. We put $k=\operatorname{rank} H_{n-2}(M)$. If M is of type (O+I), $q=\operatorname{rank}\phi$, $p=k-q$, and we fix the homotopy invariant q. If M is of type II, $k=2r$. If M is of type (O+II), $2r=\operatorname{rank}\phi$, $p=k-2r$, and we fix the homotopy invariant r. If $\pi_n(SO_{n-1})$ has several direct summands, for example, if $\alpha_i=\alpha_1^i+\alpha_2^i$, $i=1, 2$, we denote $V\binom{\alpha_1}{\alpha_2}$ by $V\begin{pmatrix}\alpha_1^1 & \alpha_2^1\\ \alpha_1^2 & \alpha_2^2\end{pmatrix}$.

Table 1

n ($\geqq 6$)	Type O
$4t-1$	$A_a\#(k-1)(S^{n+1}\times S^{n-2})$, $a\geqq 0$ $t=2\Longrightarrow$ a: even $\geqq 0$
$4t$ (t: odd)	$k(S^{n+1}\times S^{n-2})$
$4t$ (t: even)	$A_{(0,b)}\#(k-1)(S^{n+1}\times S^{n-2})$, $b=0, 1$
$4t+1$ (t: odd)	$A_{(a,0)}\#(k-1)(S^{n+1}\times S^{n-2})$, $a=0, 1$
$4t+1$ (t: even)	$A_{(a,0,b)}\#(k-1)(S^{n+1}\times S^{n-2})$, $a, b=0, 1$ $A_{(1,0,0)}\#A_{(0,0,1)}\#(k-2)(S^{n+1}\times S^{n-2})$
$4t+2$ ($t\geqq 2$)	$A_a\#(k-1)(S^{n+1}\times S^{n-2})$, $a=0, 1, 2, 4$
6	$k(S^7\times S^4)$

Table 2

$n\ (\geqq 6)$	Type I	Type (O+I)
$4t-1$	$t \geqq 3 \Longrightarrow$ Nothing	$t \geqq 3 \Longrightarrow$ Nothing
	$t=2 \Longrightarrow kB_c$, c: odd>0	$p(S^8 \times S^5) \# qB_c$, c: odd>0
$4t$ (t: odd)	kB_1	$p(S^{n+1} \times S^{n-2}) \# qB_1$
$4t$ (t: even)	$kB_{(1,0)}$, $kB_{(1,1)}$ $(k-1)B_{(1,0)} \# B_{(1,1)}$, $k \geqq 2$ $(k-2)B_{(1,0)} \# 2B_{(1,1)}$, $k \geqq 3$	$p(S^{n+1} \times S^{n-2}) \# qB_{(1,0)}$ $p(S^{n+1} \times S^{n-2}) \# qB_{(1,1)}$ $p(S^{n+1} \times S^{n-2}) \# (q-1)B_{(1,0)} \# B_{(1,1)}$, $q \geqq 2$ $p(S^{n+1} \times S^{n-2}) \# (q-2)B_{(1,0)} \# 2B_{(1,1)}$, $q \geqq 3$ $A_{(0,1)} \# (p-1)(S^{n+1} \times S^{n-2}) \# qB_{(1,0)}$
$4t+1$ (t: odd)	$kB_{(0,1)}$	$p(S^{n+1} \times S^{n-2}) \# qB_{(0,1)}$
$4t+1$ (t: even)	$kB_{(0,1,0)}$, $kB_{(0,1,1)}$ $(k-1)B_{(0,1,0)} \# B_{(0,1,1)}$, $k \geqq 2$ $(k-2)B_{(0,1,0)} \# 2B_{(0,1,1)}$, $k \geqq 3$	$p(S^{n+1} \times S^{n-2}) \# qB_{(0,1,0)}$ $p(S^{n+1} \times S^{n-2}) \# qB_{(0,1,1)}$ $p(S^{n+1} \times S^{n-2}) \# (q-1)B_{(0,1,0)} \# B_{(0,1,1)}$, $q \geqq 2$ $p(S^{n+1} \times S^{n-2}) \# (q-2)B_{(0,1,0)} \# 2B_{(0,1,1)}$, $q \geqq 3$ $A_{(0,0,1)} \# (p-1)(S^{n+1} \times S^{n-2}) \# qB_{(0,1,0)}$
$4t+2$	Nothing	Nothing

Table 3

n ($\geqq 6$)	Type II	Type (O+II)
$4t-1$	$V\begin{pmatrix} d \\ 0 \end{pmatrix} \# (r-1) V\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $d \geqq 0$	$A_a \# (p-1)(S^{n+1} \times S^{n-2}) \# r V\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $a \geqq 0$ $p(S^{n+1} \times S^{n-2}) \# V\begin{pmatrix} d \\ 0 \end{pmatrix} \# (r-1) V\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $d > 0$
	$t=2 \Longrightarrow d$: even $\geqq 0$	$t=2 \Longrightarrow a, d$: even, $a \geqq 0$, $d > 0$
$4t$ (t: odd)	$r V\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$p(S^{n+1} \times S^{n-2}) \# r V\begin{pmatrix} 0 \\ 0 \end{pmatrix}$
$4t$ (t: even)	$V\begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix} \# (r-1) V\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $d=0, 1$	$A_{(0,b)} \# (p-1)(S^{n+1} \times S^{n-2}) \# r V\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $b=0, 1$ $p(S^{n+1} \times S^{n-2}) \# V\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \# (r-1) V\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$
$4t+1$ (t: odd)	$V\begin{pmatrix} d & 0 \\ d & 0 \end{pmatrix} \# (r-1) V\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $d=0, 1$	$A_{(a,0)} \# (p-1)(S^{n+1} \times S^{n-2}) \# r V\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $a=0, 1$ $p(S^{n+1} \times S^{n-2}) \# V\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \# (r-1) V\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$4t+1$ (t: even)	$V\begin{pmatrix} d00 \\ d00 \end{pmatrix}\#(r-1)V\begin{pmatrix} 000 \\ 000 \end{pmatrix}$, $V\begin{pmatrix} 001 \\ 00d \end{pmatrix}\#(r-1)V\begin{pmatrix} 000 \\ 000 \end{pmatrix}$, $V\begin{pmatrix} 100 \\ 100 \end{pmatrix}\#V\begin{pmatrix} 001 \\ 00d \end{pmatrix}\#(r-2)V\begin{pmatrix} 000 \\ 000 \end{pmatrix}$, where $d=0, 1$.	$P(S^{n+1}\times S^{n-2})\#$ (Manifolds of type II). $A_{(1,0,0)}\#(p-1)(S^{n+1}\times S^{n-2})\#V\begin{pmatrix} 00d \\ 000 \end{pmatrix}\#(r-1)V\begin{pmatrix} 000 \\ 000 \end{pmatrix}$, $A_{(0,0,1)}\#(p-1)(S^{n+1}\times S^{n-2})\#V\begin{pmatrix} d00 \\ d00 \end{pmatrix}\#(r-1)V\begin{pmatrix} 000 \\ 000 \end{pmatrix}$, $A_{(1,0,1)}\#(p-1)(S^{n+1}\times S^{n-2})\#V\begin{pmatrix} d00 \\ d00 \end{pmatrix}\#(r-1)V\begin{pmatrix} 000 \\ 000 \end{pmatrix}$, where $d=0, 1$. $A_{(1,0,0)}\#A_{(0,0,1)}\#(p-2)(S^{n+1}\times S^{n-2})\#rV\begin{pmatrix} 000 \\ 000 \end{pmatrix}$
$4t+2$ ($t\geqq 2$)	$V\begin{pmatrix} d \\ d \end{pmatrix}\#(r-1)V\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $d=0, 4$ $V\begin{pmatrix} d \\ 0 \end{pmatrix}\#(r-1)V\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $d=1, 2$	$p(S^{n+1}\times S^{n-2})\#V\begin{pmatrix} d \\ d \end{pmatrix}\#(r-1)V\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $d=0, 4$ $p(S^{n+1}\times S^{n-2})\#V\begin{pmatrix} d \\ 0 \end{pmatrix}\#(r-1)V\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $d=1, 2$ $A_a\#(p-1)(S^{n+1}\times S^{n-2})\#rV\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $a=1, 2, 4$
6	$rV\begin{pmatrix} 0 \\ 0 \end{pmatrix}$	$p(S^7\times S^4)\#rV\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

The homotopy groups $\pi_n(SO_{n-1})$ $(n\geqq 6)$ are given as follows (Kervaire [5], Paechter [8]) and are identified with those groups under some bases (cf. § 2).

$n\ (\geqq 7)$	$8s-1$	$8s$	$8s+1$	$8s+2$	$8s+3$	$8s+4$	$8s+5$	$8s+6$
$\pi_n(SO_{n-1})$	Z	Z_2+Z_2	$Z_2+Z_2+Z_2$	Z_8	Z	Z_2	Z_2+Z_2	Z_8 ,

and $\pi_6(SO_5)=0$.

The type of a handlebody W of $\mathscr{H}(2n, k, n+1)$ $(n\geqq 6)$ is defined by the bilinear form λ of the corresponding $(H; \lambda, \alpha)$-system. (See [4] p. 222.) We have

Theorem 1′. *Let $\bar{A}_\alpha$, $\bar{B}_\beta$ be the $(n-1)$-disk bundles over $(n+1)$-spheres associated with A_α, B_β respectively. In the above tables, if we replace $S^{n+1}\times S^{n-2}$, A_α, B_β, $V\binom{\alpha_1}{\alpha_2}$, and $\#$ respectively by $S^{n+1}\times D^{n-1}$, $\bar{A}_\alpha$, $\bar{B}_\beta$, $W\binom{\alpha_1}{\alpha_2}$, and the boundary connected sum operation $\natural$, then Table* 1, *Table* 2, *and Table* 3 *give the complete classification of handlebodies of $\mathscr{H}(2n, k, n+1)$ $(n\geqq 6)$ up to diffeomorphism.*

Theorem 2. *In Theorem* 1, *the representation of M is unique* $\bmod\ \theta_{2n-1}$ *in each of the following cases when*

(i) *M is of type* O,

(ii) *M is of type* I, *$n\neq 8s$, and $n\neq 8s+1$,*

(iii) *M is of type* (O+I), *$n\neq 8s$, and $n\neq 8s+1$,*

(iv) *M is of type* II *and $n=4t-1$ or $8s+4$ or* 6,

(v) *M is of type* (O+II) *and $n=4t-1$ or $8s+4$ or* 6,

and especially, in the above (i)–(v),

(vi) *when $n=4t-1$ or* 6.

Corollary 3. *Let $n=4t-1$ $(t\geqq 2)$ and let M be a simply connected $(2n-1)$-manifold satisfying* (H_1), *and* (H_2) *if t is odd. Then M is determined by Adem's secondary cohomology operation Φ: $H^{n-2}(M; Z_2)\rightarrow H^{n+1}(M; Z_2)$ and the Pontrjagin class $P_t(M)$ up to diffeomorphism* $\bmod\ \theta_{2n-1}$.

1. Proofs of the Main Theorems

Let M be a simply connected $(2n-1)$-manifold $(n\geqq 6)$ satisfying the hypotheses (H_1), (H_2). Then there exists a handlebody W of $\mathscr{H}(2n, k, n+1)$, where $k=\operatorname{rank} H_{n-2}(M)$, and a homotopy $(2n-1)$-sphere Σ such that $M=\partial W \# \Sigma$ (Ishimoto [3], p. 509).

Let $W=D^{2n}\underset{\{f_i\}}{\cup}\{\bigcup_{i=1}^{k} D_i^{n+1}\times D_i^{n-1}\}$ and let $\lambda_{ij}\in Z_2\cong\pi_n(S^{n-2})$ $(n\geqq 6)$ be the linking element (Haefliger [2]) defined by $f_j(S_j^n\times o)$ in $S^{2n-1}-f_i(S_i^n\times o)$ if $i\neq j$, and defined by $S_i'^n$ in $S^{2n-1}-f_i(S_i^n\times o)$ slightly moved from $f_i(S_i^n\times o)$ if $i=j$. Let $\varepsilon_i\in H^{n-2}(\partial W; Z_2)$, $i=1, 2,\dots, k$, be the canonical generators which are dual to the homology classes $(x_i\times S_i^{n-2})\in H_{n-2}(\partial W; Z_2)$, $x_i\in\partial D_i^{n+1}$, $S_i^{n-2}=\partial D_i^{n-1}$, respectively. Then we have the relation $\lambda_{ij}=<\Phi\varepsilon_i\cup\varepsilon_j, [\partial W]_2>$ for all i, j, where $[\partial W]_2$ denotes the mod 2 fundamental class of $H_{2n-1}(\partial W; Z_2)$ (cf. [4], Lemma 8.2 and Remark 1 of p. 251). Let λ: $H_{n+1}(W)\times H_{n+1}(W)\to Z_2\cong\pi_{n+1}(S^{n-1})$ be the corresponding pairing of W and let $\{e_1,\dots, e_k\}$ be the canonical base of $H_{n+1}(W)$. Then the relation $\lambda(e_i, e_j)=S\lambda_{ij}$ holds by Lemma 7 of Wall [11]. So that, we have the following commutative diagram:

$$\begin{array}{lcl} H_{n+1}(W)\times H_{n+1}(W) & \searrow^{\lambda} & \\ i_*\times i_*\uparrow\cong & & \\ H_{n+1}(\partial W)\times H_{n+1}(\partial W) & \longrightarrow & Z_2 \\ D\times D\uparrow\cong & & \\ H^{n-2}(\partial W)\times H^{n-2}(\partial W) & \nearrow_{\Phi} & \end{array}$$

where i_* is the isomorphism induced from the inclusion map i and D denotes the Poincaré duality. (cf. Theorem 8.3 of [4]). Thus, the type of W defined by the bilinear form λ of the corresponding $(H; \lambda, \alpha)$-system coincides with that of M. Therefore, we have Theorem 1 by the complete classification of the handlebodies of $\mathscr{H}(2n, k, n+1)$ $(n\geqq 6)$ up to diffeomorphism, which has been performed in the following sections, using Wall's classification theorem [11]. Theorem 1$'$ is the collection of the results.

If the membrane W of M is unique up to diffeomorphism, then also

M up to diffeomorphism $\bmod\,\theta_{2n-1}$. If W_i, $i=1, 2$, are handlebodies of type O of $\mathscr{H}(2n, k, n+1)$ $(n\geqq 6)$ and if ∂W_1 is diffeomorphic to ∂W_2 $\bmod\,\theta_{2n-1}$, then W_1 is diffeomorphic to W_2, similarly as Theorem 9.1 of [4]. So that, if M is of type O, the representation of M in Table 1 is unique $\bmod\,\theta_{2n-1}$.

Let ξ be an orientable $(4t-2)$-plane bundle over the $4t$-sphere $(t\geqq 2)$ with the characteristic element $\gamma\in Z\cong\pi_{4t-1}(SO_{4t-2})$. Then, the Pontrjagin class $P_t(\xi)$ satisfies the relation $P_t(\xi)=\pm(c_1\gamma)\cdot\bar{\mu}$, where

$$c_1=\begin{cases} 24 & \text{if } t=2, \\ 2(2t-1)! & \text{if } t \text{ is odd } \geqq 3, \\ (2t-1)! & \text{if } t \text{ is even } \geqq 4, \end{cases}$$

and $\bar{\mu}$ is the fundamental class of $H^{n+1}(S^{n+1}; Z)$. For, since $P_t(\xi)$ $=P_t(\xi\oplus\varepsilon)=\pm c(S\gamma)\cdot\bar{\mu}$ where c is the number defined in [4] (p. 254) or [10] (p. 378) and $S\colon \pi_n(SO_{n-1})\to\pi_n(SO_n)$ is the suspension homomorphism, the relation is obtained by the fact that $S\gamma=\pm\gamma$ if $t\geqq 3$ and $S\gamma=\pm 2\gamma$ if $t=2$, which is known from the following exact sequence

$$\begin{array}{ccccccc} \pi_{4t-1}(SO_{4t-2}) & \xrightarrow{S} & \pi_{4t-1}(SO_{4t-1}) & \longrightarrow & \pi_{4t-1}(S^{4t-2}) & \xrightarrow{\partial} & \pi_{4t-2}(SO_{4t-2}), \\ \wr\Vert & & \wr\Vert & & \wr\Vert & & \\ Z & & Z & & Z_2 & & \end{array}$$

where $\pi_{4t-2}(SO_{4t-2})\cong Z_4$ if $t\geqq 3$, $\pi_6(SO_6)\cong 0$, and $\partial(1)=2$ if $t\geqq 3$. (See [4] Lemma 2.1.)

Let $n=4t-1$ $(t\geqq 2)$ and let W be a handlebody of $\mathscr{H}(2n, k, n+1)$ with the system $(H; \lambda, \alpha)$. Then, similarly as Lemma 9.2 of [4], we have $\alpha=\pm\dfrac{1}{c_1}\langle P_t(W),\ \rangle=\pm\dfrac{1}{c_1}\langle P_t(\partial W), i_*^{-1}(\ )\rangle$, where i_* is the isomorphism induced from the inclusion map $i\colon \partial W\Rightarrow W$. If $\partial W_1\#\sum_1$ $=\partial W_2\#\sum_2$, where $W_i\in\mathscr{H}(2n, k, n+1)$, $i=1, 2$, and $\sum_i$ are homotopy $(2n-1)$-spheres, there exists a homeomorphism $g\colon \partial W_1\to\partial W_2$ such that $g^*(\tau(\partial W_2))=\tau(\partial W_1)$ (Shiraiwa [9]). So that we know the uniqueness of the representation of M $\bmod\,\theta_{2n-1}$ when $n=4t-1$ $(t\geqq 2)$ (cf. Theorem 9.3 of [4]).

This completes the proof of Theorem 2. The corollary is clear from the above.

2. Calculations of ∂ and π

Let $\partial_n: \pi_{n+1}(S^{n-1})(\cong Z_2)\to\pi_n(SO_{n-1})$ be the boundary homomorphism in the homotopy exact sequence of the fibering $SO_{n-1}\to SO_n\to S^{n-1}$, and let $\pi_n: \pi_n(SO_{n-1})\to\pi_n(S^{n-2})(\cong Z_2)$ be the homomorphism induced by the projection of SO_{n-1} to $S^{n-2}=SO_{n-1}/SO_{n-2}$. We note that the suffix "n" of ∂_n and π_n implies that we consider at $\pi_n(SO_{n-1})$, though it is irregular use.

The groups $\pi_n(SO_{n-1})$, which were calculated by Kervaire [5], are given previously in the table. Using Kervaire [5] and Paechter [8], we can find the bases of the groups $\pi_n(SO_{n-1})$ such that the following relations hold under the identification of the groups, where 1 denotes the (standard) generators of the cyclic groups Z_2, Z_8, and Z.

Lemma 2.1.

(i) $\partial_{4t-1}=\partial_{4t}=0$ *for* $t\geqq 1$.

(ii) $\partial_{4t+1}\neq 0$ *for* $t\geqq 1$, *more precisely,*

$\partial_{8s+1}(1)=(1, 0, 0)\in Z_2+Z_2+Z_2$ *for* $s\geqq 1$,

and $\partial_{8s+5}(1)=(1, 0)\in Z_2+Z_2$ *for* $s\geqq 0$.

(iii) $\partial_{4t+2}(1)=4\in Z_8$ *for* $t\geqq 2$, *and* $\partial_6=0$.

Lemma 2.2.

(i) $\pi_{4t-1}=0$ *for* $t\geqq 3$, *and* $\pi_7(1)=1$

(ii) $\pi_{4t}\neq 0$ *for* $t\geqq 1$, *more precisely,*

$\pi_{8s}(1, 0)=1$, $\pi_{8s}(0, 1)=0$ *for* $s\geqq 1$,

and $\pi_{8s+4}(1)=1$ *for* $s\geqq 0$.

(iii) $\pi_{4t+1}\neq 0$ *for* $t\geqq 1$, *more precisely,*

$\pi_{8s+1}(1, 0, 0)=\pi_{8s+1}(0, 0, 1)=0$, $\pi_{8s+1}(0, 1, 0)=1$, *for* $s\geqq 1$,

and $\pi_{8s+5}(1, 0)=0$, $\pi_{8s+5}(0, 1)=1$ *for* $s\geqq 0$.

(iv) $\pi_{4t+2}=0$ *for* $t\geqq 1$.

Proof. These lemmas are obtained, except precise informations of π_{8s}, π_{4t+1}, and ∂_{4t+1}, by the results of Kervaire [5], using the homotopy exact sequence of the fibering $SO_{n-1}\to SO_n\to S^{n-1}$.

If $n=8s+4$ $(s\geqq 1)$, the generator of $\pi_n(SO_{n-1})$ is unique. If $n=4t-1$

$(t\geqq 1)$ or $4t+2$ $(t\geqq 1)$, we need not choose a special generator of $\pi_n(SO_{n-1})$ since the lemmas remain valid for any choice of the generator of $\pi_n(SO_{n-1})$.

Let $n=8s$ $(s\geqq 2)$. By Kervaire [5], there is the sequence

$$0\longrightarrow \pi_{8s+1}(V_{m,m-8s+i})\xrightarrow{\partial_*}\pi_{8s}(SO_{8s-i})\longrightarrow \pi_{8s}(SO_m)\longrightarrow 0$$

which is exact and splits for $i\leqq 4$, $s\geqq 2$, where m is to be large. Since $\pi_{8s+1}(V_{m,m-8s+4})=0$, $\pi_{8s}(SO_{8s-4})\cong\pi_{8s}(SO_m)\cong Z_2$. Let θ_1 be the generator of $\pi_{8s}(SO_{8s-4})\cong Z_2$ and ω_1 be the image of θ_1 by the suspension homomorphism $\pi_{8s}(SO_{8s-4})\to\pi_{8s}(SO_{8s-1})$. Let μ be the generator of $\pi_{8s+1}(V_{m,m-8s+1})\cong Z_2$ given by Paechter [8] and let $\xi=\partial_*(\mu)$. Then, $\{\xi, \omega_1\}$ forms a base of $\pi_{8s}(SO_{8s-1})\cong Z_2+Z_2$ for $s\geqq 2$. We adopt this. Thus, $\pi_{8s}(\omega_1)=0$, and $\pi_{8s}(\xi)\neq 0$ since $\pi_{8s}\neq 0$.

Let $n=8$, and let ν_5 be the generator of the 2-primary component of $\pi_8(S^5)$. We note that $q_*\colon \pi_8(SO_6)\to\pi_8(S^5)$, the homomorphism induced from the projection $q\colon SO_6\to S^5$, is an isomorphism. It is well known that $\pi_8(SO_7)\cong Z_2+Z_2$ is generated by the homotopy class $(\rho_7\circ\eta_7)$ and $i_*(q_*^{-1}\nu_5)$, where $\rho_7(c)c'=c\cdot c'\cdot\bar{c}$ for Cayley numbers $c\in S^7$, $c'\in S^6$, $\eta_7=E^5\eta_2$ ($\eta_2\colon S^3\to S^2$ is the Hopf map), and $i\colon SO_6\to SO_7$ is the inclusion map. We adopt $\{(\rho_7\circ\eta_7), i_*(q_*^{-1}\nu_5)\}$ as the base of $\pi_8(SO_7)$. Then, $\pi_8(i_*(q_*^{-1}\nu_5))=0$, and $\pi_8((\rho_7\circ\eta_7))\neq 0$ since $\pi_8\neq 0$.

Let $n=4t+1$ $(t\geqq 1)$. Let $\{\mu'_1, \mu'_2\}$, $\{\mu_1, \mu_2\}$, and μ'' be the generators of $\pi_{8s+6}(V_{m,m-8s-3})\cong Z_2+Z_2$ $(s\geqq 1)$, $\pi_{4t+2}(V_{m,m-4t})\cong Z_2+Z_2$, and $\pi_{4t+2}(V_{m,m-4t-1})\cong Z_2$ respectively which are given by Paechter [8], and denote, both by μ', the generators of $\pi_6(V_{m,m-3})\cong Z_2$ and $\pi_{8s+2}(V_{m,m-8s+1})\cong Z_2$ $(s\geqq 1)$ also given by Paechter [8], where m is sufficiently large and μ'_1, μ_1 correspond respectively to the generators $(i_{8s+4,4}\circ h_{8s+3,8s+6})$ $(s\geqq 1)$, $(i_{4t+1,3}\circ h_{4t,4t+2})$ of Paechter [8]. Then, examining those generators, we know that $p'_*(\mu'_1)=0$, $p'_*(\mu'_2)=\mu_1$, $p'_*(\mu')=\mu_1$, $p_*(\mu_1)=0$, and $p_*(\mu_2)=\mu''$, where $p'\colon V_{m,m-4t+1}\to V_{m,m-4t}$, $p\colon V_{m,m-4t}\to V_{m,m-4t-1}$ are the projections. In the homotopy exact sequence of the fibering $SO_{4t}\to SO_m\to V_{m,m-4t}$, let $\xi_1=\partial_*(\mu_1)$ and $\xi_2=\partial_*(\mu_2)$.

If $t=2s+1$ $(s\geqq 0)$, there are the following exact sequences

$$0=\pi_{8s+6}(SO_m)\longrightarrow\pi_{8s+6}(V_{m,m-8s-i})\xrightarrow[\cong]{\partial_*}\pi_{8s+5}(SO_{8s+i})\longrightarrow\pi_{8s+5}(SO_m)=0,$$

$i=3, 4, 5$, where m is sufficiently large. We adopt $\{\xi_1, \xi_2\}$ as the base of $\pi_{8s+5}(SO_{8s+4})\cong Z_2+Z_2$ $(s\geqq 0)$. Then, we know the precise correspondence of the homomorphisms

$$\pi_{8s+5}(SO_{8s+3})\xrightarrow{i'_*}\pi_{8s+5}(SO_{8s+4})\xrightarrow{i_*}\pi_{8s+5}(SO_{8s+5})$$

equivalent to p'_* and p_* respectively, where i', i are inclusion maps.

If $t=2s$ $(s\geqq 1)$, by Kervaire [5] we have the following sequence which is exact and splits for $s\geqq 2$, $i\leqq 3$ and $s=1$, $i\leqq 2$:

$$0\longrightarrow\pi_{8s+2}(V_{m,m-8s+i})\xrightarrow{\partial_*}\pi_{8s+1}(SO_{8s-i})\longrightarrow\pi_{8s+1}(SO_m)\longrightarrow 0,$$

where m is to be large. Since $\pi_{8s+2}(V_{m,m-8s+3})=0$ for $s\geqq 1$ and $\pi_{10}(V_{m,m-6})=0$, we know that $\pi_{8s+1}(SO_{8s-3})\cong\pi_{8s+1}(SO_m)\cong Z_2$ $(s\geqq 2)$ and $\pi_9(SO_6)\cong\pi_9(SO_m)\cong Z_2$. Denote those generators both by θ_2 and let $\omega_2\in\pi_{8s+1}(SO_{8s})$ $(s\geqq 1)$ be the image of θ_2 by the suspension homomorphism. Then, $\{\xi_1, \xi_2, \omega_2\}$ forms a base of $\pi_{8s+1}(SO_{8s})\cong Z_2+Z_2+Z_2$ for $s\geqq 1$, and we adopt this. So that, by p'_* and p_*, we know the precise correspondence of the homomorphisms

$$\pi_{8s+1}(SO_{8s-1})\xrightarrow{i'_*}\pi_{8s+1}(SO_{8s})\xrightarrow{i_*}\pi_{8s+1}(SO_{8s+1}),$$

where i', i are inclusion maps.

Thus, the precise correspondences of ∂_{4t+1} and π_{4t+1} is known by the following exact sequences

$$\begin{array}{ccccccc} & & & \pi_{4t+1}(SO_{4t-1}) & & & \\ & & & \downarrow i'_* & & & \\ 0\longrightarrow & \pi_{4t+2}(S^{4t}) & \xrightarrow{\partial_{4t+1}} & \pi_{4t+1}(SO_{4t}) & \xrightarrow{i_*} & \pi_{4t+1}(SO_{4t+1}) & \longrightarrow 0, \\ & \wr\Vert & & \downarrow \pi_{4t+1} & & & \\ & Z_2 & & \pi_{4t+1}(S^{4t-1})\cong Z_2 & & & \\ & & & \downarrow & & & \\ & & & 0 & & & \end{array}$$

and this completes the proof.

3. Classification of Handlebodies of Type O

Let W be a handlebody of $\mathscr{H}(2n, k, n+1)$, $n\geqq 6$. W is of type O if and only if the bilinear form λ of the corresponding $(H; \lambda, \alpha)$-system is trivial. So that, classifying the handlebodies of type O up to diffeomorphism comes to classifying the homomorphisms $\alpha: H \to \pi_n(SO_{n-1})$ up to equivalence, where H is a free abelian group of rank k and the homomorphisms $\alpha_i: H \to \pi_n(SO_{n-1})$, $i=1, 2$, are equivalent if and only if there exists an isomorphism $h: H \to H$ such that $\alpha_1 = \alpha_2 \circ h$.

Theorem 3.1. *The handlebody W of type O of $\mathscr{H}(2n, k, n+1)$ $(n \geqq 6)$ is uniquely represented up to diffeomorphism as follows:*

(i) *If $n=4t-1$ $(t \geqq 2)$,*

$$W = \bar{A}_a \natural (k-1)(S^{n+1} \times D^{n-1}),$$

where $a \in Z \cong \pi_{4t-1}(SO_{4t-2})$, $a \geqq 0$, especially $a \in 2Z$, $a \geqq 0$, if $t=2$.

(ii) *In the case when $n=4t$ $(t \geqq 2)$, if $n=8s+4$ $(s \geqq 1)$,*

$$W = k(S^{n+1} \times D^{n-1}),$$

and if $n=8s$ $(s \geqq 1)$,

$$W = \bar{A}_{(0,b)} \natural (k-1)(S^{n+1} \times D^{n-1}),$$

where $(0, b) \in Z_2 + Z_2 \cong \pi_{8s}(SO_{8s-1})$.

(iii) *In the case when $n=4t+1$ $(t \geqq 2)$, if $n=8s+5$ $(s \geqq 1)$,*

$$W = \bar{A}_{(a,0)} \natural (k-1)(S^{n+1} \times D^{n-1}),$$

where $(a, 0) \in Z_2 + Z_2 \cong \pi_{8s+5}(SO_{8s+4})$, and if $n=8s+1$ $(s \geqq 1)$,

$$W = \bar{A}_{(a,0,b)} \natural (k-1)(S^{n+1} \times D^{n-1}),$$

$$\text{or } W = \bar{A}_{(1,0,0)} \natural A_{(0,0,1)} \natural (k-2)(S^{n+1} \times D^{n-1}),$$

where $(a, 0, b)$, $(1, 0, 0)$, $(0, 0, 1) \in Z_2 + Z_2 + Z_2 \cong \pi_{8s+1}(SO_{8s})$.

(iv) *In the case when $n=4t+2$ $(t \geqq 1)$, if $t \geqq 2$,*

$$W = \bar{A}_a \natural (k-1)(S^{n+1} \times D^{n-1}),$$

where $a=0, 1, 2, 4 \in Z_8 \cong \pi_{4t+2}(SO_{4t+1})$, and if $t=1$,

$$W = k(S^7 \times D^5).$$

Proof. Since $s\pi\alpha(u_i)=\lambda(u_i, u_i)=0$ for each basis element u_i of H, $\alpha(H)$ is contained in $\operatorname{Ker}\pi$, where $S: \pi_n(S^{n-2})\to\pi_{n+1}(S^{n-1})$ $(n\geqq 6)$ is the suspension isomorphism. $\operatorname{Ker}\pi$ is known by Lemma 2.2, and we can simplify and characterize α by replacing the basis of H. Those are similar to that of Theorem 3.1 of [4]. Only a difference is the case when $n=4t+2$ $(t\geqq 2)$. In this case $\operatorname{Ker}\pi=Z_8$. If $\alpha(H)\subset\{0, 2, 4, 6\}\cong Z_4$, the case is similar to [4]. If $\alpha(H)\not\subset\{0, 2, 4, 6\}$, there exists a basis $\{u_1,\dots, u_k\}$ of H such that $\alpha(u_1)=1$ and $\alpha(u_i)=0$ for $i\geqq 2$. So that we have the result.

4. Classification of Handlebodies of Type I

In this section, we classify the handlebodies of type I of $\mathscr{H}(2n, k, n+1)$ $(n\geqq 6)$ up to diffeomorphism, that is, the $(H; \lambda, \alpha)$-systems of type I with $\operatorname{rank} H=k$ up to isomorphism. If $(H; \lambda, \alpha)$ is a system of type I, there is a basis of H which is orthogonal with respect to λ.

Theorem 4.1. *Let $n=4t-1$ $(t\geqq 2)$. If $t\geqq 3$, the handlebodies of type* I *of $\mathscr{H}(2n, k, n+1)$ do not exist. If $t=2$, i.e. $n=7$, the handlebody W of type* I *of $\mathscr{H}(2n, k, n+1)$ is uniquely represented up to diffeomorphism as $W=k\bar{B}_c$, where c is a positive odd integer of $\pi_7(SO_6)\cong Z$.*

Proof. The proof is quite similar to that of Theorem 4.1 of [4].

Theorem 4.2. *If $n=8s+4$ $(s\geqq 1)$, the handlebody W of type* I *of $\mathscr{H}(2n, k, n+1)$ is unique up to diffeomorphism and is represented as $W=k\bar{B}_1$, where $1\in Z_2\cong\pi_{8s+4}(SO_{8s+3})$.*

Proof. We know that $\alpha(v_i)=1$ for any orthogonal basis $\{v_i\}$ of H since $\lambda(v_i, v_i)=s\pi\alpha(v_i)=1$ and π_{8s+4} is an isomorphism by Lemma 2.2. So that, the $(H; \lambda, \alpha)$-system of type I is unique up to isomorphism.

Theorem 4.3. *If $n=8s$ $(s\geqq 1)$, the handlebodies of type* I *of $\mathscr{H}(2n, k, n+1)$ are uniquely represented up to diffeomorphism as follows:*

(i) $k\bar{B}_{(1,0)}$,

(ii) $k\bar{B}_{(1,1)}$,

(iii) $(k-1)\bar{B}_{(1,0)}\natural\bar{B}_{(1,1)}$ $(k\geqq 2)$,

(iv) $(k-2)\bar{B}_{(1,0)}\natural 2\bar{B}_{(1,1)}$ $(k\geqq 3)$,

where the characteristic elements belong to $\pi_{8s}(SO_{8s-1})\cong Z_2+Z_2$.

Proof. Let W be a handlebody of type I and $(H; \lambda, \alpha)$ the corresponding system. Since $\partial_{4t}=0$ by Lemma 2.1, $\alpha: H\to Z_2+Z_2\cong\pi_{8s}(SO_{8s-1})$ is a homomorphism. By Lemma 2.2, $\pi_{8s}^{-1}(1)$ consists of $(1, 0)$ and $(1, 1)$. So that, W is diffeomorphic to a boundary connected sum of some copies of $\bar{B}=\bar{B}_{(1,0)}$ and $\bar{B}'=\bar{B}_{(1,1)}$. Let $\alpha=(\alpha^{(1)}, \alpha^{(2)})$, $\alpha^{(i)}=p_i\circ\alpha$ $(i=1, 2)$, where p_i is the projection of Z_2+Z_2 to the i-th direct summand. Then, using the homomorphism $\alpha^{(1)}, \alpha^{(2)}$, the $(H; \lambda, \alpha)$-systems are classified up to isomorphism, similarly as in Theorem 4.5 of [4] (We note that Assertion 1, 2, and 3 of Theorem 4.5 of [4] are shown by $\alpha^{(2)}, \alpha^{(3)}$ of $\alpha=(\alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)})$.)

Theorem 4.4. *If* $n=8s+5$ $(s\geqq 1)$, *the handlebody* W *of type* I *of* $\mathscr{H}(2n, k, n+1)$ *is unique up to diffeomorphism and is represented as* $W=k\bar{B}_{(0,1)}$, *where* $(0, 1)\in Z_2+Z_2\cong\pi_{8s+5}(SO_{8s+4})$.

Proof. Since $\partial_{8s+5}(1)=(1, 0)$ and $\pi_{8s+5}^{-1}(1)=\{(0, 1), (1, 1)\}$, the situation is quite similar to that of Theorem 4.4 of [4].

Theorem 4.5. *If* $n=8s+1$ $(s\geqq 1)$, *the handlebodies of type* I *of* $\mathscr{H}(2n, k, n+1)$ *are uniquely represented up to diffeomorphism as follows*:

(i) $k\bar{B}_{(0,1,0)}$,

(ii) $k\bar{B}_{(0,1,1)}$,

(iii) $(k-1)\bar{B}_{(0,1,0)}\natural\bar{B}_{(0,1,1)}$ $(k\geqq 2)$,

(iv) $(k-2)\bar{B}_{(0,1,0)}\natural 2\bar{B}_{(0,1,1)}$ $(k\geqq 3)$,

where the characteristic elements belong to $\pi_{8s+1}(SO_{8s})\cong Z_2+Z_2+Z_2$.

Proof. By Lemma 2.1 and Lemma 2.2, $\partial_{8s+1}(1)=(1, 0, 0)$ and $\pi_{8s+1}^{-1}(1)=\{(\gamma, 1, \delta); \gamma, \delta=0 \text{ or } 1\}$. So that the situation is quite similar to that of Theorem 4.5 of [4].

Theorem 4.6. *If* $n=4t+2$ $(t\geqq 1)$, *there are no handlebodies of type* I *of* $\mathscr{H}(2n, k, n+1)$.

Proof. Since $\pi_{4t+2}=0$ by Lemma 2.2 and $\lambda(v_i, v_i)=s\pi\alpha(v_i)=1$ for any orthogonal basis $\{v_i\}$ of H, there arises a contradiction for any $(H; \lambda, \alpha)$-system of type I if $n=4t+2$, $t \geqq 1$.

5. Classification of Handlebodies of Type (O+I)

In this section, we classify the handlebodies of type (O+I) of $\mathscr{H}(2n, k, n+1)$ $(n \geqq 6)$ up to diffeomorphism, that is $(H; \lambda, \alpha)$-systems of type (O+I) with $\operatorname{rank} H=k$ up to isomorphism. For a handlebody W of type (O+I) of $\mathscr{H}(2n, k, n+1)$ $(n \geqq 6)$ and the corresponding system $(H; \lambda, \alpha)$, $q=\operatorname{rank} \lambda$ and $p=k-q$ are the diffeomorphism invariants of W, more precisely, the homotopy invariants of ∂W. We call $\operatorname{rank} \lambda$ briefly the *rank* of W.

Theorem 5.1. *Let* $n=4t-1$ $(t \geqq 2)$. *If* $t \geqq 3$, *the handlebodies of type* (O+I) *of* $\mathscr{H}(2n, k, n+1)$ *do not exist. If* $t=2$, *i.e.* $n=7$, *the handlebody* W *of type* (O+I) *of* $\mathscr{H}(14, k, 8)$ *with rank* q *is uniquely represented up to diffeomorphism as* $W=p(S^8 \times D^6) \natural q\bar{B}_c$, $p+q=k$, *where* c *is a positive odd integer of* $\pi_7(SO_6) \cong Z$.

Proof. Since $\pi_{4t-1}=0$ $(t \geqq 3)$ we have the former half of the theorem, and since $\partial_{4t-1}=0$ $(t \geqq 1)$ the latter half similarly to Theorem 5.2 of [4].

Theorem 5.2. *If* $n=8s+4$ $(s \geqq 1)$, *the handlebody* W *of type* (O+I) *of* $\mathscr{H}(2n, k, n+1)$ *with rank* q *is unique up to diffeomorphism and is represented as* $W=p(S^{n+1} \times D^{n-1}) \natural q\bar{B}_1$, $p+q=k$, *where* 1 *is the generator of* $\pi_n(SO_{n-1}) \cong Z_2$.

Proof. The handlebody of type O and the handlebody of type I are unique up to diffeomorphism by Theorem 3.1 and Theorem 4.2. Since W is the sum of such handlebodies, we have the result.

Theorem 5.3. *If* $n=8s$ $(s \geqq 1)$, *the handlebodies of type* (O+I) *of* $\mathscr{H}(2n, k, n+1)$ *with rank* q *are uniquely represented up to diffeomorphism as follows*:

(i) $p(S^{n+1} \times D^{n-1}) \natural q\bar{B}_{(1,0)}$,

(ii) $p(S^{n+1} \times D^{n-1}) \natural q\bar{B}_{(1,1)}$,

(iii) $p(S^{n+1}\times D^{n-1})\natural(q-1)\bar{B}_{(1,0)}\natural\bar{B}_{(1,1)}$ $(q\geqq 2)$,

(iv) $p(S^{n+1}\times D^{n-1})\natural(q-2)\bar{B}_{(1,0)}\natural 2\bar{B}_{(1,1)}$ $(q\geqq 3)$,

(v) $\bar{A}_{(0,1)}\natural(p-1)(S^{n+1}\times D^{n-1})\natural q\bar{B}_{(1,0)}$,

where $p+q=k$ *and the characteristic elements belong to* $\pi_{8s}(SO_{8s-1})\cong Z_2+Z_2$.

Proof. The proof is similar to that of Theorem 5.4 of [4].

Theorem 5.4. *If* $n=8s+5$ $(s\geqq 1)$, *the handlebody* W *of type* (O+I) *of* $\mathscr{H}(2n, k, n+1)$ *with rank* q *is unique up to diffeomorphism and is represented as* $W=p(S^{n+1}\times D^{n-1})\natural q\bar{B}_{(0,1)}$, $p+q=k$, *where* $(0, 1)\in Z_2+Z_2\cong\pi_{8s+5}(SO_{8s+4})$.

Proof. By Theorem 3.1 and Theorem 4.4, a handlebody of type (O+I) of $\mathscr{H}(2n, k, n+1)$ with rank q has a representation such as $p(S^{n+1}\times D^{n-1})\natural q\bar{B}_{(0,1)}$ or $\bar{A}_{(1,0)}\natural(p-1)(S^{n+1}\times D^{n-1})\natural q\bar{B}_{(0,1)}$. Let $\{u_1,\cdots, u_p; v_1\cdots, v_q\}$, $p+q=k$, be the admissible basis of H which corresponds to the latter representation. Then, $\alpha(u_1)=(1, 0)$, $\alpha(u_i)=(0, 0)$ if $i>1$, $\alpha(v_1)=\cdots=\alpha(v_q)=(0, 1)$. Replace u_1 by $u'_1=u_1+2v_1$. Then, by Lemma 2.1, $\alpha(u'_1)=(0, 0)$. So that, there exists an admissible basis $\{u'_1, u_2,\cdots, u_p; v_1,\cdots, v_q\}$ of H which corresponds to the former representation. This implies that those representations are equivalent.

Theorem 5.5. *If* $n=8s+1$ $(s\geqq 1)$, *the handlebodies of type* (O+I) *of* $\mathscr{H}(2n, k, n+1)$ *with rank* q *are uniquely represented up to diffeomorphism as follows*:

(i) $p(S^{n+1}\times D^{n-1})\natural q\bar{B}_{(0,1,0)}$,

(ii) $p(S^{n+1}\times D^{n-1})\natural q\bar{B}_{(0,1,1)}$,

(iii) $p(S^{n+1}\times D^{n-1})\natural(q-1)\bar{B}_{(0,1,0)}\natural\bar{B}_{(0,1,1)}$ $(q\geqq 2)$,

(iv) $p(S^{n+1}\times D^{n-1})\natural(q-2)\bar{B}_{(0,1,0)}\natural 2\bar{B}_{(0,1,1)}$ $(q\geqq 3)$,

(v) $\bar{A}_{(0,0,1)}\natural(p-1)(S^{n+1}\times D^{n-1})\natural q\bar{B}_{(0,1,0)}$,

where $p+q=k$ *and the characteristic elements belong to* $\pi_{8s+1}(SO_{8s})\cong Z_2+Z_2+Z_2$.

Proof. Since $\partial_{8s+1}(1)=(1, 0, 0)$, $\operatorname{Ker}\pi_{8s+1}$ is generated by $\{(1, 0, 0), (0, 0, 1)\}$, and $\pi_{8s+1}^{-1}(1)=\{(\gamma, 1, \delta); \gamma, \delta=0, \text{ or } 1\}$, the proof is quite similar to that of Theorem 5.4 of [4].

Since $\pi_{4t+2}=0$ $(t\geqq 1)$ we also have

Theorem 5.6. *If $n=4t+2$ $(t\geqq 1)$, there are no handlebodies of type* (O+I) *of $\mathscr{H}(2n, k, n+1)$.*

6. Classification of Handlebodies of Type II

In this section, we classify the handlebodies of type II of $\mathscr{H}(2n, k, n+1)$ $(n\geqq 6)$ up to diffeomorphism, that is, the $(H; \lambda, \alpha)$-systems of type II with $\operatorname{rank} H=k$ up to isomorphism. If $(H; \lambda, \alpha)$ is a system of type II with $\operatorname{rank} H=k$, then $k=2r$ and H has a basis symplectic with respect to λ.

Theorem 6.1. *If $n=4t-1$ $(t\geqq 2)$, the handlebody W of type* II *of $\mathscr{H}(2n, k, n+1)$ is represented uniquely up to diffeomorphism as*

$$W=W\begin{pmatrix} d \\ 0 \end{pmatrix} \natural (r-1) W\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad d\geqq 0,$$

where $k=2r$ and $d\in Z\cong\pi_{4t-1}(SO_{4t-2})$, especially $d\in 2Z$ if $t=2$.

Proof. Since $\partial_{4t-1}=0$ $(t\geqq 1)$, $\alpha: H\to\pi_{4t-1}(SO_{4t-2})\cong Z (t\geqq 2)$ is a homomorphism, and since $s\pi\alpha(e_i)=s\pi\alpha(f_i)=0$ and $\operatorname{Ker}\pi_7=2Z$, we have the theorem by Lemma 6.1 of [4].

Theorem 6.2. *If $n=8s+4$ $(s\geqq 1)$, the handlebody W of type* II *of $\mathscr{H}(2n, k, n+1)$ is unique up to diffeomorphism and is represented as $W=rW\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $k=2r$.*

Proof. Since $\pi_{8s+4}\colon \pi_{8s+4}(SO_{8s+3})=Z_2\to\pi_{8s+4}(S^{8s+2})=Z_2$, $s\geqq 1$, is an isomorphism by Lemma 2.2, $\alpha(e_i)=\alpha(f_j)=0$ for all $i, j=1, 2,\cdots, r$. So that we have the theorem.

Theorem 6.3. *If $n=8s$ $(s\geqq 1)$, the handlebody W of type* II *of $\mathscr{H}(2n, k, n+1)$ is represented uniquely up to diffeomorphism as*

$$W=W\begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix} \natural (r-1) W\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

where $k=2r$ *and* $(0, d)\in Z_2+Z_2\cong\pi_{8s}(SO_{8s-1})$.

Proof. Since $\partial_{4t}=0$ $(t\geqq 1)$, α is a homomorphism. The image of α is in $\mathrm{Ker}\,\pi_{8s}=o+Z_2\subset Z_2+Z_2=\pi_{8s}(SO_{8s-1})$. So that we have the theorem by Lemma 6.1 of [4].

Theorem 6.4. *If* $n=8s+5$ $(s\geqq 1)$, *the handlebody* W *of type* II *of* $\mathcal{H}(2n, k, n+1)$ *is uniquely represented up to diffeomorphism as*

$$W=W\begin{pmatrix} d\,0 \\ d\,0 \end{pmatrix}\natural(r-1)W\begin{pmatrix} 0\,0 \\ 0\,0 \end{pmatrix},$$

where $k=2r$ *and* $(d, 0)\in Z_2+Z_2\cong\pi_{8s+5}(SO_{8s+4})$.

Proof. Since $\mathrm{Ker}\,\pi_{8s+5}=Z_2+0\subset Z_2+Z_2=\pi_{8s+5}(SO_{8s+4})$ and $\partial_{8s+5}(1)=(1, 0)\in\mathrm{Ker}\,\pi_{8s+5}$ by Lemma 2.1 and Lemma 2.2, we know that $\alpha(H)\subset\mathrm{Ker}\,\pi_{8s+5}$. So that α is regarded as a quadratic form over Z_2, and is classified by the Arf invariant.[2]

Theorem 6.5. *If* $n=8s+1$ $(s\geqq 1)$, *the handlebodies of type* II *of* $\mathcal{H}(2n, k, n+1)$ *are uniquely represented up to diffeomorphism as follows*:

(i) $W\begin{pmatrix} d\,0\,0 \\ d\,0\,0 \end{pmatrix}\natural(r-1)W\begin{pmatrix} 0\,0\,0 \\ 0\,0\,0 \end{pmatrix}$,

(ii) $W\begin{pmatrix} 0\,0\,1 \\ 0\,0\,d \end{pmatrix}\natural(r-1)W\begin{pmatrix} 0\,0\,0 \\ 0\,0\,0 \end{pmatrix}$,

(iii) $W\begin{pmatrix} 0\,0\,1 \\ 0\,0\,d \end{pmatrix}\natural W\begin{pmatrix} 1\,0\,0 \\ 1\,0\,0 \end{pmatrix}\natural(r-2)W\begin{pmatrix} 0\,0\,0 \\ 0\,0\,0 \end{pmatrix}$,

where $k=2r$, *and* $(0, 0, 1), (1, 0, 0), (0, 0, d)$, *and* $(d, 0, 0)$ *belong to* $Z_2+Z_2+Z_2\cong\pi_{8s+1}(SO_{8s})$.

Proof. Since $\mathrm{Ker}\,\pi_{8s+1}=Z_2+o+Z_2\subset Z_2+Z_2+Z_2=\pi_{8s+1}(SO_{8s})$ and $\partial_{8s+1}=(1, 0, 0)\in\mathrm{Ker}\,\pi_{8s+1}$, we know that $\alpha(H)\subset\mathrm{Ker}\,\pi_{8s+1}$. So that, we have the theorem similarly to Theorem 6.5 of [4].

Theorem 6.6. *If* $n=4t+2$ $(t\geqq 1)$, *the handlebodies of type* II *of* $\mathcal{H}(2n, k, n+1)$ *are represented uniquely up to diffeomorphism as follows*: *If* $t\geqq 2$,

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2) See, for example, Browder [1] p. 55.

(i) $W\binom{d}{d}\natural(r-1)W\binom{0}{0}$, $\quad d=0, 4$,

(ii) $W\binom{2}{0}\natural(r-1)W\binom{0}{0}$,

(iii) $W\binom{1}{0}\natural(r-1)W\binom{0}{0}$,

and if $t=1$, *i.e.* $n=6$,

(iv) $rW\binom{0}{0}$,

where $k=2r$ *and the characteristic elements* 0, 1, 2, *and* 4 *belong to* $Z_8 \cong \pi_{4t+2}(SO_{4t+1})$.

Proof. Let $(H; \lambda, \alpha)$ be a system of type II with rank $H=k=2r$, and let $\{e_1, f_1, \dots, e_r, f_r\}$ be a symplectic base of H. If $\{\alpha(e_1), \alpha(f_1), \cdots, \alpha(e_r), \alpha(f_r)\} \subset \{0, 2, 4, 6\} \subset Z_8$, then $\alpha(H) \subset \{0, 2, 4, 6\} \cong Z_4$ since $\partial_{4t+2}=4$ $(t \geqq 2)$ by Lemma 2.1. So the situation is quite similar to that of Theorem 6.7 of [4], and we have the results (i), (ii).

If $\alpha(H) \not\subset \{0, 2, 4, 6\}$, we may assume that $\{\alpha(e_i), \alpha(f_i)\} \not\subset \{0, 2, 4, 6\}$, $i=1, 2, \cdots, s$, $s \geqq 1$, and $\{\alpha(e_j), \alpha(f_j)\} \subset \{0, 2, 4, 6\}$, $j=s+1, s+2, \cdots, r$. Performing some elementary transformations to $\{e_i, f_i\}$, we may assume that $(\alpha(e_i), \alpha(f_i))=(1, 0)$, $i=1, 2, \cdots, s$, and $(\alpha(e_j), \alpha(f_j))=(0, 0)$, or $(2, 0)$, or $(4, 4)$, $j=s+1, s+2, \cdots, r$. But, each pair $(\alpha(e_j), \alpha(f_j))=(2, 0)$ or $(4, 4)$ can be killed using a certain pair $(\alpha(e_i), \alpha(f_i))=(1, 0)$ by adopting the new basis elements $e'_j=e_j-2e_i, f'_j=f_j$, or $e'_j=e_j-4e_i, f'_j=f_j-4f_i$. So that, there exists a symplectic base $\{e_1, f_1, \cdots, e_r, f_r\}$ of H such that each pair $(\alpha(e_i), \alpha(f_i))=(0, 0)$, or $(1, 0)$, $i=1, 2, \cdots, r$. If $(\alpha(e_i), \alpha(f_i))=(\alpha(e_j), \alpha(f_j))=(1, 0)$, $i \neq j$, let $e'_i=e_j+2(f_i-f_j)$, $f'_i=f_i-f_j$, $e'_j=e_i-e_j+2(f_i-f_j)$, and $f'_j=f_i$. Then, we have $(\alpha(e_i), \alpha(f_i))=(1, 0)$, $(\alpha(e_j), \alpha(f_j))=(0, 0)$.

Thus, if $\alpha(H) \not\subset \{0, 2, 4, 6\}$, there exists a symplectic base $\{e'_1, f'_1, \cdots, e'_r, f'_r\}$ of H such that $(\alpha(e'_1), \alpha(f'_1))=(1, 0)$, and $(\alpha(e'_2), \alpha(f'_2))=\cdots=(\alpha(e'_r), \alpha(f'_r))=(0, 0)$. This implies that the corresponding handlebody is diffeomorphic to (iii).

If $t=2$, we have (iv) since $\pi_6(SO_5) \cong 0$. This completes the proof.

7. Classification of Handlebodies of Type (O+II)

In this section, we classify the handlebodies of type (O+II) of

$\mathscr{H}(2n, k, n+1)$ $(n\geqq 6)$ up to diffeomorphism, that is, the $(H; \lambda, \alpha)$-systems of type (O+II) with $\operatorname{rank} H=k$ up to isomorphism. For a handlebody W of type (O+II) of $\mathscr{H}(2n, k, n+1)$ $(n\geqq 6)$ and the corresponding system $(H; \lambda, \alpha)$, $2r=\operatorname{rank}\lambda$ and $p=k-2r$ are the diffeomorphism invariants of W, more precisely, the homotopy invariants of ∂W. We call $\operatorname{rank}\lambda$ briefly the *rank* of W.

Theorem 7.1. *If* $n=4t-1$ $(t\geqq 2)$, *the handlebodies of type* (O+II) *of* $\mathscr{H}(2n, k, n+1)$ *with rank* $2r$ *are uniquely represented up to diffeomorphism as follows*:

(i) $\bar{A}_a \natural (p-1)(S^{n+1}\times D^{n-1})\natural rW\begin{pmatrix}0\\0\end{pmatrix}$, $a\geqq 0$, $p+2r=k$,

where $a\in Z\cong\pi_{4t-1}(SO_{4t-2})$.

(ii) $p(S^{n+1}\times D^{n-1})\natural W\begin{pmatrix}d\\0\end{pmatrix}\natural(r-1)W\begin{pmatrix}0\\0\end{pmatrix}$, $d>0$, $p+2r=k$,

where $d\in Z\cong\pi_{4t-1}(SO_{4t-2})$.

In (i) *and* (ii), *if* $t=2$ *then* a *and* d *are even.*

Proof. Since $\partial_{4t-1}=0$, $\pi_{4t-1}=0$ $(t\geqq 3)$, and $\pi_7(1)=1$, the proof is quite similar to that of Theorem 7.2 of [4].

Theorem 7.2. *If* $n=8s+4$ $(s\geqq 1)$, *the handlebody* W *of type* (O+II) *of* $\mathscr{H}(2n, k, n+1)$ *with rank* $2r$ *is unique up to diffeomorphism and is represented as*

$$W=p(S^{n+1}\times D^{n-1})\natural rW\begin{pmatrix}0\\0\end{pmatrix}, \qquad p+2r=k.$$

Proof. If $n=8s+4$ $(s\geqq 1)$, the handlebodies of type O and type II are respectively unique up to diffeomorphism by Theorem 3.1 and Theorem 6.2. So that, we have the result.

Theorem 7.3. *If* $n=8s$ $(s\geqq 1)$, *the handlebodies of type* (O+II) *of* $\mathscr{H}(2n, k, n+1)$ *with rank* $2r$ *are uniquely represented up to diffeomorphism as follows*:

(i) $\bar{A}_{(0,b)}\natural(p-1)(S^{n+1}\times D^{n-1})\natural rW\begin{pmatrix}0&0\\0&0\end{pmatrix}$,

(ii) $p(S^{n+1}\times D^{n-1})\natural W\begin{pmatrix}0&1\\0&0\end{pmatrix}\natural(r-1)W\begin{pmatrix}0&0\\0&0\end{pmatrix}$,

where $p+2r=k$ *and* $(0, b), (0, 1)\in Z_2+Z_2\cong\pi_{8s}(SO_{8s-1})$.

Proof. By Theorem 3.1 and Theorem 6.3, the handlebody of type (O+II) with *rank* $2r$ has a representation such as

$$\bar{A}_{(0,b)}\natural(p-1)(S^{n+1}\times D^{n-1})\natural W\begin{pmatrix}0 & d\\ 0 & 0\end{pmatrix}\natural(r-1)W\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix},$$

where $(0, b), (0, d)\in Z_2+Z_2\cong\pi_{8s}(SO_{8s-1})$ and $p+2r=k$. Since $\partial_{8s}=0$, α is a homomorphism, and $\alpha(H)\subset\operatorname{Ker}\pi_{8s}=\{(0, 0), (0, 1)\}\cong Z_2$ for any $(H; \lambda, \alpha)$-system of type (O+II). Let $(H; \lambda, \alpha)$ be a system of type (O+II) with *rank* $2r$ and let $\{u_1,\cdots, u_p; e_1, f_1,\cdots, e_r, f_r\}$ be an admissible base of H $(p+2r=k)$. If $\alpha(u_1)=\alpha(e_1)=(0, 1)$, let $e'_1=e_1+u_1$. Then $\alpha(e'_1)=(0, 0)$. So that, any case can be reduced to one of the following three:

(1) α is the zero homomorphism.

(2) $\alpha(u_1)=(0, 1)$ and α takes $(0, 0)$ for any other basis elements.

(3) $\alpha(e_1)=(0, 1)$ and α takes $(0, 0)$ for any other basis elements.

These three are independent. Because, if the case (2) is equivalent to (3), that is, if there are the two admissible bases $\{u_1,\cdots, u_p; e_1, f_1,\cdots, e_r, f_r\}$, $\{u'_1,\cdots, u'_p; e'_1, f'_1,\cdots, e'_r, f'_r\}$ of H satisfying (2), (3) respectively, then, by Lemma 7.1 of [4], there exists an unimodular matrix T such that $(u'_1,\cdots, u'_p; e'_1, f'_1,\cdots, e'_r, f'_r)^t=T(u_1,\cdots, u_p; e_1, f_1,\cdots, e_r, f_r)^t$,

$$T=\begin{pmatrix}\overbrace{M}^{p} & \overbrace{O}^{2r}\\ * & L\end{pmatrix}\begin{matrix}\}\,p\\ \}\,2r\end{matrix}\quad(\operatorname{mod}2),$$

and L is mod 2 symplectic. But, since $\alpha^{(2)}(u'_i)=t_{i1}=0$ (mod 2), $|M|=0$ (mod 2). This contradicts to $|T|=1$. So that, the $(H; \lambda, \alpha)$-systems corresponding to the above cases are independent up to isomorphism. This completes the proof.

Let $n=4t+1$ $(t\geqq 2)$. If $t=2s+1$ $(s\geqq 1)$, then $\partial_{8s+5}(1)=(1, 0)\in Z_2+Z_2\cong\pi_{8s+5}(SO_{8s+4})$, $\operatorname{Ker}\pi_{8s+5}\cong Z_2+0$, and so $\alpha(H)\subset\operatorname{Ker}\pi_{8s+5}$ for any $(H; \lambda, \alpha)$-system of type (O+II). So that, the situation is quite similar to that of Theorem 7.3 of [4]. If $t=2s$ $(s\geqq 1)$, then $\partial_{8s+1}(1)=(1, 0, 0)\in Z_2+Z_2+Z_2\cong\pi_{8s+1}(SO_{8s})$, $\operatorname{Ker}\pi_{8s+1}=Z_2+o+Z_2$, and so $\alpha(H)\subset\operatorname{Ker}\pi_{8s+1}$

for any $(H;\lambda,\alpha)$-system of type (O+II). So that, the situation is quite similar to that of Theorem 7.5 of [4]. Thus, we have the following theorems.

Theorem 7.4. *If $n=8s+5$ $(s\geqq 1)$, the handlebodies of type* (O+II) *of $\mathscr{H}(2n, k, n+1)$ with rank $2r$ are uniquely represented up to diffeomorphism as follows:*

(i) $\bar{A}_{(a,0)}\natural(p-1)(S^{n+1}\times D^{n-1})\natural rW\begin{pmatrix}0\,0\\0\,0\end{pmatrix}$,

(ii) $p(S^{n+1}\times D^{n-1})\natural W\begin{pmatrix}1\,0\\1\,0\end{pmatrix}\natural(r-1)W\begin{pmatrix}0\,0\\0\,0\end{pmatrix}$,

where $p+2r=k$ and $(a, 0)$, $(1, 0)$ belong to $Z_2+Z_2\cong\pi_{8s+5}(SO_{8s+4})$.

Theorem 7.5. *If $n=8s+1$ $(s\geqq 1)$, the handlebodies of type* (O+II) *of $\mathscr{H}(2n, k, n+1)$ with rank $2r$ are uniquely represented up to diffeomorphism as follows:*

(i) $p(S^{n+1}\times D^{n-1})\natural W_1$, *where W_1 is a handlebody of type* II *of $\mathscr{H}(2n, 2r, n+1)$,*

(ii) $\bar{A}_{(1,0,0)}\natural(p-1)(S^{n+1}\times D^{n-1})\natural W\begin{pmatrix}0\,0\,d\\0\,0\,0\end{pmatrix}\natural(r-1)W\begin{pmatrix}0\,0\,0\\0\,0\,0\end{pmatrix}$,

(iii) $\bar{A}_{(0,0,1)}\natural(p-1)(S^{n+1}\times D^{n-1})\natural W\begin{pmatrix}d\,0\,0\\d\,0\,0\end{pmatrix}\natural(r-1)W\begin{pmatrix}0\,0\,0\\0\,0\,0\end{pmatrix}$,

(iv) $\bar{A}_{(1,0,1)}\natural(p-1)(S^{n+1}\times D^{n-1})\natural W\begin{pmatrix}d\,0\,0\\d\,0\,0\end{pmatrix}\natural(r-1)W\begin{pmatrix}0\,0\,0\\0\,0\,0\end{pmatrix}$,

(v) $\bar{A}_{(1,0,0)}\natural\bar{A}_{(0,0,1)}\natural(p-2)(S^{n+1}\times D^{n-1})\natural rW\begin{pmatrix}0\,0\,0\\0\,0\,0\end{pmatrix}$,

where $p+2r=k$ and the characteristic elements belong to $Z_2+Z_2+Z_2\cong\pi_{8s+1}(SO_{8s})$.

If $n=4t+2$ $(t\geqq 1)$, the situation is slightly different form that of [4] since $\pi_{4t+2}(SO_{4t+1})\cong Z_8$.

Theorem 7.6. *If $n=4t+2$ $(t\geqq 1)$, the handlebodies of type* (O+II) *of $\mathscr{H}(2n, k, n+1)$ with rank $2r$ are uniquely represented up to diffeomorphism as follows:*

(i) $p(S^{n+1}\times D^{n-1})\natural W\begin{pmatrix}d\\d\end{pmatrix}\natural(r-1)W\begin{pmatrix}0\\0\end{pmatrix}$, $\quad d=0, 4$,

(ii) $\bar{A}_4\natural(p-1)(S^{n+1}\times D^{n-1})\natural rW\begin{pmatrix}0\\0\end{pmatrix}$,

(iii) $p(S^{n+1}\times D^{n-1})\natural W\binom{2}{0}\natural(r-1)W\binom{0}{0}$,

(iv) $\bar{A}_2\natural(p-1)(S^{n+1}\times D^{n-1})\natural rW\binom{0}{0}$,

(v) $p(S^{n+1}\times D^{n-1})\natural W\binom{1}{0}\natural(r-1)W\binom{0}{0}$,

(vi) $\bar{A}_1\natural(p-1)(S^{n+1}\times D^{n-1})\natural rW\binom{0}{0}$,

where during (i)–(vi), $t\geqq 2$, $p+2r=k$, *and the characteristic elements belong to* $Z_8\cong\pi_{4t+2}(SO_{4t+1})$, *and*

(vii) $p(S^7\times D^5)\natural rW\binom{0}{0}$, $p+2r=k$, *if* $t=1$ *i.e.* $n=6$.

Proof. Let $t\geqq 2$. Then $\partial_{4t+2}(1)=4\in Z_8\cong\pi_{4t+2}(SO_{4t+1})$. Let $(H;\lambda,\alpha)$ be a system of type (O+II) with *rank* $2r$. If $\alpha(H)\subset\{0, 2, 4, 6\}$ $(\cong Z_4)\subset Z_8$, then the situation is quite similar to that of Theorem 7.7 of [4]. So that, we have the results (i)–(iv). Let $\alpha(H)\not\subset\{0, 2, 4, 6\}$ and let $\{u_1,\cdots, u_p; e_1, f_1,\cdots, e_r, f_r\}$ be an admissible base of H. Then,

(a) $\alpha(\{u_1,\cdots, u_p\})\not\subset\{0, 2, 4, 6\}$,

or (b) $\alpha(\{e_1, f_1,\cdots, e_r, f_r\})\not\subset\{0, 2, 4, 6\}$.

If (a), we may assume that $\alpha(u_1)=1$, $\alpha(u_i)=0$ for $i\geqq 2$. If (b), we may assume that $(\alpha(e_1), \alpha(f_1))=(1, 0)$ and $(\alpha(e_j), \alpha(f_j))=(0, 0)$ for all $j\geqq 2$, as in the proof of Theorem 6.6.

If (a) and (b), replacing e_1 by $e'_1=e_1-u_1$, there is an admissible base $\{u'_1,\cdots, u'_p; e'_1, f'_1,\cdots, e'_r, f'_r\}$ of H such that $\alpha(u'_1)=1$, $\alpha(u'_i)=0$ for $i\geqq 2$, and $(\alpha(e'_j), \alpha(f'_j))=(0, 0)$ for all j. If (a) and not (b) i.e. $\alpha(\{e_1, f_1,\cdots, e_r, f_r\})\subset\{0, 2, 4, 6\}$, we may assume that $\alpha(u_1)=1$, $\alpha(u_i)=0$ for $i\geqq 2$ and $(\alpha(e_j), \alpha(f_j))=(0, 0)$, or $(2, 0)$, or $(4, 4)$ for all j by some elementary transformations of symplectic bases. Then, by replacing e_j or f_j by $e'_j=e_j+lu_1$ or $f'_j=f_j+mu_1$ (l, m: integers), there is also an admissible base $\{u'_1,\cdots, u'_p; e'_1, f'_1,\cdots, e'_r, f'_r\}$ of H such that $\alpha(u'_1)=1$, $\alpha(u'_i)=0$ for $i\geqq 2$, and $(\alpha(e'_j), \alpha(f'_j))=(0, 0)$ for all j. If not (a) but (b), we may assume that $\alpha(u_1)=0$, or 2, or 4, $\alpha(u_i)=0$ for $i\geqq 2$, and $(\alpha(e_1), \alpha(f_1))=(1, 0)$, $(\alpha(e_j), \alpha(f_j))=(0, 0)$ for $j\geqq 2$. Then, by replacing u_1 by $u'_1=u_1+2le_1$ (l: integer), there is an admissible base $\{u'_1,\cdots, u'_p; e'_1, f'_1,\cdots, e'_r, f'_r\}$ of H such that $\alpha(u'_i)=0$ for all i and $(\alpha(e_1), \alpha(f_1))=(1, 0)$, $(\alpha(e_j), \alpha(f_j))=(0, 0)$ for $j\geqq 2$.

Thus, for any $(H; \lambda, \alpha)$-system of type (O+II) with *rank* $2r$, there is an admissible base $\{u_1, \cdots, u_p; e_1, f_1, \cdots, e_r, f_r\}$ of H such that

(1) $\alpha(u_1)=1$, $\alpha(u_i)=0$ for $i \geqq 2$, and $(\alpha(e_j), \alpha(f_j))=(0, 0)$ for all j,

or (2) $\alpha(u_i)=0$ for all i, and $(\alpha(e_1), \alpha(f_1))=(1, 0)$, $(\alpha(e_j), \alpha(f_j))=(0, 0)$ for $j \geqq 2$.

Now, we can show that the cases (1) and (2) are independent of each other, using Lemma 7.1 of [4] as in the proof of Theorem 7.3. So that, the two $(H; \lambda, \alpha)$-systems corresponding respectively to the cases (1) and (2) are not isomorphic. Thus, we have the results (v), (vi).

If $t=1$, since $\pi_6(SO_5)=0$, we have the result (vii). This completes the proof.

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