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On the Diameter of Certain Riemannian Manifolds

By

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§1. Introduction

The purpose of this paper is to estimate the lower bound for the diameters of certain Riemannian manifolds.

Let M be a compact connected C^∞ Riemannian manifold and K its sectional curvature. Let p be a point of M and G_p the group of all isometries of M which fix the point p. We denote by $d(M)$ the diameter of M.

We shall prove:

Theorem A. *Assume* $K \leqq 1$ *and there exists a point* p *in* M *such that* $\dim G_p \geqq 1$. *Then* $d(M) \geqq \pi/2$. *If the equality* $d(M) = \pi/2$ *holds, then* $K \equiv 1$. *Therefore, in case of even dimension, the equality holds if and only if* M *is isometric to the real projective space of constant curvature* 1.

From this theorem we obtain an alternative proof of the following

Corollary (cf. [2]). *Assume* $K \leqq 0$. *Then* $\dim G_p = 0$ *for any point* p *of* M.

We shall also prove:

Theorem B. *Assume* $0 < K \leqq 1$ *and* M *is of even dimension, then* $d(M) \geqq \pi/2$. *The equality* $d(M) = \pi/2$ *holds if and only if* M *is isometric to the real projective space of constant curvature* 1.

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Remark 1.1. The assumption $K \leqq 1$ is not essential. Since M is compact, in order to fulfill the assumption, we have only to change the metric by a constant factor.

Remark 1.2. In Theorem B, the assumption that M is of even dimension cannot be eliminated. In fact, even if M is of constant curvature 1, $d(M)$ can be less than $\pi/2$ (cf. [8]).

Next we shall study the relation between the cut locus and the conjugate locus. And we shall prove:

Theorem A'. *Assume* $K \leqq 1$ *and there exists a point* p *in* M *such that* $\dim G_p \geqq 1$. *Further assume that* M *is simply connected and that* $\dim M$ *is* 3 *or* 4. *Then* $d(M) \geqq \pi$.

Corresponding to Theorem B, the following theorem is known.

Theorem B'. *Assume* $0 < K \leqq 1$ *and* M *is simply connected and of even dimension. Then* $d(M) \geqq \pi$.

Remark 1.3. If M is of dimension 2, we need less assumption. In fact, we have: *If* M *is of dimension* 2 *and simply connected and if* $K \leqq 1$, *then* $d(M) \geqq \pi$ (cf. Theorem 6.1).

Note: In this paper, we assume that the manifold M is always compact and $\dim M \geqq 2$.

§2. Examples

The complex n-space $\mathbf{C}^n$ $(n \geqq 2)$ is considered to be the Euclidean $2n$-space by the usual hermitian inner product. Let S^{2n-1} be the unit sphere in $\mathbf{C}^n$. Then S^{2n-1} is of constant curvature 1, concerning to the induced metric by the inclusion $S^{2n-1} \subset \mathbf{C}^n$. And the unitary transformations $U(n)$ act on S^{2n-1} as isometries.

(1) Lens spaces. Let m be an integer greater than 1 and let $r_1, \ldots, r_n$ be primitive m-th roots of 1 (not necessarily distinct). We define a unitary matrix U by

$$\begin{pmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_n \end{pmatrix}.$$

Let Γ be the subgroup of $U(n)$ generated by U. Then Γ is a finite group which acts freely on S^{2n-1}. Hence $M=S^{2n-1}/\Gamma$ is a manifold which inherits the Riemannian structure of constant curvature 1. M is called a lens space. Let G be the subgroup of $U(n)$ consisting of matrices

$$\begin{pmatrix} 1 & & & 0 \\ & \lambda & & \\ & & \ddots & \\ 0 & & & \lambda \end{pmatrix}$$

such that $\lambda \in \boldsymbol{C}$ and $|\lambda|=1$ and let p be a point of M which is represented by the vector $(1, 0, \dots, 0) \in \boldsymbol{C}^n$. Since G is contained in the normalizer of Γ, each element of G is considered to be an isometry of M and fixes the point p. Therefore we obtain $\dim G_p \geqq \dim G = 1$. As for the lens spaces, $d(M)=\pi/2$ (cf. [8]).

Remark 2.1. If M is of constant curvature 1 with $d(M)=\pi/2$ and if $\dim M$ is 3 or 5, then the fundamental group is cyclic (cf. [8]). Hence M is a lens space.

(2) Let Γ be a subgroup of $U(4)$ generated by the following matrices

$$A=\begin{pmatrix} e^{2/3\, i} & & & 0 \\ & e^{-2/3\, i} & & \\ & & e^{2/3\, i} & \\ 0 & & & e^{-2/3\, i} \end{pmatrix}$$

and

$$B=\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Then Γ is a non-commutative finite group which acts freely on S^7. Hence $M=S^7/\Gamma$ is also a Riemannian manifold of constant curvature 1. we put

$$G=\left\{\begin{pmatrix}1 & & & 0\\ & 1 & & \\ & & \lambda & \\ 0 & & & \lambda\end{pmatrix};\ \lambda\in\boldsymbol{C}\ \text{ and }\ |\lambda|=1\right\}.$$

Then each element of G acts on M as an isometry and fixes the point p which is represented by the vector $(1, 0, 0, 0)\in\boldsymbol{C}^4$. Hence $\dim G_p \geqq \dim G=1$. In this case, we also have $d(M)=\pi/2$ (cf. [8]). However M is not a lens space since Γ is not cyclic.

§3. Preliminaries

We denote by $<\ ,\ >$ the Riemannian metric of M. We denote by $\dot{c}(t)$ the velocity vector of a curve $c: [a, b]\to M$ at $c(t)$ and by $L(c)$ the length of c. We denote by $d(p, q)$ the distance between two points p and q of M. We denote by T_pM the tangent space to M at p and by $\exp_p$ the exponential mapping of T_pM to M. We put $B_p(r)=\{v\in T_pM;\ \|v\|<r\}$ and $S_p(r)=\{v\in T_pM;\ \|v\|=r\}$. Then, by the theorem of Morse-Schoenberg, we know that if $K\leqq 1$, then the map $\exp_p|_{B_p(\pi)}$ is of maximal rank at any point p of M. The following proposition is known (cf. [1] p. 149, Theorem 2).

Proposition 3.1. *Assume that the map $\exp_p|_{B_p(r)}$ is of maximal rank. Let $\gamma: [a, b]\to B_p(r)$ be a piecewise C^∞ curve such that $\gamma(a)=0$. Then we have $L(\exp_p\circ\gamma)\geqq\|\gamma(b)\|$. The equality $L(\exp_p\circ\gamma)=\|\gamma(b)\|$ holds if and only if $\exp_p\circ\gamma$ is a geodesic with an appropriate change of parametrization.*

Making use of this proposition, we obtain

Lemma 3.2. *Assume that the map $\exp_p|_{B_p(r)}$ is of maximal rank. Let $c: [a, b]\to M$ be a piecewise C^∞ curve emanating from p. Assume that the following* (i) *or* (ii) *holds.*

(i) $L(c)<r$.

(ii) *$L(c)\leqq r$ and c is not a geodesic with any reparametrization. Then there exists a unique piecewise C^∞ curve $\gamma: [a, b]\to B_p(r)$ such that $\gamma(a)=0$ and $\exp_p\circ\gamma=c$.*

γ is called the lift of c to T_pM.

Proof. Since the map $\exp_p|_{B_p(r)}$ is of maximal rank, γ is unique if it exists. By our assumption, there is a number $s\,(a<s\leqq b)$ such that there is the lift γ_s of $c|_{[a,s]}$ to T_pM. Let s_0 be the least upper bound of such s. We note that, for any s and $s'\,(s<s')$, $\gamma_s=\gamma_{s'}$ on $[a, s]$. Therefore the curve $\gamma\colon [a, s_0)\to B_p(r)$ can be defined by $\gamma(t)=\gamma_s(t)$ for $a\leqq t\leqq s<s_0$. By Proposition 3.1 and our assumption, we easily see that $\gamma(s)\,(s\to s_0)$ converges to a point of $B_p(r)$. Hence it follows that there is the lift γ_{s_0} of $c|_{[a,s_0]}$ to T_pM. So, if $s_0<b$, the lift γ_{s_0} can be extended, which is contradictory to the definition of s_0. Therefore $s_0=b$ and γ_{s_0} is the lift of c to T_pM. Q.E.D.

Lemma 3.3. *Assume $K\leqq 1$. Let v and w be tangent vectors contained in $B_p(\pi)$ such that $<v, w>=0$. Let $\gamma\colon [a, b]\to B_p(\pi)$ be a piecewise C^∞ curve connecting v and w. Then $L(\exp_p\circ\gamma)\geqq \|v\|$ if $\|v\|\leqq\pi/2$.*

Proof. Let S^n $(n=\dim M)$ denote the n-sphere of constant curvature 1. Let p_0 be a point of S^n and $I\colon T_pM\to T_{p_0}S^n$ a linear isometry. Then, by Rauch's comparison theorem (cf. [1], p. 250, Theorem 14), we have

$$\text{(1)}\qquad \|\dot{\widehat{(\exp_p\circ\gamma)}}(t)\|\geqq\|\dot{\widehat{(\exp_{p_0}\circ I\circ\gamma)}}(t)\| \qquad (a\leqq t\leqq b).$$

Hence it follows that

$$\text{(2)}\qquad L(\exp_p\circ\gamma)\geqq L(\exp_{p_0}\circ I\circ\gamma).$$

On the other hand, it is clear that

(3) If $\|Iv\|\leqq\pi/2$,

$$d(\exp_{p_0}\circ Iv,\ \exp_{p_0}\circ Iw)\geqq d(\exp_{p_0}\circ Iv,\ p_0)=\|Iv\|.$$

The lemma follows from (2) and (3). Q.E.D.

Remark 3.4. In the proof above, if we further assume that $\|v\| =L(\exp_p\circ\gamma)=\pi/2$, then it follows from (2) and (3) that

$$\text{(4)}\qquad L(\exp_p\circ\gamma)=L(\exp_{p_0}\circ I\circ\gamma)=d(\exp_{p_0}\circ Iv,\ \exp_{p_0}\circ Iw).$$

(4) combined with (1) yields

$$\|\dot{\widehat{(\exp_p\circ\gamma)}}(t)\| = \|\dot{\widehat{(\exp_{p_0}\circ I\circ\gamma)}}(t)\| \qquad (a\leqq t\leqq b). \tag{5}$$

Hence we see from (4) that $\exp_{p_0}\circ I\circ\gamma$ is a shortest geodesic with an appropriate change of parametrization. And if $\exp_p\circ\gamma$ is a geodesic with the original parametrization, so is $\exp_{p_0}\circ I\circ\gamma$.

Lemma 3.5. *Assume $K\leqq 1$. Let $\alpha\colon \mathbf{R}\to S_q(\pi/2)$ be a curve such that $a=\exp_q\circ\alpha$ is a geodesic with $\|\dot{a}(t)\|=1$ $(t\in\mathbf{R})$. We put $\exp_q\alpha(0)=p$. Let $v\in T_pM$ be the initial tangent of the geodesic $\exp_q(1-t)\alpha(0)$ $(0\leqq t\leqq 1)$. Then $\exp_p\{\lambda\dot{a}(0)+\mu v;\ \lambda,\ \mu\in\mathbf{R},\ \|\lambda\dot{a}(0)+\mu v\|=\pi$ and $\mu\geqq 0\}$ is a single point.*

Proof. First we note that $<\dot{a}(0), v>=0$. Let $(\varepsilon_1, \varepsilon_2)$ be the connected component of $\{t\in\mathbf{R};\ \alpha(t)\neq-\alpha(0)\}$ which contains 0. We define a family of curves $c_t\colon [0, 1]\to M$ $(\varepsilon_1<t<\varepsilon_2)$ by

$$c_t(s)=\begin{cases}\exp_p 2sv & (0\leqq s\leqq 1/2)\\ \exp_q(2s-1)\alpha(t) & (1/2\leqq s\leqq 1).\end{cases}$$

Then, by Lemma 3.2, there are lifts γ_t of c_t to T_pM. Let S^n $(n=\dim M)$ be the n-sphere of constant curvature 1. Let p_0 be a point of S^n and $I\colon T_pM\to T_{p_0}S^n$ a linear isometry. We define $b_t\colon [1/2, 1]\to S^n$ by $b_t=\exp_{p_0}\circ I\circ\gamma_t|_{[1/2,1]}$. Then by the definition of c_t and b_t, it is clear that

$$\dot{b}_t(1/2)\neq-\dot{b}_0(1/2) \qquad (\varepsilon_1<t<\varepsilon_2). \tag{6}$$

On the other hand, we have

$$L(\exp_p\circ\gamma_t|_{[1/2,1]})=\|\gamma_t(1/2)\|=\pi/2.$$

Hence it is easily seen from Remark 3.4 that:

(7) b_t is the shortest geodesic connecting $\exp_{p_0}\circ Iv$ and $\exp_{p_0}\circ I(t\dot{a}(0))$;

$$\|\dot{b}_t(s)\|=\pi\ (\varepsilon_1<t<\varepsilon_2 \ \text{ and } \ 1/2\leqq s\leqq 1); \tag{8}$$

(9) The angle between two initial vectors $\dot{b}_t(1/2)$ and $\dot{b}_0(1/2)$ is equal to $\|t\dot{a}(0)\|=|t|$.

By the definition of ε_1 and ε_2 and by (8), we have

$$(10) \qquad \lim_{t\to\varepsilon_1+0} \dot{b}_t(1/2) = -\dot{b}_0(1/2) = \lim_{t\to\varepsilon_2-0} \dot{b}_t(1/2).$$

It follows from (6), (9) and (10) that $-\varepsilon_1=\varepsilon_2=\pi$. We fix a vector $\xi_\lambda=\lambda\dot{a}(0)+\mu v$ ($\mu\geqq 0$ and $\|\xi_\lambda\|=\pi$) and first study the case that $\lambda>0$. We obtain from (7) that the half line $\boldsymbol{R}_+ I\xi_\lambda$ ($\boldsymbol{R}_+=\{r\in\boldsymbol{R};\ r\geqq 0\}$) and $I\circ\gamma_t|_{[1/2,1]}$ have a single intersection which we denote by $I\circ\gamma_t(s_t)$ where $1/2<s_t\leqq 1$. Then it is clear from (9) and (10) that

$$\lim_{t\to\pi-0} s_t=1 \quad \text{and} \quad \lim_{t\to\pi-0} \|I\circ\gamma_t(s_t)\|=\pi.$$

Hence we have

$$d(\exp_p\xi_\lambda, \exp_p\pi\dot{a}(0)) = \lim_{t\to\pi-0} d(\exp_p\circ\gamma_t(s_t), \exp_p t\dot{a}(0))$$

$$\leqq \lim_{t\to\pi-0} L(\exp_p\circ\gamma_t|_{[s_t,1]}) = \lim_{t\to\pi-0} (1-s_t)\pi=0.$$

In case that $\lambda<0$, by the similar argument we see that

$$\exp_p\xi_\lambda=\exp_p(-\pi\dot{a}(0)).$$

By the continuity of the map $\exp_p$, we obtain

$$\lim_{\lambda\to-0} \exp_p\xi_\lambda=\exp_p\xi_0=\lim_{\lambda\to+0} \exp_p\xi_\lambda,$$

proving the lemma. Q.E.D.

We define the cut locus $C(p)$ of p in T_pM by

$$C(p)=\left\{ v\in T_pM;\ \begin{array}{ll} d(p, \exp_p tv)=\|tv\| & (0\leqq t\leqq 1) \\ d(p, \exp_p tv)<\|tv\| & (1<t) \end{array} \right\}$$

and the set $\exp_p C(p)$ is also called the cut locus of p and denoted by $C(p)$.

The following theorems are known.

Theorem 3.6 (cf. [7]). *Assume $K\leqq 1$ and there is a point p in M such that $C(p)=S_p(\pi/2)$. Then M is isometric to the real projective space of constant curvature* 1.

Theorem 3.7 (cf. [6]). *Assume $K \leqq 1$ and there is a point p in M such that $C(p) = S_p(\pi)$ and $\exp_p C(p)$ is a single point. Then M is isometric to the sphere of constant curvature* 1.

In this paper, we use a weaker version of Theorem 3.7:

Theorem 3.8. *Assume $K \leqq 1$ and there is a point p in M such that $\exp_p S_p(\pi)$ is a single point. Then $K \equiv 1$.*

This theorem can be proved by the similar argument to that of section 2 in [6].

§4. Fixed Points of Isometries

In this section, p denotes an arbitrary, but fixed point of M and we assume that $\dim G_p \geqq 1$ and $K \leqq 1$. G_p is sometimes considered to be a subgroup of the linear orthogonal transformations $O(T_pM)$ by the linear isotropy representation. Let G denote the identity component of G_p and let F be the set of points of M which are fixed by G.

We note that F is a finite union of totally geodesic closed submanifolds of M.

Lemma 4.1. *Let x and y be two points of F and $c\colon [0, 1] \to M$ a geodesic connecting x and y such that $L(c) < \pi$. Then c is contained in F.*

Proof. Since $\exp_x \dot{c}(0) = y \in F$, $\exp_x \dot{c}(0) = G(\exp_x \dot{c}(0)) = \exp_x G\dot{c}(0)$. Hence it follows from the theorem of Morse-Schoenberg with $\|\dot{c}(0)\| = L(c) < \pi$ that $\dot{c}(0)$ is fixed by G, which implies that $\dot{c}(0)$ is tangent to F. Since F is totally geodesic, c is contained in F. Q.E.D.

Proposition 4.2. *Let q be a point of F and v a tangent vector to M at q such that $\|v\| \leqq \pi/2$ and $\langle v, T_qF \rangle = 0$. Then $d(F, \exp_q v) = \|v\|$. Therefore the inequality $\|w\| \geqq \pi/2$ holds for any $w \in C(q) \cap (T_qF)^{\perp}$.*

Proof. Suppose that $d(F, \exp_q v) < \|v\|$. Then there is a point $q' \in F$ such that $d(q', \exp_q v) = d(F, \exp_q v)$. Let $a\colon [0, 1] \to M$ be a shortest geodesic from $\exp_q v$ to q'. We define a curve $c\colon [0, 1] \to M$ by

$$c(t)=\begin{cases}\exp_q 2tv & (0\leqq t\leqq 1/2)\\ a(2t-1) & (1/2\leqq t\leqq 1).\end{cases}$$

Then we obtain $L(c)<\pi$. Hence by Lemma 3.2, there is the lift γ of c to T_qM. Since $\|\gamma(1)\|<\pi$ and $\exp_q\gamma(1)\in F$, it follows from Lemma 4.1 that the geodesic $\exp_q s\gamma(1)$ $(0\leqq s\leqq 1)$ is contained in F, i.e., $\gamma(1)\in T_qF$. Applying Lemma 3.3, we see $L(a)=L(\exp_q\gamma|_{[1/2,1]})\geqq\|v\|$, which contradicts our hypothesis. Q.E.D.

Corollary 4.3. *Let q be a point of F. If $d(M)=\pi/2$, then $C(q)\cap(T_qF)^{\perp}=S_q(\pi/2)\cap(T_qF)^{\perp}$.*

We put $S_q^F=S_q(\pi/2)\cap(T_qF)^{\perp}$.

Lemma 4.4. *If $d(M)=\pi/2$, then $\exp_p(S_p^F)=\exp_q(S_q^F)$ for any point q of F.*

Proof. Let x be a point of $\exp_q S_q^F$. Then, by our assumption $d(M)=\pi/2$ and by Proposition 4.2, we get $d(F,x)=d(p,x)=\pi/2$. Hence the shortest geodesic from x to p is normal to F at p and its length is $\pi/2$. It means that x is contained in $\exp_p S_p^F$. So we have $\exp_q S_q^F\subset\exp_p S_p^F$. In the same way, we get $\exp_q S_q^F\supset\exp_p S_p^F$. Q.E.D.

Lemma 4.5. *Assume $d(M)=\pi/2$. Let $a:\boldsymbol{R}\to F$ be a geodesic such that $a(0)=p$. Let v be a vector in S_p^F and put $q=\exp_p v$. Then there is a curve $\alpha:\boldsymbol{R}\to S_q(\pi/2)$ such that $\exp_q\circ\alpha=a$.*

Proof. We define a sphere bundle S^F over F by $S^F=\bigcup_{x\in F}S_x^F$, with the projection π induced by the projection of the tangent bundle TM of M. We define a map $\varphi: S^F\to M$ by $\varphi|_{S_x^F}=\exp_x|_{S_x^F}$. Since the map $\varphi|_{S_x^F}$ is of maximal rank, Lemma 4.4 implies that the subset $\varphi^{-1}(q)=\{w\in S^F;\ \varphi(w)=q\}$ of S^F is a submanifold and $\pi:\varphi^{-1}(q)\to F$ is a covering. Hence there is a curve $\alpha_1:\boldsymbol{R}\to\varphi^{-1}(q)$ such that $\pi\circ\alpha_1=a$. It is clear that the curve $\alpha:\boldsymbol{R}\to S_q(\pi/2)$ defined by the equation

$$\exp_q t\alpha(s)=\exp_{a(s)}(1-t)\alpha_1(s)\qquad(0\leqq t\leqq 1\ \text{ and }\ s\in\boldsymbol{R})$$

has the required property. Q.E.D.

Proof of Theorem A. The inequality $d(M)\geqq\pi/2$ follows from Proposition 4.2. Hence we assume $d(M)=\pi/2$ and derive $K\equiv 1$.

Case 1. The case where $\dim F=0$. By Corollary 4.3, we have $C(p) =S_p(\pi/2)$. So the assertion follows from Theorem 3.6.

Case 2. The case where $\dim F\geqq 1$. Let v be a tangent vector to M at p with $<v, T_pF>=0$ and $\|v\|=\pi$. Let w be a tangent vector to F at p with $\|w\|=\pi$. And let $a: \boldsymbol{R}\to F$ be a geodesic to which w is tangent at $a(0)=p$. Then Lemma 3.5, combined with Lemma 4.5, implies that

$$\{\exp_p v\}=\{\exp_p(\cos\theta w+\sin\theta v);\ 0\leqq\theta\leqq\pi\}=\{\exp_p w\}\,.$$

Since any tangent vector u to M at p with $\|u\|=\pi$ can be written in the form $\cos\theta w+\sin\theta v$ $(0\leqq\theta\leqq\pi)$, we obtain that $\exp_p S_p(\pi)$ is a single point. Hence the assertion follows from Theorem 3.8. Q. E. D.

Proof of Corollary to Theorem A. Suppose that both $K\leqq 0$ and $\dim G_p\geqq 1$ are satisfied. We consider the metric $r<\ ,\ >$ $(r>0)$ and denote by $d_r(M)$ the diameter of M concerning to the metric. Then, by Theorem A, we have $d_r(M)\geqq\pi/2$. On the other hand, it is clear that $\lim_{r\to 0} d_r(M)=0$, which is a contradiction. Q. E. D.

§5. Riemannian Manifolds of Even-Dimension and Positive Curvature

In this section, M denotes a Riemannian manifold of even dimension with $0<K\leqq 1$.

The following theorems are known.

Theorem 5.1 (cf. [3]). *If M is simply connected, then $d(p, C(p))\geqq\pi$ for any point p of M.*

Theorem 5.2 (cf. [10]). *If M is orientable, then it is simply connected.*

By virtue of the theorems above, we obtain

Theorem 5.3. $d(p, C(p))\geqq\pi/2$ *for any point p of M.*

Proof. According to Theorem 5.1, we may assume that M is not simply connected. Then Theorem 5.2 implies that the fundamental group of M is $\boldsymbol{Z}_2$. Suppose that there is a vector $v \in C(p)$ $(\subset T_pM)$ such that $\|v\| < \pi/2$. Then there is a vector w $(w \neq v)$ in $C(p)$ such that $\exp_p w = \exp_1 v$. Let $\pi: \tilde{M} \to M$ be the Riemannian universal covering and put $\pi^{-1}(p) = \{p_1, p_2\}$. We define a curve $c: [0, 1] \to \tilde{M}$ emanating from p_1 by

$$\pi \circ c(t) = \begin{cases} \exp_p 2tv & (0 \leqq t \leqq 1/2) \\ \exp_p (2-2t)w & (1/2 \leqq t \leqq 1). \end{cases}$$

Then it is clear by Theorem 5.1 that $c(1) = p_2$, which gives $d(p_1, p_2) \leqq \|v\| + \|w\| < \pi$. Let $c_1: [0, 1] \to \tilde{M}$ be a shortest geodesic from p_1 to p_2. We define another geodesic c_2 emanating from p_1 by

$$\pi \circ c_2(t) = \pi \circ c_1(1-t) \qquad (0 \leqq t \leqq 1).$$

Then clearly c_1 and c_2 are distinct geodesics from p_1 to p_2 with $L(c_1) = L(c_2) = d(p_1, p_2)$. Hence it follows that $d(p_1, C(p_1)) \leqq d(p_1, p_2) < \pi$, which is contradictory to Theorem 5.1. Q.E.D.

Proof of Theorem B. By Theorem 5.3, we get the inequalities $d(M) \geqq d(p, C(p)) \geqq \pi/2$ for any point p of M. Hence the theorem follows from Theorem 3.6. Q.E.D.

§6. The Relations between the Cut Locus and the Conjugate Locus

Let p be a point of M. The conjugate locus $Q(p)$ of p in T_pM is defined to be the set of tangent vectors where the exponential mapping $\exp_p: T_pM \to M$ is not of maximal rank.

In case of dimension 2, the following theorem is known.

Theorem 6.1 (cf. [5], [9]). *Let M be a simply connected 2-dimensional Riemannian manifold. Then $C(p)$ and $Q(p)$ have an intersection for any point p of M.*

However, in case of $\dim M \geqq 3$, the assertion in this theorem is false (cf. [11]). Hence we need some further hypothesis.

Theorem 6.2 (cf. [9]). *Let M be a simply connected 3-dimensional Riemannian manifold. Assume that there is a point p in M such that* $\dim G_p \geqq 1$. *Then $C(p)$ and $Q(p)$ have an intersection.*

Theorem 6.3. *Let M be a 4-dimensional Riemannian manifold. Assume that there is a point p in M such that* $\dim G_p \geqq 1$ *and $C(p) \cap Q(p) = \emptyset$. Then the Euler-Poincaré characteristic $\chi(M)$ of M is less than 2.*

Proof. Let G be a one-parameter subgroup of G_p. Let F be the set of points of M which are fixed by G. Then F is a finite union of totally geodesic closed submanifolds of M with even codimensions and $\chi(F) = \chi(M)$ (cf. [4]). Let x be a point of F. Let $c: [0, 1] \to M$ be the shortest geodesic from p to x. Since $C(p) \cap Q(p) = \emptyset$ and $x = G \circ \exp_p \dot{c}(0) = \exp_p \circ G \circ \dot{c}(0)$, c is contained in F. Hence F is connected.

Case 1. The case where $\dim F = 0$. In this case, we have $F = \{p\}$. Hence it follows that $\chi(M) = \chi(F) = 1$.

Case 2. The case where $\dim F = 2$. We consider F a Riemannian manifold with the metric induced by the inclusion $F \subset M$. And we define the cut locus and the conjugate locus of p in T_pF which we denote by $C_F(p)$ and $Q_F(p)$ respectively. As in the beginning of this proof, every shortest geodesic from p to a point F is contained in F. Hence it follows that $C_F(p) = C(p) \cap T_pF$. Since $Q(p) \supset Q_F(p)$, we get $C_F(p) \cap Q_F(p) = \emptyset$. So we can apply Theorem 6.1 and obtain that F is not simply connected. Then it is clear that $\chi(M) = \chi(F) \leqq 1$. Q.E.D.

Corollary 6.4. *Let M be a simply connected 4-dimensional Riemannian manifold. Assume that there is a point p in M such that* $\dim G_p \geqq 1$. *Then $C(p) \cap Q(p) \neq \emptyset$.*

Proof. We put $b_i = \dim H_i(M; \mathbf{R})$. Since M is connected and simply connected, $b_0 = b_4 = 1$ and $b_1 = 0$. By the Poincaré duality, we see $b_3 = b_1$. After all we have

$$\chi(M) = b_0 - b_1 + b_2 - b_3 + b_4 = 2 + b_2 \geqq 2.$$

Hence the assertion is clear. Q.E.D.

As to the manifolds of positive curvature and even dimension, we know

Theorem 6.5 (cf. [3]). *Let M be a simply connected even dimensional Riemannian manifold with strictly positive curvature. Then there is a point p in M such that $C(p) \cap Q(p) \neq \emptyset$.*

Proof of Theorem A′ and Theorem B′. By the theorem of Morse-Schoenberg, $Q(p) \cap B_p(\pi) = \emptyset$ if $K \leqq 1$. Hence our assertion $d(M) \geqq \pi$ follows from Corollary 6.4 and Theorem 6.5. Q.E.D.

References

[1] Bishop, R. L. and Crittenden, R. J., *Geometry of Manifolds*, Academic Press, New York, 1964.

[2] Bochner, S., Vector fields and Ricci curvature, *Bull. Amer. Math. Soc.*, **52** (1946), 776–797.

[3] Klingenberg, W., Contributions to Riemannian geometry in the large, *Ann. of Math.*, **69** (1959), 654–666.

[4] Kobayashi, S., Fixed points of isometries, *Nagoya Math. J.*, **13** (1958), 63–68.

[5] Myers, S. B., Connection between differential geometry and topology, *Duke Math. J.*, **1** (1935), 376–391.

[6] Nakagawa, H., Riemannian manifolds with many geodesic loops, *J. Math. Soc. Japan*, **20** (1968), 648–654.

[7] Nakagawa, H. and Shiohama, K., On Riemannian manifolds with certain cut loci, *Tôhoku Math. J.*, **22** (1970), 14–23.

[8] Sakai, T. and Shiohama, K., On the structure of positively curved manifolds with certain diameter, *Math. Z.*, **127** (1972), 75–82.

[9] Sugahara, K., On the cut locus and the topology of Riemannian manifolds, *J. Math. Kyoto Univ.*, **14** (1974), 391–411.

[10] Synge, J. L., On the connectivity of spaces of positive curvature, *Quart. J. Math. (Oxford Ser.)*, **7** (1936), 316–320.

[11] Weinstein, A. D., The cut locus and conjugate locus of a Riemannian manifold, *Ann. of Math.*, **87** (1968), 29–41.