Publ. RIMS, Kyoto Univ.
11 (1976), 749-773

An Aspect of Quasi-Invariant Measures on $\mathbf{R}^\infty$

By

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Introduction

The study of translationally quasi-invariant measures μ on an infinite-dimensional vector space is essentially different from the same study on a finite-dimensional vector space. In the finite-dimensional case, we can characterize μ as the Lebesgue measure modulo equivalence of absolute continuity. However, in the infinite-dimensional case the situation is more complicated and difficult, owing to the fact that there exist many extremal measures. For example, in a rigged Hilbert space $E \subset H \subset E^*$, we can construct various kinds of ergodic quasi-invariant measures which are singular with respect to each other. See [6]. So we want to study an aspect of quasi-invariant measures. As a special but essential case we shall here discuss the translationally quasi-invariant measures on $\mathbf{R}^\infty$ which are of the type of countably infinite products of one-dimensional probability measures. The main result is a characterization of l^2-quasi-invariant measures in terms of its second moment. The author thanks Prof. H. Yoshizawa for the many valuable comments and thanks Prof. Y. Yamasaki for his useful suggestions.

§ 1. Preliminary Discussions

Throughout this paper, we shall only consider probability measures which are defined on the usual Borel σ-field $\mathfrak{B}(\mathbf{R}^\infty)$ on $\mathbf{R}^\infty$. First we shall prepare some basic concepts and theorems for our later discussions. Let μ be a measure on $\mathfrak{B}(\mathbf{R}^\infty)$ and $t=(t_1, t_2,\dots, t_n,\dots) \in \mathbf{R}^\infty$. We define a transformed measure μ_t such that,

Communicated by K. Itô, May 23, 1975.
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$$\mu_t(A)=\mu(A-t), \qquad \text{for all} \quad A\in\mathfrak{B}(\mathbf{R}^\infty).$$

Definition 1.1. *A measure μ is called t-quasi-invariant, if and only if $\mu_t \cong \mu$ holds.*

Here the symbol $\cong$ means the equivalence relation of the absolute continuity. We put $T_\mu=\{t\in\mathbf{R}^\infty | \mu_t\cong\mu\}$.

Definition 1.2. *Let Φ be a subset of $\mathbf{R}^\infty$. A measure μ is called Φ-quasi-invariant (strictly-Φ-quasi-invariant), if and only if $\Phi\subset T_\mu$ $(\Phi=T_\mu)$ holds, respectively.*

From now on, the measure μ is always assumed to be the product-measure of one-dimensional probability measures. More exactly,

$d\mu(x)=\otimes_{j=1}^\infty f_j(x_j)dx_j$, $x=(x_1, x_2,\dots)\in\mathbf{R}^\infty$, where $f_j(u)>0$, for Lebesgue-a.e. u and $\int_{-\infty}^{\infty} f_j(u)du=1$. ...... ($\Pi$).

Clearly, $T_\mu\supset\mathbf{R}_0^\infty$ holds, where $\mathbf{R}_0^\infty=\{x=(x_1, x_2,\dots)\in\mathbf{R}^\infty | x_n=0$ except finite numbers of $n.\}$. T_μ forms an additive group, but does not necessarily form a vector space. We shall give a counter example for it in the last part of thissection.

Now let $f_n(u)$ be as in (Π). Then $\sqrt{f_n}\in L^2_{du}(\mathbf{R}^1)$, which is the class of all square summable functions with respect to the one-dimensional Lebesgue measure du. Let $\mathscr{F}$ be the Fourier transformation on $L^2_{du}(\mathbf{R}^1)$, $\mathscr{F}(f)(v)=\int\exp(2\pi ivu)f(u)du$, and we put $\mathscr{F}(\sqrt{f_n})=g_n$, for all n. First we shall state a simple criterion for equivalence of measures.

Theorem 1.1. *Let μ be a measure as in (Π), and $d\mu_1(x)=\otimes_{j=1}^\infty f_j^1(x_j)dx_j$ be an another measure also as in (Π). Then in order that $\mu_1\cong\mu$ holds, it is necessary and sufficient that,*

$$\sum_{j=1}^\infty\left\{1-\int_{-\infty}^{\infty}\sqrt{f_j(u)}\,\sqrt{f_j^1(u)}du\right\}<\infty .$$

Especially, if $\mu_1=\mu_t$ for some $t=(t_1, t_2,\dots)\in\mathbf{R}^\infty$, the above inequality becomes

$$\sum_{j=1}^\infty\left\{1-\int_{-\infty}^{\infty}\sqrt{f_j(u)}\sqrt{f_j(u-t_j)}\,du\right\}<\infty, \text{ or equivalently},$$

$$\sum_{j=1}^\infty\left\{1-\int_{-\infty}^{\infty}\exp(2\pi it_jv)|g_j(v)|^2dv\right\}<\infty .$$

Proof. The assertion of the theorem is the special case in the general equivalence criterion in [10]. It is a typical application of the martingale convergence theorem. We omit it.

If μ is a measure as in (Π) and for all n, f_n is identical with the same function f, we say that μ is a stationary measure with f. Then,

Proposition 1.1. *Let μ be a stationary measure with f. Then we conclude that $T_\mu \subset l^2$.*

Proof. Let $t=(t_1, t_2,\ldots)\in T_\mu$, and we put $\mathscr{F}(\sqrt{f})=g$. Then by the above theorem, we have

$$\sum_{j=1}^\infty \left\{1-\int \exp(2\pi i t_j v)|g(v)|^2 dv\right\}<\infty .$$

First we shall show that $t_j\to 0$, as $j\to\infty$. For it, we put

$$H(s)=\int(1-\exp(2\pi i s v))|g(v)|^2 dv, \qquad \text{for} \quad s\in \mathbf{R}^1 .$$

Then $H(0)=0, \lim_{s\to\pm\infty} H(s)=1$ due to the Riemann-Lebesgue theorem, and $0<H(s)<1$, for $0<{}^\forall|s|<\infty$. It follows that, for an arbitrary $\varepsilon>0$, $\inf_{|s|\geqq\varepsilon} H(s)>0$. Suppose $\{t_j\}$ does not converge to 0. Then there exist some $\varepsilon_0>0$ and subsequence $\{t_{j_n}\}$ such that, $|t_{j_n}|>\varepsilon_0$, for all n. Consequently,

$$\infty> \sum_{n=1}^\infty\int(1-\exp(2\pi i t_{j_n} v))|g(v)|^2 dv=\sum_{n=1}^\infty H(t_{j_n})$$

$$=\sum_{n=1}^\infty \inf_{|s|\geqq\varepsilon_0} H(s)=\infty .$$

We reach to a contradiction.

As $1-\cos(x)=O(x^2)$ at $x=0$, and

$$\infty>\sum_{j=1}^\infty H(t_j)=\sum_{j=1}^\infty \int(1-\cos 2\pi t_j v)|g(v)|^2 dv>$$

$\sum_{j=1}^\infty \int_{-K}^{K}(1-\cos 2\pi t_j v)|g(v)|^2 dv$, for all $K>0$, so $\sum_{j=1}^\infty t_j^2\int_{-K}^{K} v^2|g(v)|^2 dv<\infty$. If we take K so large that, $\int_{-K}^{K} v^2|g(v)|^2 dv>0$, it follows that $\sum_{j=1}^\infty t_j^2 <\infty$. Q. E. D.

Now we give an example for the fact that T_μ is not necessarily a vector space.

Example 1.1. We put $f_c(u)=1/4A_c\ (1+\cos 2\pi u)^2 \exp(-4\pi^2 cu^2)$, for a positive constant c, where A_c is a normalizing constant such that,

$$A_c=1/16\sqrt{\pi c}\ \{3+\exp(-1/c)+4\exp(-1/4c)\}.$$

Some calculations show that,
$g_c(v)=\mathscr{F}(\sqrt{f_c})(v)=1/4\sqrt{2\pi c A_c}\,\{\exp(-(v-1)^2/2c)+\exp(-(v+1)^2/2c)+2\exp(-v^2/2c)\}$, and that

$$\int \exp(iyv)|g_c(v)|^2 dv=1/16A_c\sqrt{\pi c}\ \exp(-cy^2/4)\{\cos y+2+\exp(-1/c)+4\exp(-1/4c)\cos(y/2)\}.$$

Now we define a measure μ such that,

$d\mu(x)=\otimes_{j=1}^{\infty} f_{c_j}(x_j)dx_j$, where $\{c_j\}$ is taken as $\sum_{j=1}^{\infty} c_j<\infty$.
Then in order that $t=(t_1, t_2,\dots)\in T_\mu$, it is necessary and sufficient that,

$\sum_{j=1}^{\infty} 1/B_{c_j}\{3+\exp(-1/c_j)+4\exp(-1/4c_j)-\exp(-\pi^2 t_j^2 c_j)(\cos(2\pi t_j)+2+\exp(-1/c_j)+4\exp(-1/4c_j)\cos(\pi t_j))\}<\infty$, where we put $B_{c_j}=16\sqrt{\pi c_j}A_{c_j}$ (hence $\lim_j B_{c_j}=3$).

As $\sum_{j=1}^{\infty} c_j<\infty$ (hence $\sum_{j=1}^{\infty}\exp(-1/4c_j)<\infty$), so the above inequality is equivalent that,

$$\sum_{j=1}^{\infty}\{3-\exp(-\pi^2 t_j^2 c_j)(\cos(2\pi t_j)+2)\}<\infty.$$

If we put $t_j=1$ for all j, then it yields

$$\sum_{j=1}^{\infty} 3\{1-\exp(-\pi^2 c_j)\}\leqq \sum_{j=1}^{\infty} 3\pi^2 c_j<\infty.$$

On the other hand if we put $t_j=1/2$ for all j, then it yields

$$\sum_{j=1}^{\infty}\{3-\exp(-\pi^2 c_j/4)\}=\infty.$$

Therefore we assured that $e=(1, 1,\dots, 1,\dots)\in T_\mu$, while $1/2\ e\notin T_\mu$.

§2. General Aspect of Quasi-Invariant Measures

In this section, we shall mostly consider the l^p-quasi-invariance

$(0<p\leqq\infty)$ of measures, where $l^p=\{x=(x_1, x_2,\dots)|\sum_{j=1}^{\infty}|x_j|^p<\infty\}$. It will be turned out that l^p-quasi-invariant measures actually exist, but strictly-l^p-quasi-invariant measure does not exist except for $0<p\leqq 2$. Further from another criterions (compactness, e.t.c.) we shall see that the case $p=2$ is a worth special interest.

Let μ be a measure on $\mathfrak{B}(\mathbf{R}^\infty)$ as in (Π), and put $\mathscr{F}(\sqrt{f_n})=g_n$ for each n. Then $\int|g_n(v)|^2dv=1$, due to the Plancherel's theorem, and we can construct a probability measure ν on $\mathfrak{B}(\mathbf{R}^\infty)$ such that,

$$d\nu(x)=\otimes_{j=1}^{\infty}|g_j(x_j)|^2dx_j\,.$$

The measure ν is called an adjoint measure of μ. Formally μ and ν is related as follows. We use the duality bracket $<x, y>=\sum_{j=1}^{\infty}x_jy_j$, for $x=(x_1, x_2,\dots)\in\mathbf{R}^\infty$ and for $y=(y_1, y_2,\dots)\in\mathbf{R}_0^\infty$, and $\frac{d\mu_t}{d\mu}(x)$ means the Radon-Nikodym derivative.

$$\int_{\mathbf{R}^\infty}\sqrt{\frac{d\mu_t}{d\mu}}(x)d\mu(x)=\int_{\mathbf{R}^\infty}\exp\,(2\pi i<x, t>)d\nu(x)\,.$$

Now let $a=(a_1, a_2,\dots)\in\mathbf{R}^\infty$, and we set $H_a=\{x\in\mathbf{R}^\infty|\sum_{j=1}^{\infty}x_j^2a_j^2<\infty\}$. Unless otherwise stated, we fix these symbols. The quasi-invariance of the measure μ concerns with the smoothness of each function f_n, while the support of the measure ν concerns with the decreasing order at infinity of each function $|g_n|^2$. As Fourier transformation reflects these two properties, we can settle this point in a following lemma.

Lemma 2.1. (*Foundamental*) *Let* $t=(t_1, t_2,\dots)\in\mathbf{R}^\infty$. *Then in order that* $ct\in T_\mu$ *holds for all* $c\in\mathbf{R}^1$, *it is necessary and sufficient that* $\nu(H_t)=1$.

Proof. First we shall assume that $ct\in T_\mu$, for all $c\in\mathbf{R}^1$. Then we can define a one-parameter unitary group $\{U_c\}_{c\in\mathbf{R}^1}$ on $L_\mu^2(\mathbf{R}^\infty)$ such that,

$$U_c: f(x)\in L_\mu^2(\mathbf{R}^\infty)\longrightarrow\sqrt{\frac{d\mu_{ct}}{d\mu}}(x)f(x-ct)\,.$$

$$\int\sqrt{\frac{d\mu_{ct}}{d\mu}}(x)d\mu(x)=\prod_{j=1}^{\infty}\int\exp\,(2\pi ict_jx_j)d\nu(x)$$

holds by the martingale convergence theorem and it is the continuous function of c by the well known theorem for a one-parameter group. As the each term of the above infinite product is positive for any c, so $\sum_{j=1}^{\infty}\left\{1-\int \exp(2\pi i c t_j x_j)\,d\nu(x)\right\}$ is again a continuous function of c. We integrate it with the normalized Lebesgue measure on $[0, 1]$, then we get

$$\sum_{j=1}^{\infty}\int\left(1-\frac{\sin(2\pi t_j x_j)}{2\pi t_j x_j}\right)d\nu(x)<\infty .$$

It follows that

$$\sum_{j=1}^{\infty}\left(1-\frac{\sin(2\pi t_j x_j)}{2\pi t_j x_j}\right)<\infty \qquad (\text{equivalently, } \textstyle\sum_{j=1}^{\infty} t_j^2 x_j^2<\infty)$$

holds for ν-a.e. $x=(x_1, x_2, \ldots)$. It shows that $\nu(H_t)=1$. From the above argument, *we remark that* $\sum_{j=1}^{\infty} t_j x_j$ *converges in law for* ν, *and that the independence of each* x_j *derives that it converges almost surely for* ν. Conversely, suppose that $\nu(H_t)=1$, then

$$0<\int_{\mathbf{R}^\infty}\exp\left(-\sum_{j=1}^{\infty} t_j^2 x_j^2/2\right)d\nu(x)=\lim_N \prod_{j=1}^{N}\int \exp(-t_j^2 v^2/2)|g_j(v)|^2 dv$$

and therefore,

$$\sum_{j=1}^{\infty}\left\{1-\int \exp(-t_j^2 v^2/2)|g_j(v)|^2 dv\right\}<\infty .$$

Let n be the one-dimensional Gaussian probability measure with mean 0 and variance 1. Then the above inequality yields,

$$\sum_{j=1}^{\infty}\int\{1-\exp(it_j uv)\}|g_j(v)|^2 dv\,dn(u)<\infty .$$

It follows that,

$$\sum_{j=1}^{\infty}\int\{1-\exp(it_j uv)\}|g_j(v)|^2 dv<\infty, \qquad \text{for } n\text{-a.e. } u .$$

We set

$$A=\left\{u\in\mathbf{R}^1 \,\middle|\, \sum_{j=1}^{\infty}\int\{1-\exp(it_j uv)\}|g_j(v)|^2 dv<\infty\right\}.$$

Then $n(A)=1$, and clearly A forms an additive group. Suppose $A\neq\mathbf{R}^1$, then there exists some s such that $(s+A)\cap A=\emptyset$, and therefore $n(A\cup(s+A))=2$. It contradicts the total mass of n, so $A=\mathbf{R}^1$. It follows that $ct\in T_\mu$ holds for all c. Q.E.D.

Using the above fundamental lemma, we can actually construct l^p-quasi-invariant measures as follows. First we put for $u>0$,

$$K_0(u)=\int_0^\infty \exp(-u\cosh t)dt,$$

which is the modified Bessel function. And for each $1\leqq p\leqq\infty$, we take and fix a sequence $\beta=\{\beta_j\}\in l^p$, whose all components are positive numbers. Further let q be a conjugate exponent of p, $1/p+1/q=1$. Now we define a measure μ_q such that,

$$d\mu_q(x)=\otimes_{j=1}^\infty 4\,\beta_j/\pi\; K_0^2(2\pi\beta_j|x_j|)dx_j, \qquad \text{for each } (p,q) \text{ and } \beta\in l^p.$$

Noting that

$\mathscr{F}(2K_0(2\pi a|u|)(v)=(v^2+a^2)^{-1/2}$, for an arbitrary real constant a, we get for the adjoint measure ν_q of μ_q,

$$d\nu_q(x)=\otimes_{j=1}^\infty\; \beta_j/\pi\;(x_j^2+\beta_j^2)^{-1}dx_j.$$

Proposition 2.1. *If* $a=(a_1,a_2,\ldots)\in l^q$, *then* $\nu_q(H_a)=1$.

Proof. $\nu_q(H_a)=1$ is equivalent that,

$$\sum_{j=1}^\infty\int_{-\infty}^\infty \{1-\exp(-a_j^2\beta_j^2u^2)\}1/\pi\;(1+u^2)^{-1}du<\infty. \tag{1}$$

In order to assure (1), we put for $s\in\mathbf{R}^1$,

$$w(s)=1/\pi\int\{1-\exp(-s^2u^2)\}(1+u^2)^{-1}du,$$

and estimate the order of $w(s)$ at $s=0$. Then after some calculations, we can derive that $w(s)=O(|s|)$ at $s=0$, so the convergence of (1) is equivalent to $\sum_{j=1}^\infty|a_j|\beta_j<\infty$. Clearly this inequality is satisfied by the assumptions for a and for β. Q.E.D.

Combining Proposition 2.1 together with Lemma 2.1, μ_q is actually the l^q-quasi-invariant measure for each $q\geqq 1$. Later we shall give examples of strictly-l^q-quasi-invariant measures for $0<p\leqq 2$.

The following definition and lemma are essentially due to L. Shwarz, [7]. We list them here in a partially different but special form of the original one.

Definition 2.1. *Let m be a measure on $\mathfrak{B}(\mathbf{R}^\infty)$ such that, $m=\otimes_{j=1}^\infty m_j$, where m_j is the probability measure on $\mathfrak{B}(\mathbf{R}^1)$ for each j, and Φ be a subspace of $\mathbf{R}^\infty$.*

(a) *If for an arbitrary element $t=(t_1, t_2,\dots)\in\Phi$, $\sum_{j=1}^\infty t_j x_j$ converges for m-a.e. $x=(x_1, x_2,\dots)$, then we say that m is a type Φ.*

(b) *Conversely, if a following assertion holds, we say that m is a cotype Φ.*

Let $t=(t_1, t_2,\dots)\in\mathbf{R}^\infty$. If $\sum_{j=1}^\infty t_j x_j$ converges for m-a.e. x, then it follows that $t\in\Phi$.

(c) *If* (a) *and* (b) *are both satisfied, we say that m is a special type Φ.*

After these definitions, we can state the following corollary of Lemma 2.1.

Corollary. *Let μ be a measure as in (Π), and ν be the adjoint measure of μ. Then,*

(a) *$T_\mu\supset l^p$ is equivalent that ν is a type l^p.*

(b) *if $T_\mu=l^p$, then ν is a special type l^p.*

(c) *if ν is a special type l^p, and T_μ forms a vector space, then $T_\mu=l^p$.*

Proof is derived from the consideration and the remark of Lemma 2.1.

Now if m is a type l^p $(p>0)$, we can define a following operator T from l^p to $\mathrm{Mes}(\mathbf{R}^\infty, m, \mathbf{R}^1)$ such that,

$$T\colon t=(t_1, t_2,\dots)\in l^p \longrightarrow \sum_{j=1}^\infty t_j x_j\in \mathrm{Mes}(\mathbf{R}^\infty, m, \mathbf{R}^1),$$

where the last symbol means the class of all real-valued m-measurable functions defined on $\mathbf{R}^\infty$.

Lemma 2.2. *Let m be a measure on $\mathfrak{B}(\mathbf{R}^\infty)$ as in Definition 2.1. For $1 \leqq p \leqq \infty$,*
(a) *if m is a type l^p, then the above mapping T is the continuous operator from l^p to $\mathrm{Mes}(\mathbf{R}^\infty, m, \mathbf{R}^1)$ equipped with the topology of convergence in probability.*
(b) *if m is a special type l^p, and is the symmetric measure, then the map T is the homeomorphic operator from l^p to $\mathrm{Mes}(\mathbf{R}^\infty, m, \mathbf{R}^1)$ equipped with the same one.*

Proof is stated in [7], [8]. So we omit it. But it is an application of Baire's theorem and closed graph theorem.

Here we shall discuss strictly-l^p-quasi-invariant measures $(0 < p \leqq \infty)$ on $\mathfrak{B}(\mathbf{R}^\infty)$.

Proposition 2.2. *There does not exist any strictly-l^∞-quasi-invariant measure as in (Π) on $\mathfrak{B}(\mathbf{R}^\infty)$.*

Proof. Suppose the contrary case, namely let μ be a measure as in (Π), and be the strictly-l^∞-quasi-invariant measure. Then the adjoint measure ν of μ is the special type l^∞ in virtue of the corollary of Lemma 2.1. Applying Theorem 1.1 for an element $(s, s, \dots, s, \dots) \in l^\infty$ $(s \in \mathbf{R}^1)$, we get

$$\sum_{j=1}^{\infty}\left\{1-\int \exp(2\pi i s x_j) d\nu(x)\right\} < \infty .$$

Therefore,

$$\lim_j \int \exp(2\pi i s x_j) d\nu(x) = 1 .$$

It follows that for ν, $\{x_j\}$ converges in law to the Dirac measure, equivalently it converges to 0 in probability. Let T be the same meaning as in Lemma 2.2, in which we shall put ν for m. Then an element $e_j = (0, 0, \dots, 0, \overset{j}{\overbrace{1}}, 0, \dots)$ corresponds to x_j by the map T, and it is a homeomorphic operator in virtue of Lemma 2.2. Therefore by the above argument, e_j must tend to $0 = (0, 0, \dots)$, which is a contradiction. Q.E.D.

Lemma 2.3. *Let α be a probability measure on $\mathfrak{B}(\mathbf{R}^1)$ and put*

$$\Phi(s)=\int_{-\infty}^{\infty}(1-\exp(-s^2u^2))d\alpha(u), \qquad \textit{for} \quad s\in\mathbf{R}^1 .$$

Then,

$$\Phi(\lambda s)\geqq\lambda^2\Phi(s), \qquad \textit{for all} \quad |\lambda|\leqq 1.$$

Proof. It is derived from the following elementary inequality.

$$1-\exp(-cv)\geqq c(1-\exp(-v)), \quad \text{for} \quad 0\leqq{}^{\forall}c\leqq 1 \quad \text{and for} \quad {}^{\forall}v\geqq 0.$$

Q.E.D.

Proposition 2.3. *For $2<p<\infty$, there does not exist any strictly-l^p-quasi-invariant measure as in (Π) on $\mathfrak{B}(\mathbf{R}^\infty)$.*

Proof. Suppose the contrary case, namely let μ be a measure as in (Π) and be the strictly-l^p-quasi-invariant measure. Then from Lemma 2.1, $\sum_{j=1}^{\infty}\int(1-\exp(-b_j^2x_j^2))d\nu(x)<\infty$ holds if and only if $\{b_j\}\in l^p$. If necessary, we divide each b_j by a suitable normalizing constant, and apply Lemma 2.3. Then it follows that,

$$\sum_{j=1}^{\infty}b_j^2\int(1-\exp(-x_j^2))d\nu(x)<\infty, \qquad \text{for all} \quad \{b_j\}\in l^p. \tag{3}$$

Suppose $\inf_j\int(1-\exp(-x_j^2))d\nu(x)=0$. Then a suitable subsequence $\{j_n\}$ exists such that,

$$\sum_{n=1}^{\infty}\int(1-\exp(-x_{j_n}^2))d\nu(x)<\infty .$$

So putting $e=(0, 0,\dots, 0, \overset{j_1}{\overbrace{1}}, 0,\dots, 0, \overset{j_n}{\overbrace{1}}, 0,\dots)$, the above inequality shows that μ is e-quasi-invariant. But e does not belong to any l^p $(\infty>p>0)$. So it contradicts the assumption of quasi-invariance. Consequently $\inf_j\int(1-\exp(-x_j^2))d\nu(x)>0$, and from (3) we get $\sum_{j=1}^{\infty}b_j^2<\infty$ for all $\{b_j\}\in l^p$. Again we reach to a contradiction. Q.E.D.

On the other hand, in the case of $0<p\leqq 2$, strictly-l^p-quasi-invariant measures actually exist as follows. Let r be a real number such that $2r>1$ and we put $f_r(u)=\frac{4\pi}{\gamma_r\Gamma^2(r/2)}|\pi u|^{r-1}K^2_{\frac{r-1}{2}}(2\pi|u|)$, where $K_{\frac{r-1}{2}}$ is again the modified Bessel function and γ_r is the normalizing constant.

$$\gamma_r=\sqrt{\pi}\,\Gamma(r-1/2)/\Gamma(r).$$

We define a measure μ_r on $\mathfrak{B}(\mathbf{R}^\infty)$ such that,

$$d\mu_r(x)=\otimes_{j=1}^{\infty} f_r(x_j)dx_j,$$

namely μ_r is a stationary measure with f_r. By the well known formula for the Fourier transformation, we obtain for the adjoint measure ν_r

$$d\nu_r(x)=\otimes_{j=1}^{\infty} 1/\gamma_r\ (1+x_j^2)^{-r}dx_j.$$

Therefore from Theorem 1.1, $t=(t_1, t_2,\dots)\in T_\mu$ holds, if and only if

$$\sum_{j=1}^{\infty}\int\frac{(1-\cos 2\pi t_j u)}{\gamma_r(1+u^2)^r}du<\infty. \tag{4}$$

Using the result of the stationary case (Proposition 1.1), it is necessary that $\lim_j t_j=0$ for (4). So putting

$$W(s)=\int_{-\infty}^{\infty}1/\gamma_r\ (1-\cos(2\pi su))(1+u^2)^{-r}du,$$

and estimating the order of it at $s=0$, we get

(a) $W(s)=O(|s|^{2r-1})$, if $2r-1<2$

(b) $W(s)=O(|s|^2\log|s|)$, if $2r-1=2$

(c) $W(s)=O(s^2)$, if $2r-1>2$.

Finally we have the following result

(A) $T_{\mu_r}=l^{2r-1}$, if $2r-1<2$

(B) $T_{\mu_r}=l^{2-}$, if $2r-1=2$, where $l^{2-}=\{x\in\mathbf{R}^\infty\mid \sum_{j=1}^{\infty}x_j^2<\infty$ and $\sum_{j=1}^{\infty}x_j^2|\log|x_j||<\infty\}$

(C) $T_{\mu_r}=l^2$, if $2r-1>2$.

Lemma 2.4. *For $1\leqq p\leqq\infty$, let μ be a measure as in (II) and be the l^p-quasi-invariant measure. Then for $a=(a_1, a_2,\dots)\in l^p$,*

$$\sum_{j=1}^{\infty}\left\{1-\int\sqrt{f_j(u-a_j)}\sqrt{f_j(u)}\,du\right\}$$

is a continuous function of a with respect to the natural topology of

l^p. *Especially, for* $j=1, 2,\dots$, $\|\sqrt{f_j(u-s)}-\sqrt{f_j(u)}\|_{L^2}$ *is an equicontinuous function of* $s\in\mathbf{R}^1$.

Proof. Let ν be the adjoint measure of μ. $(d\nu(x)=\otimes_{j=1}^{\infty}|g_j(x_j)|^2dx_j)$. Then ν is the type l^p and a function

$$\hat{\nu}(a)=\int\exp(2\pi i\textstyle\sum_{j=1}^{\infty}a_jx_j)d\nu(x)=\prod_{j=1}^{\infty}\int\exp(2\pi ia_jv)\,|g_j(v)|^2dv$$

is the continuous function of $a\in l^p$, in virtue of Lemma 2.2. As $\hat{\nu}(a)$ is always positive, and the infinite product converges uniformly in a neighbourhood of each point,

$$\sum_{j=1}^{\infty}\left\{1-\int\exp(2\pi ia_jv)\,|g_j(v)|^2dv\right\}=\sum_{j=1}^{\infty}\left\{1-\int\sqrt{f_j(u-a_j)}\sqrt{f_j(u)}du\right\}$$

is also continuous. The assertion of the last part is an easy consequence of the above argument. Q.E.D.

Now we shall discuss the compactness of the set $\{\sqrt{f_j(u)}\}$ in $L^2_{du}(\mathbf{R}^1)$. The following proposition is found in [9]. But we list it for reference.

Proposition 2.4. *Let* $d\xi$ *be the Lebesgue measure on* $\mathbf{R}^N$, *and* $L^p_{d\xi}(\mathbf{R}^N)$ $(1\leqq p<\infty)$ *be the Banach space of (classes of) functions* f *such that* $|f|^p$ *is Lebesgue integral. Then a subset* A *of* $L^p_{d\xi}(\mathbf{R}^N)$ *is totally bounded if and only if it has the following three properties.*
(a) *A is bounded in* $L^p_{d\xi}(\mathbf{R}^N)$ *(in the sense of the* L^p *norm).*
(b) *A is equismall at infinity, i.e., to every* $\varepsilon>0$, *there is* $\rho>0$, *such that, for all* $f\in A$

$$\int_{\|\xi\|>\rho}|f(\xi)|^pd\xi<\varepsilon.$$

(c) *To every* $\varepsilon>0$, *there is* $\delta>0$ *such that, for all* $a\in\mathbf{R}^N$ *such that* $\|a\|<\delta$ *and for all* $f\in A$,

$$\int|f(\xi-a)-f(\xi)|^p\,d\xi\leqq\varepsilon.$$

Applying it to the present case, from Lemma 2.4,

Proposition 2.5. *For* $1\leqq p\leqq\infty$, *let* μ *be a measure as in* (Π), *and be the l^p-quasi-invariant measure. If $\{\sqrt{f_j(u)}\}$ is equismall at infinity, then it is the totally bounded set in L^2_{du}.*

Remark. The equismall property of $\{\sqrt{f_j(u)}\}$ is the same as the uniform tightness of the set of the measures $\{f_j(u)du\}$. It depends on the continuity of μ. For example, if μ can be regarded as a continuous cylindrical measure on l^p $(1 \leqq p < \infty)$, then the uniform tightness condition is satisfied.

Conversely,

Proposition 2.6. *Let μ be a measure as in (Π), and T_μ be a vector space. Suppose that $\{\sqrt{f_j(u)}\}$ is a totally bounded set in $L^2_{du}(\mathbf{R}^1)$, then we conclude that $T_\mu \subset l^2$.*

Proof. Let $dv(x) = \otimes_{j=1}^{\infty} |g_j(x_j)|^2 dx_j$ be the adjoint measure of μ. Suppose that $a = (a_1, a_2, \ldots) \in T_\mu$. Then in virtue of Lemma 2.1,

$$\sum_{j=1}^{\infty} \int (1 - \exp(-s^2 a_j^2 v^2)) |g_j(v)|^2 dv < \infty, \qquad \text{for all} \quad s \in \mathbf{R}^1 . \tag{5}$$

First we shall assume that $\sup_j |a_j| = \infty$. Then, as $1 - \exp(-u^2)$ is a monotone increasing function, there exist subsequence $\{j_n\}$ such that,

$$\sum_{n=1}^{\infty} \int (1 - \exp(-v^2)) |g_{j_n}(v)|^2 dv < \infty ,$$

hence

$$\lim_n \int (1 - \exp(-v^2)) |g_{j_n}(v)|^2 dv = 0 . \tag{6}$$

By the assumption, $\{g_j(v)\}$ is the totally bounded set, and therefore if necessary taking a subsequence of $\{j_n\}$, we can suppose that $g_{j_n}(v)$ converges to some $g_0(v)$ in $L^2_{dv}(\mathbf{R}^1)$. Then from (6), we get

$$\int (1 - \exp(-v^2)) |g_0(v)|^2 dv = 0 ,$$

and we reach to a contradiction. So $\sup_j |a_j| < \infty$. Using Lemma 2.3, from (5) it follows that

$$\sum_{j=1}^{\infty} a_j^2 \int (1-\exp(-v^2)) |g_j(v)|^2 dv < \infty .$$

Again the compactness of $\{g_j(v)\}$ implies

$$\inf_j \int (1-\exp(-v^2)) |g_j(v)|^2 dv > 0, \text{ so } \sum_{j=1}^{\infty} a_j^2 < \infty . \qquad \text{Q.E.D.}$$

Even if a measure μ is strictly-l^2-quasi-invariant, $\{\sqrt{f_j(u)}\}$ is not necessarily a totally bounded set.

Example 2.1. We set μ such that,

$d\mu(x) = \otimes_{j=1}^{\infty} 1/\sqrt{2\pi} \exp(-(x_j - r_j)^2/2) dx_j$, where the positive sequence $\{r_j\}$ is taken such that $\lim_j r_j = \infty$.

Then $T_\mu = l^2$ holds but the set $\{\exp(-(u-r_j)^2/4)\}$ is not totally bounded.

Example 2.2.

$d\mu(x) = \otimes_{j=1}^{\infty} 1/\sqrt{2\pi} \exp(-(x_j^2 + r_j^2)/2) \cosh(r_j x_j) dx_j$, where $\{r_j\}$ is a positive sequence such that $\sum_{j=1}^{\infty} \exp(-r_j^2/2 + r_j) < \infty$.

μ has the same properties as in above, but we remark that it is a symmetric measure.

Roughly speaking, the compact case is possible to arise only in that of $T_\mu \subset l^2$. On the other hand,

Proposition 2.7. *For $p > 2$, let μ be a measure as in (II), and be the l^p-quasi-invariant measure. Then $\{\sqrt{f_j(u)}\}$ is a discrete set in $L^2_{du}(\mathbf{R}^1)$.*

Proof. Suppose the contrary case, namely we shall assume that a suitable subsequence $\{\sqrt{f_{j_n}(u)}\}$ converges to some $\sqrt{f_0(u)} \in L^2_{du}(\mathbf{R}^1)$. If necessary, again we take a subsequence of $\{j_n\}$ such that,

$$\sum_{n=1}^{\infty} \| \sqrt{f_{j_n}(u)} - \sqrt{f_0(u)} \|_{L^2}^2 < \infty .$$

Then by the assumption, also for $\{j_n\}$,

$$\sum_{n=1}^{\infty} \int \{1 - \sqrt{f_{j_n}(u-a_n)} \sqrt{f_{j_n}(u)}\} du < \infty, \quad \text{for all} \quad a = (a_1, a_2, \dots) \in l^p .$$

As

$$\|\sqrt{f_0(u-s)}-\sqrt{f_0(u)}\|_{L^2}^2 \leqq 8\|\sqrt{f_0(u)}-\sqrt{f_{j_n}(u)}\|_{L^2}^2+2\|\sqrt{f_{j_n}(u-s)} -\sqrt{f_{j_n}(u)}\|_{L^2}^2,$$

we have

$$\sum_{j=1}^{\infty}\left\{1-\int\sqrt{f_0(u)}\sqrt{f_0(u-a_j)}\,du\right\}<\infty, \quad \text{for all} \quad a=(a_1, a_2,\dots)\in l^p .$$

Even if $\sqrt{f_0(u)}$ vanishes on a set with positive Lebesgue measure, the proof of Proposition 1.1 is still valid for the present case. Therefore we conclude that $a\in l^2$. But it contradicts the fact, $l^2 \subsetneqq l^p$. Q.E.D.

§3. Characterization of l^2-Quasi-Invariant Measures

In the former sections, we have discussed the aspect of (mainly, l^p-) quasi-invariant measures, and showed that the case $p>2$ and $p\leqq 2$ present the different situations. So we wish to consider the case of $T_\mu \supseteqq l^2$ (especially $T_\mu = l^2$) and to characterize it in terms of $\{\sqrt{f_j(u)}\}$. First we shall consider a stationary measure.

Theorem 3.1. *Let μ be a measure as in (Π), and be the stationary measure with f. We put $\mathscr{F}(\sqrt{f})=g$. Then in order that $T_\mu \supseteqq l^2$ holds (automatically, $T_\mu = l^2$ holds due to Proposition 1.1), it is necessary and sufficient that*

$$\int v^2 |g(v)|^2 dv<\infty . \tag{7}$$

(It is equivalent to $\left\|\frac{d}{du}\sqrt{f(u)}\right\|_{L^2}<\infty$.)

Proof. Let ν be the adjoint measure of μ,

$$d\nu(x)=\otimes_{j=1}^{\infty}|g(x_j)|^2 dx_j.$$

First we shall prove sufficiency. Assume that (7) is satisfied. Then for an arbitrary element $a=(a_1, a_2,\dots)\in l^2$, we have

$$\infty > \sum_{j=1}^{\infty} a_j^2 \int v^2 |g(v)|^2 dv=\int \sum_{j=1}^{\infty} a_j^2\, x_j^2\, d\nu(x),$$

and it follows that $\nu(H_a)=1$. Consequently, from Lemma 2.1, we conclude that $T_\mu \supseteqq l^2$.

Conversely, suppose to be $T_\mu = l^2$. Then from Lemma 2.4, for an arbitrary $\varepsilon_0>0$, there exists $\delta_0>0$ such that,

$$\sum_{j=1}^{\infty}\int(1-\cos(a_j v))\,|g(v)|^2 dv<\varepsilon_0, \qquad \text{for all} \quad \|a\|^2=\sum_{j=1}^{\infty}a_j^2 \leqq \delta_0^2 .$$

In this inequality, we shall put $a_j=\delta_0/\sqrt{n}$ for $1\leqq j\leqq n$ and $a_j=0$ for $j\geqq n+1$. Then for any n,

$$n\int(1-\cos(\delta_0 v/\sqrt{n}))\,|g(v)|^2 dv<\varepsilon_0.$$

So, letting n tend to infinity and applying Lebesgue-Fatou's lemma, it follows that,

$$\delta_0^2/2\int v^2\,|g(v)|^2 dv \leqq \varepsilon_0, \text{ which shows the necessity.} \qquad \text{Q.E.D.}$$

Let μ be a measure as in (Π). i.e.,

$$d\mu(x)=\otimes_{j=1}^{\infty} f_j(x_j)dx_j, \quad \text{and} \quad \mathscr{F}(\sqrt{f_j})=g_j .$$

Assume that μ is the l^2-quasi-invariant measure. Then from the result of the above stationary case, it seems that,

$$\sup_j \int v^2 |g_j(v)|^2 dv<\infty .$$

But it is false for the general μ. Even in the case of $T_\mu=l^2$, we have a following example.

Example 3.1. Let $K_0(u)$ be the modified Bessel function, and γ be a constant such that $0<\gamma<1$. We put

$$f_\gamma(u)=1/n(\gamma)\ \{(1-\gamma)2/\sqrt{\pi}\ K_0(2\pi|u|)+\gamma/\sqrt{2}\ \exp(-|x|/2)\}^2 ,$$

where $n(\gamma)$ is the normalizing constant. And we define a measure μ on $\mathfrak{B}(\mathbf{R}^\infty)$ such that, $d\mu(x)=\otimes_{j=1}^{\infty} f_{\gamma_j}(x_j)dx_j$, where $\{\gamma_j\}$ is taken such that $\sum_{j=1}^{\infty}(1-\gamma_j)^2<\infty$. Then some calculation shows that $d\mu(x)\cong\otimes_{j=1}^{\infty}1/2\exp(-|x_j|)dx_j$, and the later measure is strictly-l^2-quasi-invariant. Therefore the same holds for μ. On the other hand, as

$$g_\gamma(v)=\mathscr{F}(\sqrt{f_\gamma})(v)=1/\sqrt{n(\gamma)}\{(1-\gamma)/\sqrt{\pi}(1+v^2)^{-1/2}+2\sqrt{2}\gamma(1+16\pi^2v^2)^{-1}\},$$

and

$$|g_\gamma(v)|^2\geqq(1-\gamma)^2/\pi n(\gamma)\cdot(1+v^2)^{-1},$$

so

$$\int v^2|g_\gamma(v)|^2dv=\infty, \qquad \text{for any} \quad \gamma>0.$$

However the above conjecture is modified as a following theorem.

Theorem 3.2. *Let μ be a measure as in (Π). Then in order that μ is an l^2-quasi-invariant measure, it is necessary and sufficient that there exists some measure M on $\mathfrak{B}(\mathbf{R}^\infty)$, which has following three properties.*

(a) $dM(x)=\otimes_{j=1}^\infty F_j(x_j)dx_j$, $F_j(u)>0$ *for Lebesgue-a.e. u*

(b) $M\simeq\mu$ *(in the sense of Definition* 1.1)

(c) $\sup_j\int v^2|G_j(v)|^2dv<\infty$, *where* $G_j=\mathscr{F}(\sqrt{F_j})$.

Proof. First we shall prove sufficiency. Clearly the equivalence of measures does not change the set T_μ, namely if $\mu\simeq M$ holds, then $T_\mu=T_M$. So we have only to check that M itself is the l^2-quasi-invariant measure. Now using (c) in place of (7) in Theorem 3.1, we reach to the desired conclusion in a similar way with it. The necessity of the proof is derived from the following two lemmas. From now on we shall use a symbol $*$ for convolution operation. Let μ be a measure as in (Π), and $a=(a_1, a_2,\dots)\in\mathbf{R}^\infty$. We put

$$\sqrt{h_j(u)}=\begin{cases}1/n(a_j)\{\sqrt{f_j(u)}*1/\sqrt{2\pi}|a_j|\exp(-u^2/2a_j^2)\}, & \text{if } a_j\neq0\\ \sqrt{f_j(u)}, & \text{if } a_j=0,\end{cases}$$

where $n(a_j)$ is the normalizing constant such that

$$\int h_j(u)du=1.$$

Using the above $\{h_j(u)\}$, we define a measure such that,

$$d\mu^a(x) = \otimes_{j=1}^{\infty} h_j(x_j)dx_j\,.$$

Then,

Lemma 3.1. *If* μ *is* l^2*-quasi-invariant and* $a=(a_1, a_2,\dots)\in l^2$, *then* $\mu \simeq \mu^a$.

Proof. First we shall put $w_j(u) = n(a_j)\sqrt{h_j}(u)$. Then,

$$\begin{aligned}\|w_j - \sqrt{f_j}\|_{L^2}^2 &\leqq 1/\sqrt{2\pi}\int|\sqrt{f_j(u-a_js)} - \sqrt{f_j(u)}|^2 \exp(-s^2/2)duds \\ &= 2/\sqrt{2\pi}\int(1-\exp(2\pi i a_j sv))\,|g_j(v)|^2 \exp(-s^2/2)dvds \\ &= 2\int(1-\exp(-2\pi^2 a_j^2 v^2))\,|g_j(v)|^2 dv\,.\end{aligned}$$

So $\sum_{j=1}^{\infty}\|w_j - \sqrt{f_j}\|_{L^2}^2 < \infty$, in virtue of Lemma 2.1. Especially

$$\sum_{j=1}^{\infty}(1-n(a_j))^2 = \sum_{j=1}^{\infty}(1-\|w_j\|_{L^2})^2 < \infty\,.$$

Therefore,

$$\begin{aligned}\sum_{j=1}^{\infty}\|\sqrt{h_j} - \sqrt{f_j}\|_{L^2}^2 &\leqq \sum_{j=1}^{\infty}(\|\sqrt{h_j} - w_j\|_{L^2} + \|w_j - \sqrt{f_j}\|_{L^2})^2 \\ &\leqq 2\sum_{j=1}^{\infty}\{(1-n(a_j))^2 + \|w_j - \sqrt{f_j}\|_{L^2}^2\} < \infty\,.\end{aligned}$$

It follows that μ and μ^a are equivalent with each other. Q. E. D.

We note that

$$\mathscr{F}(\sqrt{h_j})(v) = 1/n(a_j)\; g_j(v)\exp(-2\pi^2 a_j^2 v^2)\,.$$

Lemma 3.2. *Let* m *be a measure on* $\mathfrak{B}(\mathbf{R}^\infty)$ *such that* $m = \otimes_{j=1}^{\infty} m_j$, *where each* m_j *is the probability measure on* $\mathfrak{B}(\mathbf{R}^1)$. *Suppose that* $m(H_a)=1$ *for all* $a\in l^2$. *Then there exists some* $\delta=\{\delta_j\}\in l^2$ *such that*

$$\sup_j \int u^2 \exp(-\delta_j^2 u^2)dm_j(u) < \infty\,.$$

Proof. As $\sum_{j=1}^{\infty}\int(1-\exp(-a_j^2u^2))dm_j(u) < \infty$, so we can put

$$W(a)=\sum_{j=1}^{\infty}\int(1-\exp(-a_j^2u^2))dm_j(u), \qquad \text{for all} \quad a=(a_1, a_2,\dots)\in l^2\,.$$

First, we shall claim that W is continuous and is bounded on the unit sphere of l^2. For, let n_{∞} be the canonical Gaussian measure on $\mathfrak{B}(\mathbf{R}^{\infty})$. Namely, $dn_{\infty}(x)=\otimes_{j=1}^{\infty}1/\sqrt{2\pi}\exp(-x_j^2/2)dx_j$, and we put $dn(u)=1/\sqrt{2\pi}\exp(-u^2/2)du$ as for the measure on $\mathfrak{B}(\mathbf{R}^1)$. Then,

$$W(a)=\sum_{j=1}^{\infty}\int(1-\exp(\sqrt{2}\,ia_jus))dm_j(u)dn(s)\,.$$

Now, from m_j and n, we define a new probability measure λ_j on $\mathfrak{B}(\mathbf{R}^1)$ such that,

$$\lambda_j(A)=n\otimes m_j\{(s, u)\in\mathbf{R}^2|su\in A\}, \qquad \text{for all} \quad A\in\mathfrak{B}(\mathbf{R}^1)\,.$$

And we define a measure λ on $\mathfrak{B}(\mathbf{R}^{\infty})$ such that, $\lambda=\otimes_{j=1}^{\infty}\lambda_j$. Then λ is the symmetric measure and the above equality can be written as

$$W(a)=\sum_{j=1}^{\infty}\int(1-\exp(\sqrt{2}\,i\,a_jx_j))d\lambda(x)\,.$$

From this it follows that λ is the type l^2, and therefore $W(a)$ is the continuous function by Lemma 2.4. Consequently, for any given $\varepsilon>0$, there exists $0<\delta<1$ such that, $W(a)<\varepsilon$, for all $\|a\|\leqq\delta$. So $W(\delta a)<\varepsilon$ for all $\|a\|\leqq 1$. Applying Lemma 2.3 for W, we conclude that, $W(a)\leqq\varepsilon/\delta^2$, for all $\|a\|\leqq 1$. From now on we put $R=\varepsilon/\delta^2$. Let $K>2$ and we put for each j

$$t_j=\inf\left\{t>0|\int_{(-t,t)}u^2dm_j(u)>KR\right\} \quad \text{and} \quad s_j=1/t_j\,.$$

If the above set is empty, we put $t_j=\infty$ and $s_j=0$. Then if $s_j\neq 0$, we have

$$(8) \qquad \int_{[-t_j,t_j]}u^2dm_j(u)\geqq KR \quad \text{and} \quad \int_{(-t_j,t_j)}u^2dm_j(u)\leqq KR\,.$$

Secondly, we shall claim that $\sum_{j=1}^{\infty}s_j^2\leqq 1$. For, suppose the contrary case, namely there exists some n such that $\sum_{j=1}^{n}s_j^2>1$. Without loss of generality, we can assume that $s_j\neq 0$ for $1\leqq j\leqq n$. In the definition of W, we put $a_j=s_j(s_1^2+\cdots+s_n^2)^{-1/2}$ for $1\leqq j\leqq n$ and $a_j=0$ for $j\geqq n+1$. Then it yields

$$\sum_{j=1}^{n}\int(1-\exp(-s_j^2u^2(s_1^2+\cdots+s_n^2)^{-1}))dm_j(u)\leqq R.$$

Using Lemma 2.3, it follows that

$$\sum_{j=1}^{n}\int(1-\exp(-s_j^2u^2))dm_j(u)\leqq R(s_1^2+\cdots+s_n^2).$$

Therefore,

$$\sum_{j=1}^{n} s_j^2/2\int_{[-t_j,t_j]}u^2dm_j\,(u)\leqq\sum_{j=1}^{n}\int_{[-t_j,t_j]}(1-\exp(-s_j^2u^2))dm_j(u)$$
$$\leqq R(s_1^2+\cdots+s_n^2).$$

From (8), it follows that $1/2\ KR\leqq R$, which contradicts the choice of K. As $\{s_j\}\in l^2$, so we get

$$\sum_{j=1}^{\infty}\int(1-\exp(-s_j^2u^2))dm_j(u)<\infty,$$

and therefore

$$\sum_{j=1}^{\infty}\int_{|u|\geqq t_j}dm_j(u)<\infty.$$

Lastly, we put for each j $\delta_j=\left\{\int_{|u|\geqq t_j}dm_j(u)\right\}^{1/2}$. Then $\{\delta_j\}\in l^2$ and,

$$\int u^2\exp(-\delta_j^2u^2)dm_j(u)=\int_{(-t_j,t_j)}u^2\exp(-\delta_j^2u^2)dm_j(u)$$
$$+\int_{|u|\geqq t_j}u^2\exp(-\delta_j^2u^2)dm_j(u)\leqq KR+1/\delta_j^2\int_{|u|\geqq t_j}dm_j(u)=KR+1.$$

Q.E.D.

Proof of the necessity of Theorem 3.2. Let μ be an l^2-quasi-invariant measure and ν be the adjoint measure of it. Then in virtue of Lemma 2.1, $\nu(H_a)=1$ holds for all $a\in l^2$. So we can take a sequence $\delta=\{\delta_j\}$ such that,

$$\sup_j\int v^2\exp(-\delta_j^2v^2)\,|g_j(v)|^2dv<\infty.$$

In the notation of Lemma 3.1, putting $a_j=\delta_j/\sqrt{2}\,\pi$ and constructing a measure $d\mu_1^\delta(x)=\otimes_{j=1}^\infty h_j(x_j)dx_j$, then $\mu_1^\delta\cong\mu$ holds and

$$\sup_j \int v^2|\mathscr{F}(\sqrt{h_j})(v)|^2dv=\sup_j 1/n(a_j)^2\int v^2\exp(-\delta_j^2v^2)\,|g_j(v)|^2dv<\infty\,,$$

as

$$\lim_j n(a_j)^2=\lim_j n(\delta_j/\sqrt{2}\,\pi)^2=1\,.$$

Therefore μ_1^δ is the desired one M. Q.E.D.

The case $T_\mu=l^2$ is settled in the following theorem.

Theorem 3.3. *Let μ be a measure as in (Π), and we put $\mathscr{F}(\sqrt{f_j})=g_j$. Then in order that μ is the strictly-l^2-quasi-invariant measure, it is necessary and sufficient that*

(a) *T_μ is a vector space*

(b) *there exists some measure M on $\mathfrak{B}(\mathbf{R}^\infty)$ which has the three properties in Theorem* 3.2.

(c) $\inf_j\int(1-\exp(-v^2))\,|g_j(v)|^2dv>0$.

Proof. First, we shall consider the necessity. Then (a) is trivial, and (b) is the consequence of Theorem 3.2. While (c) is shown in a quite similar way with Proposition 2.3. Conversely, suppose that the three conditions are satisfied. Then $T_\mu\supseteqq l^2$ holds by Theorem 3.2. Now if there exists $a=(a_1, a_2,\dots)\in T_\mu\backslash l^2$, then from (a),

$$\sum_{j=1}^\infty\int(1-\exp(-a_j^2v^2))\,|g_j(v)|^2dv<\infty\,.$$

It follows that,

$$\inf_j\int(1-\exp(-v^2))\,|g_j(v)|^2dv\sum_{j=1}^\infty\min(a_j^2,\,1)<\infty\,,$$

and therefore $\sum_{j=1}^\infty a_j^2<\infty$. But it contradicts the choice of a. Q.E.D.

Remark. Even if (a) and (b) are satisfied for μ, the condition $\inf_j\int v^2|g_j(v)|^2dv>0$ is necessary but not sufficient for $T_\mu=l^2$. We have a counter example for it.

Lastly, we shall discuss the relation of the support to the quasi-invariance.

Lemma 3.3. *For* $1 \leqq p < \infty$, *let* μ *be a measure as in* (Π), *and be the* l^p*-quasi-invariant measure. If* $\mu(H_a)=1$ *for some* $a=(a_1, a_2, \dots) \in \mathbf{R}^\infty$, *then we conclude that* $a \in l^q$. (q *is the conjugate exponent of* p).

Proof. Let $\bar{\mu}$ be a measure on $\mathfrak{B}(\mathbf{R}^\infty)$ such that,

$$\bar{\mu}(A)=\mu(-A) \qquad \text{for all} \quad A \in \mathfrak{B}(\mathbf{R}^\infty).$$

The convolution of μ with $\bar{\mu}$ defines a new measure μ^S, namely

$$\mu^S(A)=\int \mu(A-x)d\bar{\mu}(x), \qquad \text{for all} \quad A \in \mathfrak{B}(\mathbf{R}^\infty).$$

We can easily check that μ^S is also the product-measure and symmetric one. Moreover, $\mu^S(H_a)=1$ and $T_{\mu^S} \supset T_\mu \supset l^p$ hold. For,

$$\mu^S(H_a)=\int_{\mathbf{R}^\infty} \mu(H_a-x)d\bar{\mu}(x)=\int_{H_a} \mu(H_a-x)d\bar{\mu}(x)=\mu(H_a)=1.$$

And if $t=(t_1, t_2, \dots) \in T_\mu$ and $\mu^S(A)=0$ for some $A \in \mathfrak{B}(\mathbf{R}^\infty)$, then it follows that $\mu(A-x)=0$ (hence, $\mu(A-x-t)=0$) holds for $\bar{\mu}$-a.e. x. Therefore,

$$\mu^S(A-t)=\int \mu(A-t-x)d\bar{\mu}(x)=0.$$

As the converse assertion holds in a similar way, we conclude that

$$T_{\mu^S} \supset T_\mu.$$

Now, $\mu^S(H_a)=1$ is equivalent that $\sum_{j=1}^\infty a_j^2 x_j^2 < \infty$ for μ^S-a.e. x, which yields that $\sum_{j=1}^\infty a_j x_j$ converges for μ^S-a.e. x due to the Kolmogorov-Khintchine's theorem and the symmetry of μ^S. Therefore if μ is l^p-quasi-invariant, then we conclude that for all $\{h_j\} \in l^p$, both $\sum_{j=1}^\infty a_j x_j$ and $\sum_{j=1}^\infty a_j(x_j+h_j)$ converges for μ^S-a.e. x. It follows that, for any $\{h_j\} \in l^p$, $\sum_{j=1}^\infty a_j h_j$ converges, which shows $\{a_j\} \in l^q$. Q.E.D.

According to the above lemma, we are specially interested in a following measure μ.

(*) μ is as in (Π), and l^2-quasi-invariant

(**) for any $a=(a_1, a_2,\dots)\in l^2$, $\mu(H_a)=1$.

For example, a measure which can be regarded as a continuous cylindrical measure on l^2 satisfies (**) due to Minlos.

Theorem 3.4. *Let μ be a measure as in (Π). Then in order that μ has the properties (*) and (**), it is necessary and sufficient that there exists a measure M on $\mathfrak{B}(\mathbf{R}^\infty)$, $dM(x)=\otimes_{j=1}^\infty F_j(x_j)dx_j$, $\mathscr{F}(\sqrt{F_j})=G_j$ such that,*

(a) *M has the three properties in Theorem 3.2*

(b) $\sup_j \int u^2 F_j(u)du<\infty$.

(*We can characterize it in terms of the uniform boundness of $\left\|\frac{d}{du}\sqrt{F_j(u)}\right\|_{L^2}$ and $\|u\sqrt{F_j(u)}\|_{L^2}$.*)

Further, $T_\mu=l^2$ holds under the condition (*) and (**).

Proof. First we shall prove that $T_\mu=l^2$. The proof $T_\mu\subseteqq l^2$ is derived as below from the consideration in Lemma 3.3. Using the same notation in it, we can assure that for all $a=(a_1, a_2,\dots)\in l^2$, $\sum_{j=1}^\infty a_jx_j$ converges for μ^S-a.e. x. Therefore if $t=(t_1, t_2,\dots)\in T_\mu$, then both $\sum_{j=1}^\infty a_jx_j$ and $\sum_{j=1}^\infty a_j(x_j+t_j)$ converges for μ^S-a.e. x. Hence $\sum_{j=1}^\infty a_jt_j$ converges for any $a\in l^2$, which shows that $t\in l^2$. Combining it with the assumption $T_\mu\supseteqq l^2$, we conclude that $T_\mu=l^2$. Secondly, we shall prove sufficiency of the former part of this assertion. For (*), it is a consequence of Theorem 3.2. Since $M\cong\mu$, for (**) we have only to check that $M(H_a)=1$ for all $a\in l^2$. Now

$$\sum_{j=1}^\infty\int a_j^2x_j^2dM(x)=\sum_{j=1}^\infty a_j^2\int u^2F_j(u)\,du\leqq \sup_j\int u^2F_j(u)\,du\,\sum_{j=1}^\infty a_j^2<\infty\,.$$

It follows that $\sum_{j=1}^\infty a_j^2x_j^2<\infty$ for M-a.e. x, which shows $M(H_a)=1$. Lastly, we shall prove the necessity. Using Lemma 3.2 for μ, it asserts that there exists a sequence $\sigma=\{\sigma_n\}\in l^2$ such that,

$$\sup_j\int u^2\exp(-\sigma_j^2u^2)f_j(u)du<\infty\,. \tag{9}$$

Putting $k_j(u)=1/n(\sigma_j)\exp(-\sigma_j^2u^2)f_j(u)$, where $n(\sigma_j)$ is the normalizing constant such that,

$$\int k_j(u)du = 1 \quad \text{(and becomes } \lim_j n(\sigma_j) = 1\text{)},$$

we define a new measure μ_1 on $\mathfrak{B}(\mathbf{R}^\infty)$ such that,

$$d\mu_1(x) = \otimes_{j=1}^{\infty} k_j(x_j)dx_j.$$

As $\mu(H_\sigma)=1$ derives that $\mu_1 \cong \mu$, so μ_1 is also l^2-quasi-invariant. Using Theorem 3.2 for μ_1, it follows that there exists a sequence $\delta = \{\delta_j\} \in l^2$ such that,

$$\sup_j \int v^2 |G_j(v)|^2 dv < \infty, \quad \text{where} \quad G_j = \mathscr{F}(\sqrt{F_j}),$$

$$\sqrt{F_j(u)} = 1/n(\delta_j)\{\sqrt{k_j(u)} * 1/\sqrt{2\pi}\delta_j \exp(-u^2/2\delta_j^2)\}, \text{ and}$$

$n(\delta_j)$ is the normalizing constant such that $\int F_j(u)du = 1$ (and becomes $\lim_j n(\delta_j) = 1$).
Now the measure $dM(x) = \otimes_{j=1}^{\infty} F_j(x_j)dx_j$ is the desired one. Because we have only to check that

$$\sup_j \int u^2 F_j(u)du < \infty . \tag{10}$$

Some calculations derive that

$$\int u^2 F_j(u)du \leqq 1/n(\delta_j)^2 \left\{\int u^2 k_j(u)du + \delta_j^2\right\}.$$

Combining it with (9), we can assure (10). Q.E.D.

As for the stationary case,

Theorem 3.5. *Let μ be a stationary measure with f, and we put $\mathscr{F}(\sqrt{f}) = g$. Then μ is l^2-quasi-invariant and $\mu(H_a)=1$ for all $a \in l^2$, if and only if*

$$\int u^2 f(u)du < \infty \quad \textit{and} \quad \int v^2 |g(v)|^2 dv < \infty .$$

Proof. First we shall consider the necessity. The last inequality has already proven, so we have only to check the first one. Applying Lemma 3.2, there exist $\{\delta_j\} \in l^2$ and a positive constant R such that,

$$\int u^2 \exp(-\delta_j^2 u^2) f(u) du \leqq R, \qquad \text{for all} \quad j.$$

Letting j tend to infinity and using Lebesgue-Fatou's lemma, we get

$$\int u^2 f(u) du \leqq R.$$

The proof of the sufficiency is carried out in a quite similar way with it in Theorem 3.4. Q.E.D.

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