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On Invariants $G(\sigma)$ and $\Gamma(\sigma)$ for an Automorphism Group of a von Neumann Algebra

By

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Abstract

An invariant Γ for an automorphism group of a factor given by Connes is generalized to a general von Neumann algebra and the relation between Γ and a characterization of an inner automorphism group of a von Neumann algebra due to Borchers are discussed.

§ 1. Introduction

Let G be a locally compact abelian group, dt a Haar measure on G, $\widehat{G}$ the dual of G and $\langle t, \gamma\rangle$ the value of $\gamma\in\widehat{G}$ at $t\in G$. For $g\in L^1(G)$ and $\gamma\in\widehat{G}$

$$\hat{g}(\gamma)\equiv\int_G g(t)\overline{\langle t, \gamma\rangle}dt$$

and $\Gamma(g)\equiv\{\gamma\in\widehat{G}: \hat{g}(\gamma)=0\}$.

Let M be a von Neumann algebra, M_* the predual of M and Aut M the group of automorphisms of M. A homomorphism σ of G into Aut M satisfying that the functions $t\in G\mapsto\phi(\sigma_t(x))$ are continuous for all $x\in M$ and $\phi\in M_*$ is called a representation of G on M. Let $\mathrm{Rep}(G, M)$ denote the set of all representations of G on M. For a finite measure μ on G (resp. $g\in L^1(G)$), $\sigma\in\mathrm{Rep}(G, M)$ and $x\in M$ let

$$\sigma(\mu)x\equiv\int_G\sigma_t(x)\mu(dt)\quad\left(\text{resp. }\sigma(g)x\equiv\int_G g(t)\sigma_t(x)dt\right).$$

Let $\mathrm{sp}\,\sigma$ denote the intersection of $\Gamma(g)$ with $\sigma(g)=0$ and $\mathrm{sp}_\sigma(x)$ the intersection of $\Gamma(g)$ with $\sigma(g)x=0$. For a closed subset E of $\widehat{G}$, $M^\sigma(E)$

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denotes the set of all $x \in M$ with $\mathrm{sp}_\sigma(x) \subset E$. Let $M^\sigma \equiv M^\sigma(\{0\})$, $Z(M) \equiv M \cap M'$ and $Z(M^\sigma) \equiv M^\sigma \cap (M^\sigma)'$. For projections e and f in M^σ, $\bar{e}$ denotes the carrier in $Z(M)$ of e and ${}^e\sigma^f$ the restriction defined by

$$ {}^e\sigma_t^f(x) \equiv \sigma_t(x), \quad x \in eMf, $$

in particular, $\sigma^e \equiv {}^e\sigma^e$ or σ^e is the restriction of σ to M_e. Furthermore, $\mathrm{sp}\,{}^e\sigma^f$ denotes the intersection of $\Gamma(g)$ with ${}^e\sigma^f(g) = 0$, where ${}^e\sigma^f(g)$ is defined similarly as above.

Definition 1.1. $G(\sigma)$ (resp. $K(\sigma)$) denotes the set of all $t \in G$ such that σ_t is implemented by a unitary in M^σ (resp. M).

Then $G(\sigma)$ and $K(\sigma)$ are subgroups of G and $G(\sigma) \subset K(\sigma)$. The following definition is essentially due to Connes, [6].

Definition 1.2. $\Gamma_0(\sigma)$ (resp. $\Gamma_1(\sigma)$) denotes the intersection of all $\mathrm{sp}\,\sigma^e$, $e \in M^\sigma$ with $e \neq 0$ (resp. $\bar{e} = 1$).

Then $\Gamma_0(\sigma)$ is a closed subgroup of $\widehat{G}$ and $\Gamma_0(\sigma) \subset \Gamma_1(\sigma)$. There is no difference between $\Gamma_0(\sigma)$ and $\Gamma_1(\sigma)$ if M is a factor.

The main purpose of this paper is to show the relations among the following four conditions for a closed subgroup Ξ of $\widehat{G}$:

(A) for any non zero projection f in $Z(M^\sigma)$ and for any neighbourhood V of 0 there exists a non zero projection e in $Z(M^\sigma)$ such that $e \leqq f$ and $\Xi \subset \mathrm{sp}\,\sigma^e \subset \Xi + V$;

(B) $G(\sigma^e) = \Xi^\perp$ for all non zero e in $Z(M^\sigma)$;

(B′) $G(\sigma^e) = \Xi^\perp$ and $\Xi \subset \mathrm{sp}\,\sigma^e$ for all non zero e in $Z(M^\sigma)$;

(C) $\Gamma_0(\sigma) = \Gamma_1(\sigma) = \Xi$.

Using these conditions, we can state our main theorem.

Theorem 1.1. (i) *The condition* (A) *implies the condition* (B′), *and the condition* (B′) *implies the condition* (C).

(ii) *If* $\widehat{G}/\Xi$ *is compact, then conditions* (A), (B′) *and* (C) *are equivalent.*

(iii) *If* G *satisfies the second axiom of countability,* $\Xi = \{0\}$ *and*

M_* is separable, then conditions (A) *and* (B) *are equivalent. In this case, σ is inner.*

The implications (A)$\Rightarrow$(B) and (B$'$)$\Rightarrow$(C) are proved in Section 2 by similar techniques as Borchers, [4]. The implication (C)$\Rightarrow$(A) for a discrete $\Xi^{\perp}$ is proved in Section 3 by similar ideas as Connes, [6]. The statement (iii) implies the following corollary.

Corollary 1.1. *If G satisfies the second axiom of countability, $G(\sigma)=G$ and M_* is separable, then σ is inner.*

The ergodicity of σ implies the equivalence of conditions (A), (B) and (C), whenever $\widehat{G}/\Xi$ is compact (Remark 3.2).

Let $W(M)$ be the set of semi-finite, faithful and normal weights on M_+. For $\phi\in W(M)$, Δ_ϕ and σ^ϕ denote the modular operator and modular automorphism of ϕ, respectively.

Definition 1.3. $\mathrm{S}(M)$ denotes the intersection of all spectrum of Δ_ϕ, $\phi\in W(M)$.

Theorem 1.2. *If $\Gamma_0(\sigma^\phi)=\Gamma_1(\sigma^\phi)\neq\{0\}$, then* $\log(\mathrm{S}(M_e)\setminus\{0\})$ $=\Gamma_0(\sigma^\phi)$ *for any non zero e in $Z(M)$.*

Finally, in Section 5 we shall give a characterization of an unbounded derivation which corresponds to a representation of $\boldsymbol{R}$ on M.

§ 2. Proof of (i) in Theorem 1.1

The condition (A) implies the existence of a projection e_0 in M^σ (or $Z(M^\sigma)$) such that $\Xi\subset\mathrm{sp}\ \sigma^{e_0}\subset\Xi+V$ and $\bar{e}_0=1$. For this, let $\mathscr{F}$ be the family of sets of non zero projections e in M^σ (or $Z(M^\sigma)$) such that $\Xi\subset\mathrm{sp}\ \sigma^e\subset\Xi+V$ and their central carriers in M are mutually orthogonal. Then $\mathscr{F}$ is a non empty ordered set by set inclusion. Here we apply Zorn's lemma to $\mathscr{F}$ and obtain a maximal set $\{e_\alpha\}\in\mathscr{F}$. We complete the proof by defining e_0 by $\sum e_\alpha$. Therefore the condition (A) implies that $\Xi\subset\Gamma_0(\sigma)\subset\Gamma_1(\sigma)\subset\bigcap\{\Xi+V$: *as above*$\}=\Xi$, which implies the condition (C).

It is known that $K(\sigma^e)=K(\sigma^{\bar{e}})$ for $e\in M^\sigma$, [4, Lemma 5.7; 6, Lemma 1.5.2].

Lemma 2.1. *$G(\sigma^e)=G(\sigma^{\bar{e}})$ for $e\in M^\sigma$.*

Proof. Since $G(\sigma^{\bar{e}})\subset G(\sigma^e)$, it suffices to show the converse inclusion for a non zero e in M^σ. Suppose that $t\in G(\sigma^e)$. Then we have a unitary v on $e\mathcal{H}$ such that $v\in M_e{}^\sigma$ and $\sigma_t{}^e(x)=vxv^*$ for $x\in M_e$. Define an operator u on $\bar{e}\mathcal{H}$ by $uy\xi\equiv\sigma_t{}^{\bar{e}}(y)v\xi$ for $y\in M_{\bar{e}}$ and $\xi\in e\mathcal{H}$. Since

$$(uy\xi|uz\eta)=(\sigma_t{}^e(z^*y)v\xi|v\eta)=(y\xi|z\eta)$$

whenever ξ, $\eta\in e\mathcal{H}$, u is a unitary in $M_{\bar{e}}$ such that $u_e=v$ and $\sigma_t{}^{\bar{e}}(y)=uyu^*$ for $y\in M_{\bar{e}}$. Since $v\in M^\sigma\cap M_e$, we have $e\sigma_s{}^{\bar{e}}(u)=\sigma_s{}^{\bar{e}}(u)e$ and $(\sigma_s{}^{\bar{e}}(u))_e=v$ for all $s\in G$. For $s\in G$, $y\in M_{\bar{e}}$ and $\xi\in e\mathcal{H}$, we have

$$\begin{aligned}yv\xi&=y\sigma_s{}^{\bar{e}}(u)\xi=\sigma_s{}^{\bar{e}}(\sigma^{\bar{e}}_{-s}(y)u)\xi\\&=\sigma_s{}^{\bar{e}}(u\sigma^{\bar{e}}_{-t-s}(y))\xi=\sigma_s{}^{\bar{e}}(u)\sigma^{\bar{e}}_{-t}(y)\xi=\sigma_s{}^{\bar{e}}(u)u^*yv\xi\,.\end{aligned}$$

Therefore $\sigma_s{}^{\bar{e}}(u)=u$ for all $s\in G$ and hence $u\in M^\sigma\cap M_{\bar{e}}$. Consequently we have $G(\sigma^e)\subset G(\sigma^{\bar{e}})$. Q.E.D.

This lemma implies the equivalence between the conditions (B) and

(B_1) $G(\sigma^f)=\Xi^\perp$ for all non zero f in $Z(M)\cap Z(M^\sigma)$.

Let τ be a representation of $\boldsymbol{Z}$ defined by $\tau_n=\sigma_{nt}$ for some fixed $t\in G$ in this section. $\langle t, \mathrm{sp}_\sigma(x)\rangle$ denotes the set of all $\langle t,\gamma\rangle$ with $\gamma\in\mathrm{sp}_\sigma(x)$. In the following lemmas we shall identify the dual of $\boldsymbol{Z}$ with the unit circle $\boldsymbol{T}$.

Lemma 2.2. *$\langle t, \mathrm{sp}_\sigma(x)\rangle^-=\mathrm{sp}_\tau(x)$ for all $x\in M$.*

Proof. Suppose that $\gamma\in\mathrm{sp}_\sigma(x)$. Let $g\in l^1(\boldsymbol{Z})$ with $\tau(g)x=0$. By setting $\mu\equiv\sum_{n\in\boldsymbol{Z}}g(n)\delta_{nt}$, we have $\sigma(\mu)x=\tau(g)x=0$ and hence

$$\hat{g}(\langle t,\gamma\rangle)=\hat{\mu}(\gamma)=0\,.$$

Therefore we have $\langle t,\gamma\rangle\in\mathrm{sp}_\tau(x)$.

Choose any $g\in l^1(\boldsymbol{Z})$ so that $\hat{g}$ vanishes on a neighbourhood V of $\langle t, \mathrm{sp}_\sigma(x)\rangle^-$. Setting $\mu\equiv\sum_n g(n)\delta_{nt}$, we have

$$\hat{\mu}(\gamma)=\hat{g}(\langle t,\gamma\rangle)=0$$

on the neighbourhood $\{\gamma\in\widehat{G}:\langle t,\gamma\rangle\in V\}$ of $\mathrm{sp}_\sigma(x)$, and so $\tau(g)x=\sigma(\mu)x=0$. Consequently, we have

$$\mathrm{sp}_\tau(x)\subset\langle t,\mathrm{sp}_\sigma(x)\rangle^- . \qquad \text{Q.E.D.}$$

We shall also identify the dual of Z with $(-\pi,\pi]$ and denote $[-\delta,\delta]$ by I_δ. For a projection $e\in Z(M^\tau)$ and a closed subset E of $(-\pi,\pi]$, let $\phi(E,e)$ denote the projection onto the closed subspace spanned by $M^\tau(E)e\mathcal{H}$. Since Lemma 2.2 implies

$$M^\sigma(\{\gamma\in\widehat{G}\colon\langle t,\gamma\rangle\in E\})=M^\tau(E),$$

we have $\phi(E,e)\in Z(M^\sigma)$ for $e\in Z(M^\sigma)$.

The following Lemmas 2.3 and 2.4 have been obtained by Borchers, [4], while we shall give their proofs for completeness.

Assume that $\langle t,\mathrm{sp}\sigma\rangle^-\subset(-2\pi/3,2\pi/3)$ in Lemmas 2.3, 2.4 and 2.5.

Lemma 2.3. *There exists a family* $\{p(\delta)\colon\delta\in(0,2\pi/3)\}$ *of projections in* $Z(M^\sigma)$ *such that* $p(\delta)$ *is increasing in* δ, $\mathrm{sp}\,\tau^{p(\delta)}\subset I_\delta$ *and* $\overline{p(\delta)}=1$.

Proof. We shall define p_n by induction. Put $\delta_n\equiv2^{-n+2}\pi/3$ and $p_1\equiv1$. Then $\mathrm{sp}\,\tau^{p_1}\subset(-2\pi/3,2\pi/3)$ by assumption. Assume that $p_j\in Z(M^\sigma)$, $p_{j-1}\geq p_j$, $\mathrm{sp}\,\tau^{p_j}\subset I_{\delta_j}$ and $\bar{p}_j=1$ for $j=2,3,\cdots,n$. Put

$$\delta\equiv2^{-1}\inf\{\varepsilon>0\colon\mathrm{sp}\ \tau^{p_n}\subset I_\varepsilon\}\ \ (<\pi/3),$$

$$p\equiv p_n\phi([\delta,2\delta+\varepsilon],p_n)\in Z(M^\sigma),$$

$$p_{n+1}\equiv p+p_n(1-\bar{p})\in Z(M^\sigma)$$

for some $\varepsilon\in(0,2\pi-6\delta)$. Then $\delta\leq\delta_{n+1}, p_{n+1}\leq p_n$ and $\overline{p_{n+1}}=1$. Since $p_n(1-\bar{p})\phi([\delta,2\delta+\varepsilon],p_n)=0$, we have

$$\mathrm{sp}^{p_n(1-\bar{p})}\tau^{p_n}\cap(\delta,2\delta+\varepsilon)=\emptyset .$$

Since $\mathrm{sp}\,\tau^q\subset\mathrm{sp}^q\tau^{p_n}$ and $\mathrm{sp}\,\tau^q=-\mathrm{sp}\ \tau^q$ for $q\equiv p_n(1-\bar{p})$,

$$\mathrm{sp}\,\tau^{p_n(1-\bar{p})}\subset\mathrm{sp}\,\tau^{p_n}\backslash\{(-2\delta-\varepsilon,-\delta)\cup(\delta,2\delta+\varepsilon)\}\subset I_\delta .$$

Since $\mathrm{sp}^f\tau^{\phi(E,e)}\subset\mathrm{sp}^f\tau^e-E$ for $e,f\in Z(M^\tau)$ in general,

$$\mathrm{sp}\ \tau^{p} \subset \mathrm{sp}\ {}^{p_n}\tau^{\phi([\delta, 2\delta+\varepsilon], p_n)} \cap \mathrm{sp}\ {}^{\phi([\delta, 2\delta+\varepsilon], p_n)}\tau^{p_n}$$

$$\subset (I_{2\delta} - [\delta, 2\delta+\varepsilon]) \cap (I_{2\delta} + [\delta, 2\delta+\varepsilon]) \subset I_\delta .$$

Consequently, we have

$$\mathrm{sp}\ \tau^{p_{n+1}} = \mathrm{sp}\ \tau^{p} \cup \mathrm{sp}\ \tau^{p_n(1-\bar{p})} \subset I_\delta \subset I_{\delta_{n+1}} .$$

Putting $p(\delta) \equiv p_{n+1}$ for $\delta \in [\delta_{n+1}, \delta_n)$, $n \in \boldsymbol{N}$, we have a family $\{p(\delta): \delta \in (0, 2\pi/3)\}$ with the desired property. Q.E.D.

Lemma 2.4. *For any projection e in $Z(M^\sigma)$, put*

$$S(e) \equiv \cap \{\mathrm{sp}^{e}\tau^{p(\delta)} : \delta > 0\}.$$

Then for any e_1, e_2, e_α, e, f in $Z(M^\sigma)$ and any closed subset E of $(-\pi, \pi]$, it holds that

(a) $S(e_1) \subset S(e_2)$ *if* $e_1 \leq e_2$;

(b) $S(p(\delta)) \subset I_\delta$;

(c) $S(\phi(E, e)) \subset S(e) + E$;

(d) $S(e) = \emptyset$ *if and only if* $e = 0$;

(e) $(\cup S(e_\alpha))^- = S(\sup e_\alpha)$;

(f) $\mathrm{sp}\ {}^{e}\tau^{f} \subset S(e) - S(f)$; *and*

(g) $eM^\tau(E)f \subset M^\tau(\{S(e) - S(f)\} \cap E)$.

Proof. (a) and (b) are obvious.

(c) We have

$$S(\phi(E, e)) \equiv \bigcap_{\delta>0} \mathrm{sp}\ {}^{\phi(E, e)}\tau^{p(\delta)} \subset \bigcap_{\delta>0} (\mathrm{sp}\ {}^{e}\tau^{p(\delta)} + E).$$

Since E is compact, it follows that

$$\bigcap_{\delta>0} (\mathrm{sp}\ {}^{e}\tau^{p(\delta)} + E) = \bigcap_{\delta>0} \mathrm{sp}\ {}^{e}\tau^{p(\delta)} + E \equiv S(e) + E .$$

(d) $e = 0$ clearly implies $S(e) = \emptyset$. *By compactness*, $S(e) = \emptyset$ implies $\mathrm{sp}^{e}\tau^{p(\delta)} = \emptyset$ for some $\delta > 0$. Therefore $eMp(\delta) = \{0\}$. Since $\overline{p(\delta)} = 1$, $e = 0$.

(e) For any $\delta > 0$ and $\varepsilon \in (0, \delta)$, since $\mathrm{sp}\ \tau^{p(\delta)} \subset I_\delta$, $p(\delta)\phi(I_\delta, p(\varepsilon))$ is the carrier in $Z(M_{p(\delta)})$ of $p(\varepsilon)$ and hence $p(\delta) \leq \phi(I_\delta, p(\varepsilon))$. Therefore

$$\mathrm{sp}\ {}^{e}\tau^{p(\delta)} \subset \bigcap_{\varepsilon>0} \mathrm{sp}\ {}^{e}\tau^{\phi(I_\delta, p(\varepsilon))} \subset S(e) + I_\delta . \tag{2·1}$$

Therefore

$$S(\sup e_\alpha) = \bigcap_{\delta>0} (\bigcup_\alpha \operatorname{sp} {}^{e_\alpha}\tau^{p(\delta)})^- \subset \bigcap_{\delta>0} (\bigcup_\alpha (S(e_\alpha) + I_\delta))^-$$

$$\subset \bigcap_{\delta>0} (\bigcup_\alpha S(e_\alpha) + I_\delta)^- = \bigcap_{\delta>0} \{(\bigcup_\alpha S(e_\alpha))^- + I_\delta\} = (\bigcup_\alpha S(e_\alpha))^- .$$

The converse inclusion is clear from (a).

(f) and (g) From (2·1) it follows that $eMp(\delta) \subset M^\tau(S(e) + I_\delta)$ and $p(\delta)Mf \subset M^\tau(-S(f) + I_\delta)$. Therefore

$$eMp(\delta)Mf \subset M^\tau(S(e) + I_\delta)M^\tau(-S(f) + I_\delta)$$

$$\subset M^\tau(S(e) - S(f) + I_{2\delta}).$$

Since $\overline{p(\delta)} = 1$, $Mp(\delta)M$ is weakly total in M. Therefore

$$eMf \subset M^\tau(S(e) - S(f) + I_{2\delta}),$$

and hence

$$\operatorname{sp} {}^e\tau^f \subset S(e) - S(f) + I_{2\delta} .$$

By the arbitrariness of $\delta > 0$, we have (f) and

$$eM^\tau(E)f \subset M^\tau(S(e) - S(f)) \cap M^\tau(E)$$

$$= M^\tau(\{S(e) - S(f)\} \cap E),$$

which is (g). Q.E.D.

Lemma 2.5. *Let*

$$e(\lambda) \equiv \sup\{e \in Z(M^\sigma) : S(e) \subset (-\pi, \lambda]\}.$$

Then $\{e(\lambda) : \lambda \in (-\pi, \pi]\}$ *is a spectral resolution of the identity which satisfies*

(h) $S(e(\lambda, \mu]) \subset [\lambda, \mu]$.

Proof. It is clear that $e(\lambda)$ is increasing in λ. Since $S(e(\lambda)) \subset (-\pi, \lambda]$ by (e), we have

$$S(\lim_{\mu \downarrow \lambda} e(\mu)) \subset \bigcap_{\mu > \lambda} S(e(\mu)) \subset \bigcap_{\mu > \lambda} (-\pi, \mu] = (-\pi, \lambda],$$

and hence $\lim_{\mu \downarrow \lambda} e(\mu) \leq e(\lambda)$. Therefore $e(\lambda)$ is right continuous in λ.

Since $\overline{\langle t, \operatorname{sp}\sigma\rangle} \subset (-\pi, \pi)$ by assumption, it follows from (d) that

$$\lim_{\lambda \downarrow -\pi} e(\lambda) = 0 \quad \text{and} \quad \lim_{\lambda \uparrow \pi} e(\lambda) = 1 .$$

(h) If $\alpha \in (-\pi, \lambda)$, there is a $\delta > 0$ with $\alpha + I_{2\delta} \in (-\pi, \lambda)$, and hence $S(\phi(\alpha + I_\delta, p(\delta))) \subset (-\pi, \lambda)$ by (c). It follows that $e((\lambda, \mu])\phi(\alpha + I_\delta, p(\delta)) = 0$ and hence $\alpha \notin \operatorname{sp}^{e(\lambda,\mu]}\tau^{p(\delta)}$. Therefore $\alpha \notin S(e(\lambda, \mu])$.

Q.E.D.

Proof of (A)$\Rightarrow$(B). Suppose that $t \in \Xi^\perp$. The condition (A) assures the existence of a projection $q \in Z(M^\sigma)$ with $\bar{q} = 1$ and $\langle t, \operatorname{sp}\sigma^q\rangle^- \subset (-2\pi/3, 2\pi/3)$. For the proof of $\Xi^\perp \subset G(\sigma)$ we may assume by Lemma 2.1 that $\langle t, \operatorname{sp}\sigma\rangle^- \subset (-2\pi/3, 2\pi/3)$.

Using a spectral resolution $\{e(\lambda) : \lambda \in (-\pi, \pi]\}$ obtained in Lemma 2.5, we define a unitary $u \in Z(M^\sigma)$ and a representation ρ of $\boldsymbol{Z}$ by

$$u \equiv \int_{-\pi}^{\pi} \exp(-i\lambda) e(d\lambda), \quad \rho_n \equiv (\operatorname{Ad} u)^n.$$

We shall show that $M^\tau(E) \subset M^\rho(E)$ for any closed E. Then, by [2], we have $\rho = \tau$, and so, $t \in G(\sigma)$.

Assume that $\operatorname{sp}_\tau(x) \subset E$ and $g \in l^1(\boldsymbol{Z})$ such that $\hat{g}$ vanishes on a neighbourhood of E. It follows from (g) in Lemma 2.4 and (h) in Lemma 2.5 that

$$\begin{aligned} \rho(g)x &= \sum_{n \in \boldsymbol{Z}} g(n) u^n x u^{*n} \\ &= \sum_n g(n) \iint \exp\{i(\mu - \lambda)n\} e(d\lambda) x e(d\mu) \\ &= \iint \hat{g}(\lambda - \mu) e(d\lambda) x e(d\mu) = 0 . \end{aligned}$$

Therefore $\operatorname{sp}_\rho(x) \subset E$.

Since $\Xi \subset \Gamma_1(\sigma)$ by the condition (A), the converse inclusion is clear from the following lemma, which is a partial generalization of [6, Theorem 2.3.1] for a factor.

According to [6, Lemma 2.3.8] we know that the spectrum $\operatorname{Sp}(\sigma_t)$ of σ_t on M as a Banach space is the closure $\langle t, \operatorname{sp}\sigma\rangle^-$ of $\{\langle t, \gamma\rangle : \gamma \in \operatorname{sp}\sigma\}$.

Lemma 2.6. $G(\sigma)\subset\Gamma_1(\sigma)^\perp$.

Proof. Suppose that $\sigma_t(x)=uxu^*$ for all $x\in M$ with $u\in M^\sigma$. Choose any $\varepsilon>0$. Let $\mathcal{F}$ be the family of sets of $(e_\alpha,\lambda_\alpha)$ of spectral projections e_α of u and complex numbers λ_α of modulus 1 such that

(a) $\|ue_\alpha-\lambda_\alpha e_\alpha\|<\varepsilon$; and

(b) $\bar{e}_\alpha$'s are mutually orthogonal.

Since $\mathcal{F}$ is ordered by set inclusion, we have a maximal set $F\in\mathcal{F}$ by Zorn's lemma, say $F=\{(e_\alpha,\lambda_\alpha):(\mathrm{a}),(\mathrm{b})\}$. By maximality, $\sum\bar{e}_\alpha=1$. Let $e\equiv\sum e_\alpha$ and $v\equiv\sum\lambda_\alpha{}^{-1}u\bar{e}_\alpha$. Then $e\in M^\sigma$, v is a unitary in M^σ and

$$\sigma_t(x)=uxu^*=vxv^*$$

for $x\in M$. Since $\|ve-e\|<\varepsilon$ and

$$\mathrm{Sp}(\sigma_t{}^e)\subset\{\lambda\mu^{-1}\colon\ \lambda,\mu\in\mathrm{Sp}(v_e)\},$$

$\mathrm{Sp}(\sigma_t{}^e)$ is included in $\{z\in\mathbf{C}\colon |z|=1,\ |z-1|<2\varepsilon\}$. If $\gamma\in\mathrm{sp}\,\sigma^e$, then $|\langle t,\gamma\rangle-1|<2\varepsilon$ by [6, Lemma 2.3.8]. Therefore $|\langle t,\gamma\rangle-1|<2\varepsilon$ for $\gamma\in\Gamma_1(\sigma)$. Since ε is arbitrary, $t\in\Gamma_1(\sigma)^\perp$. Q.E.D.

Remark 2.1. If $\sigma\in\mathrm{Aut}\,M$ satisfies $\|\sigma-1\|<3^{1/2}$ and if G is an abelian subgroup of Aut M containing σ, then there exists a unitary $u\in M$ such that $\sigma=\mathrm{Ad}\,u$ and $\rho(u)=u$ for all $\rho\in G$.

Remark 2.2. Let $\sigma\in\mathrm{Rep}(G,M)$. Under the condition (A), if G satisfies the first axiom of countability, we can define $S(e)$, $e\in Z(M^\sigma)$ as a subset of $\widehat{G}/G(\sigma)^\perp$ and then $e(\dot{\gamma})\in Z(M^\sigma)$, $\dot{\gamma}\in\widehat{G}/G(\sigma)^\perp$ as a spectral measure u_s:

$$u_s\equiv\int\overline{\langle s,\dot{\gamma}\rangle}e(d\dot{\gamma}),\quad \sigma_s=\mathrm{Ad}\,u_s$$

for all $s\in G(\sigma)$.

Remark 2.3. Let $\sigma\in\mathrm{Rep}(G,M)$. If G is discrete, then $G(\sigma)=\Gamma_1(\sigma)^\perp$.

Proof of the implication (B$'$)$\Rightarrow$(C). From the condition (B) and

Lemma 2.6, we have $\Gamma_1(\sigma)\subset\Xi$. From the remaining condition of (B′), we have $\Xi\subset\Gamma_0(\sigma)$. Therefore $\Gamma_0(\sigma)=\Gamma_1(\sigma)=\Xi$. Q.E.D.

§ 3. Proofs of (ii) and (iii) in Theorem 1.1

In the following we denote the carrier projection of x by $\mathrm{s}(x)$ and the carrier of $\hat{g}$ for $g\in L^1(G)$ by car $\hat{g}$.

Lemma 3.1. *For any compact neighbourhood U of* 0 *in $\widehat{G}$ and for any projections e_1 and e_2 in M^σ* (resp. $Z(M^\sigma)$) *with $\bar{e}_1=\bar{e}_2=1$ there exist projections f_1 and f_2 in M^σ* (resp. $Z(M^\sigma)$) *such that $\bar{f}_1=\bar{f}_2=1$, $f_1\leq e_1$, $f_2\leq e_2$, $\mathrm{sp}\,\sigma^{f_1}\subset U+\mathrm{sp}\,\sigma^{f_2}$ and $\mathrm{sp}\,\sigma^{f_2}\subset U+\mathrm{sp}\,\sigma^{f_1}$.*

Proof. Since $\bar{e}_1=\bar{e}_2=1$, there exists a non zero $x_0\in M$ *such that* $x_0=e_1x_0e_2$. There exists a $g_0\in L^1(G)$ with $\mathrm{car}\,\hat{g}_0-\mathrm{car}\,\hat{g}_0\subset U$ and $\sigma(g_0)x_0\neq 0$. Put $y_0\equiv\sigma(g_0)x_0$. Then $e_1y_0e_2=y_0$ and $\mathrm{sp}_\sigma(y_0)-\mathrm{sp}_\sigma(y_0)\subset U$. Let $f_1^o\equiv\sup\{\mathrm{s}(\sigma_t(y_0^*)):t\in G\}$ and $f_2^o\equiv\sup\{\mathrm{s}(\sigma_t(y_0)):t\in G\}$. Then we have projections f_j^o in M^σ such that $0<f_j^o\leq e_j$ for $j=1,2$.

Let $\mathcal{F}$ be the family of sets of $(x_\alpha, g_\alpha)\in M\times L^1(G)$ such that

(a) $x_\alpha=e_1x_\alpha e_2\neq 0$;

(b) $\mathrm{car}\,\hat{g}_\alpha-\mathrm{car}\hat{g}_\alpha\subset U$; and

(c) projections $\bar{f}_1^\alpha$ are mutually orthogonal, where $f_1^\alpha\equiv\sup\ \{\mathrm{s}(\sigma_t(y_\alpha^*)):\ t\in G\}$ and $y_\alpha\equiv\sigma(g_\alpha)x_\alpha$.

Since $\mathcal{F}$ is ordered by set inclusion, we have a maximal set $F\in\mathcal{F}$ by Zorn's lemma, say $F=\{(x_\alpha, g_\alpha)\in M\times L^1(G)$: (a), (b), (c)$\}$. By maximality, $\sum\bar{f}_1^\alpha=1$. Let $f_2^\alpha\equiv\sup\{\mathrm{s}(\sigma_t(y_\alpha)):\ t\in \mathrm{G}\}$ and $f_j\equiv\sum f_j^\alpha$ for $j=1,2$. Since $\mathrm{s}(\sigma_t(y_\alpha^*))\sim\mathrm{s}(\sigma_t(y_\alpha))$ in M for each $t\in G$, we have $\bar{f}_1^\alpha=\bar{f}_2^\alpha$ and $\bar{f}_1=\bar{f}_2=1$.

Suppose that $\gamma\in\mathrm{sp}\,\sigma^{f_1}$. For any compact neighbourhood V of γ there exists a non zero x in $M^\sigma(V)$ with $x=f_1^\alpha xf_1^\alpha$ for some α. Since $x=f_1^\alpha xf_1^\alpha$, it follows that $\sigma_{t_2}(y_\alpha^*)x\sigma_{t_1}(y_\alpha)\neq 0$ for some t_1 and t_2 in G. Put $y\equiv\sigma_{t_2}(y_\alpha^*)x\sigma_{t_1}(y_\alpha)$. Since $\mathrm{sp}_\sigma(y)\subset V-U$ and $y=f_2yf_2$, we have $M^\sigma(V-U)\cap M_{f_2}\neq\{0\}$. Since $V\cap(U+\mathrm{sp}\,\sigma^{f_2})\neq\emptyset$ and $U+\mathrm{sp}\,\sigma^{f_2}$ is closed, $\gamma\in U+\mathrm{sp}\,\sigma^{f_2}$.

The remaining inclusion is proved similarly as above. Q.E.D.

Lemma 3.2. *$\Gamma_1(\sigma^e)=\Gamma_1(\sigma^{\bar{e}})$ for $e\in M^\sigma$.*

Lemma 3.3. *Let $\mathcal{F}(\sigma)$ be the set of all $\mathrm{sp}\,\sigma^e+V$ for e in M^σ (or $Z(M^\sigma)$) with $\bar{e}=1$ and compact neighbourhoods V of 0 in $\widehat{G}$. Then $\mathcal{F}(\sigma)$ is a filter base and $\Gamma_1(\sigma)=\cap\{F: F\in\mathcal{F}(\sigma)\}$.*

These two lemmas are proved by combining Lemma 3.1 and similar arguments as the proofs of [6, Lemmas 2.3.3 and 2.3.4].

We are now ready to give a sufficient condition for a problem of Borchers which is proposed in the final remark in [4].

Proof of the implication (C)$\Rightarrow$(A) *in* (ii). Since $\Gamma_0(\sigma)=\Xi$, it follows that $\Xi\subset\mathrm{sp}\,\sigma^e$ for all non zero e in M^σ (or $Z(M^\sigma)$).

Suppose that f is a non zero projection in M^σ (or $Z(M^\sigma)$). For any ε in $(0,1)$ and $t_j\in\Xi^\perp$ for $j=1,2,\cdots,n$, V denotes the set of $\gamma\in\widehat{G}$ such that $1-\varepsilon<\mathrm{Re}\langle t_j,\gamma\rangle$ for all $j=1,2,\cdots,n$. Let ϕ be the quotient mapping of $\widehat{G}$ onto $\widehat{G}/\Xi$.

Since $\Gamma_0(\sigma)\subset\Gamma_0(\sigma^f)\subset\Gamma_1(\sigma^f)=\Gamma_1(\sigma^f)\subset\Gamma_1(\sigma)$, we have $\Gamma_0(\sigma^f)=\Gamma_1(\sigma^f)=\Xi$. By restricting our argument to M_f, we may assume that $f=1$ for the moment. Since $\mathcal{F}(\sigma)$ in Lemma 3.3 is a filter base and $\widehat{G}/\Xi$ is compact, $\{\phi(F): F\in\mathcal{F}(\sigma)\}$ is also a filter base of compact sets. Since $t_j\in\Xi^\perp$ for $j=1,2,\cdots,n$, we have $V+\Xi=V$ and hence $\phi^{-1}(\phi(V))=V$. Since $\mathrm{sp}\,\sigma+\Gamma_0(\sigma)=\mathrm{sp}\,\sigma$, we have $\phi^{-1}(\phi(F))=F$ for all $F\in\mathcal{F}(\sigma)$. Hence Lemma 3.3 implies that the intersection of all $\phi(F)$, $F\in\mathcal{F}(\sigma)$ is zero. Since $\widehat{G}/\Xi$ is compact, $\phi(F)$ converges to 0 and hence there exists an $F\in\mathcal{F}(\sigma)$ such that $\phi(F)\subset\phi(V)$ or $F\subset V$.

Consequently, $\mathrm{sp}\,\sigma^e\subset V$ for some non zero e in M^σ (or $Z(M^\sigma)$) with $e\leq f$. Q.E.D.

The case $\Xi=\{0\}$ is a special case where $\Xi^\perp$ is not discrete.

Making a slight modification of [13, Theorem 5.2], we have the following lemma.

Lemma 3.4. *If G satisfies the second axiom of countability, a Borel multiplier $\alpha\in Z^2(G,\mathbf{T})$ with $\alpha(s,t)=\alpha(t,s)$ for $s,t\in G$ is triv-*

ial, namely, $\alpha \in B^2(G, \boldsymbol{T})$.

Proof. Let $G^\alpha \equiv G \times \boldsymbol{T}$ be the extension of G by α, that is, the product is defined by

$$(3\cdot 1) \qquad (t_1, \lambda_1)(t_2, \lambda_2) \equiv (t_1 + t_2, \alpha(t_1, t_2)\lambda_1\lambda_2)$$

for $(t_j, \lambda_j) \in G^\alpha$. G^α is given the product Borel structure of $G \times \boldsymbol{T}$. Since G satisfies the second axiom of countability and $\alpha(s, t) = \alpha(t, s)$ for s, $t \in G$, G^α is a locally compact abelian group with respect to the Weil topology. Let j be an injection of $\boldsymbol{T}$ to G^α such that $j(\lambda) = (0, \lambda)$ for $\lambda \in \boldsymbol{T}$. Since $j(\boldsymbol{T})$ is a topological subgroup of G^α which is standard and j is a Borel measure isomorphism, j is a homeomorphism by a Mackey's theorem [13, Theorem 2.2]. Let $\widehat{G}^\alpha$ and $\widehat{j(\boldsymbol{T})}$ denote the duals of G^α and $j(\boldsymbol{T})$, respectively. Let l be a mapping of $j(\boldsymbol{T})$ to $\boldsymbol{T}$ such that l: $(0, \lambda) \in j(\boldsymbol{T}) \to \lambda \in \boldsymbol{T}$. Then $l \in \widehat{j(\boldsymbol{T})}$. Since $\widehat{G}^\alpha / j(\boldsymbol{T})^\perp$ is isomorphic to $\widehat{j(\boldsymbol{T})}$, we have the corresponding $\chi^* \in \widehat{G}^\alpha / j(\boldsymbol{T})^\perp$ to $l \in \widehat{j(\boldsymbol{T})}$. If $\chi \in \chi^*$, then $\chi \in \widehat{G}^\alpha$ and $\chi = l$ on $j(\boldsymbol{T})$. Put $\beta(t) \equiv \chi((t, 1))$. From $(3\cdot 1)$ we have $\beta(t_1)\beta(t_2) = \beta(t_1 + t_2)\alpha(t_1, t_2)$. Q.E.D.

Remark 3.1. This lemma is partly generalized as the following. Every symmetric (i.e., $\alpha(s, t) = \alpha(t, s)$) multiplier is trivial for an abelian discrete group. For this we have only to assume the product topology on $G^\alpha = G \times \boldsymbol{T}$ in the above proof.

Proof of (iii) *in Theorem* 1.1. We have only to prove that the condition (B) implies the condition (A). Since $0 \in \mathrm{sp}_{\sigma^e}(1_e)$ for any non zero e in M^σ (or $Z(M^\sigma)$), we have $\Xi = \{0\} \subset \mathrm{sp}\, \sigma^e$.

Suppose that V is a neighbourhood of Ξ. Since $0 \in \Xi$, we may choose an open neighbourhood U of 0 with $(U - U)^- \subset V$. Since $G(\sigma) = G$ and M_* is separable, it follows from Lemma 3.4 and [8, 11, 12] that there exists a strongly continuous unitary representation u of G in M^σ such that $\sigma_t(x) = u_t x u_t^*$ for $x \in M$ and $t \in G$. By virtue of Stone's theorem, we have a spectral resolution

$$u_t = \int_\Gamma \overline{\langle t, \gamma \rangle} e(d\gamma),$$

where $e(d\gamma)$ is a spectral projection measure on $\widehat{G}$. Utilizing a $\gamma_0\in\widehat{G}$ with $e(U+\gamma_0)f\neq 0$, we define a projection e by $e(U+\gamma_0)f$. Then $e\in M^\sigma$ (or $Z(M^\sigma)$) and $0<e\leq f$. For all $g\in L^1(G)$ with car $\hat{g}\subset\widehat{G}\backslash(U-U)$ we have

$$e(\sigma(g)x)e=\int_G g(t)\,eu_txu_t{}^*e\,dt$$
$$=\int_U\int_U \hat{g}(\gamma-\gamma')e(d\gamma)xe(d\gamma')=0$$

for all $x\in M$. Therefore $\mathrm{sp}\,\sigma^e\subset(U-U)^-\subset V$. Q.E.D.

Proof of Corollary 1.1. It is immediate from (iii) of Theorem 1.1.

The following proposition generalizes [6, Theorem 2.4.1].

Proposition 3.1. (i) *If* $Z(M^\sigma)\subset Z(M)$, *then* $\mathrm{sp}\,\sigma=\Gamma_1(\sigma)$. *In particular, if* M^σ *is a factor,* $\Gamma_0(\sigma)=\Gamma_1(\sigma)$.

(ii) *If* $\Gamma_0(\sigma)=\Gamma_1(\sigma)=\Xi$ *and* Ξ *is discrete, then* $\mathrm{sp}\,\sigma=\Xi$ *is necessary and sufficient for* $Z(M^\sigma)\subset Z(M)$.

Proof. (i) Let $e\in M^\sigma$ and f the carrier in $Z(M^\sigma)$. If $\gamma\in\mathrm{sp}\,\sigma^f$, there exists an $x\in M$ such that $\mathrm{sp}_\sigma(x)\cap(V+\gamma)\neq\emptyset$ for any neighbourhood V of 0. Since f is the carrier in $Z(M^\sigma)$ of e, there exist y and z in M^σ with $ezxye\neq 0$. Since $\mathrm{sp}_\sigma(ezxye)=\mathrm{sp}_\sigma(x)$, $\gamma\in\mathrm{sp}\,\sigma^e$ and hence $\mathrm{sp}\,\sigma^f\subset\mathrm{sp}\,\sigma^e$, which implies $\mathrm{sp}\,\sigma^e=\mathrm{sp}\,\sigma^f$. Consequently, $\Gamma_1(\sigma)=\bigcap\{\mathrm{sp}\,\sigma^f: f\in Z(M^\sigma)$ and $\bar{f}=1\}$. Since $Z(M^\sigma)\subset Z(M)$, we have $\mathrm{sp}\,\sigma=\Gamma_1(\sigma)$.

(ii) By (i) we have only to show the sufficiency. First we shall show that if e_1 and e_2 in M^σ have mutually orthogonal central carriers, then $\bar{e}_1\bar{e}_2=0$. Suppose that $e_1\in M^\sigma$, $e_2\in M^\sigma$ and $\bar{e}_1\bar{e}_2\neq 0$. Then there exists a non zero $x\in M$ such that $e_1xe_2=x$ and $\mathrm{sp}_\sigma(x)=\{\gamma\}$ for some $\gamma\in\Xi$. Put $e_3\equiv s(x)$. Since $\Gamma_1(\sigma^{e_3})=\Xi$ by Lemma 3.2 and since $\Gamma_1(\sigma^{e_3})=\mathrm{sp}\,\sigma^{e_3}$ from the assumption that $\Xi=\mathrm{sp}\,\sigma$, it follows that there exists a non zero $y\in M_{e_3}$ with $\mathrm{sp}_\sigma(y)=\{-\gamma\}$. Then $e_1xye_2=xy\neq 0$ and $xy\in M^\sigma$. Thus the product of central carriers of e_1 and e_2 is non zero.

Now, suppose $e\in Z(M^\sigma)$. Since $e(1-e)=0$, we have $\bar{e}\,\overline{(1-e)}=0$ from the above. Since $1=e+(1-e)\leq\bar{e}+\overline{(1-e)}\leq 1$, we have $e=\bar{e}$, namely, $e\in Z(M)$. Q.E.D.

Remark 3.2. Assume that $\widehat{G}/\Xi$ is compact. If M^{σ} is a factor, then the conditions (A), (B) and (C) are equivalent. For this we have only to show that (B) implies (C). Since $G(\sigma)$ is discrete by assumption, there exists by Lemma 4.4 a $\sigma'\in\mathrm{Rep}(G, M)$ such that $\sigma'\sim\sigma$, $Z(M^{\sigma'})=Z(M^{\sigma})$ and $G(\sigma)=(\mathrm{sp}\,\sigma')^{\perp}$. If M^{σ} is a factor, then $M^{\sigma'}$ is a factor and hence $\Gamma_0(\sigma')=\Gamma_1(\sigma')=\mathrm{sp}\,\sigma'$ by Proposition 3.1. Therefore $\mathrm{sp}\,\sigma'$ is a group and hence $\Xi=G(\sigma)^{\perp}=\mathrm{sp}\,\sigma'=\Gamma_1(\sigma')$ by (B). Since $\Gamma_0(\sigma)=\Gamma_1(\sigma)$ by Proposition 3.1 and $\Gamma_1(\sigma)=\Gamma_1(\sigma')$ by Lemma 4.3, we have $\Gamma_0(\sigma)=\Gamma_1(\sigma)=\Xi$, which is (C).

§ 4. Proof of (B)$\Rightarrow$(C) and S-Set

In the following $I_{\varepsilon}(\lambda)$ denotes the ε-neighbourhood of $\lambda\in\boldsymbol{R}$, namely, the open interval $(\lambda-\varepsilon, \lambda+\varepsilon)$.

Proposition 4.1. *For a $\sigma\in\mathrm{Rep}(\boldsymbol{R}, M)$ there exist projections q_0, q_{∞} in $Z(M^{\sigma})\cap Z(M)$ and an increasing left continuous spectral resolution $\{p(\lambda)\in Z(M^{\sigma})\cap Z(M): \lambda>0\}$ of $q_{\infty}-q_0$ such that*

(i) $\Gamma_0(\sigma^{q_0})=\Gamma_1(\sigma^{q_0})=\{0\}$;

(ii) $\Gamma_0(\sigma^{1-q_{\infty}})=\Gamma_1(\sigma^{1-q_{\infty}})=\boldsymbol{R}$;

(iii) *for any non zero $\lambda\in\mathrm{Sp}\,h$ ($h\equiv\int\lambda p(d\lambda)$), $\varepsilon_0\in(0, \lambda)$ and $\varepsilon\in(0, \varepsilon_0)$ there exists a non zero projection $e\in Z(M^{\sigma})$ majorized by $p(\lambda+\varepsilon_0)-p(\lambda-\varepsilon_0)$ satisfying that $\mathrm{sp}\,\sigma^{f}\subset I_{\varepsilon}(\lambda)\boldsymbol{Z}\cup I_{\varepsilon}(0)$ and $\mathrm{sp}\,\sigma^{f}\cap nI_{\varepsilon}(\lambda)\neq\emptyset$ for all $n\in\boldsymbol{N}$ and $f\in Z(M^{\sigma})$ with $0<f\leq e$;*

(iv) *if $\lambda\in\mathrm{Sp}\,h$ and $\lambda>0$, then $\lambda\boldsymbol{Z}\subset\Gamma_1(\sigma)$; and*

(v) *if $\lambda\in\mathrm{Sp}\,h$ is isolated and $q_{\lambda}\equiv p(\lambda+0)-p(\lambda)$, then $\Gamma_0(\sigma^{q_{\lambda}})=\Gamma_1(\sigma^{q_{\lambda}})=\lambda\boldsymbol{Z}$.*

Proof. (i) Let $\mathscr{F}_0$ be the set of all projections $p\in Z(M^{\sigma})\cap Z(M)$ such that for any $e\in Z(M^{\sigma})\cap Z(M)$ with $0<e\leq p$ and for any $\lambda>0$ and $\delta\in(0, \lambda/2)$ there exists a projection $f\in Z(M^{\sigma})$ with $0<f\leq e$ and $\mathrm{sp}\,\sigma^{f}\cap(\delta, \lambda-\delta)=\emptyset$. Put $q_0\equiv\sup\{p: p\in\mathscr{F}_0\}$. Using Zorn's lemma, we have a projection $e_0\in Z(M^{\sigma})$ such that $\bar{e}_0=q_0$ and $\mathrm{sp}\,\sigma^{e_0}\cap(\delta, \lambda-\delta)=\emptyset$.

Therefore

$$\Gamma_1(\sigma^{q_0}) \subset \bigcap_{\lambda>0,\, 0<\delta<\lambda/2} \bigcap \boldsymbol{R}\backslash\{(-\lambda+\delta, -\delta)\cup(\delta, \lambda-\delta)\} = \{0\},$$

and hence $\Gamma_0(\sigma^{q_0}) = \Gamma_1(\sigma^{q_0}) = \{0\}$.

(ii) Let $\mathcal{F}_\infty$ be the set of all projections $e \in Z(M^\sigma)$ such that $\mathrm{sp}\,\sigma^e \neq \boldsymbol{R}$. Put $q_\infty \equiv \sup\{\bar{e} : e \in \mathcal{F}_\infty\}$. Then $q_0 \leq q_\infty$ and $\Gamma_0(\sigma^{1-q_\infty}) = \Gamma_1(\sigma^{1-q_\infty}) = \boldsymbol{R}$. Moreover, $q_\infty \in Z(M^\sigma) \cap Z(M)$.

For the proof of the remaining part we must prepare the following two lemmas. Before going into the proof we recall that if $\mathrm{sp}\,\sigma \cap (\delta, \lambda-\delta) \neq \emptyset$ with $0<2\delta<\lambda$ then for any $\varepsilon>0$ there exist a $\lambda_0 \in (\delta, \lambda-\delta)$ and a non zero $x \in M$ such that

$$\mathrm{sp}_\sigma(x) \subset I_\varepsilon(\lambda_0) \cap (\delta, \lambda-\delta).$$

Lemma 4.1. *Assume that* $q_0 = 1-q_\infty = 0$. *For any* $\lambda>0$ *and* $\delta \in (0, \lambda/2)$ *let* $\mathcal{F}_{\lambda,\delta}$ *be the set of all projections e in* $Z(M^\sigma) \cap Z(M)$ *such that if f is a projection in* $Z(M^\sigma)$ *with* $0<f\leq e$ *then* $\mathrm{sp}\,\sigma^f \cap (\delta, \lambda-\delta) \neq \emptyset$. *If*

$$p(\lambda) \equiv \sup_{0<2\delta<\lambda} \sup\{e : e \in \mathcal{F}_{\lambda,\delta}\},$$

then $\{p(\lambda) : \lambda>0\}$ *is an increasing and left continuous spectral resolution of the identity.*

Proof. In the following we denote $\sup\{e : e \in \mathcal{F}_{\lambda,\delta}\}$ by $p(\lambda, \delta)$.

Since $p(\lambda, \delta) \leq p(\mu, \delta)$ for $0<\lambda\leq\mu$, it follows that $p(\lambda) \leq p(\mu)$ for $0<\lambda\leq\mu$. Therefore $p(\lambda)$ is increasing in $\lambda>0$.

Since $\mathcal{F}_{\lambda,\varepsilon+\delta} \subset \mathcal{F}_{\lambda-\varepsilon,\delta}$, it follows that

$$\begin{aligned}\lim_{\varepsilon\downarrow 0} p(\lambda-\varepsilon) &= \sup_{\varepsilon>0} p(\lambda-\varepsilon) = \sup_{\varepsilon>0}\sup_{\delta>0} p(\lambda-\varepsilon, \delta) \\ &\geq \sup_{\varepsilon>0}\sup_{\delta>0} p(\lambda, \varepsilon+\delta) = p(\lambda),\end{aligned}$$

and hence that $p(\lambda)$ is left continuous.

Putting $p_\infty \equiv 1-\lim_{\lambda\to\infty} p(\lambda)$, we have $p_\infty \in Z(M^\sigma) \cap Z(M)$ and

$$p_\infty = \inf_{\lambda>0}\inf_{\delta>0} (1-p(\lambda, \delta)).$$

Suppose that $p_\infty \neq 0$. If e is a projection in $Z(M^\sigma) \cap Z(M)$ with $0 < e \leq p_\infty$, then for any $\lambda > 0$ and $\delta \in (0, \lambda/2)$ there exists a projection $f \in Z(M^\sigma)$ with $0 < f \leq e$ and $\mathrm{sp}\, \sigma^f \cap (\delta, \lambda - \delta) = \emptyset$. Thus $p_\infty \leq q_0$. Since $q_0 = 0$ by assumption, it follows that $\lim_{\lambda\to\infty} p(\lambda) = 1$.

Putting $p_0 \equiv \lim_{\lambda\to 0} p(\lambda)$, we have $p_0 \in Z(M^\sigma) \cap Z(M)$ and

$$p_0 = \inf_{\lambda > 0} \sup \{p(\lambda, \delta) : 0 < 2\delta < \lambda\}.$$

Suppose that $p_0 \neq 0$. If e is a projection in $Z(M^\sigma)$ with $0 < e \leq p_0$, then for any $\lambda > 0$ there exist a $\delta \in (0, \lambda/2)$ and a projection $e_0 \in Z(M^\sigma)$, $0 < e_0 \leq ep(\lambda, \delta)$ such that $\mathrm{sp}\, \sigma^f \cap (\delta, \lambda - \delta) \neq \emptyset$ whenever $f \in Z(M^\sigma)$ and $0 < f \leq e_0$.

For any $\mu > 0$ and its ε-neighbourhood $I_\varepsilon(\mu) \subset \boldsymbol{R}_+$ we shall show $I_\varepsilon(\mu) \cap \mathrm{sp}\, \sigma^e$ is non empty. For a given ε we have a positive $\lambda < \varepsilon$, for which we get a δ and a projection e_0 as above. Choose an $n \in \boldsymbol{N}$ so that $\mu \leq n\delta$ and put $\eta \equiv \delta/n$. Since $\mathrm{sp}\, \sigma^{e_0} \cap (\delta, \lambda - \delta) \neq \emptyset$, we have a non zero $x_1 \in M_{e_0}$ and a $\lambda_1 \in (\delta, \lambda - \delta)$ satisfying

$$\mathrm{sp}_\sigma(x_1) \subset I_\eta(\lambda_1) \cap (\delta, \lambda - \delta).$$

Let f_1 be the carrier in $Z(M^\sigma)$ of

$$\sup\{\sigma_t(s(x_1)) : t \in \boldsymbol{R}\},$$

where $\mathrm{s}(x_1)$ denotes the carrier of x_1. Since $f_1 \in Z(M^\sigma)$ and $0 < f_1 \leq e_0$, we have $\mathrm{sp}\, \sigma^{f_1} \cap (\delta, \lambda - \delta) \neq \emptyset$. Therefore we have a non zero $x_2 \in M_{f_1}$ and a $\lambda_2 \in (\delta, \lambda - \delta)$ satisfying $\mathrm{sp}_\sigma(x_2) \subset I_\eta(\lambda_2) \cap (\delta, \lambda - \delta)$. Since $x_2 \in M_{f_1}$, we have a $v_1 \in M^\sigma$ and a $t_1 \in \boldsymbol{R}$ with

$$y_2 \equiv \sigma_{t_1}(x_1) v_1 x_2 \neq 0\,.$$

Let f_2 be the carrier in $Z(M^\sigma)$ of

$$\sup\{\sigma_t(\mathrm{s}(y_2)) : t \in \boldsymbol{R}\}.$$

Since $f_2 \in Z(M^\sigma)$ and $0 < f_2 \leq e_0$, we have $\mathrm{sp}\, \sigma^{f_2} \cap (\delta, \lambda - \delta) \neq \emptyset$. We repeat the similar argument as above and obtain sets $\{x_1, \cdots, x_n\} \subset M$ and $\{\lambda_1, \cdots, \lambda_n\} \subset (\delta, \lambda - \delta)$ satisfying

$$\mathrm{sp}_\sigma(x_j) \subset I_\eta(\lambda_j) \cap (\delta, \lambda - \delta)$$

for $j = 1, \cdots, n$. Since f_j is the carrier in $Z(M^\sigma)$ of

$$\sup\{\sigma_t(\mathrm{s}(y_j)) : t\in \boldsymbol{R}\}$$

and $x_{j+1}\in M_{f_j}$ for $j=2, \cdots, n$, we have sets $\{v_1, \cdots, v_{n-1}\}\subset M^\sigma$ and $\{t_1, \cdots, t_{n-1}\}\subset \boldsymbol{R}$ satisfying

$$y_k\equiv\sigma_{t_{k-1}}(\cdots\sigma_{t_2}(\sigma_{t_1}(x_1)v_1x_2)v_2x_3\cdots)v_{k-1}x_k\neq 0$$

for all $k=2, \cdots, n$. Since $\mathrm{sp}_\sigma(y_k)\subset\{\sum_{j=1}^k \mathrm{sp}_\sigma(x_j)\}^-$, we have

$$\mathrm{sp}_\sigma(y_k)\subset \sum_{j=1}^k I_\eta(\lambda_j)\subset I_{n\eta}(\sum_{j=1}^k \lambda_j).$$

Since $\mu\leq n\delta<\sum_{j=1}^n \lambda_j$, $\delta<\lambda_j<\lambda-\delta$ and $\lambda<\varepsilon$, there exists an $m\in \boldsymbol{N}$, $m<n$ with $\mu-\varepsilon<\sum_{j=1}^m \lambda_j<\mu$. Since $\delta=n\eta$, we have $I_\delta(\sum_{j=1}^{m+1}\lambda_j)\subset I_\varepsilon(\mu)$, and hence

$$y_{m+1}\in M_{e_0}\subset M_e \quad \text{and} \quad \mathrm{sp}_\sigma(y_{m+1})\subset I_\varepsilon(\mu).$$

Since ε can be arbitrarily small, it follows that $\mu\in\mathrm{sp}\,\sigma^e$. Since $\mu(>0)$ is arbitrary, $\mathrm{sp}\,\sigma^e=\boldsymbol{R}$. The arbitrariness of $e\in Z(M^\sigma)$ with $0<e\leq p_0$ implies that $\Gamma_0(\sigma^{p_0})=\boldsymbol{R}$. Since $q_\infty=1$ by assumption, we have a contradiction. Thus $p_0=0$, namely, $\lim_{\lambda\to 0}p(\lambda)=0$. Q.E.D.

The idea of the following lemma is essentially due to Borchers, [4].

Lemma 4.2. *For any* $\varepsilon\in(0, \lambda/2)$ *let* p *be a non zero projection in* $Z(M^\sigma)$ *satisfying* $\mathrm{sp}\,\sigma^e\cap I_\varepsilon(\lambda)\neq\emptyset$ *for all* $e\in Z(M^\sigma)$ *with* $0<e\leq p$. *For a non zero projection* q *in* $Z(M^\sigma)$ *if* $l(q)$ *is defined by*

$$\sup\{k(e) : e\in Z(M^\sigma),\ 0<e\leq q\},$$

where $k(e)$ *denotes the supremum length of subintervales of* $(0, \lambda+\varepsilon)\backslash\mathrm{sp}\,\sigma^e$, *then*

(i) $\mathrm{sp}\,\sigma^e\cap nI_\varepsilon(\lambda)\neq\emptyset$ *for all* $e\in Z(M^\sigma)$ *with* $0<e\leq p$ *and* $n\in\boldsymbol{Z}$; *and*

(ii) *for any* $\delta>0$ *there exists a projection* e *in* $Z(M^\sigma)$ *such that* $0<e\leq p$ *and* $\mathrm{sp}\,\sigma^e\subset I_\delta(l(p))\boldsymbol{Z}\cup I_\delta(0)$.

Proof. (i) We shall use an induction argument. Suppose that $\mathrm{sp}\,\sigma^e\cap nI_\varepsilon(\lambda)\neq\emptyset$ for some $n>0$. For a non zero x in $M^{\sigma^e}(nI_\varepsilon(\lambda))$ let $f\equiv\sup\{\sigma_t(\mathrm{s}(x)) : t\in G\}$. Then $0<f\leq e\leq p$. Since $\mathrm{sp}\,\sigma^f\cap I_\varepsilon(\lambda)\neq\emptyset$ by as-

sumption, we have a non zero y in $M^{\sigma^f}(I_\varepsilon(\lambda))$. Since $0<f\leq e$ and $\sigma_t^e(x)y\neq 0$ for some $t\in G$, we have $\mathrm{sp}\,\sigma^e\cap(n+1)I_\varepsilon(\lambda)\neq\emptyset$.

(ii) We have nothing to prove if $\delta>2^{-1}l(p)$. For any positive $\delta\leq 2^{-1}l(p)$ there exists a projection $e\in Z(M^\sigma)$ such that $0<e\leq p$ and $k(e)>l(p)-\delta$. Put $l\equiv l(p)$. We have then a subinterval $(2^{-1}\delta, l-2^{-1}\cdot\delta)+\lambda_0$ of $(0,\lambda+\varepsilon)\backslash\mathrm{sp}\,\sigma^e$ for some $\lambda_0\in\boldsymbol{R}$. We shall show by induction that $(n-1)l+n\delta,\ n(l-\delta))\cap\mathrm{sp}\,\sigma^e=\emptyset$ for all $n\in\boldsymbol{N}$ with $n<(2\delta)^{-1}l$. For $n=1$ we assume the contrary. Let $\phi(E,e)$ for $e\in M^\sigma$ denote the projection onto the subspace spanned by $M^\sigma(E)e\mathcal{H}$. For any $\lambda\in(\delta, l-\delta)\cap\mathrm{sp}\,\sigma^e$ we have

$$e'\equiv e\phi([-\delta',\delta']+\lambda, e)$$

for $\delta'\in(0,\min\{\lambda-\delta, l-\delta-\lambda\})$. Since $\lambda\in\mathrm{sp}\,\sigma^e$, we have $e'\neq 0$ and $\mathrm{sp}\,\sigma^{e'}\subset\mathrm{sp}\,\sigma^e\cap(\mathrm{sp}\,\sigma^e-I_{\delta'}(\lambda)^-)$, for $\mathrm{sp}^e\sigma^{\phi(E,e)}\subset\mathrm{sp}\,\sigma^e-E$ with $E\equiv I_{\delta'}(\lambda)^-$. Therefore $\mathrm{sp}\,\sigma^{e'}$ is disjoint from

$$(\lambda_0+2^{-1}\delta, \lambda_0+l-2^{-1}\delta)\cup(\lambda_0+2^{-1}\delta-\lambda+\delta', \lambda_0+l-2^{-1}\delta-\lambda-\delta')$$
$$=(\lambda_0-\lambda+2^{-1}\delta+\delta', \lambda_0+l-2^{-1}\delta).$$

Since $\lambda_0+l-2^{-1}\delta>0$ and $0\in\mathrm{sp}\,\sigma^{e'}$, $\lambda_0-\lambda+2^{-1}\delta+\delta'>0$. The length l' of the interval on the right hand side is $l-\delta+\lambda-\delta'$. Since $l<l'\leq l(e')$, we have a contradiction. Thus $\mathrm{sp}\,\sigma^e$ is disjoint from $(\delta, l-\delta)$. Suppose that the result is true for $n>1$ ($n<(2\delta)^{-1}l-1$). If $(nl+(n+1)\delta,\ (n+1)(l-\delta))\cap\mathrm{sp}\,\sigma^e\neq\emptyset$, then

$$f\equiv e\phi([-\delta'',\delta'']+\mu, e)$$

is non zero for any fixed

$$\mu\in(nl+(n+1)\delta,\ (n+1)(l-\delta))\cap\mathrm{sp}\,\sigma^e$$

$$\delta''\in(0,\min\{\mu-nl-(n+1)\delta,\ (n+1)(l-\delta)-\mu\}).$$

Since $\mu\in\mathrm{sp}\,\sigma^e$, we have $f\neq 0$ and $\mathrm{sp}\,\sigma^f\subset\mathrm{sp}\,\sigma^e\cap(\mathrm{sp}\,\sigma^e-I_{\delta''}(\mu)^-)$. Therefore $\mathrm{sp}\,\sigma^f$ is disjoint from

$$(-l+\delta,\ -\delta)\cup((n-1)l+n\delta-\mu+\delta'', n(l-\delta)-\mu-\delta'')$$
$$=((n-1)l+n\delta-\mu+\delta'',\ -\delta),$$

whose length is larger than l, for $\mathrm{sp}\,\sigma^f=-\mathrm{sp}\,\sigma^f$. This contradicts with the fact that $l(f)\leq l(p)\equiv l$. Thus the result is true for $n+1$.

Q.E.D.

Proof of Proposition 4.1. (*Continued*). By our previous proofs of (i) and (ii) we may assume that $q_0=1-q_\infty=0$ in the ramaining part of the proof.

(iii) Suppose that λ is a non zero element of Sp h. For any $\varepsilon\in(0, \varepsilon_0/3)$

$$p\equiv p(\lambda+\varepsilon)-p(\lambda-\varepsilon)>0.$$

Since $p(\lambda+\varepsilon)=\sup_{\delta>0}p(\lambda+\varepsilon, \delta)$, there exists a projection q in $Z(M^\sigma)\cap Z(M)$ such that $qp(\lambda-\varepsilon)=0$ and $0<q\leq p(\lambda+\varepsilon, \delta)$ for some $\delta\in(0, \varepsilon/2)$. Since $qp(\lambda-\varepsilon)=0$, there exists a projection $e'\in Z(M^\sigma)$ with $0<e'\leq q$ and $\mathrm{sp}\,\sigma^{e'}\cap(\delta, \lambda-\varepsilon-\delta)=\emptyset$. On the other hand, since $0<q\leq p(\lambda+\varepsilon, \delta)$, if $f\in Z(M^\sigma)$ and $0<f\leq q$ then $\mathrm{sp}\,\sigma^f\cap(\delta, \lambda+\varepsilon-\delta)\neq\emptyset$. Therefore, if $f\in Z(M^\sigma)$ and $0<f\leq e'$, then $\mathrm{sp}\,\sigma^f\cap I\neq\emptyset$ with $I\equiv(\lambda-2^{-1}3\varepsilon, \lambda+\varepsilon)$ and hence $\mathrm{sp}\,\sigma^f\cap nI\neq\emptyset$ by (i) in Lemma 4.2. Furthermore, we can define $l(e')$ by the same way as in Lemma 4.2. It follows from the above that $\lambda-2\varepsilon<l(e')<\lambda+\varepsilon$. By virtue of (ii) in Lemma 4.2 we have a projection $e\in Z(M^\sigma)$ with $0<e\leq e'$ and $\mathrm{sp}\,\sigma^e\subset I_\varepsilon(l(e'))\boldsymbol{Z}\cup I_\varepsilon(0)$. Then for any $f\in Z(M^\sigma)$ with $0<f\leq e$ we have

$$\mathrm{sp}\,\sigma^f\subset I_\varepsilon(l(e'))\boldsymbol{Z}\cup I_\varepsilon(0)\subset I_{3\varepsilon}(\lambda)\boldsymbol{Z}\cup I_{3\varepsilon}(0).$$

Considering 3ε as ε, we have (iii).

(iv) Suppose that $\lambda\in$ Sp h and $\lambda>0$. We shall use the same notation as in (iii). For any $\varepsilon\in(0, \varepsilon_0)$ let $\mathscr{F}$ be the set of all projections $e\in Z(M^\sigma)$ satisfying the same condition as in (iii). Put $e_\varepsilon\equiv\sup\{e: e\in\mathscr{F}\}$. For any $p\in Z(M^\sigma)$ with $\bar{p}=1$ we set $p_\varepsilon\equiv p\bar{e}_\varepsilon$. By means of Lemma 3.1 since $\bar{p}_\varepsilon=\bar{e}_\varepsilon$ there exist projections e_1 and e_2 in $Z(M^\sigma)$ such that $e_1\leq e_\varepsilon$, $e_2\leq p_\varepsilon$, $\bar{e}_1=\bar{e}_2=\bar{e}_\varepsilon$ and $\mathrm{sp}\,\sigma^{e_1}\subset\mathrm{sp}\,\sigma^{e_2}+I_\varepsilon(0)$. This inclusion relation and the condition in (iii) imply that $(\mathrm{sp}\,\sigma^{e_2}+I_\varepsilon(0))\cap nI_\varepsilon(\lambda)\neq\emptyset$ for all $n\in\boldsymbol{N}$. Since $e_2\leq p$, we have

$$(\mathrm{sp}\,\sigma^p+I_\varepsilon(0))\cap nI_\varepsilon(\lambda)\neq\emptyset$$

for all $n\in\boldsymbol{N}$. Since ε is arbitrary, $\lambda\boldsymbol{Z}\subset\mathrm{sp}\,\sigma^p$ and hence $\lambda\boldsymbol{Z}\subset\Gamma_1(\sigma)$.

(v) Since λ is isolated in $\mathrm{Sp}\, h$, $\mathrm{Sp}\, h \cap (\lambda - \varepsilon_0, \lambda + \varepsilon_0) = \{\lambda\}$ for some $\varepsilon_0 > 0$. For any $\varepsilon \in (0, \varepsilon_0)$ let $\mathscr{F}_\lambda$ be the family of sets of non zero projections e' in $Z(M^\sigma)$ with mutually orthogonal carriers in $Z(M)$ majorized by $q_\lambda \equiv p(\lambda + \varepsilon) - p(\lambda)$ satisfying that if $f \in Z(M^\sigma)$ and $0 < f \leq e'$ then $\mathrm{sp}\, \sigma^f \subset I_\varepsilon(\lambda) \boldsymbol{Z} \cup I_\varepsilon(0)$ and $\mathrm{sp}\, \sigma^f \cap n I_\varepsilon(\lambda) \neq \emptyset$ for all $n \in \boldsymbol{N}$. Since $\mathscr{F}_\lambda$ is ordered by set inclusion and inductive, we have a maximal $\{e_\iota : \iota \in I\} \in \mathscr{F}_\lambda$ by Zorn's lemma. Put $p_\lambda \equiv \sup\{\bar{e}_\iota : \iota \in I\}$. If $q_\lambda - p_\lambda \neq 0$, there exists by (iii) a non zero projection $e'' \in Z(M^\sigma)$ satisfying the same condition as $e_\iota \in \mathscr{F}_\lambda$ and $\bar{e}'' \bar{e}_\iota = 0$ for all $\iota \in I$, which contradicts with the maximality of $\mathscr{F}_\lambda$. Thus $q_\lambda = p_\lambda$. Putting $e \equiv \sup\{e_\iota : \iota \in I\}$, we have $\bar{e} = q_\lambda$ and $\mathrm{sp}\, \sigma^e \subset I_\varepsilon(\lambda) \boldsymbol{Z} \cup I_\varepsilon(0)$. Since ε is arbitrary, we have $\Gamma_1(\sigma^{q_\lambda}) \subset \lambda \boldsymbol{Z}$.

Suppose that $\Gamma_0(\sigma^{q_\lambda}) \neq \lambda \boldsymbol{Z}$. Since $\Gamma_0(\sigma^{q_\lambda})$ is a subgroup, there exists a projection $e_\lambda \in Z(M^\sigma)$ such that $0 < e_\lambda \leq q_\lambda$ and $\mathrm{sp}\, \sigma^{e_\lambda} \cap I_\delta(\lambda) = \emptyset$ for some $\delta > 0$. Here we may assume that the above ε is less than $\delta/2$. Since $\bar{e}_\lambda \leq \bar{e}$, it follows from Lemma 3.1 that there are projections e_1 and e_2 in $Z(M^\sigma)$ such that $e_1 \leq e_\lambda$, $0 < e_2 \leq e$ and $\mathrm{sp}\, \sigma^{e_2} \subset \mathrm{sp}\, \sigma^{e_1} + I_{\delta/2}(0)$. This inclusion relation contradicts with the fact that

$$I_{\delta/2}(\lambda) \cap \mathrm{sp}\, \sigma^{e_2} \neq \emptyset, \quad I_{\delta/2}(\lambda) \cap (\mathrm{sp}\, \sigma^{e_1} + I_{\delta/2}(0)) = \emptyset\,.$$

Thus $\Gamma_0(\sigma^{q_\lambda}) = \lambda \boldsymbol{Z}$ and hence (v) follows. Q.E.D

Remark 4.1. Let G be the additive group $\boldsymbol{R}$ and $\sigma \in \mathrm{Rep}(\boldsymbol{R}, M)$. If $\mathrm{sp}\, \sigma$ is compact, then $\Gamma_0(\sigma) = \Gamma_1(\sigma) = \{0\}$.

From the above proposition we have the following one.

Proposition 4.2. *The condition* (B) *implies the condition* (C), *if one of the following two assumptions is satisfied*:

(i) $G = G(\sigma)$; *and*

(ii) G *is the additive group* $\boldsymbol{R}$ *or* $\boldsymbol{Z}$ *with the usual topology,* $G(\sigma) \neq \{0\}$ *and* $\Gamma_0(\sigma) \neq \{0\}$.

For any σ and σ' in $\mathrm{Rep}(G, M)$, $\sigma \sim \sigma'$ if there exists a strongly continuous mapping u of G to the unitaries in M such that $u_{s+t} = u_s \sigma_s(u_t)$ and $\sigma_t'(x) = u_t \sigma_t(x) u_t^*$ for $s, t \in G$ and $x \in M$. This equivalence relation

"$\sim$" is called an "exterior equivalence" by Connes. The following lemma follows immediately from Lemma 3.1.

Lemma 4.3. *If* $\sigma\sim\sigma'$, *then* $\Gamma_1(\sigma)=\Gamma_1(\sigma')$.

The following lemma is used to relate the $\Gamma_0(\sigma)$ with the algebraic invariant $\mathrm{S}(M)$ which was defined in Section 1 for a general von Neumann algebra M.

Lemma 4.4. *Assume either that* $G(\sigma)$ *is discrete or that* $G(\sigma)$ *is closed and satisfies the second axiom of countability and* M_* *is separable. Then there exists a* $\sigma'\in\mathrm{Rep}(G, M)$ *such that* $\sigma'\sim\sigma$, $M^\sigma\subset M^{\sigma'}$, $Z(M^{\sigma'})\subset Z(M^\sigma)$ *and* $G(\sigma)=G(\sigma')=(\mathrm{sp}\,\sigma')^\perp$.

Proof. By similar discussions as in the proof of Lemma 3.4 and Remark 3.1, we have a strongly continuous unitary representation v of $G(\sigma)$ in $Z(M^\sigma)$ such that $\sigma_s(x)=v_sxv_s{}^*$ for $x\in M$ and $s\in G(\sigma)$. Since $G(\sigma)$ is a closed subgroup of G, it follows from [6, Lemma 3.3.12] that there exists a strongly continuous unitary representation u of G in $Z(M^\sigma)$ such that $u_s=v_s$ for $s\in G(\sigma)$. Define a $\sigma'\in\mathrm{Rep}(G, M)$ by $\sigma_t'(x)\equiv u_t{}^*\sigma_t(x)u_t$ for $t\in G$ and $x\in M$. Since $u_{t+s}=\sigma_t(u_s)u_t$, we have $\sigma'\sim\sigma$ and

$$G(\sigma)=\{t\in G:\sigma_t'=1\}\subset G(\sigma').$$

Since $u_t\in M^\sigma\subset M^{\sigma'}$, if $y\in Z(M^{\sigma'})$, then $\sigma_t(y)=u_t\sigma_t'(y)u_t{}^*=y$ and hence $y\in M^\sigma$. Since $M^\sigma\subset M^{\sigma'}$, $y\in(M^{\sigma'})'\subset(M^\sigma)'$ and hence $y\in Z(M^\sigma)$. Thus $Z(M^{\sigma'})\subset Z(M^\sigma)$. If $\sigma_s'(x)=wxw^*$ for $s\in G(\sigma')$ and a unitary $w\in Z(M^{\sigma'})$, then $u_sw\in Z(M^\sigma)$ and hence $G(\sigma')\subset G(\sigma)$. Thus $G(\sigma)=G(\sigma')$.

Since $G(\sigma')=(\mathrm{sp}\,\sigma')^\perp$ is clear, we complete the proof.

Proof of Proposition 4.2. (i) If $G(\sigma^e)=G$, then $\Gamma_0(\sigma)\subset\Gamma_1(\sigma)\subset\{0\}$ by Lemma 2.6, which implies the condition (C).

(ii) By (i) it suffices to consider the case $G(\sigma)\neq G$.

The case where $G(\sigma)=(\mathrm{sp}\,\sigma)^\perp$. Suppose that $G=\boldsymbol{R}$ (resp. $\boldsymbol{Z}$). Since $\Xi\equiv G(\sigma)^\perp$ is discrete and $G(\sigma)\neq G$ by assumption, there exists a generator $\gamma\in\boldsymbol{R}$ (resp. $[0, 2\pi)$) of Ξ. Here we apply Proposition 4.1 to

σ. Since $G(\sigma)\neq\{0\}$ by assumption, $q_\infty=1$. Since $\Gamma_0(\sigma)\neq\{0\}$ by assumption, we have $q_0=0$. Since Ξ is discrete, we have a partition $\{p_n: n\in\boldsymbol{N}\}$ (resp. $\{p_n: n|m\}$, $m\equiv 2\pi/\gamma$) in $Z(M^\sigma)\cap Z(M)$ of the identity such that

$$\Gamma_0(\sigma^{p_n})=\Gamma_1(\sigma^{p_n})=n\gamma\boldsymbol{Z} \ \ (\text{resp. } \{n\gamma: n|m\}),$$

which is the condition (C) for σ^{p_n} over M_{p_n}. Since $\widehat{G}/n\gamma\boldsymbol{Z}$ is compact, the conditions (B′) and (C) are equivalent over M_{p_n} by (ii) of Theorem 1.1 and hence $G(\sigma^e)^\perp=n\gamma\boldsymbol{Z}$ (resp. $\{n\gamma: n|m\}$) for all non zero $e\in Z(M^\sigma)$ with $e\leq p_n$. On the other hand, the condition (B) implies $G(\sigma^e)^\perp=\Xi$ for all $e\in Z(M^\sigma)$. Thus $p_n=1$ and $p_m=0$ $(m\neq n)$ for some $n\in\boldsymbol{N}$, namely, $\Xi=n\gamma\boldsymbol{Z}$. Since γ is a generator of Ξ, $n=1$. Thus $\Gamma_0(\sigma)=\Gamma_1(\sigma)=\Xi$.

The general case. For a given σ we choose a σ' as in Lemma 4.4. Then $G(\sigma)=G(\sigma')=(\operatorname{sp}\sigma')^\perp$ and hence $\Gamma_0(\sigma')=\Gamma_1(\sigma')=G(\sigma')^\perp$ from the above. Since $G(\sigma')=G(\sigma)=\Xi^\perp$ by the condition (B), we have $\Gamma_0(\sigma')=\Gamma_1(\sigma')=\Xi$. Therefore $\Gamma_1((\sigma')^e)=\Xi$ for all non zero $e\in Z(M^\sigma)$. Since $M^\sigma\subset M^{\sigma'}$ by Lemma 4.4, if e is a projection in $Z(M^\sigma)$, $\Gamma_1(\sigma^e)=\Gamma_1((\sigma')^e)$ by Lemma 4.3, and hence $\Gamma_0(\sigma)=\Gamma_1(\sigma)=\Xi$, which is the condition (C). Q.E.D.

Proof of Theorem 1.2. Since $\widehat{G}/\Gamma_0(\sigma^\phi)$ is compact and $\Gamma_0(\sigma^\phi)=\Gamma_1(\sigma^\phi)$ by assumption, the condition (B′) holds by (ii) of Theorem 1.1 and hence $G(\sigma)=\Gamma_0(\sigma^\phi)^\perp$ is discrete. Applying Lemma 4.4, we have a $\sigma\in\operatorname{Rep}(\boldsymbol{R}, M)$ such that $\sigma\sim\sigma^\phi$ and $\operatorname{sp}\sigma\subset G(\sigma^\phi)^\perp$. The condition (B′) implies that

$$G(\sigma^\phi)=G((\sigma^\phi)^e)=\Gamma_0(\sigma^\phi)^\perp \quad\text{and}\quad \Gamma_0(\sigma^\phi)\subset\operatorname{sp}(\sigma^\phi)^e$$

for all non zero e in $Z(M^{\sigma^\phi})$ and hence that

$$\operatorname{sp}\sigma\subset G(\sigma^\phi)^\perp=\Gamma_0(\sigma^\phi)\subset\operatorname{sp}(\sigma^\phi)^e.$$

Since $\sigma\sim\sigma^\phi$ implies $\sigma=\sigma^\psi$ for some $\psi\in W(M)$ by [6, Theorem 1.2.4], we have

$$(4\cdot1)\qquad \bigcap\{\operatorname{sp}\sigma^\psi: \psi\in W(M)\}\subset\Gamma_0(\sigma^\phi).$$

On the other hand, since $\sigma^\phi\sim\sigma^\psi$ for all ψ in $W(M)$, it follows from Lemma 4.3 that

$$\Gamma_0(\sigma^\phi) \subset \Gamma_1(\sigma^\phi) = \Gamma_1(\sigma^\psi) \subset \mathrm{sp}\, \sigma^\psi$$

and hence

$$\Gamma_0(\sigma^\phi) = \bigcap \{\mathrm{sp}\, \sigma^\psi : \psi \in W(M)\}. \tag{4·2}$$

Since $\mathrm{sp}\, \sigma^\psi = \log\, (\mathrm{Sp}(\Delta_\psi) \cap \boldsymbol{R}_+{}^*)$ by [6, Lemma 3.2.2] and $Z(M) \subset M^{\sigma^\phi}$ for $\psi \in W(M)$, we have a desired result.

§ 5. Unbounded Derivation

Before going into the definition of a derivation, we recall that $\sigma(g)$ is σ-weakly continuous on M for $g \in L^1(G)$. For the sake of completeness we shall give a slightly different proof from [2, Proposition 1.4].

Let M_1 be the unit ball of M with the σ-weak topology. Choose a compact $K \subset G$ for a given $\varepsilon > 0$ such that

$$\int_{G \backslash K} |g(t)|\, dt < \varepsilon\, .$$

Since the dual representation σ' on M_* of $\sigma \in \mathrm{Rep}(G, M)$ is strongly continuous [1, Proposition 1 in § 6], for any $t_j \in G$, $\varepsilon > 0$ and $\phi \in M_*$ there exists a neighbourhood V_j of t_j such that

$$\sup_{t \in V_j} \sup_{x \in M_1} |\langle (\sigma_t - \sigma_{t_j})(x), \phi \rangle| < 2^{-1}\varepsilon\, .$$

Since K is compact, we can find a finite covering V_j, $j = 1, \cdots, n$ of K. Since σ_{t_j} is σ-weakly continuous, there exists a neighbourhood N_j of 0 in M_1 such that $|\langle \sigma_{t_j}(N_j), \phi \rangle| < 2^{-1}\varepsilon$. Set $N \equiv \bigcap_{j=1}^n N_j$. Since $t \in K$ belongs to some V_j,

$$|\langle \sigma_t(x), \phi \rangle| \leq |\langle (\sigma_t - \sigma_{t_j})(x), \phi \rangle| + |\langle \sigma_{t_j}(x), \phi \rangle| < \varepsilon$$

for all $x \in N$. Therefore

$$\begin{aligned} |\langle \sigma(g)x, \phi \rangle| &\leq \varepsilon \int_K |g(t)|\, dt + 2\|\phi\| \int_{G \backslash K} |g(t)|\, dt \\ &\leq (\|g\|_1 + 2\|\phi\|)\, \varepsilon \end{aligned}$$

for all $x \in N$. Thus $\phi \circ \sigma(g)$ is σ-weakly continuous on M_1 and hence on M by Banach's theorem. Consequently, $\sigma(g)$ is σ-weakly continuous.

Now we shall generalize the concept of a derivation of M to the unbounded case as the following, [7].

Definition 5.1. A linear operator δ on M is called a *self-adjoint derivation* of M if the domain $\mathrm{D}(\delta)$ of δ is a σ-weakly dense *-subalgebra of M and

$$\delta(xy)=\delta(x)y+x\delta(y),\ \delta(x^*)=-\delta(x)^*$$

for all $x,\ y\in\mathrm{D}(\delta)$. In addition, δ is said to be *spatial* (resp. *inner*) if there exists a self-adjoint operator h (resp. $h_\eta M$) whose domain is invariant under $\mathrm{D}(\delta)$ and which satisfies

$$\delta(x)=\overline{hx-xh}\equiv\overline{[h,x]}$$

for all $x\in\mathrm{D}(\delta)$.

For a linear operator δ on a Banach space E, an $x\in E$ is *analytic* (resp. *entire*) for δ if the function $t\in\boldsymbol{R}\mapsto\sum_{n=0}^{\infty}(n!)^{-1}t^n\delta^n x\in E$ exists and is analytic in some neighbourhood of 0 (resp. *entire*). For a representation σ of $\boldsymbol{R}$ on E, an $x\in E$ is *analytic* (resp. *entire*) for σ if the function $t\mapsto\sigma_t(x)$ is analytic in some neighbourhood of 0 (resp. *entire*).

If σ_t, $t\in\boldsymbol{R}$ is a strongly continuous one parameter group of uniformly bounded operators on E, then

$$x_\lambda\equiv\left(\frac{1}{2\pi\lambda^2}\right)^{1/2}\int_{\boldsymbol{R}}\sigma_t(x)\exp\left(-\frac{t^2}{2\lambda^2}\right)dt \tag{5·1}$$

for $x\in E$, are entire for σ and x is the limit of x_λ as $\lambda\to0$. Furthermore if δ is the generator of σ, then

$$\sum_{n=0}^{\infty}\frac{t^n}{n!}\|\delta^n(x_\lambda)\|<+\infty,\ \text{for all } t\in\boldsymbol{R}.$$

In the following a linear operator δ on M is said to be *σ-weakly closed* if the graph of δ in $M\oplus M$ is σ-weakly closed.

Proposition 5.1. *Let $\sigma\in\mathrm{Rep}(\boldsymbol{R},M)$ and δ be a linear operator on M whose domain $\mathrm{D}(\delta)$ is the set of $x\in M$ for which $t^{-1}(\sigma_t(x)-x)$ is σ-weakly convergent as $t\downarrow0$, and*

$$\delta x=\lim_{t\downarrow0}(it)^{-1}(\sigma_t(x)-x)$$

for all $x\in\mathrm{D}(\delta)$. Then

(i) $D(\delta)$ *is a σ-weakly dense *-subalgebra of M and δ is a self-adjoint σ-weakly closed derivation of M;*

(ii) *for any non-zero real number λ, $\lambda-i\delta$ has the σ-weakly continuous inverse $(\lambda-i\delta)^{-1}$ and $\|(\lambda-i\delta)^{-1}\|\leq|\lambda|^{-1}$;*

(iii) *the set of entire elements for δ is σ-weakly dense in M;*

(iv) *δ is spatial(resp. inner) if and only if σ is spatial(resp. inner); and*

(v) *the infinitesimal generator of the dual representation σ' of σ is the dual of δ.*

Conversely, if δ is a self-adjoint σ-weakly closed derivation of M and if for any non-zero real number λ, $\lambda-i\delta$ has an inverse and $\|(\lambda-i\delta)^{-1}\|\leq|\lambda|^{-1}$, then there exists a unique representation $\sigma\in\mathrm{Rep}(\boldsymbol{R}, M)$ of which δ is an infinitesimal generator.

Proof. (i, ii) It is clear that $D(\delta)$ is a *-subalgebra of M and δ is a self-adjoint derivation of M. Define ϕ_λ for $\lambda>0$ by

$$\phi_\lambda=\int_0^\infty \lambda\sigma_t \exp(-\lambda t)\,dt\,.$$

Applying the same argument as the one parameter semi-group theory on a Banach space, we know that the range of ϕ_λ coincides with $D(\delta)$, that $\lambda^{-1}\phi_\lambda=(\lambda-i\delta)^{-1}$ and that $\phi_\lambda(x)$ converges σ-weakly to x as $\lambda\to\infty$ for $x\in M$. Therefore $D(\delta)$ is σ-weakly dense in M. Since ϕ_λ is σ-weakly continuous as shown at the begining of this Section, δ is σ-weakly closed.

(iii) x_λ in $(5\cdot 1)$ is entire for δ and σ-weakly converges to x as $\lambda\to\infty$. Therefore we conclude (iii).

(iv) Suppose that σ is spatial(resp. inner). There exists a self-adjoint operator h (resp. $h\eta M$) such that $\sigma_t(x)=u_t x u_t^*$ and $u_t=\exp(ith)$. Since

$$(it)^{-1}(\sigma_t(x)-x)\xi=u_t x(it)^{-1}(u_t^*-1)\xi+(it)^{-1}(u_t-1)x\xi\,,$$

if $x\in D(\delta)$ and ξ is in the domain $D(h)$ of h, then $x\xi\in D(h)$ and $\delta(x)\xi=[h, x]\xi$. Since $D(h)$ is dense in $\mathcal{H}$, we have $\delta x=\overline{[h, x]}$.

Conversely, suppose that δ is spatial(resp. inner). Let h be a self-adjoint operator which induces δ as in Definition 5.1. Put $u_t\equiv\exp(ith)$. Denote by $\mathcal{H}^{(e)}$ (resp. $M^{(e)}$) the set of entire elements for u(resp. δ).

We shall show by induction that $x\mathcal{H}^{(e)}\subset \mathrm{D}(h^n)$ for $n\in \boldsymbol{N}$ and $x\in M^{(e)}$. By the assumption for h, $x\mathcal{H}^{(e)}\subset \mathrm{D}(h)$. If $x\mathcal{H}^{(e)}\subset \mathrm{D}(h^n)$, then

$$\delta^n x\xi=\sum_{k=0}^{n}\binom{n}{k}h^k x(-h)^{n-k}\xi\ ,$$

for $\xi\in\mathcal{H}^{(e)}$. Since $(\delta^n x)\xi\in \mathrm{D}(h)$, we know that

$$h^n x\xi=(\delta^n x)\xi-\sum_{k=0}^{n-1}\binom{n}{k}h^k x(-h)^{n-k}\xi$$

is in $\mathrm{D}(h)$ and hence $x\xi\in \mathrm{D}(h^{n+1})$.

If $x\in M^{(e)}$ and ξ, $\eta\in\mathcal{H}^{(e)}$, then

$$\begin{aligned}(\sigma_t(x)\xi|\eta) &= \sum_{n=0}^{\infty}(n!)^{-1}(it)^n((\delta^n x)\xi|\eta)\\ &= \sum_{n=0}^{\infty}\frac{(it)^n}{n!}\left(\sum_{k=0}^{n}\binom{n}{k}h^k x(-h)^{n-k}\xi\,\middle|\,\eta\right)\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(x\frac{(it)^{n-k}}{(n-k)!}(-h)^{n-k}\xi\,\middle|\,\frac{(it)^k}{k!}(-h)^k\eta\right).\end{aligned}$$

Since $\xi,\eta\in\mathcal{H}^{(e)}$, the right hand side is absolutely convergent. Therefore

$$\begin{aligned}(\sigma_t(x)\xi|\eta) &= (x\sum_{n=0}^{\infty}(n!)^{-1}(-ith)^n\xi|\sum_{m=0}^{\infty}(m!)^{-1}(-ith)^m\eta)\\ &= (x(\exp(-ith))\xi|\exp(-ith)\eta)\\ &= ((\exp(ith))x(\exp(-ith))\xi|\eta).\end{aligned}$$

Since $\mathcal{H}^{(e)}$ is dense in $\mathcal{H}$ and $M^{(e)}$ is σ-weakly dense in M, we have $\sigma_t(x)=(\exp(ith))x(\exp(-ith))$ for $x\in M$.

(v) Let δ' and ${}^t\delta$ be the infinitesimal generator of the dual σ' on M_* of σ and the dual of δ, respectively. For $\lambda>0$, the dual of $(\lambda-i\delta)^{-1}$ is $(\lambda-i^t\delta)^{-1}$. Since $\delta'\subset{}^t\delta$, $(\lambda-i\delta')^{-1}\subset(\lambda-i^t\delta)^{-1}$. Since the domain of $(\lambda-i\delta')^{-1}$ is M_*, we have $(\lambda-i\delta')^{-1}=(\lambda-i^t\delta)^{-1}$ and hence, $\delta'={}^t\delta$.

Suppose that δ is a self-adjoint σ-weakly closed derivation of M and that $\|(\lambda-i\delta)^{-1}\|\leq|\lambda|^{-1}$ for any $\lambda\neq 0$. Denote by δ' the dual of δ on M_*. Since $\|(\lambda-i\delta')^{-1}\|\leq|\lambda|^{-1}$, by the Hille-Yosida theorem, δ' is the generator of a strongly continuous contraction one parameter group σ' on M_*. The dual σ of σ' is a σ-weakly continuous contraction one parameter group on M. Moreover (v) is valid for σ and the generator of σ is

σ-weakly closed. Since the bidual of a closed linear map is itself, the generator of σ is the dual of δ', namely, δ. Therefore we have for any entire elements x and y,

$$\sigma_t(x)\sigma_t(y) = \sum_{n=0}^{\infty}\frac{(it)^n}{n!}\delta^n(x)\sum_{n=0}^{\infty}\frac{(it)^n}{n!}\delta^n(y) = \sum_{n=0}^{\infty}\frac{(it)^n}{n!}\delta^n(xy) = \sigma_t(xy),$$

$$\sigma_t(x^*) = \sum_{n=0}^{\infty}\frac{(it)^n}{n!}\delta^n(x^*) = \sum_{n=0}^{\infty}\frac{(-it)^n}{n!}\delta^n(x)^* = \sigma_t(x)^*.$$

Since (iii) is valid for δ, we conclude the multiplicativity and self-adjointness of σ_t. Therefore σ_t is a *-automorphism. Q.E.D.

Remark 5.1. In the above proposition $M^{(e)}$ is a core of δ with respect to the σ-weak topology on M. Indeed, if $x \in \mathrm{D}(\delta)$, then x_λ defined by (5·1) converges σ-weakly to x. Furthermore $\delta(x_\lambda) = (\delta x)_\lambda$ converges σ-weakly to δx as $\lambda \to 0$.

Lemma 5.1. *If δ is the infinitesimal generator of $\sigma \in \mathrm{Rep}(\boldsymbol{R}, M)$, then* $\mathrm{Sp}\,\delta = \mathrm{sp}\,\sigma$.

Proof. Suppose that $\lambda \in \mathrm{sp}\,\sigma = -\mathrm{sp}\,\sigma$ and $\langle t, \lambda\rangle \equiv \exp(it\lambda)$ for $t \in \boldsymbol{R}$. Define a function $g \in \mathrm{L}^1(\boldsymbol{R})$ for any $\alpha > 0$ by

$$g(t) \equiv \exp(-\alpha t)\overline{\langle t, \lambda\rangle} \qquad (t>0);\ g(t) \equiv 0 \qquad (t \leq 0).$$

Since $i(\delta - \lambda)$ is the infinitesimal generator of a one parameter group $t \mapsto \overline{\langle t, \lambda\rangle}\sigma_t$, we have

$$\sigma(g) = \int_0^\infty \exp(-\alpha t)\overline{\langle t, \lambda\rangle}\sigma_t dt = -i(\lambda - i\alpha - \delta)^{-1}$$

and $\hat{g}(-\lambda) = \alpha^{-1}$. Therefore, by [6, Lemma 2.3.6], we have $\|(\lambda - i\alpha - \delta)^{-1}\| \geq \alpha^{-1}$ and hence

$$\lim_{\alpha \downarrow 0} \|(\lambda - i\alpha - \delta)^{-1}\| = \infty\,. \tag{5·3}$$

Consequently, $\lambda \in \mathrm{Sp}\,\delta$.

Assume that

$$\lim_{\alpha \downarrow 0} \|(\lambda - i\alpha - \delta)^{-1}\| < \infty\,.$$

By the resolvent equation, $(\lambda - i\alpha - \delta)^{-1}$ converges in norm to a bounded operator ρ as $\alpha\downarrow 0$ and $\rho=(\lambda-\delta)^{-1}$. Therefore $\lambda\in\mathrm{Sp}\,\delta$ implies (5.3) and hence that there exist for any $\varepsilon>0$ a positive $\alpha\in\boldsymbol{R}$ and a non zero $y\in M$ such that $2\alpha<\varepsilon$ and

$$\varepsilon\|(\lambda-i\alpha-\delta)^{-1}y\|>2\|y\|.$$

By putting $x=\|z\|^{-1}z$ for $z=(\lambda-i\alpha-\delta)^{-1}y$, we have

$$\|(\lambda-\delta)x\|\leq\|(\lambda-i\alpha-\delta)x\|+\|i\alpha x\|<\varepsilon.$$

From the equation

$$\overline{\langle t, x\rangle}\sigma_t(x)-x=\int_0^t\overline{\langle s, \lambda\rangle}\sigma_s\circ(i(\delta-\lambda))(x)ds,$$

it follows that

$$\|\sigma_t(x)-\langle t, \lambda\rangle x\|<|t|\varepsilon.$$

Therefore, by [6, Lemma 2.3.6], we have $-\lambda\in\mathrm{sp}\,\sigma=-\mathrm{sp}\,\sigma$. Q.E.D.

Lemma 5.1 and Theorem 1.1 give following corollaries. It is clear that $x\in M^\sigma$ if and only if $\delta x=0$. Therefore the restriction δ^e of δ to M_e is a derivation corresponding to σ^e. For a derivation δ, we denote by M^δ the set $\{x\in M: \delta x=0\}$.

Corollary 5.1. *Let δ be a derivation of M which is the infinitesimal generator of a representation in* $\mathrm{Rep}(\boldsymbol{R}, M)$. *The following conditions are equivalent for $\lambda>0$:*

(i) $\cap\{\mathrm{Sp}\,\delta^e: e\in M^\delta, e\neq 0\}=\cap\{\mathrm{Sp}\,\delta^e: e\in M^\delta, \bar{e}=1\}=\lambda\boldsymbol{Z}$; *and*

(ii) *for any non zero projection f in $Z(M^\delta)$ and for any neighbourhood V of* 0, *there exists a non zero projection e in $Z(M^\delta)$ such that $e\leq f$ and $\lambda\boldsymbol{Z}\in\mathrm{Sp}\,\delta^e\subset\lambda\boldsymbol{Z}+V$.*

Corollary 5.2. *Let δ be a derivation of M which is the infinitesimal generator of a representation in* $\mathrm{Rep}(\boldsymbol{R}, M)$. *If M_* is separable, then the following conditions are equivalent*:

(i) *δ is inner*; *and*

(ii) *for any non zero projection f in $Z(M^\delta)$ and for any $\varepsilon>0$ there exists a non zero projection e in $Z(M^\delta)$ such that $e\leq f$ and* $\|\delta^e\|\leq\varepsilon$.

Since the separability of M_* is unnecessary for the implication (ii) $\Rightarrow$(i) in Corollary 5.2, we have Corollary 5.3, which is a restatement of a result of Borchers [3, Theorem]. We shall restate it more precisely.

Corollary 5.3. *Let δ be a derivation of M which is the infinitesimal generator of a representation in* Rep$(\boldsymbol{R}, M)$. *If there is a non negative self-adjoint operator k implementing δ, then δ is inner, and a self-adjoint operator $h\eta M$ implementing δ is uniquely determined by the condition that $2\|he\| = \|\delta^e\|$ for all $e \in Z(M^\delta)$. In particular,* Sp $(he) + \|he\| \subset \mathrm{Sp}\,\delta^e \cap \boldsymbol{R}_+$.

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