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Existence and Stability of Almost Periodic Solutions in Almost Periodic Systems

By

Fumio NAKAJIMA*

§ 1. Introduction

We shall consider the existence of almost periodic solutions of the almost periodic system of the form

$$(1\cdot1)\qquad \dot{x}_i=\sum_{j=1}^{n} a_{ij}(t)x_j^k \quad (\cdot=d/dt) \quad \text{for } 1\leqq i\leqq n\,,$$

where x_j are real, k is a positive integer and $a_{ij}(t)$ are almost periodic functions of t. Under some conditions on $a_{ij}(t)$, Theorem 1 shows that the trivial solution of the first approximation of system $(1\cdot1)$ is uniformly asymptotically stable in a subspace Π of $\boldsymbol{R}^n$ (see Definition 2). Using this fact, we obtain a nontrivial almost periodic solution of system $(1\cdot1)$ which is uniformly asymptotically stable in a compact set Ω and whose module is contained in the module of $a_{ij}(t)$. This is shown in Theorem 2.

Especially, in the case where $a_{ij}(t)$ in system $(1\cdot1)$ are constants, the system governs one of mathematical models of gas dynamics (cf. [2, p. 104]) and has been studied by Jenks [4]. One of Jenks' results is a special case of Theorem 2.

We denote by $\boldsymbol{R}^n$ the real Euclidean n-space. Let $\boldsymbol{R}=(-\infty,\infty)$ and $\boldsymbol{R}^+=[0,\infty)$. For x in $\boldsymbol{R}^n$, let $|x|$ be the Euclidean norm of x and x_i be the i-th component. We let

$$D=\{x\in\boldsymbol{R}^n;\, x_i\geqq0 \quad \text{for } 1\leqq i\leqq n\}$$

$$\Omega=\{x\in D;\, \sum_{i=1}^{n} x_i=1\}$$

and

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* Mathematical Institute of Tôhoku University, Sendai.

$$\Pi=\{x\in \boldsymbol{R}^n\,;\sum_{i=1}^{n}x_i=0\}.$$

For a continuous function $f(t)$ on $\boldsymbol{R}$ with values in $\boldsymbol{R}^n$, $H(f)$ denotes the set of all functions $g(t)$ such that for some sequence t_k,

$$f(t+t_k)\to g(t) \text{ in } \boldsymbol{R} \text{ as } k\to\infty\,,$$

where the symbol"$\to$" stands for the uniform convergence on any compact set in $\boldsymbol{R}$. Clearly $f\in H(f)$.

Definition 1. An $n\times n$ matrix $A(t)=(a_{ij}(t))$ is said to be irreducible if for any two nonempty disjoint subsets I and J of the set of n integers $\{1,\cdots,n\}$ with $I\cup J=\{1,\cdots,n\}$, there exists an i in I and a j in J such that

$$a_{ij}(t)\not\equiv 0\,.$$

In the case where $A(t)$ is scalar, $A(t)$ is said to be irreducible if $A(t)\not\equiv 0$. Otherwise, $A(t)$ is said to be reducible, and we can assume without loss of generality that $A(t)$ takes the form of

$$A(t)=\left(\begin{array}{c|c} * & 0 \\ \hline * & B(t)\end{array}\right),$$

where $B(t)$ is zero or a square irreducible matrix.

We shall define stability properties. Here we denote by $x(t,t_0,x_0)$ the solution of system $(1\cdot 1)$ with initial condition (t_0,x_0).

Definition 2. Let $x(t)$ be a solution of system $(1\cdot 1)$ defined on $\boldsymbol{R}$ and K be a subset of $\boldsymbol{R}^n$.

(i) $x(t)$ is said to be uniformly stable (*u. s* for short) in K on $\boldsymbol{R}^+$ if for each $\varepsilon>0$ there exists a $\delta(\varepsilon)>0$ such that

$$|x(t)-x(t,t_0,x_0)|<\varepsilon \text{ for } t\geqq t_0$$

whenever $x_0\in K$ and $|x(t_0)-x_0|<\delta(\varepsilon)$ at some t_0 in $\boldsymbol{R}^+$.

(ii) $x(t)$ is said to be uniformly asymptotically stable (*u. a. s* for short) in K on $\boldsymbol{R}^+$ if it is *u. s* in K on $\boldsymbol{R}^+$ and if there exists a $\delta_0>0$ and, for each $\varepsilon>0$ there exists a $T(\varepsilon)>0$ such that

$$|x(t)-x(t,t_0,x_0)|<\varepsilon \text{ for } t\geqq t_0+T(\varepsilon)$$

whenever $x_0\in K$ and $|x(t_0)-x_0|<\delta_0$ at some t_0 in $\boldsymbol{R}^+$.

(iii) $x(t)$ is said to be *u. a. s* in the whole K on $\boldsymbol{R}^+$ if it is *u. s* in

K on $\boldsymbol{R}^+$ and if for each $\varepsilon>0$ and $r>0$ there exists a $T(\varepsilon, r)>0$ such that

$$|x(t)-x(t, t_0, x_0)|<\varepsilon \text{ for } t \geqq t_0+T(\varepsilon, r)$$

whenever $x_0 \in K$ and $|x(t_0)-x_0|<r$ at some t_0 in $\boldsymbol{R}^+$.

When $\boldsymbol{R}^+$ in the definitions (i), (ii) and (iii) is replaced by $\boldsymbol{R}$, we say that $x(t)$ is *u. s* in K on $\boldsymbol{R}$, *u. a. s* in K on $\boldsymbol{R}$ and *u. a. s* in the whole K on $\boldsymbol{R}$, respectively. Clearly Definition 2 agrees with the definitions of the usual stability properties in the case where $K=\boldsymbol{R}^n$.

§ 2. Linear Systems

Consider the system

$$(2\cdot1) \qquad \dot{x}=A(t)x, \; x \in \boldsymbol{R}^n,$$

where the $n \times n$ matrix $A(t)=(a_{ij}(t))$ is bounded and uniformly continuous on $\boldsymbol{R}$. We shall state a generalization of one of Jenks' result (Corollary 3 in [4]).

Theorem 1. *Assume that system* $(2\cdot1)$ *satisfies the following conditions*;

(i) $\sum_{i=1}^{n} a_{ij}(t)=0$ *for* $1 \leqq j \leqq n$,

(ii) $a_{ij}(t) \geqq 0$ *for* $i \neq j$,

(iii) *each element in* $H(A)$ *is irreducible.*

Then the trivial solution of system $(2\cdot1)$ *is u. a. s in* Π *on* $\boldsymbol{R}$.

To prove this theorem, first of all we shall prove the following lemmas.

Lemma 1. *Consider the n-system* $\dot{x}_i=f_i(t, x)$, $1 \leqq i \leqq n$, *where* $f_i(t, x)$ *is continuous on* $\boldsymbol{R} \times \boldsymbol{R}^n$, *and assume that the initial value problem has a unique solution.*

(I) *If* $\sum_{i=1}^{n} f_i(t, x)=0$, *then the set* Π *is invariant.*

(II) *If* $f_i(t, x) \geqq 0$ *for* $x_i=0$ *and all* $x_j \geqq 0$, *then the set* D *is positively invariant, and in addition, if* $\sum_{i=1}^{n} f_i(t, x)=0$, *then the set*

Ω is positively invariant.

The proof of the above lemma is obvious (for example, see [5, p. 270]). Obviously the assumptions in (I) and (II) hold for system $(2\cdot1)$ satisfying conditions (i) and (ii) of Theorem 1.

Lemma 2. *The trivial solution of system $(2\cdot1)$ is u. s in Π on $\boldsymbol{R}$ and also u. s on $\boldsymbol{R}$, if conditions (i) and (ii) in Theorem 1 are satisfied.*

By using theorems in [3, p. 58], we can easily prove Lemma 2.

Lemma 3. *If each element in $H(A)$ is irreducible, then the each element in $H(A)$, say $B(t)=(b_{ij}(t))$, has the property that for any two nonempty disjoint subsets I and J of the set of n integers $\{1,\cdots,n\}$ with $I\cup J=\{1,\cdots,n\}$, there exists an $i\in I$ and a $j\in J$ such that*

$$\overline{\lim_{t\to-\infty}}|b_{ij}(t)|\neq0\,.$$

Proof. Suppose not. Then there exists a $B(t)=(b_{ij}(t))$ in $H(A)$ and two nonempty disjoint subsets I and J of $\{1,\cdots,n\}$ with $I\cup J=\{1,\cdots,n\}$ such that

$$\lim_{t\to-\infty}b_{ij}(t)=0 \text{ for all } i\in I \text{ and all } j\in J.$$

Since $B(t)$ is bounded and uniformly continuous on $\boldsymbol{R}$, there exists a sequence t_k, $t_k\to-\infty$ as $k\to\infty$, such that

$$B(t+t_k)\to C(t) \text{ in } \boldsymbol{R} \text{ as } k\to\infty\,,$$

where $C(t)=(c_{ij}(t))\in H(A)$. Clearly

$$c_{ij}(t)=\lim_{k\to\infty}b_{ij}(t+t_k)=0 \text{ for } t\in\boldsymbol{R} \text{ and } i\in I, j\in J.$$

This shows the reducibility of $C(t)$. This contradiction proves the lemma.

Lemma 4. *Assume that conditions (ii) and (iii) in Theorem 1 are satisfied for system $(2\cdot1)$ and consider the system*

$$(2\cdot2)\qquad \dot{x}=B(t)x,\ B\in H(A).$$

Let $x(t)$ be a nontrivial solution of system $(2\cdot2)$ such that

$$x(t)\in D \text{ on } \boldsymbol{R}.$$

Then there exists a constant $c>0$ *such that*

$$x_i(t)/|x(t)|\geqq c \text{ for } t\in R \text{ and } 1\leqq i\leqq n .$$

Proof. First of all, letting $x(t)=(x_1(t),\cdots,x_n(t))$ be a solution of system $(2\cdot2)$ such that $x(t)\in D$ on $\boldsymbol{R}$, we shall show that if $x_i(t_0)=0$ at some $t_0\in\boldsymbol{R}$, then

$$x_i(t)=0 \text{ for all } t\leqq t_0 .$$

Since $x_i(t)$ satisfies the equation

$$\dot{x}_i=b_{ii}(t)x_i+\sum_{j\neq i} b_{ij}(t)x_j ,$$

where $(b_{ij}(t))=B(t)$, and since $\sum_{j\neq i} b_{ij}(t)x_j\geqq 0$, we have

$$(2\cdot3) \qquad \dot{x}_i(t)\geqq b_{ii}(t)x_i(t),$$

which implies

$$x_i(t)\leqq x_i(t_0)\exp\left(\int_{t_0}^{t} b_{ii}(s)\,ds\right) \text{ for } t\leqq t_0 .$$

Thus we obtain

$$x_i(t)=0 \text{ for } t\leqq t_0 ,$$

because $x_i(t_0)=0$ and $x_i(t)\geqq 0$ on $\boldsymbol{R}$.

Now suppose that Lemma 4 is not true. Then for some B in $H(A)$, the corresponding system $(2\cdot2)$ has a nontrivial solution $x(t)$, $x(t)\in D$ on $\boldsymbol{R}$, such that for some sequence t_k,

$$(2\cdot4) \qquad x_1(t_k)/|x(t_k)|\to 0 \text{ as } k\to\infty .$$

Setting $\phi_k(t)=x(t+t_k)/|x(t_k)|$, $\phi_k(t)$ satisfies

$$\dot{x}=B(t+t_k)x$$

and

$$\phi_k(t)\in D \text{ on } \boldsymbol{R},\ |\phi_k(0)|=1 .$$

Since the sequence $\{\phi_k(0)\}$ is bounded, $\{\phi_k(t)\}$ is uniformly bounded and equicontinuous on each finite interval in $\boldsymbol{R}$, and hence Ascoli-Arzela's theorem gives us a convergent subsequence of ϕ_k, which is again denoted by ϕ_k, such that

$$\phi_k(t)\to y(t) \text{ in } \boldsymbol{R} \text{ for some function } y(t) \text{ as } k\to\infty .$$

We can also assume that

$$B(t+t_k) \to C(t) \text{ in } \boldsymbol{R} \text{ as } k\to\infty ,$$

where $C(t)=(c_{ij}(t))\in H(A)$. Therefore, $y(t)$ is the solution of the system

$$\dot{y}=C(t)y , \tag{2·5}$$

$y(t)\in D$ on $\boldsymbol{R}$ and $|y(0)|=1$. Moreover (2·4) implies that $y_1(0)=0$. Thus, as was proved above, we have

$$y_1(t)=0 \text{ for } t\leqq 0 .$$

For this $y(t)$, we define two subsets I and J of $\{1, \cdots, n\}$ by $I=\{1\leqq i\leqq n; y_i(t)\equiv 0$ for $t\leqq N_i$, where N_i may depend on $y_i(t)\}$ and $J=\{1\leqq i\leqq n; y_i(t)>0$ on $\boldsymbol{R}\}$. Then $I\cup J=\{1, \cdots, n\}$, $\{1\}\in I$ and $J\neq\emptyset$ since $y(t)\neq 0$. By Lemma 3,

$$\overline{\lim_{t\to-\infty}}|c_{i_0,j_0}(t)|\neq 0 \text{ for some } i_0\in I \text{ and some } j_0\in J. \tag{2·6}$$

Now the i_0-th equation of system (2·5) takes the form of

$$\dot{y}_{i_0}(t)=\sum_{k\in I}c_{i_0,k}(t)y_k(t)+\sum_{k\in J}c_{i_0,k}(t)y_k(t),$$

and hence

$$\sum_{k\in J}c_{i_0,k}(t)y_k(t)=0 \quad \text{for } t<\min_{i\in I} N_i , \tag{2·7}$$

because of the definition of the set I. Since each term in the left hand side is nonnegative, all of them are equal to zero. Therefore

$$c_{i_0,j_0}(t)y_{j_0}(t)=0 \text{ for } t<\min_{i\in I} N_i ,$$

which implies, by (2·6),

$$y_{j_0}(t_0)=0 \text{ at some } t_0 .$$

This contradicts the definition of the set of J. The proof is completed.

The following proposition is an immediate result of Lemma 4.

Proposition 1. *Under conditions* (*ii*) *and* (*iii*) *in Theorem* 1, *system* (2·2) *has no nontrivial solution* $x(t)$ *such that*

$$x(t)\in\partial D \text{ on } \boldsymbol{R} ,$$

where $\partial D=\{x\in D; x_i=0$ for some i, $1\leqq i\leqq n\}$.

Lemma 5. *Consider a nonhomogeneous system corresponding to system* (2·1)

$$\dot{x}=A(t)x+f(t) \tag{2·8}$$

and assume that $A(t)$ *satisfies all conditions in Theorem* 1. *If* $f(t)$ *is bounded and continuous on* $\boldsymbol{R}^+$ *with values in* $\boldsymbol{R}^n$ *and if the integral of* $\sum_{i=1}^n f_i(t)$ *is bounded on* $\boldsymbol{R}^+$, *then all solutions of system* (2·8) *are bounded on* $\boldsymbol{R}^+$.

Proof. It is sufficient to show that (2·8) has at least one bounded solution on $\boldsymbol{R}^+$, because the trivial solution of (2·1) is uniformly stable by Lemma 2. We shall consider the system with real parameter ε

$$\dot{x}=A(t)x+\varepsilon f(t) \tag{2·9}$$

and show that for a sufficiently small ε, system (2·9) has a bounded solution on $\boldsymbol{R}^+$, which implies the existence of a bounded solution on $\boldsymbol{R}^+$ for system (2·8) by replacing x in (2·9) with x/ε.

For a $0<\delta<1/\sqrt{n}$ and for the n-vector e each of whose components is 1, let D' be a convex cone defined by

$$D'=\{x\in\boldsymbol{R}^n;\langle e,x\rangle\geqq|e|\cdot|x|\cdot\delta\},$$

where $\langle\,,\rangle$ denotes the inner product and $|x|^2=\langle x,x\rangle$. Then clearly $D\subset D'$. Every solution $x(t)$ of (2·9) satisfies

$$\frac{d}{dt}\sum_{i=1}^n x_i(t)=\varepsilon\sum_{i=1}^n f_i(t)$$

because of condition (i). By integrating the both sides,

$$\sum_{i=1}^n x_i(t)=\varepsilon\int_0^t\sum_{i=0}^n f_i(s)\,ds+1 \text{ for } x(0)\in\Omega\,.$$

When $x(t)\in D'$, we have

$$\sum_{i=1}^n x_i(t)=\langle x(t),e\rangle\geqq|x(t)|\cdot|e|\cdot\delta\,,$$

and hence

$$(1-\varepsilon M)/|e|\leqq|x(t)|\leqq(1+\varepsilon M)/(|e|\cdot\delta), \tag{2·10}$$

where

$$M=\sup_{t>0}\int_0^t\sum_{i=1}^n f_i(s)\,ds\,.$$

Therefore, in order to show the boundedness of $x(t)$ with $x(0)$ in Ω, it is sufficient to prove that $x(t)\in D'$ on $\boldsymbol{R}^+$ if ε is sufficiently small.

Now suppose that for each solution $x_\varepsilon(t)$ of $(2\cdot 9)$ with $x_\varepsilon(0)$ in Ω, there exists an $s_\varepsilon>0$ such that

$$x_\varepsilon(s_\varepsilon)\in\partial D'\,.$$

We can assume that

$$x_\varepsilon(t_\varepsilon)\in\partial D \text{ at some } t_\varepsilon,\ 0\leqq t_\varepsilon<s_\varepsilon$$

and

$$x_\varepsilon(t)\in\overline{D'-D} \text{ for } t_\varepsilon\leqq t\leqq s_\varepsilon\,,$$

where ∂K and $\overline{K}$ denote the boundary and the closure of the set K, respectively. If we set $y_\varepsilon(t)=x_\varepsilon(t+t_\varepsilon)$, $y_\varepsilon(t)$ is a solution of the system

$$\dot y=A(t+t_\varepsilon)y+\varepsilon f(t+t_\varepsilon)$$

such that $y_\varepsilon(0)\in\partial D$, $y_\varepsilon(\tau_\varepsilon)\in\partial D'$ at $\tau_\varepsilon=s_\varepsilon-t_\varepsilon>0$ and $y_\varepsilon(t)\in\overline{D'-D}$ for $0\leqq t\leqq\tau_\varepsilon$. Thus, by $(2\cdot 10)$,

$$(1-\varepsilon M)/|e|\leqq|y_\varepsilon(t)|\leqq(1+\varepsilon M)/(|e|\cdot\delta) \text{ for } 0\leqq t\leqq\tau_\varepsilon\,.$$

The same argument involving Ascoli-Arzela's theorem as in the proof of Lemma 4 enables us to assume that

$$y_\varepsilon(t)\to z(t) \text{ in } \boldsymbol{R} \text{ for some function } z(t) \text{ as } \varepsilon\to 0$$

and

$$A(t+t_\varepsilon)\to B(t) \text{ in } \boldsymbol{R} \text{ for some } B(t) \text{ in } H(A) \text{ as } \varepsilon\to 0\,.$$

Therefore $z(t)$ satisfies $\dot z=B(t)z$ and clearly, for $\tau=\underline{\lim}_{\varepsilon\to 0}\tau_\varepsilon$,

$$(2\cdot 11)\qquad z(t)\in\overline{D'-D} \text{ for } 0\leqq t<\tau$$

and

$$(2\cdot 12)\qquad 1/|e|\leqq|z(t)|\leqq 1/(|e|\cdot\delta) \text{ for } 0\leqq t<\tau\,.$$

Moreover we have $z(0)\in\partial D$, which implies by Lemma 1 that

$$z(t)\in D \text{ on } \boldsymbol{R}^+\,.$$

From this and $(2\cdot 11)$ it follows that

$$(2\cdot 13)\qquad z(t)\in D\cap\overline{D'-D}=\partial D \text{ for } 0\leqq t<\tau\,.$$

Now we show that $\tau=\infty$. In fact, if $\tau<\infty$, we have

$$y_\varepsilon(\tau_\varepsilon)\to z(\tau) \text{ as } \varepsilon\to 0\,.$$

Thus $z(\tau)\in\partial D'$ because $y_\varepsilon(\tau_\varepsilon)\in\partial D'$, and hence

$$z(\tau)\in\partial D'\cap\partial D=\{0\},$$

which contradicts $(2\cdot 12)$. Therefore $(2\cdot 12)$ and $(2\cdot 13)$ hold for $\tau=\infty$. Moreover this enables us to assume that

$$1/|e|\leqq|z(t)|\leqq 1/(|e|\cdot\delta) \text{ for all } t\in\boldsymbol{R}$$

and

$$z(t)\in\partial D \text{ for all } t\in\boldsymbol{R}\,,$$

because $H(B)$ is compact in the sense of the convergence "$\to$". This contradicts the conclusion in Proposition 1. This proves that $x(t)\in D'$ on $\boldsymbol{R}^+$ if ε is sufficiently small. The proof is completed.

Lemma 6. *If for each B in $H(A)$, the trivial solution of the system*

$$\dot{x}=B(t)x$$

is u. s on $\boldsymbol{R}$ and u. a. s on $\boldsymbol{R}^+$, then the trivial solution of system $(2\cdot 1)$ *is u. a. s on $\boldsymbol{R}$.*

Proof. Let $x(t, t_0, x_0)$ be the solution of $(2\cdot 1)$. Since the trivial solution of $(2\cdot 1)$ is *u. s* on $\boldsymbol{R}$, as is seen from Definition 2, (ii) it is sufficient to show that for each $\varepsilon>0$ there exists a $T(\varepsilon)>0$ such that

$$|x(t, t_0, x_0)|<\delta(\varepsilon) \text{ for some } t,\ t_0\leqq t\leqq t_0+T(\varepsilon),$$

whenever $t_0\in\boldsymbol{R}$ and $|x_0|<\delta_0=\delta(1)$, where $\delta(\cdot)$ is the number in Definition 2, (i).

Now suppose that there exists an $\varepsilon>0$ and sequences $\{t_k\}$ in $\boldsymbol{R}$ and $\{x_k\}$ in $\boldsymbol{R}^n$ such that $|x_k|\leqq\delta_0$ and

$$|x(t, t_k, x_k)|\geqq\delta(\varepsilon) \text{ for all } t,\ t_k\leqq t\leqq t_k+k\,.$$

Since $|x_k|<\delta_0=\delta(1)$,

$$\delta(\varepsilon)\leqq|x(t, t_k, x_k)|\leqq 1 \text{ for } t_k\leqq t\leqq t_k+k\,.$$

Setting $\phi_k(t)=x(t+t_k, t_k, x_k)$, $\phi_k(t)$ satisfies

$$\dot{x}=A(t+t_k)x$$

and

$$\delta(\varepsilon)\leqq|\phi_k(t)|\leqq 1 \text{ for } 0\leqq t\leqq k.$$

We can assume that

$$\phi_k(t)\to y(t) \text{ in } \boldsymbol{R}^+ \text{ for some function } y(t) \text{ as } k\to\infty$$

and

$$A(t+t_k)\to B(t) \text{ in } \boldsymbol{R} \text{ for some } B(t) \text{ in } H(A) \text{ as } k\to\infty .$$

Therefore $y(t)$ is a solution of the system

$$(2\cdot 14) \qquad \dot{y}=B(t)y$$

and

$$\delta(\varepsilon)\leqq|y(t)|\leqq 1 \text{ on } \boldsymbol{R}^+ .$$

On the other hand, we have

$$y(t)\to 0 \text{ as } t\to\infty ,$$

because the trivial solution of $(2\cdot 14)$ is *u. a. s* on $\boldsymbol{R}^+$. Therefore there arises a contradiction. Thus the proof is completed.

Now we are in position to prove Theorem 1. On the set Π which is invariant for system $(2\cdot 1)$, the system is written as the $(n-1)$-system

$$(2\cdot 15) \qquad \dot{x}'=A'(t)x'$$

where $x'=(x_1, \cdots, x_{n-1})\in\boldsymbol{R}^{n-1}$ and $A'(t)$ is an $(n-1)\times(n-1)$ matrix whose (i,j) element is given by $a_{ij}(t)-a_{in}(t)$ for $1\leqq i,\ j\leqq n-1$.

First of all we shall show that for each B' in $H(A')$, the system

$$(2\cdot 16) \qquad \dot{x}'=B'(t)x'$$

has an exponential dichotomy on $\boldsymbol{R}^+$, and as is well known (cf. [6]), it is equivalent to show that the system

$$(2\cdot 17) \qquad \dot{x}'=B'(t)x'+f'(t)$$

possesses at least one bounded solution on $\boldsymbol{R}^+$ for any continuous bounded function $f'(t)$ on $\boldsymbol{R}^+$. For each $B'(t)$ in $H(A')$ there corresponds some $B(t)=(b_{ij}(t))$ in $H(A)$ such that the (i, j) element of $B'(t)$ is equal to $b_{ij}(t)-b_{in}(t)$ for $1\leqq i,\ j\leqq n-1$. For $f'(t)=(f_1(t),\ \cdots,\ f_{n-1}(t))$, let $g(\mathrm{t})$ be defined by

$$g_i(t)=f_i(t) \text{ for } 1\leqq i\leqq n-1\,,$$

$$g_n(t)=-\sum_{i=1}^{n-1} f_i(t)\,.$$

Obviously $g(t)$ and the integral of $\sum_{i=1}^{n} g_i(t)$ $(=0)$ are bounded on $\boldsymbol{R}^+$. Applying Lemma 5 to the n-system

$$\dot{x}=B(t)x+g(t)\,, \tag{2·18}$$

we obtain the bounded solution $x(t)$ on $\boldsymbol{R}^+$ with $x(0)=0$, and $(d/dt)\sum_{i=1}^{n} x_i(t)=0$ which yields

$$\sum_{i=1}^{n} x_i(t)=0\,.$$

Hence we can verify that $x'(t)=(x_1(t),\ \cdots,\ x_{n-1}(t))$ is a bounded solution on $\boldsymbol{R}^+$ of system (2·17).

The exponential dichotomy of (2·16) implies further that the trivial solution is *u. a. s* on $\boldsymbol{R}^+$, because the trivial solution is *u. s* on $\boldsymbol{R}$ by Lemma 2. Therefore it follows from Lemma 6 that the trivial solution of (2·15) is *u. a. s* on $\boldsymbol{R}$, i. e., the trivial solution of (2·1) is *u. a. s* in Π on $\boldsymbol{R}$. The proof is completed.

§ 3. Nonlinear System

We shall consider the nonlinear almost periodic system of the type

$$\dot{x}_i=\sum_{j=1}^{n} a_{ij}(t)g_j(x_j) \text{ for } 1\leqq i\leqq \mathrm{n}\,, \tag{3·1}$$

where $A(t)=(a_{ij}(t))$ is almost periodic function of t with conditions

(i) $$\sum_{i=1}^{n} a_{ij}(t)=0 \text{ for } 1\leqq j\leqq n$$

and

(ii) $$a_{ij}(t)\geqq 0 \text{ for } i\neq j\,.$$

In addition, assume that $g_j(u)$ are continuously differentiable for $u\geqq 0$, $g_j(0)=0$ and $\dot{g}_j(u)>0$ for $u>0$.

Theorem 2. *Under the assumptions above, system* $(3\cdot1)$ *possesses a nontrivial almost periodic solution in* Ω *whose module is contained in the module of* $A(t)$. *In addition to the assumptions above, if* $A(t)$ *is irreducible, then the above almost periodic solution is unique in* Ω, *which remains in* Ω^0 *on* $\boldsymbol{R}$, *and it is u. a. s in the whole* Ω *on* $\boldsymbol{R}$, *where* $\Omega^0=\{x\in\Omega; x_i>0 \text{ for all } i, 1\leqq i\leqq n\}$, *and if* $A(t)$ *is reducible, then at least one of the above almost periodic solutions* $p(t)$ *satisfies that* $p(t)\in\partial\Omega$ *on* $\boldsymbol{R}$, *where* $\partial\Omega=\{x\in\Omega; x_i=0 \text{ for some } i, 1\leqq i\leqq n\}$.

Remark. As will be seen from the module containment, the above almost periodic solution is a critical point in the case where $A(t)$ is a constant. Hence Theorem 2 is a generalization of one of Jenks' results (Theorem 2 in [4]).

To prove the theorem, first of all we shall prove the following lemmas.

Lemma 7. *Consider the linear system*

$$(3\cdot2) \qquad \dot{x}=M(t)x$$

and its perturbed system

$$(3\cdot3) \qquad \dot{x}=M(t)x+f(t,x),$$

where $M(t)$ *and* $f(t,x)$ *are continuous with respect to its arguments, respectively, and* $f(t,x)=o(|x|)$ *uniformly for* $t\in\boldsymbol{R}$. *Assume that the set* Π *is invariant for both systems* $(3\cdot2)$ *and* $(3\cdot3)$. *If the trivial solution of system* $(3\cdot2)$ *is u. a. s in* Π *on* $\boldsymbol{R}$, *then the trivial solution of system* $(3\cdot3)$ *has also the same stability property.*

Proof. Let $x'=(x_1,\cdots,x_{n-1})$ for $x=(x_1,\cdots,x_n)\in\boldsymbol{R}^n$. Then there are positive constants c_1 and c_2 such that

$$(3\cdot4) \qquad c_1|x'|\leqq|x|\leqq c_2|x'| \text{ for } x \text{ in } \Pi,$$

because $x_n=-(x_1+\cdots+x_{n-1})$. On the set Π, systems $(3\cdot2)$ and $(3\cdot3)$ are written as

$$(3\cdot5) \qquad \dot{x}'=M'(t)x'$$

and

$$(3\cdot6)\qquad \dot{x}'=M'(t)x'+g(t,x')$$

respectively, where the (i,j) element of $M'(t)$, $1\leqq i,\ j\leqq n-1$, is given by $m_{ij}(t)-m_{in}(t)$ for $M(t)=(m_{ij}(t))$ and $g(t,x')=o(|x'|)$ uniformly for $t\in\boldsymbol{R}$. Inequality $(3\cdot4)$ shows that the trivial solution of $(3\cdot2)$ is *u. a. s* in Π if and only if the trivial solution of $(3\cdot5)$ is *u. a. s*, and we have also the same equivalence between $(3\cdot3)$ and $(3\cdot6)$. As is well known, if the trivial solution of $(3\cdot5)$ is *u. a. s*, then the trivial solution of $(3\cdot6)$ has also the same stability property. Thus our assertion is clear.

The following lemma is obtained by the slight modification of Seifert's result [8].

Lemma 8. *Consider the almost periodic system*

$$(3\cdot7)\qquad \dot{x}=f(t,x),\ t\in\boldsymbol{R},\ x\in\boldsymbol{R}^n,$$

where $f(t,x)$ is almost periodic in t uniformly for $x\in\boldsymbol{R}^n$ and for a constant $L>0$, $|f(t,x)-f(t,y)|\leqq L|x-y|$ for $t\in\boldsymbol{R}$ and $x,y\in\Omega$. Assume that the set Ω is positively invariant for system $(3\cdot7)$ and all solutions in Ω on $\boldsymbol{R}$ are u. a. s in Ω on $\boldsymbol{R}$. Then the set of such solutions is finite and consists of only almost periodic solutions $\phi_1,\cdots,\phi_m$ which satisfy

$$|\phi_i(t)-\phi_j(t)|\geqq\beta \text{ on } \boldsymbol{R} \text{ for } i\neq j \text{ and some constant } \beta>0.$$

Now we shall prove Theorem 2. Since the last statements of Theorem 2 are alternative, under each assumption of these satements we shall prove the existence of almost periodic solutions in Ω and the module containment.

First of all, we shall consider the case where $A(t)$ is irreducible. Since system $(3\cdot1)$ satisfies the conditions of Lemma 1, the set Ω is positively invariant, namely, $y(t)\in\Omega$ on $\boldsymbol{R}^+$ for a solution $y(t)$ of $(3\cdot1)$ with $y(0)\in\Omega$, and furthermore we can assume that

$$y(t)\in\Omega \text{ on } \boldsymbol{R}$$

because of the almost periodicity of $A(t)$. We shall show that this

$y(t)$ is $u.a.s$ in Ω on $\boldsymbol{R}$. If we set $x=y(t)+z$ in system $(3\cdot1)$, then $z\in\Pi$ for x in Ω and

$$(3\cdot8) \qquad \dot{z}_i=\sum_{j=1}^{n} a_{ij}(t)\dot{g}_j(y_j(t))z_j+o(|z|) \quad \text{for } 1\leqq i\leqq n\,,$$

and Π is invariant for the above system. Considering the first approximation of system $(3\cdot8)$

$$(3\cdot9) \qquad \dot{z}=M(t)z\,,$$

where $M(t)=(m_{ij}(t))$ is defined by $m_{ij}(t)=a_{ij}(t)\dot{g}_j(y_j(t))$, condition (i) implies that Π is also invariant for $(3\cdot9)$. Then, by Lemma 6, if the trivial solution of $(3\cdot9)$ is shown to be $u.a.s$ in Π on $\boldsymbol{R}$, then the trivial solution of $(3\cdot8)$ has the same stability, and consequently $y(t)$ is $u.a.s$ in Ω on $\boldsymbol{R}$. Therefore it is sufficient to show that the trivial solution of $(3\cdot9)$ is $u.a.s$ in Π on $\boldsymbol{R}$.

Clearly $M(t)$ is bounded and uniformly continuous on $\boldsymbol{R}$, and we have

$$\sum_{i=1}^{n} m_{ij}(t)=\sum_{i=1}^{n} a_{ij}(t)\dot{g}_j(y_j(t))=0 \quad \text{for } 1\leqq j\leqq n$$

and

$$(3\cdot10) \qquad m_{ij}(t)=a_{ij}(t)\dot{g}_j(y_j(t))\geqq 0 \quad \text{for } i\neq j$$

because of conditions (i) and (ii), respectively. Thus $M(t)$ satisfies conditions (i) and (ii) in Theorem 1. Condition (iii) of Theorem 1 will be verified in the following way. Applying the same argument as in the proof of Lemma 4 to system $(3\cdot1)$, we can see that there exists a constant $c>0$ such that

$$(3\cdot11) \qquad 1\geqq y_i(t)\geqq c \quad \text{for } t\in\boldsymbol{R} \text{ and } 1\leqq i\leqq n\,,$$

and hence there is a constant $c'>0$ such that

$$\dot{g}_i(y_i(t))\geqq c' \quad \text{for } t\in\boldsymbol{R} \text{ and } 1\leqq i\leqq n\,.$$

Therefore, $(3\cdot10)$ implies

$$m_{ij}(t)\geqq c'a_{ij}(t) \quad \text{for } i\neq j\,,$$

which guarantees that each element of $H(M)$ is irreducible, because $A(t)$ is irreducible and almost periodic. Thus it follows from Theorem 1 that the trivial solution of (3·9) is *u. a. s* in Π on $\boldsymbol{R}$, i.e., all solutions of system (3·1) in Ω on $\boldsymbol{R}$ are *u. a. s* in Ω on $\boldsymbol{R}$. Therefore Lemma 8 concludes that system (3·1) possesses an almost periodic solution in Ω which remains in Ω^0 by (3·11), and the set of solutions in Ω on $\boldsymbol{R}$ is finite and consists of only almost periodic solutions $\phi_1, \cdots, \phi_m$ which satisfy

$$|\phi_i(t)-\phi_j(t)|\geqq\beta \text{ on } \boldsymbol{R} \text{ for } i\neq j \text{ and some constant } \beta>0 .$$

Next we shall show that there exists a $T>0$ such that each solution $x(t, t_0, x_0)$ of (3·1) with $x_0\in\Omega$ satisfies that for some ϕ_j and the constant δ_0 of Definition 2, (ii),

$$|x(t, t_0, x_0)-\phi_j(t)|<\delta_0 \text{ at some } t,\ t_0\leqq t\leqq t_0+T ,$$

which implies

$$(3\cdot 12) \qquad |x(t, t_0, x_0)-\phi_j(t)|\to 0 \text{ as } t\to\infty ,$$

because ϕ_j is *u. a. s* in Ω. Suppose that this is not true. Then there exists a small constant $\alpha>0$ less than β and sequences $\{t_k\}$ in $\boldsymbol{R}$ and $\{x_k\}$ in Ω such that

$$|x(t, t_k, x_k)-\phi_j(t)|\geqq\alpha \text{ for all } t\in[t_k, t_k+k] \text{ and all } j,\ 1\leqq j\leqq m.$$

Since $f(t, x)$ is almost periodic in t, we can choose a sequence $\{\tau_k\}$, $t_k+k/2<\tau_k<t_k+k$, such that

$$f(t+\tau_k, x)\to f(t, x) \text{ in } \boldsymbol{R}\times\Omega \text{ as } k\to\infty .$$

If we set $\psi_j(t, k)=\phi_j(t+\tau_k)$ for $1\leqq j\leqq m$ and $\psi_{m+1}(t, k)=x(t+\tau_k, t_k, x_k)$, these $m+1$ functions $\psi_j(t, k)$ satisfy

$$\dot{x}=f(t+\tau_k, x)$$

and

$$\psi_j(t, k)\in\Omega \text{ on } \boldsymbol{R} \text{ for } 1\leqq j\leqq m ,$$

$$\psi_{m+1}(t, k)\in\Omega \text{ for } t\geqq -k/2\geqq t_k-\tau_k ,$$

because $\psi_{m+1}(t_k-\tau_k, k)=x_k\in\Omega$. Moreover,

$$|\psi_j(0, k)-\psi_i(0, k)|\geqq\alpha \text{ for } i\neq j,\ 1\leqq i,\ j\leqq m+1 .$$

We may assume that $\psi_j(t, k)\to\psi_j(t)$ in $\boldsymbol{R}$ for some function ψ_j, $1\leqq j\leqq m+1$, as $k\to\infty$. Therefore ψ_j $(j=1,\cdots,n)$ are solutions of system (3·1), because $f(t+\tau_k, x)\to f(t, x)$ in $\boldsymbol{R}\times\Omega$ as $k\to\infty$, and

$$\psi_i(t)\in\Omega \text{ on } \boldsymbol{R} \text{ for } 1\leqq i\leqq m+1$$

$$|\psi_j(0)-\psi_i(0)|>\alpha \text{ for } i\neq j,\ 1\leqq i,\ j\leqq m+1\,.$$

which shows that system (3·1) has $m+1$ distinct solutions in Ω on $\boldsymbol{R}$. This is a contradiction. Therefore, $\phi_j(t)$ is *u. a. s* in the whole Ω on $\boldsymbol{R}$, if the uniqueness of ϕ_j is shown.

Now we shall prove the uniqueness of ϕ_j. Suppose $\phi_1\neq\phi_j$ for $j\geqq 2$ and set

$$S_1=\{x_0\in\Omega;\ |x(t, 0, x_0)-\phi_1(t)|\to 0 \text{ as } t\to\infty\}$$

and

$$S_2=\{x_0\in\Omega;\ |x(t, 0, x_0)-\phi_j(t)|\to 0 \text{ for some } \phi_j, j\geqq 2, \text{ as } t\to\infty\}.$$

Then S_1 and S_2 are open sets in Ω, and moreover these sets are nonempty and disjoint, because $|\phi_1(t)-\phi_j(t)|\geqq\alpha$ on $\boldsymbol{R}$ for $j\geqq 2$. On the other hand, (3·12) shows that $\Omega=S_1\cup S_2$, which contradicts the connectedness of Ω. Thus the uniqueness of an almost periodic solution is proved, and moreover, as is seen from [7], this uniqueness guarantees the module containment of the almost periodic solution.

Now consider the case where $A(t)$ is reducible. We can assume that $A(t)$ takes the form of

$$A(t)=\left(\begin{array}{c|c} * & 0 \\ \hline * & B(t)\end{array}\right),$$

where $B(t)$ is zero or a square irreducible matrix of order m, $2\leqq m\leqq n-1$. If $B(t)$ is zero, system (3·1) obviously has the constant solution $p(t)$ in $\partial\Omega$ such that $p_i(t)=0$ for $1\leqq i\leqq n-1$ and $p_n(t)=1$. In the latter case, if we set in system (3·1)

$$x_k=0 \text{ for } 1\leqq k\leqq m-n \text{ and } y_i=x_{n-m+i} \text{ for } 1\leqq i\leqq m\,,$$

then system (3·1) is reduced to the lower dimensional system

$$\dot{y}_i=\sum_{j=1}^{m} b_{ij}(t)g_j(y_j) \text{ for } 1\leqq i\leqq m\,, \tag{3·13}$$

where $B(t)=(b_{ij}(t))$. Since $B(t)$ is irreducible, the above system (3·13) has an almost periodic solution $y(t)$ such that

$$y_i(t) > 0 \text{ for } 1 \leqq i \leqq m \text{ and } \sum_{i=1}^{m} y_i(t) = 1$$

and furthermore the module of $y(t)$ is contained in the module of $B(t)$, i.e., of the module of $A(t)$. Thus, system $(3\cdot 1)$ has an almost periodic solution $p(t)$ in $\partial\Omega$ on $\boldsymbol{R}$ such that $p_i(t) = 0$ for $1 \leqq i \leqq n-m$ and $p_i(t) = y_{i-n+m}(t)$ for $n-m+1 \leqq i \leqq n$. The proof is completed.

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