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On the Dual of Besov Spaces

By

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§ 1. Introduction and Main Results

This paper is a supplement to the author's paper [6]. Here we shall discuss the space $B^{\sigma}_{p,\infty-}(\Omega)$, a closed subspace of $B^{\sigma}_{p,\infty}(\Omega)$, and determine the dual of Besov spaces $B^{\sigma}_{p,q}(\Omega)$.

For a measure space (M, μ) and a Banach space X by $L^p(M, \mu; X)$ we denote the space of all X-valued strongly measurable functions $f(x)$ such that $\|f(x)\|_X \in L^p(M, \mu)$. For the sake of simplicity, we write $d_*y = |y|^{-m}dy$, where $y \in M \subset \boldsymbol{R}^m$, $L_*{}^p(M; X) = L^p(M, d_*y; X)$, $1 \leqq p \leqq \infty$, and by $L_*{}^{\infty-}(M; X)$ we denote the closed subspace of all functions $f \in L_*{}^{\infty}(M; X)$ which converge to zero as $|y| \to 0$ and as $|y| \to \infty$. We shall make use of the following conventions: $p < \infty - < \infty$ for real p, $1/\infty - = 1/\infty = 0$.

The space $B^{\sigma}_{p,\infty-}(\Omega; X)$ is defined as follows:

Definition 1.1. Let Ω be an open set in $\boldsymbol{R}^n$. For $0 < \sigma < 1$ $B^{\sigma}_{p,\infty-}(\Omega; X)$ is the space of all functions $f \in L^p(\Omega; X)$ such that

$$\|f(x+y) - f(x)\|_{L^p(\Omega_{1,y}; X)} |y|^{-\sigma} \in L_*{}^{\infty-}(\boldsymbol{R}^n),$$

where $\Omega_{j,y} = \{x; x, x+y, \cdots, x+jy \in \Omega\}$, and $B^1_{p,\infty-}(\Omega; X)$ is the space of all $f \in L^p(\Omega; X)$ such that

$$\|f(x+2y) - 2f(x+y) + f(x)\|_{L^p(\Omega_{2,y}; X)} |y|^{-1} \in L_*{}^{\infty-}(\boldsymbol{R}^n).$$

For $\sigma = k + \theta$, $0 < \theta \leqq 1$, k is a positive integer, $B^{\sigma}_{p,\infty-}(\Omega; X)$ is the space of all $f \in W_p{}^k(\Omega; X)$ whose all partial derivaties $D^{\alpha}f$ of order k belong, to $B^{\theta}_{p,\infty-}(\Omega; X)$. Finally, for $\sigma = k + \theta$, $0 < \theta \leqq 1$, k is a negative integer,

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$B^{\sigma}_{p,\infty-}(\Omega; X)$ is the space of all f which can be expressed as

$$f=\sum_{|\alpha|\leqq -k} D^{\alpha} f_{\alpha},\ f_{\alpha}\in B^{\theta}_{p,\infty-}(\Omega; X)^{1)}.$$

The space $B^{1}_{\infty,\infty-}(\boldsymbol{T}^1)$, where $\boldsymbol{T}^1$ is the 1-dimensional torus, is identical with the space of smooth functions due to Zygmund [7]. In his paper it is shown that the space $B^{1}_{p,\infty-}$ plays an essential rôle in problems of the theory of real functions and of trigonometric series. Our notation $L_*^{\infty-}$ is due to Komatsu [3].

As in [6] we assume throughout this paper that Ω is an open set with the cone property, and by $\Psi(x)$, t_0, b, $\mathcal{K}_j$ we denote the same things as in [6] (see p. 328, p. 329).

Now, we state our main results. By $B^{\sigma}_{p,q,\bar{\Omega}}(\boldsymbol{R}^n)$ we shall denote the closed subspace of all $f\in B^{\sigma}_{p,q}(\boldsymbol{R}^n)$ whose support is contained in $\bar{\Omega}$.

Theorem 1.2. *If* $1\leqq p\leqq\infty$, $1\leqq q\leqq\infty$, $1/p+1/p'=1$, $1/q+1/q'=1$, *and if* σ *is a real number, then there exists unique continuous bilinear form* $\langle f, g\rangle$ *on* $B^{\sigma}_{p,q}(\Omega)\times B^{-\sigma}_{p',q',\bar{\Omega}}(\boldsymbol{R}^n)$ *with the following properties*: (i) *If* $j>-\sigma$, $K\in\mathcal{K}_j$, $0<a<t_0$, $u(t,x)\in L_*^{q}([0,a]; L^p(\Omega))$, $K^0\in\mathcal{K}_0$, $f^0(x)\in L^p(\Omega)$, *and if* $g\in B^{-\sigma}_{p',q',\bar{\Omega}}(\boldsymbol{R}^n)$, *then*

$$(1\cdot 1)\quad \left\langle \int_0^a d_*t\int t^{\sigma-n}K\left(t,x,\frac{x-y}{t},y\right)u(t,y)\,dy,\ g(x)\right\rangle$$

$$=\int_0^a d_*t\int t^{\sigma-n}u(t,y)\left\langle K\left(t,x,\frac{x-y}{t},y\right), g(x)\right\rangle_x dy\,,$$

$$(1\cdot 2)\quad \left\langle \int a^{-n}K^0\left(a,x,\frac{x-y}{a},y\right)f^0(y)\,dy,\ g(x)\right\rangle$$

$$=\int f^0(y)\left\langle a^{-n}K^0\left(a,x,\frac{x-y}{a},y\right), g(x)\right\rangle_x dy\,.$$

(ii) *If* $i>\sigma$, $H\in\mathcal{K}_i$, $v(t,x)\in L_*^{q'}([0,a]; L^{p'}(\Omega))$, $H^0\in\mathcal{K}_0$, $g^0(x)\in L^{p'}(\Omega)$, *and if* $f\in B^{\sigma}_{p,q}(\Omega)$, *then*

$$(1\cdot 3)\quad \left\langle f(x),\ \int_0^a d_*t\int t^{-\sigma-n}H\left(t,y,\frac{y-x}{t},x\right)v(t,y)\,dy\right\rangle$$

$$=\int_0^a d_*t\int t^{-\sigma-n}v(t,y)\left\langle H\left(t,y,\frac{y-x}{t},x\right), f(x)\right\rangle_x dy\,,$$

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1) $\alpha=(\alpha_1,\cdots,\alpha_n)$, $|\alpha|=\alpha_1+\cdots+\alpha_n$, $D_j=\partial/\partial x_j$, $D^{\alpha}=D_1^{\alpha_1}\cdots D_n^{\alpha_n}$.

$$(1\cdot 4)\quad \left\langle f(x), \int a^{-n}H^0\left(a, y, \frac{y-x}{a}, x\right) g^0(y)\,dy \right\rangle$$

$$= \int g^0(y) \left\langle a^{-n}H^0\left(a, y, \frac{y-x}{a}, x\right), f(x)\right\rangle_x dy\,.$$

Here $\langle \varphi(x), f(x)\rangle_x$ *denotes the duality on* $\mathcal{D}(\Omega)\times\mathcal{D}'(\Omega)$.

Theorem 1.3. *Let* σ, p, q, p', q', *and* $\langle\ ,\ \rangle$ *be as in Theorem* 1.2. *Then* $B^{\sigma}_{p,q}(\Omega)$ *and* $B^{-\sigma}_{p',q',\bar\Omega}(\boldsymbol{R}^n)$ *form a dual pair with respect to* $\langle\ ,\ \rangle$: (i) *If* $g\in B^{-\sigma}_{p',q',\bar\Omega}(\boldsymbol{R}^n)$, *and if* $\langle f, g\rangle=0$ *for all* $f\in B^{\sigma}_{p,q}(\Omega)$, *then* $g=0$. (ii) *If* $f\in B^{\sigma}_{p,q}(\Omega)$, *and if* $\langle f, g\rangle=0$ *for all* $g\in B^{-\sigma}_{p',q',\bar\Omega}(\boldsymbol{R}^n)$, *then* $f=0$.

By Theorem 1.2 and Theorem 1.3 we observe that the mapping $g\to l_g$, where $l_g(f)=\langle f, g\rangle$ for all $f\in B^{\sigma}_{p,q}(\Omega)$, is a continuous injection from $B^{-\sigma}_{p',q',\bar\Omega}(\boldsymbol{R}^n)$ into $\{B_{p,q}(\Omega)\}'$ (the dual space of $B^{\sigma}_{p,q}(\Omega)$), and the mapping $f\to l_f$, where $l_f(g)=\langle f, g\rangle$ for all $g\in B^{-\sigma}_{p',q',\bar\Omega}(\boldsymbol{R}^n)$, is a continuous injection from $B^{\sigma}_{p,q}(\Omega)$ into $\{B^{-\sigma}_{p',q',\bar\Omega}(\boldsymbol{R}^n)\}'$. In [6] we proved that these mappings are surjective ([6] Theorem 9) if $1<p<\infty$, $1<q<\infty-$. But there are some other cases for which they are surjective. Namely,

Theorem 1.4. *Let* p, q, p', q', *and* σ *be as in Theorem* 1.2. (i) *The dual of* $B^{\sigma}_{p,q}(\Omega)$ *is canonically isomorphic to* $B^{-\sigma}_{p',q',\bar\Omega}(\boldsymbol{R}^n)$, *that is, the mapping* $g\to l_g$ *is an isomorphism if* (*a*) $1\leqq p<\infty$, $1\leqq q\leqq\infty-$, $q'\neq\infty-$, *or* (b) Ω *is bounded and* $p=\infty$, $1\leqq q\leqq\infty-$, $q'\neq\infty-$. (ii) *The dual of* $B^{-\sigma}_{p',q',\bar\Omega}(\boldsymbol{R}^n)$ *is canonically isomorphic to* $B^{\sigma}_{p,q}(\Omega)$, *that is, the mapping* $f\to l_f$ *is an isomorphism if* (*a*) $1\leqq p'<\infty$, $1\leqq q'\leqq\infty-$, $q\neq\infty-$, *or* (*b*) Ω *is bounded and* $p'=\infty$, $1\leqq q'\leqq\infty-$, $p=1$, $q\neq\infty-$.

Some special cases of Theorem 1.4 are proved by Flett: The proof of the fact that $\{B^{\sigma}_{p,1}(\boldsymbol{R}^n)\}'=B^{-\sigma}_{p',\infty}(\boldsymbol{R}^n)$ for $1\leqq p<\infty$, and $\{B^{\sigma}_{p,\infty-}(\boldsymbol{R}^n)\}'=B^{-\sigma}_{p,1}(\boldsymbol{R}^n)$ for $1<p<\infty$, is given in [1], and that of $\{B^{\sigma}_{p,1}(\boldsymbol{T}^1)\}'=B^{-\sigma}_{p',\infty}(\boldsymbol{T}^1)$ and $\{B^{\sigma}_{p,\infty-}(\boldsymbol{T}^1)\}'=B^{-\sigma}_{p',1}(\boldsymbol{T}^1)$ for $1\leqq p\leqq\infty$, where $\boldsymbol{T}^1$ is the one-dimensional torus (circle), is given in [2].

§ 2. The Space $L^q(M, \mu; L^p)$

We shall begin with observing some properties of the space $L^q(M, \mu; L^p)$ which is closely related to Besov spaces.

Lemma 2.1. *Let (M, μ) be a σ-finite measure space, $1 \leqq p \leqq \infty$, $\{E_k\}_{k=1,2,\cdots}$, be an increasing sequence of measurable sets of finite measure whose union is M, f be a μ-measurable function, and let $f_k(x) = \min \{|f(x)|, k\} \chi_k(x)$, where χ_k is the characteristic function of E_k. Then $\|f_k\|_{L^p(M)} \to \|f\|_{L^p(M)}$ as $k \to \infty$.*

Proof. It is obvious that $f_k(x) \to |f(x)|$ as $k \to \infty$ for every x. If $1 \leqq p < \infty$, then the assertion follows from Fatou's lemma. Assume now $p = \infty$, and let $l_k = \text{ess sup} f_k(x)$. Since $\{f_k(x)\}; k = 1, 2, \cdots$, is increasing it follows that $l = \lim l_k$ exists. The fact that $f_k(x) \leqq |f(x)|$ implies that $l \leqq \|f\|_{L^\infty(M)}$. On the other hand, it is seen that $\{x; |f(x)| = \lim f_k(x) > r\} = \cup_k \{x; f_k(x) > r\}$, which gives the converse inequality. The proof is complete.

Lemma. 2.2. *Let M be a measurable set in $\boldsymbol{R}^n$, $1 \leqq p \leqq \infty$, and let $1/p + 1/p' = 1$, $p' \neq \infty -$. If f is a measurable non-negative function such that*

$$\int_M f(x) g(x) d_* x \leqq c \|g\|_{L_*^p(M)} \tag{2·1}$$

holds for all non-negative function g in $L_*^p(M)$, then $f \in L_*^{p'}(M)$ and $\|f\|_{L_*^{p'}} \leqq c$.

Proof. Let $E_k = M \cap \{x; 1/k < |x| < k\}$, χ_k be the characteristic function of E_k, and let $f_k(x) = \min\{f(x), k\} \chi_k(x)$. By Lemma 2.1 it is sufficient to prove that $\|f_k\|_{L_*^{p'}} \leqq c$.

(i) Case $p = 1$. Let χ_{kr} be the characteristic function of the set $\{x; f_k(x) > r\}$. Substituting χ_{kr} for g, by (2·1) we have

$$\int r \chi_{kr}(x) d_* x \leqq \int \chi_{kr}(x) f(x) d_* x \leqq c \int \chi_{kr}(x) d_* x .$$

Hence, for $r > c$ the measure of the set $\{x; f_k(x) > r\}$ is equal to zero, that is, ess. $\sup f_k(x) \leqq c$.

(ii) Case $1 < p < \infty -$. Taking $g(x) = f_k(x)^{p'-1}$ we obtain

$$(\|f_k\|_{L_*^{p'}})^{p'} \leqq \int f_k(x)^{p'-1} f(x)\, d_*x \leqq c(\|f_k\|_{L_*^{p'}})^{p'-1},$$

and hence $\|f_k\|_{L_*^{p'}} \leqq c$.

(iii) $p \geqq \infty -$. Since $\chi_k \in L_*^{\infty -}$, taking $g = \chi_k$, we have by (2·1) that

$$\|f_k\|_{L_*^1} \leqq \int \chi_k(x) f(x)\, d_*x \leqq c \|\chi_k\|_{L_*^\infty} = c.$$

Lemma 2. 3. *Let* (M_1, μ) *be* σ*-finite measure space,* M_2 *be a measurable set in* $\boldsymbol{R}^n$, $1 \leqq p,\ q \leqq \infty$, $1/p + 1/p' = 1$, $1/q + 1/q' = 1$, $q' \neq \infty -$, *and let* $g(x, y)$ *be a measurable function. Then*

$$\left| \iint_{M_1 \times M_2} f(x,y)\, g(x,y)\, \mu(dx)\, d_*y \right| \leqq C \|f\|_{L_*^q(M_2; L^p(M_1))} \tag{2·2}$$

holds for all $f \in L_*^q(M_2; L^p(M_1))$ *if and only if*

$$g \in L_*^{q'}(M_2; L^{p'}(M_1)) \quad \textit{and} \quad \|g\|_{L_*^{q'}(M_2;\, L^{p'}(M_1))} \leqq C\,. \tag{2·3}$$

Proof. The fact that (2·3) implies (2·2) is a consequence of Hölder's inequality. Conversely assume that (2·2) holds for all f in $L_*^q(M_2;\ L^p(M_1))$. Let $\psi(y) = \|g(x,y)\|_{L^{p'}(M_1)}$ if $g(x,y) \in L^{p'}(M_1)$, and $\psi(y) = \infty$ otherwise. Then, it is sufficient to show that

$$\psi \in L_*^{q'}(M_2) \quad \text{and} \quad \|\psi\|_{L_*^{q'}} \leqq C\,. \tag{2·4}$$

To show this let $\{E_k\}$ be an incresing sequence of measurable set in M_1 such that $0 < \mu_1(E_k) < \infty$ and $\cup E_k = M_1$, $g_k(x,y) = \min\{|g(x,y)|, k\}$ $\chi_k(x)$, where χ_k is the characteristic function of E_k, and let $\psi_k = \|g_k(x,y)\|_{L^p(M_1)}$. Then, by Lemma 2.2 (2·4) follows from the fact that

$$\int \varphi(y)\, \psi_k(y)\, d_*y \leqq C \|\varphi\|_{L_*^q(M_2)} \tag{2·5}$$

holds for any non-negative functions φ in $L_*^q(M_2)$. First consider the case where $p = 1$. Let $0 < \eta < 1$, $E_{k,\eta} = \{(x,y)\,;\, x \in E_k,\ g_k(x,y) \geqq \eta \psi_k(y)\}$, $\sigma_k(y) = \mu\{x; (x,y) \in E_{k,\eta}\}$, and let $\chi_{k,\eta}$ be the characteristic function of $E_{k,\eta}$. For a complex number ζ let $e(\zeta) = |\zeta|/\zeta$ if $\zeta \neq 0$ and $e(0) = 0$. Let $f(x,y) = \varphi(y)\, \sigma_k(y)^{-1} \chi_{k,\eta}(x,y)\, e(g(x,y))$. Then

$$\eta\int\varphi(y)\psi_k(y)\,d_*y\leqq\iint f(x,y)\,g(x,y)\,\mu(dx)\,d_*y$$
$$\leqq C\|f\|_{L_*^q(M_2;\,L^1(M_1))}=C\|\varphi\|_{L_*^q(M_2)}\,.$$

Therefore, letting $\eta\to 1$, we obtain $(2\cdot 5)$. Next assume that $1<p<\infty$. Let $f(x,y)=\varphi(y)e(g(x,y))\{g_k(x,y)/\psi_k(y)\}^{p'-1}$ if $\psi_k(y)\neq 0$, and $f(x,y)=0$ if $\psi_k(y)=0$. Then we have

$$\int\varphi(y)\psi_k(y)\,d_*y\leqq\iint f(x,y)\,g(x,y)\,\mu(dx)\,d_*y$$
$$\leqq C\|f\|_{L_*^q(M_2:\,L^p(M_1))}=C\|\varphi\|_{L_*^q(M_2)}\,.$$

Finally let $f(x,y)=\chi_k(x)\varphi(y)e(g(x,y))$. Then we obtain $(2\cdot 5)$ for the case $p=\infty$.

Lemma 2.4. *Let (M_1,μ_1), (M_2,μ_2) be σ-finite measure spaces, $1\leqq p,q\leqq\infty$, and let X be a Banach space. Assume that $K(x,\xi,y,\eta)$ is a measurable function such that*

$$\sup_y\int|K(x,\xi,y,\eta)|\,\mu_2(d\eta),\ \sup_\eta\int|K(x,\xi,y,\eta)|\,\mu_2(dy)\leqq k(x,\xi),$$

$$\sup_x\int k(x,\xi)\,\mu_1(d\xi),\ \sup_\xi\int k(x,\xi)\,\mu_1(dx)\leqq C<\infty.$$

Then the operator defined by

$$(Tu)\,(x,y)=\iint K(x,\xi,y,\eta)\,u(\xi,\eta)\,\mu_1(d\xi)\,\mu_2(d\eta)$$

is a bounded linear operator on $L^q(M_1,\mu_1;L^p(M_2,\mu_2;X))$ with norm not greater than C.

Proof. This follows from Lemma 2.5 in [5].

Corollary 2.5. *Let (M_1,μ_1), (M_2,μ_2), p, q and K be as in Lemma 2.4, and let $1/p+1/p'=1/q+1/q'=1$. If $u\in L^q(M_1,\mu_1;L^p(M_1,\mu_2))$ and $v\in L^{q'}(M_1,\mu_1;L^{p'}(M_2,\mu_2))$, then the integral*

$$\iiiint K(x,\xi,y,\eta)\,u(x,y)\,v(\xi,\eta)\,\mu_1(dx)\,\mu_1(d\xi)\,\mu_2(dy)\,\mu_2(d\eta)$$

is absolutely convergent and its absolute value is not greater than

$C\|u\|_{L^q(M_1; L^p(M_2))}\|v\|_{L^{q'}(M_1; L^{p'}(M_2))}$.

Proof. From Lemma 2.4 it follows that

$$\int\int |K(x, \xi, y, \eta)||v(\xi, \eta)|\mu_1(d\xi)\mu_2(d\eta) \in L^{q'}(M_1; L^{p'}(M_2)).$$

Therefore, making use of Hölder's inequality twice, we obtain the desired result.

In the following of this section X will denote a Banach space and B will denote the unit ball in $\boldsymbol{R}^n$.

Lemma 2.6. *Let* $\Omega_1 \subset \boldsymbol{R}^n$, $\Omega_2 \subset \boldsymbol{R}^m$, $1 \leqq q \leqq \infty-$, $1/q+1/q'=1$, $\lambda>0$, *and let* $K(x, y)$ *be a measurable function satisfying*

$$(2\cdot6) \qquad \sup_x \|K(x, y)\|_{L_*^{q'}(\Omega_2)} = C < \infty,$$

$$(2\cdot7) \qquad \sup_x \|K(x, y)\|_{L_*^{q'}(\Omega_2 \cap r|x|^\lambda B)} = \varphi_1(r) \to 0 \text{ as } r \to 0, \text{ and}$$

$$(2\cdot8) \qquad \sup_x \|K(x, y)\|_{L_*^{q'}(\Omega_2 \setminus r|x|^\lambda B)} = \varphi_2(r) \to 0 \text{ as } r \to \infty.$$

Then the integral operator $(Tf)(x) = \int K(x, y) f(y) d_* y$ *is a bounded operator from* $L_*^q(\Omega_2; X)$ *into* $L_*^{\infty-}(\Omega_1; X)$.

Proof. By Hölder's inequality we have

$$\|Tf\|_{L_*^\infty(\Omega_1; X)} \leqq C\|f\|_{L_*^q(\Omega_2; X)},$$

and

$$\|Tf(x)\|_X \leqq \left\{ \int_{\Omega_2 \cap r|x|^\lambda B} + \int_{\Omega_2 \setminus r|x|^\lambda B} \right\} |K(x, y)| \, \|f(y)\|_X d_* y,$$

$$\leqq \varphi_1(r) \|f\|_{L_*^q(\Omega_2; X)} + C\|f\|_{L_*^q(\Omega_2 \setminus r|x|^\lambda B; X)}.$$

Since $q \leqq \infty-$, it follows that $\|f\|_{L_*^q(\Omega_2 \setminus r|x|^\lambda B)} \to 0$ as $|x| \to \infty$, and therefore $\limsup_{|x| \to \infty} \|Tf(x)\|_X \leqq \varphi_1(r) \|f\|_{L_*^q(\Omega_2; X)}$. Letting $r \to 0$, we have $\|Tf(x)\|_X \to 0$ as $|x| \to \infty$. Similarly we have $\|Tf(x)\|_X \to 0$ as $|x| \to 0$, and the proof is complete.

Corollary 2.7. *Let* $\Omega_1 \subset \boldsymbol{R}^n$, $\Omega_2 \subset \boldsymbol{R}^m$, $1 \leqq p \leqq q \leqq \infty$ *(including* $\infty-$*). And let*

$$(T_1f)(x) = \int_{|y|\geqq\rho|x|^\lambda} |x|^{\lambda\sigma}|y|^{-\sigma}f(y)\,d_*y \qquad (\lambda, \rho, \sigma>0),$$

$$(T_2f)(x) = \int_{|y|\leqq\rho|x|^\lambda} |x|^{-\lambda\sigma}|y|^{\sigma}f(y)\,d_*y \qquad (\lambda, \rho, \sigma>0)$$

and

$$(T_3f)(x) = \int \min\{|x|^\sigma|y|^{-\sigma}, |x|^{-\tau}|y|^\tau\}f(y)\,d_*y\,(\sigma, \tau>0)$$

for any $f\in L_*{}^p(\Omega_2; X)$. *Then* T_1, T_2 *and* T_3 *are bounded operators from* $L_*{}^p(\Omega_2; X)$ *into* $L_*{}^q(\Omega_1; X)$.

Proof. Let $\zeta_\sigma(t)=t^\sigma$ if $t\leqq 1$ and $\zeta_\sigma(t)=0$ if $t>1$. Then the kernel of T_1 is $\zeta_\sigma(\rho|x|^\lambda/|y|)\rho^{-\sigma}$ and that of T_2 is $\zeta_\sigma(|y|/\rho|x|^\lambda)\rho^\sigma$. It is easy to show that these kernels and $\min\{|x|^\sigma|y|^{-\sigma}, |x|^{-\tau}|y|^\tau\}$ satisfy the conditions stated in Lemma 2.6 and [5] Lemma 2.5, which gives the desired result.

§ 3. Besov Spaces $B^\sigma_{p,\infty-}$

In this section we discuss the properties of the Besov spaces $B^\sigma_{p,\infty-}$. Throughout this section X is a Banach space.

Lemma 3.1. *If* $1\leqq p\leqq\infty$, $1\leqq q_0\leqq q_1\leqq\infty$, j *is a non-negative integer,* $K(t, x, z, y)\in\mathcal{K}_j$, *and if* $\sigma<j$, *then the mapping*

$$f\mapsto t^{-\sigma}U(t, x)=t^{-\sigma}\langle t^{-n}K(t, x, (x-y)/t, y), f(y)\rangle_y$$

is a bounded linear operator from $B^\sigma_{p,q_0}(\Omega)$ *into* $L_*{}^{q_1}([0, a]; L^p(\Omega; X))$.

Proof. With the aid of Corollary 2.7, the same reasoning as in the proof of Lemma 4.1 in [6] gives the assertion.

Lemma 3.2. *If* $1\leqq p\leqq\infty$, $1\leqq q_0\leqq q_1\leqq\infty$, j *is a non-negative integer,* $K(t, x, z, y)\in\mathcal{K}_j$, *and if* $-\sigma<j$, *then the mapping*

$$u(t, x)\mapsto\int_0^a t^\sigma d_*t\int K(t, x, -z, x+tz)\,u(t, x+tz)\,dz$$

is a bounded linear operator from $L_*{}^{q_0}([0, a]; L^p(\Omega; X))$ *into* $B^\sigma_{p,q_1}(\Omega; X)$.

Proof. Corollary 2.7 and the same argument as in the proof of Lemma 4.2 in [6] give the conclusion.

By Lemma 3.1, Lemma 3.2 and the integral representation ([6] Theorem 1), we obtain the charaterization theorem for Besov spaces $B^{\sigma}_{p,\infty-}$. That is,

Theorem 3.3. *Let* $1 \leqq p \leqq \infty$, $\sigma \in \boldsymbol{R}$, *and let* j *be a non-negative integer with* $j > \sigma$. *Then* $f \in B^{\sigma}_{p,\infty-}(\Omega; X)$ *if and only if* $f \in W_p^{-\infty}(\Omega; X)$ *and* $t^{-\sigma}\langle t^{-n}K(x, (x-y)/t), f(y)\rangle_y \in L_*^{\infty-}([0, a]; L^p(\Omega; X))$ *for any* $K(x, z) \in \mathcal{K}_j$.

By Theorem 3.3 and Lemma 3.1 we have the following imbedding theorem (c.f. Remark in [6] p. 357).

Theorem 3.4. *If* $1 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty -$, *then the imbedding operator*

$$B^{\sigma}_{p,q}(\Omega; X) \to B^{\sigma}_{p,\infty-}(\Omega; X)$$

exists.

Let $1 \leqq q \leqq \infty -$, $u(t, x) \in L_*^q([0, a]; L^p(\Omega; X))$, and let for any $\varepsilon > 0$ $u_\varepsilon(t, x) = u(t, x)$ if $\varepsilon \leqq t \leqq a$ and $u(t, x) = 0$ if $t < \varepsilon$. Then $u_\varepsilon(t, x) \to u(t, x)$ in $L_*^q([0, a]; L^p(\Omega; X))$ as $\varepsilon \to 0$. This fact, Lemma 3.1, Lemma 3.2, Lemma 3.1 in [6], and Lemma 3.4 in [6], give the following theorem:

Theorem 3.5. (*Approximation*). *Let* $K(t, x, z, y) \in \mathcal{K}_0$, $\int K(0, x, z, x)\,dz = 1$, m *be a positive integer*,

$$K_m(t, x, z, y) = \sum_{|\alpha| < m} \frac{1}{\alpha!} D_z^{\alpha}\{z^{\alpha}K(t, x, z, y)\},$$

$1 \leqq p \leqq \infty$, $1 \leqq q \leqq \infty -$, *and let* $\sigma < m$. *Then for any* $f \in B^{\sigma}_{p,q}(\Omega; X)$

$$U_m(t, x) = \langle t^{-n}K(t, x, (x-y)/t, y), f(y)\rangle_y$$

converges to f *in* $B^{\sigma}_{p,q}(\Omega; X)$ *as* $t \to 0$, *and for any* $g \in B^{\sigma}_{p,q,\bar{\Omega}}(\boldsymbol{R}^n; X)$

$$V_m(t, x) = \langle t^{-n}K_m(t, y, (y-x)/t, x), g(y)\rangle_y$$

converges to g *in* $B^{\sigma}_{p,q,\Omega}(\boldsymbol{R}^n; X)$ *as* $t\to 0$.

Since for $u\in L_*^{\infty}([0,a];L^p)$ u_ε does not converge to u as $\varepsilon\to 0$ in $L_*^{\infty}([0,a];L^p)$ unless $u\in L_*^{\infty-}([0,a];L^p)$, the conclusion of Theorem 3.5 is not valid in case $q=\infty$. Consequently, the present author should have assumed that $q\leqq\infty-$ in [6] Theorem 5. Also, in the assertion that C_0^{∞} is dense in $B^{\sigma}_{p,q,\Omega}(\boldsymbol{R}^n)$ ([6] p. 368) he should have assumed that $p<\infty$ and $q\leqq\infty-$.

For a measurable set M in $\boldsymbol{R}^n$ and a Banach space X $L_*^{p,\sigma}(M;X)$ denotes the space of functions f such that $|x|^{-\sigma}f(x)\in L_*^{p}(M;X)$, and their norm is defined by

$$\|f\|_{L_*^{p,\sigma}(M;X)}=\|\,|x|^{-\sigma}f(x)\|_{L_*^{p}(M;X)}\,.$$

Let X and Y be Banach spaces contained in a Hausdorff vector space E. The mean interpolation space due to Lions and Peetre, which is denoted by $(X,Y)_{\theta,p}$, $0<\theta<1$, $1\leqq p\leqq\infty$, is the space of means

$$f=\int_0^{\infty}u(t)\,d_*t,\ \ u\in L_*^{p,-\theta}(\boldsymbol{R}_+;X)\cap L_*^{p,1-\theta}(\boldsymbol{R}_+;Y).$$

Lemma 3.6. *Let* σ *and* τ *be real numbers,* $\sigma\neq\tau$, $0<\theta<1$, $\mu=(1-\theta)\sigma+\theta\tau$, *and let* M *be a measurable set in* $\boldsymbol{R}^n$. *Then*

$$(L_*^{\infty,\sigma}(M;X),L_*^{\infty,\tau}(M;X))_{\theta,\infty-}\subset L_*^{\infty-,\mu}(M;X)$$
$$\subset(L_*^{1,\sigma}(M;X),L_*^{1,\tau}(M;X))_{\theta,\infty-}$$

with continuous injections.

Proof. Assume that $\lambda=\sigma-\tau>0$ (the case where $\lambda<0$ is discussed analogously). Let $f\in(L_*^{\infty,\sigma}(M;X),L_*^{\infty,\tau}(M;X))_{\theta,\infty-}$. Then there exists $u(t,x)$ such that

$$f(x)=\int_0^{\infty}u(t,x)\,d_*t\,,$$

and $u(t,x)\in L_*^{\infty-,-\theta}(\boldsymbol{R}_+;L_*^{\infty,\sigma}(M;X))\cap L_*^{\infty-,1-\theta}(\boldsymbol{R}_+;L_*^{\infty,\tau}(M;X))$. It follows from this that

$$\|\,|x|^{-\mu}f(x)\|_X\leqq\left\{\int_{|x|^{\lambda}}^{\infty}+\int_0^{|x|^{\lambda}}\right\}\|\,|x|^{-\mu}u(t,x)\|_X\,d_*t\,,$$

$$\leqq \int_{|x|^\lambda}^{\infty} (|x|^\lambda/t)^\theta \|t^\theta u(t, x)\|_{L_*^{\infty,\sigma}(M; X)} d_*t$$

$$+ \int_0^{|x|^\lambda} (|x|^\lambda/t)^{\theta-1} \|t^{\theta-1} u(t, x)\|_{L_*^{\infty,\tau}(M; X)} d_*t .$$

Therefore, by Corollary 2.7 we have $\|x^{-\mu} f(x)\|_X \in L_*^{\infty-}(M; X)$. Conversely, let $f \in L_*^{\infty-,\mu}(M; X)$. Let $\varphi \in C_0(\boldsymbol{R}_+)$ such that $\int_{\boldsymbol{R}_+} \varphi(t) d_*t = 1$, and define $u(t, x) = \varphi(t|x|^{-\lambda}) f(x)$. Then it is obvious that

$$f(x) = \int_0^\infty u(t, x) d_*t .$$

Since

$$\|t^\theta |x|^{-\sigma} u(t, x)\|_{L_*^1(M; X)} = \int \varphi(t|x|^{-\lambda}) (t|x|^{-\lambda})^\theta \| |x|^{-\mu} f(x)\|_X d_*x ,$$

and since the kernel $\varphi(t|x|^{-\lambda})(t|x|^{-\lambda})^\theta$ satisfies the conditions stated in Lemma 2.6 (this fact is a consequence of a simple calculation), it follows from Lemma 2.6 that $t^\theta u(t, x) \in L_*^{\infty-}(\boldsymbol{R}_+; L_*^{1,\sigma}(M; X))$. The same observation gives that $t^{\theta-1} u(t, x) \in L_*^{\infty-}(\boldsymbol{R}_+; L_*^{1,\tau}(M; X))$. Hence f is an element of $(L_*^{1,\sigma}(M; X), L_*^{1,\tau}(M; X))_{\theta,\infty-}$.

Theorem 3.7. *Let σ and τ be real numbers, $\sigma \neq \tau$, $0 < \theta < 1$, and let $\mu = (1-\theta)\sigma + \theta\tau$. Then*

$$(B^\sigma_{p,q_0}(\Omega; X), B^\tau_{p,q_1}(\Omega; X))_{\theta,\infty-} = B^\mu_{p,\infty-}(\Omega; X),$$

$$(H_p^\sigma(\Omega; X), H_p^\tau(\Omega; X))_{\theta,\infty-} = B^\mu_{p,\infty-}(\Omega; X)$$

Proof. This follows from Lemma 3.6 and Theorem 3.3 (see Proof of [6] Theorem 8).

Theorem 3.8. *If k is an integer, and if $1 \leqq p < \infty$, then $W_p^k(\Omega) \hookrightarrow B^k_{p,\infty-}(\Omega)$ with continuous injection.*

Proof. Let $k \geqq 0$, $f \in W_p^k(\Omega)$, and let $K(x, z) = \sum_{|\alpha|=k} D_z^\alpha K(x, z)$, $K_\alpha \in \mathcal{K}_1$. Then, since $\int K_\alpha(x, -z) dz = 0$,

$$\left\langle t^{-k-n} K\left(x, \frac{x-y}{t}\right), f(y)\right\rangle_y = \sum_{|\alpha|=k} \int K_\alpha(x, -z) \{f^{(\alpha)}(x+tz) - f^{(\alpha)}(x)\} dz$$

where $f^{(\alpha)}(x)=D^\alpha f(x)$. Since

$$\|f^{(\alpha)}(x+y)-f^{(\alpha)}(x)\|_{L^p}\to 0 \quad as \quad |y|\to 0,$$

it follows that $\langle t^{-k-n}K(x,(x-y)/t), f(y)\rangle_y\in L_*^{\infty-}([0,a]; L^p(\Omega))$, and hence $f\in B^k_{p,\infty-}(\Omega)$. If $k<0$, $f\in W_p^k(\Omega)$, $K\in\mathcal{K}_0$, this facts follows from the identity

$$\left\langle t^{-k-n}K\left(x, \frac{x-y}{t}\right), f(y)\right\rangle_y$$

$$=\sum_{|\alpha|\leqq -k}\int t^{-|\alpha|-k}K^{(0,\alpha)}(x,-z)\{f_\alpha(x+tz)-f_\alpha(x)\}dz,$$

where $f=\sum_{|\alpha|\leqq -k}D^\alpha f_\alpha$, $f_\alpha\in L^p(\Omega)$, and $K^{(0,\alpha)}(x,z)=D_z^\alpha K(x,z)$.

§ 4. Proof of the Main Results

To prove the main results the following is fundamental.

Lemma 4.1. *Let* $K(t,x,z,y)\in\mathcal{K}_j$, $H(t,x,z,y)\in\mathcal{K}_i$, *and let*

$$J(s,t,x,y)=s^{-n}t^{-n}\int K\left(s,z,\frac{z-x}{s},x\right)H\left(t,y,\frac{y-z}{t},z\right)dz.$$

Then,

$$\sup_y\int|J(s,t,x,y)|dx,\ \sup_x\int|J(s,t,x,y)|dy\leqq C\min(s^jt^{-j}, t^is^{-i})$$

holds for $0<s,\ t\leqq a<t_0$, *where* C *is a constant independent of* s,t.

Proof. From the identity

$$K\left(s,z,\frac{z-x}{s},x\right)=s^j\sum_{|\beta|\leqq j}(-D_z)^\beta\left\{\hat{K}_\beta\left(t,z,\frac{z-x}{t},x\right)\right\}$$

(see [6] p. 336)

it follows that

$$J(s,t,x,y)=\sum_{|\beta|\leqq j}s^{j-n}t^{-n}\int\hat{K}_\beta\left(s,z,\frac{z-x}{s},x\right)D_z^\beta\left\{H\left(t,z,\frac{z-y}{t},y\right)\right\}dz.$$

This implies that

$$\int|J(s,t,x,y)|dx\leqq C_1s^{j-n}t^{-j-n}\int_{y+tbB}dz\int_{z+sbB}dx=Cs^jt^{-j}.$$

and

$$\int |J(s,t,x,y)|\,dy \leqq C_1 s^{j-n} t^{-j-n} \int_{x+sbB} dz \int_{z+tbB} dy = C s^j t^{-j}.$$

In the same way we have

$$\int |J(s,t,x,y)|\,dy \leqq C t^i s^{-i}, \quad \int |J(s,t,x,y)|\,dx \leqq C t^i s^{-i}.$$

Lemma 4.2. *Let σ be a real number, i and j be non-negative integers, $-j<\sigma<i$, $1\leqq p\leqq\infty$, $1\leqq q\leqq\infty$, $1/p+1/p'=1$, $1/q+1/q'=1$, $K_k^1(t,x,z,y)\in\mathcal{K}_j$, $u_k(t,x)\in L_*^q([0,a];L^p(\Omega))$, $k=1,\cdots,m_1$, $H_h^1(t,x,z,y)\in\mathcal{K}_i$, $v_h(t,x)\in L_*^{q'}([0,a];L^{p'}(\Omega))$, $h=1,\cdots,l_1$, $K_k^0(t,x,z,y)\in\mathcal{K}_0$, $f_k^0(x)\in L^p(\Omega)$, $k=1,\cdots,m_0$, $H_h^0(t,x,z,y)\in\mathcal{K}_0$, $g_h^0(x)\in L^{p'}(\Omega)$, $h=1,\cdots,l_0$, and let*

$$f(x)=\sum_k \int_0^a d_*t \int f^{-n+\sigma} K_k^1\left(t,x,\frac{x-y}{t},y\right) u_k(t,y)\,dy$$

$$+\sum_k \int a^{-n} K_k^0\left(a,x,\frac{x-y}{a},y\right) f_k^0(y)\,dy,$$

$$g(x)=\sum_h \int_0^a d_*t \int t^{-n-\sigma} H_h^1\left(t,y,\frac{y-x}{t},x\right) v_h(t,y)\,dy$$

$$+\sum_h \int a^{-n} H_h^0\left(a,y,\frac{y-x}{a},x\right) g_h^0(y)\,dy.$$

Then

$$(4\cdot1)\qquad \sum_k \int_0^a d_*s \int s^\sigma u_k(s,x) \left\langle s^{-n} K_k^1\left(s,y,\frac{y-x}{s},x\right), g(y)\right\rangle_y dx$$

$$+\sum_k \int f_k^0(x) \left\langle a^{-n} K_k^0\left(a,y,\frac{y-x}{a},x\right), g(y)\right\rangle_y dx$$

$$=\sum_h \int_0^a d_*t \int t^{-\sigma} v_h(t,z) \left\langle t^{-n} H_h^1\left(t,z,\frac{z-y}{t},y\right), f(y)\right\rangle_y dz$$

$$+\sum_h \int g_h^0(x) \left\langle a^{-n} H_h^0\left(a,z,\frac{z-y}{a},y\right), f(y)\right\rangle_y dz.$$

Proof. By Lemma 2.2 Corollary 1 in [6] we have

$$\left\langle t^{-n}K_k{}^1\left(s, y, \frac{y-x}{s}, x\right), g(y)\right\rangle_y$$

$$= \sum_h \int_0^a d_*t \int s^{-n}K_k{}^1\left(s, y, \frac{y-x}{s}, x\right)dy \int t^{-n-\sigma}H_h{}^1\left(t, z, \frac{z-y}{t}, y\right)$$

$$\times v_h(t, z)\, dz$$

$$+ \sum_h \int s^{-n}K_k{}^1\left(s, y, \frac{y-x}{s}, x\right)dy \int a^{-n}H_h{}^0\left(a, z, \frac{z-y}{a}, y\right)g_h{}^0(z)\, dz$$

$$= \sum_h \int_0^a d_*t \int J_{kh}^{11}(s, t, x, z)\, t^{-\sigma}v_h(t, z)\, dz + \sum_h \int J_{kh}^{10}(s, a, x, z)g_h{}^0(z)dz\,,$$

where

$$J_{kh}^{\nu\mu}(s, t, x, z) = s^{-n}t^{-n} \int K_k{}^\nu\left(s, y, \frac{y-x}{s}, x\right)H_h{}^\mu\left(t, z, \frac{z-y}{t}, y\right)dy\,,$$

$$(\nu, \mu = 0, 1)$$

since for any $\varphi \in L^{p'}(\Omega)$ the integrals

$$\iint K_k{}^\nu\left(s, y, \frac{y-x}{s}, x\right)H_h{}^\mu\left(t, x, \frac{z-y}{t}, y\right)\varphi(x)\, dydz, \quad (\nu, \mu = 0, 1)$$

are absolutely integrable. Analogously, we can obtain

$$\left\langle a^{-n}K_k{}^0\left(a, y, \frac{y-x}{a}, x\right), g(y)\right\rangle_y$$

$$= \sum_h \int_0^a d_*t \int J_{kh}^{01}(a, t, x, z)\, t^{-\sigma}v_h(t, z)\, dz + \sum_h \int J_{kh}^{00}(a, a, x, z)\, g_h{}^0(z)\, dz\,,$$

and therefore the left hand side of (5·1) is equal to

$$\sum_{k,h} \int_0^a d_*s \int u_k(s, x)\, dx \int_0^a d_*t \int s^\sigma t^{-\sigma} J_{kh}^{11}(s, t, x, z)\, v_h(t, z)\, dz$$

$$+ \sum_{k,h} \int_0^a d_*s \int u_k(s, x)\, dx \int s^\sigma J_{kh}^{10}(s, a, x, z)\, g_h{}^0(z)\, dz$$

$$+ \sum_{k,h} \int f_k{}^0(x)\, dx \int_0^a d_*t \int t^{-\sigma} J_{kh}^{01}(a, t, x, z)\, v_h(t, z)\, dz$$

$$+ \sum_{k,h} \int f_k{}^0(x)\, dx \int J_{kh}^{00}(a, a, x, z)\, g_h{}^0(z)\, dz\,.$$

The same computation gives that its right hand side is equal to the sum of the same iterated integrals, except that the order of the integration is different. Thus, (4·1) follows from Fubini's theorem, if it is shown that the above integrals are absolutely integrable, which follows from Corollary 2.5 and Lemma 4.1 in view of the fact that

$$\int_0^\infty s^\sigma t^{-\sigma}\min(s^j t^{-j}, s^{-i}t^i)\,d_*t=\int_0^\infty s^\sigma t^{-\sigma}\min(s^j t^{-j}, s^{-i}t^i)\,d_*s$$
$$=\frac{1}{j+\sigma}+\frac{1}{i-\sigma}\,.$$

Hence the proof of the lemma is complete.

Corollary 4.3. *Let $\sigma, j, p, q, p', q', K_k^1, u_k, K_k^0, f_k^0$ be as in Lemma 4.2, and assume that*

$$\sum_k \int_0^a d_*t\int t^{\sigma-n}K_k^1\Big(t, x, \frac{x-y}{t}, y\Big)u_k(t, y)\,dy$$
$$+\sum_k \int a^{-n}K_k^0\Big(a, x, \frac{x-y}{a}, y\Big)f_k^0(y)\,dy=0$$

Then

$$\sum_k \int_0^a d_*s\int s^\sigma u_k(s, x)\Big\langle s^{-n}K_k^1\Big(s, y, \frac{y-x}{s}, x\Big), g(y)\Big\rangle_y\,dx$$
$$+\sum_k \int f_k^0(x)\Big\langle a^{-n}K_k^0\Big(a, y, \frac{y-x}{a}, x\Big), g(y)\Big\rangle_y\,dx=0$$

for every $g\in B^{-\sigma}_{p',q',\Omega}(\boldsymbol{R}^n)$.

Proof. This follows from Lemma 4.2 and the integral representation ([6] Theorem 1).

Proof of Theorem 1.2. Let $\langle f, g\rangle$ be a continuous bilinear form with the property stated in the theorem and let m, k, l, h be integers such that $m\geqq k\geqq 0$, $m-k>-\sigma$, $l\geqq h\geqq 0$, $l-h>\sigma$. Then, from the integral representation it follows that

$$f(x)=\sum_{|\alpha|=k}\int_0^a d_*t\int t^{-n}M_\alpha\Big(x, \frac{x-y}{t}\Big)\{u^\alpha(t, y)+t^k f_\infty^{(\alpha)}(y)\}\,dy$$
$$+\sum_{|\beta|\leqq h}\int_0^a d_*t\int t^{h-|\beta|-n}M^{(0,\beta)}\Big(x, \frac{x-y}{t}\Big)u_\beta(t, y)\,dy$$

$$+\sum_{|\alpha|\leqq h}\int a^{-|\beta|-n}\omega_m{}^{(0,\beta)}\left(x,\frac{x-y}{t}\right)f_\beta(y)\,dy$$

$$+\int a^{-n}\omega_m\left(x,\frac{x-y}{a}\right)f_\infty(y)\,dy\,,$$

(for the definition of u^α, u_β, f_β and f_∞ see [6] p. 344) which gives

$$(4\cdot 2)\qquad \langle f,g\rangle=\sum_{|\alpha|=k}\int_0^a d_*t\int\{u^\alpha(t,x)+t^k f_\infty{}^{(\alpha)}(x)\}\,V_\alpha(t,x)\,dx$$

$$+\sum_{|\beta|\leqq h}\int_0^a d_*t\int t^{h-|\beta|}u_\beta(t,x)\,V^\beta(t,x)\,dx$$

$$+\sum_{|\beta|\leqq h}\int f_\beta(x)\,(-1)^{|\beta|}g_\infty{}^{(\beta)}(x)\,dx+\int f_\infty(x)\,g_\infty(x)\,dx\,,$$

where

$$V_\alpha(t,x)=t^{-n}\langle M_\alpha(y,(y-x)/t),g(y)\rangle_y\,,$$

$$V^\beta(t,x)=t^{-n}\langle M^{(0,\beta)}(y,(y-x)/t),g(y)\rangle_y\,,$$

$$g_\infty(x)=a^{-n}\langle\omega_m(y,(y-x)/a),g(y)\rangle_y,\quad g_\infty{}^{(\alpha)}(x)=D^\alpha g_\infty(x)\,.$$

Hence the bilinear form is unique. Conversely, the form defined by $(4\cdot 2)$ is a continuous bilinear form on $B^\sigma_{p,q}(\Omega)\times B^{-\sigma}_{p',q',\bar\Omega}(\boldsymbol{R}^n)$ (see [6] Theorem 2 and [6] Lemma 3.3). Furthermore, it is obvious that the identities $(1\cdot 1)$ and $(1\cdot 2)$ follows from Corollary 4.3, and the identities $(1\cdot 3)$ and $(1\cdot 4)$ follows from Lemma 4.2. Thus Theorem 1.2 is completely proved.

Proof of Theorem 1. 3. (i) Let $j>-\sigma$, and let $K\in\mathcal{K}_j$. Then for any $u(t,x)\in L_*{}^q([0,a];L^p(\Omega))$

$$\int_0^a d_*t\int t^\sigma u(t,x)\left\langle t^{-n}K\left(t,y,\frac{y-x}{t},x\right),g(y)\right\rangle_y dx$$

$$=\left\langle\int_0^a d_*t\int t^{\sigma-n}K\left(t,y,\frac{y-x}{t},x\right)u(t,x)\,dx,\ g\right\rangle=0\,.$$

Hence, by Lemma 2.3 we have $\langle t^{-n}K(t,y,(y-x)/t,x),g(y)\rangle_y=0$. Also, if $K\in\mathcal{K}_0$ and if $f_0\in L^p(\Omega)$, then

$$\int f_0(x)\left\langle a^{-n}K\left(a,y,\frac{y-x}{t},x\right),g(y)\right\rangle_y dx$$

$$=\left\langle \int a^{-n}K\left(a, y, \frac{y-x}{t}, x\right) f_0(x)\,dx, g \right\rangle = 0\,,$$

and therefore $\langle a^{-n}K(a, y, (y-x)/a, x),\ g(y)\rangle_y = 0$. These facts imply that $g=0$, in view of the integral representation. The proof of part (ii) is analogous to that of (i).

Proof of Theorem 1.4. Let $\langle f, g\rangle$ be the bilinear form on $B^{\sigma}_{p,q}(\Omega) \times B^{-\sigma}_{p',q',\bar{\Omega}}(\boldsymbol{R}^n)$ defined in Theorem 1.2. It is already known that the mapping $g \mapsto l_g$ and $f \mapsto l_f$ are continuous injections, where $l_g(f) = \langle f, g\rangle$ and $l_f(g) = \langle f, g\rangle$, so that it is sufficient to prove their surjectivity.

(i) For the case where $1<p<\infty$, $1<q<\infty-$ this fact has been proved in [6].

(ii) Case $p=q=1$. Since the dual of $L^1([0, a]\times\Omega, d_*t dx)$ is $L^\infty([0, a]\times\Omega, d_*t dx)$, this fact can be proved in the same way as for the case (i).

(iii) Case $1<p<\infty$, $q=1$. Let $l\in\{B_{p,1}(\Omega)\}'$, and let J be the imbedding operator from $B^{\tau}_{p,q}(\Omega)$ into $B^{\sigma}_{p,1}(\Omega)$, where $\sigma<\tau$ and $1<q<\infty-$. Then $l\circ J \in \{B^{\tau}_{p,q}(\Omega)\}' = B^{-\tau}_{p',q',\bar{\Omega}}(\boldsymbol{R}^n)$, so that there exists $g \in B^{-\tau}_{p',q',\bar{\Omega}}(\boldsymbol{R}^n)$ such that $l\circ J = l_g$. To prove $g \in B^{-\sigma}_{p',\infty,\bar{\Omega}}(\boldsymbol{R}^n)$, let $K\in\mathcal{K}_j$, $-\sigma<j$, $u(t, x)\in L_*{}^1([0, a]; L^p(\Omega))$, $\varepsilon>0$, and define

$$u_\varepsilon(t, x) = \begin{cases} u(t, x) & \text{for} \quad t \geqq \varepsilon\,, \\ 0 & \text{for} \quad t<\varepsilon\,. \end{cases}$$

Then $t^{\sigma-\tau}u_\varepsilon(t, x) \in L_*{}^q([0, a]; L^p)$, and therefore

$$\left\langle \int_0^a d_*t \int t^{\sigma-n} K\left(t, x, \frac{x-y}{t}, y\right) u_\varepsilon(t, y)\,dy, g \right\rangle$$

$$= \int_0^a d_*t \int t^{\sigma} u_\varepsilon(t, y)\, V(t, y)\,dy\,,$$

where $V(t, y) = \langle t^{-n}K(t, x, (x-y)/t, y), g(x)\rangle_x$. Since

$$|\langle f, g\rangle| = |l\circ J(f)| \leqq C\|J(f)\|_{B^{\sigma}_{p,1}}\,,$$

we have

$$\left| \int_\varepsilon^a d_*t \int t^{\sigma} u(t, x)\, V(t, x)\,dx \right| \leqq C'\|u\|_{L_*{}^1([0,a]; L^p)}\,.$$

Letting $\varepsilon\to 0$, by Lemma 2.3 we obtain that $t^{\sigma}V(t, x) \in L_*{}^\infty([0, a]; L^{p'}(\Omega))$.

This and the fact that $g\in W_{p',\bar{\Omega}}^{-\infty}(\boldsymbol{R}^n)$ imply that $g\in B_{p',\infty,\bar{\Omega}}^{-\sigma}(\boldsymbol{R}^n)$, according to Theorem 2 in [6]. Therefore $l-l_g$ is continuous on $B_{p,1}^{\sigma}(\Omega)$. Since $l-l_g=0$ on $J(B_{p,q}^{\tau}(\Omega))$, which is dense in $B_{p,1}^{\sigma}(\Omega)$ (see [6] Theorem 5), it follows that $l=l_g$ on $B_{p,1}^{\sigma}(\Omega)$. Thus the mapping $g\mapsto l_g$ is surjective.

(iv) Case $p=1$, $1<q<\infty-$. Making use of the imbedding operator $B_{1,1}^{\sigma}(\Omega)\to B_{1,q}^{\sigma}(\Omega)$, by the same argument as in case (iii) we obtain the desired result in this case.

(v) Case $1<p<\infty$, $q=\infty-$. With the aid of the imbedding $B_{p,r}^{\sigma}\to B_{p,\infty-}^{\sigma}$, where $1<r<\infty-$, the same argument gives the desired conclusion.

(vi) Case where $p=\infty$, $1\leqq q\leqq\infty-$, and Ω is a bounded domain. Let J be the imbedding $B_{r,q}^{\tau}\to B_{\infty,q}^{\sigma}$, where $1<r<\infty$ and $\tau=\sigma+n/r$, and let $l\in\{B_{\infty,q}^{\sigma}(\Omega))'$. Then $l\circ J\in\{B_{r,q}^{\tau}(\Omega)\}'=B_{r',q',\bar{\Omega}}^{-\tau}(\boldsymbol{R}^n)$, and hence there exists $g\in B_{r',q',\bar{\Omega}}^{-\tau}(\boldsymbol{R}^n)$ such that $l\circ J=l_g$. Since $L^{r'}(\Omega)$ is continuously imbedded in $L^1(\Omega)$, it follows that $g\in B_{1,q',\bar{\Omega}}^{-\tau}(\boldsymbol{R}^n)\subset W_{1,\bar{\Omega}}^{-\infty}(\boldsymbol{R}^n)$. The remainder part of the proof is the same as in case (iii).

Similarly we can find that the mapping $f\mapsto l_f$ is surjective, and the theorem is completely proved.

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