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# Hopf Algebra Structure of mod 2 Cohomology of Simple Lie Groups

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## Introduction

The purpose of the present paper is to determine the Hopf algebra structure of the mod 2 cohomology $H^*(G; Z_2)$ of each compact connected simple Lie group $G$. For classical type $G$, the Hopf algebra $H^*(G; Z_2)$ is determined by Borel [6] and Baum-Browder [3], except the spinor groups $Spin(n)$ and the semi-spinor groups $Ss(4m)$. For exceptional type $G$, it is determined by several authors [6], [8], [9], [15], except the case $G=AdE_7=E_7/Z_2$.

In order to describe our results, we shall use the submodule $T_G{}^*$ of $H^*(G; Z_p)$ which consists of the transgressive elements with respect to the fibering

$$G \xrightarrow{\pi} G/T \xrightarrow{i} BT$$

where $T$ denotes a maximal torus of $G$.

The submodule $T_G{}^*$ enjoys the following convenient properties. Let $a_1, \cdots, a_l$ be a basis of the odd dimensional part $T_G{}^{\text{odd}}$ of $T_G{}^*$, then

$$H^*(G; Z_p) \cong \Delta(a_1, \cdots, a_l) \otimes \mathrm{Im}\, \pi^*$$

and

$$T_G{}^* = \langle a_1, \cdots, a_l \rangle + \mathrm{Im}\, \pi^+,$$

which is a part of Theorem 1.1 and the non-simply connected version of the main theorem of [16]. Furthermore, $T_G{}^*$ is natural with respect to the group homomorphism, closed under the action of the mod $p$ Steenrod algebra, and each element $x$ of $T_G{}^*$ is characterized by the diagonal

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map:

$$\bar{\phi}(x)=\phi(x)-x\otimes 1-1\otimes x\in \operatorname{Im}\pi^{+}\otimes T_G{}^{*} \quad \text{(see Theorem 2.2)}.$$

The Hopf algebra structure of $H^{*}(G;Z_p)$ for $G=Spin(n)$, $Ss(4m)$ and $AdE_7$ will be determined in § 3, § 4 and § 5 respectively. The essential part of the results are stated as follows, for the details see Theorems 3.2, 4.4 and 5.3:

$$H^{*}(Spin(n);Z_2)=\Lambda(x_i, z; 3\leq i<n, i\neq 4, 8, \cdots, 2^{s-1}),$$

$$(2^{s-1}<n\leq 2^{s}, \deg x_i=i, \deg z=2^{s}-1, x_i=0 \text{ if } i=2^{t} \text{ or } i\geq n)$$

$$\bar{\phi}(x_i)=0, \quad \bar{\phi}(z)=\sum_{i+j=2^{s-1}} x_{2i}\otimes x_{2j-1}, \quad Sq^1 z=\sum_{1<i<2^{s-1}} x_{2i}x_{2^{s}-2i};$$

$$H^{*}(Ss(4m);Z_2)=\Lambda(x_i, z; 3\leq i<4m, i\neq 4, 8, \cdots, 2^{s-1}, 2^{r}-1)$$

$$\otimes Z_2[y]/(y^{2^r}),$$

$$(4m=2^{r}\cdot\text{odd}, 2^{s-1}<4m\leq 2^{s}, \deg x_i=i, \deg z=2^{s}-1, \deg y=1,$$

$$x_i=0 \text{ if } i=2^{t} \text{ or } i\geq 4m)$$

$$\bar{\phi}(y)=0, \quad \bar{\phi}(x_i)=\sum_{1<j<i/2}\binom{i}{2j} y^{2j}\otimes x_{i-2j}+i\cdot x_i\otimes y \quad (i\neq 2^{r}-1),$$

$$\bar{\phi}(z)=\sum_{\substack{i+j+k=2^{s-1}\\ 0<i<j}}\binom{i+j}{j} y^{2i}x_{2j}\otimes x_{2k-1}+\sum_{\substack{i+j=2^{s-1}-1\\ 0<i<j}} x_{2i}x_{2j}\otimes y,$$

$$Sq^1 z=\sum_{1<i<2^{s-1}-1} x_{2i}x_{2^{s}-2i}+\sum_{1<i<2^{s-1}-1} y^{2}x_{2i}x_{2^{s}-2i-2};$$

$$H^{*}(AdE_7;Z_2)=Z_2[x_1, x_5, Sq^4x_5]/(x_1{}^4, x_5{}^4, (Sq^4x_5)^4)$$

$$\otimes\Lambda(Sq^1x_5, x_{15}, Sq^8Sq^4x_5, Sq^8x_{15}, Sq^4Sq^8x_{15}),$$

$$(\deg x_i=i)$$

$$\bar{\phi}(x_1)=\bar{\phi}(x_5)=0, \quad \bar{\phi}(x_{15})=Sq^1x_5\otimes Sq^4x_5+x_5{}^2\otimes x_5.$$

## § 1. A Transgression Theorem

Throughout the paper, $G$ denotes a compact connected Lie group, $T$ a maximal torus of $G$, and $p$ a prime. As is seen in § 2 of [16], the fibering

$$(1\cdot 1) \qquad G\xrightarrow{\pi} G/T\xrightarrow{i} BT$$

is equivalent to the principal $G$-bundle

$$(1\cdot 2) \qquad G \xrightarrow{\pi} E \xrightarrow{i} BT$$

where $E=EG\times_T G$ and $BT=EG\times_T pt$.

Denote by $T_G{}^*$ the graded submodule of $H^*(G;Z_p)$ which consists of the transgressive elements with respect to $(1\cdot 1)$ or $(1\cdot 2)$. Thus

$$(1\cdot 3) \qquad T_G{}^*=\delta^{-1}(i^*H^{*+1}(BT,pt;Z_p))$$

for the coboundary homomorphism $\delta:H^*(G;Z_p)\to H^{*+1}(E,G;Z_p)$ and the homomorphism $i^*: H^{*+1}(BT,pt;Z_p)\to H^{*+1}(E,G;Z_p)$ induced by the projection $i$. Remark that the definition of $T_G{}^*$ is independent of the choice of the maximal torus $T$ since any maximal tori are conjugate to each other and $G$ is connected. Obviously we have

$$(1\cdot 4) \qquad \operatorname{Im}\pi^+\subset \mathrm{T}_G{}^* \text{ for } \pi^*:H^*(G/T;Z_p)\to H^*(G;Z_p).$$

The following theorem has been proved for simply connected $G$ in [16].

**Theorem 1.1** *There exist elements $a_1,\cdots,a_l$ of odd degrees such that the following assertions hold*:

(i) $H^*(G;Z_p)=\Delta(a_1,\cdots,a_l)\otimes\operatorname{Im}\pi^*$ *as* $\operatorname{Im}\pi^*$*-modules.*

(ii) $T_G{}^*=\langle a_1,\cdots,a_l\rangle+\operatorname{Im}\pi^+$, $\operatorname{Im}\pi^+=\pi^*(H^+(G/T;Z_p))$.

(iii) $H^*(G/T;Z_p)\cong\operatorname{Im}\pi^*\otimes\operatorname{Im}i^*$ *as* $\operatorname{Im}i^*$*-modules.*

(iv) $\operatorname{Im}i^*\cong H^*(BT;Z_p)/(\tau(a_1),\cdots,\tau(a_l))$ *for transgression images $\tau(a_i)$ of $a_i$.*

(v) $\tau(a_i)$ *are of no relation in* $H^*(BT;Z_p)$, *i.e.*,

$$P(\operatorname{Im}i^*,t)=P(H^*(BT;Z_p),t)\prod_{i=1}^{l}(1-t^{\deg(\tau(a_i))})$$

*for the Poincaré series* $P(\sum M_n,t)=\sum\dim M_n\cdot t^n$.

Here $\Delta(a_1,\cdots,a_l)$ indicates the submodule spanned by the simple monomials $a_1{}^{\varepsilon_1}\cdots a_l{}^{\varepsilon_l}(\varepsilon_i=0$ or $1)$ which are linearly independent, and $\langle a_i\rangle$ does a submodule spanned by $a_i$. Remark that $l=\operatorname{rank} G$ since $H^*(G/T;$

$Z_p$) is finite dimensional and $H^*(BT; Z_p)$ is a polynomial algebra of $l$ generators of degree 2.

The following lemma is a special case of Theorem 1.1 of [16].

**Lemma 1.2.** *Assume that there are elements* $a_1, \cdots, a_l$ *of odd degrees and a submodule* $M^*$ *of* $H^*(G; Z_p)$ *satisfying the following*

(i) $H^*(G; Z_p) = \Delta(a_1, \cdots, a_l) \otimes M^*$ *by cup products*,

(ii) $M^* \subset \operatorname{Im} \pi^*$

*and*

(iii) $P(H^*(G/T; Z_p), t) = P(M^*, t) \cdot P(H^*(BT; Z_p), t) \cdot \prod_{i=1}^{l} (1 - t^{\deg a_i + 1})$.

*Then, by suitable change of generators* $a_1, \cdots, a_l$ *the assertions* (i)-(v) *of Theorem* 1.1 *hold.*

Next we prove

**Lemma 1.3.** (i) *Theorem* 1.1 *holds for simply connected simple* $G$.

(ii) *If Theorem* 1.1 *holds for* $G_1$ *and* $G_2$, *then it holds for* $G_1 \times G_2$.

(iii) *Let* $q$ *be a prime and let* $Z_q$ *be a cyclic subgroup of order* $q$ *contained in the center of* $G$. *If Theorem* 1.1 *holds for* $G$, *then it holds for* $G/Z_q$.

*Proof.* (i) follows from Propositions 3.1, 3.2 of [16].

(ii) is proved directly by use of the Künneth formula.

If $q \neq p$, then $H^*(G; Z_p)$ is naturally isomorphic to $H^*(G/Z_q; Z_p)$ and (iii) is trivial.

Let $q = p$, $G' = G/Z_q$ and $T' = T/Z_q$. Consider the cohomology spectral sequence associated with the upper fibering in the following diagram:

$$\begin{array}{ccccc} G & \xrightarrow{\pi_p} & G' & \xrightarrow{i_p} & BZ_p \\ \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \rho} \\ G & \xrightarrow{\pi} & G/T & \xrightarrow{i} & BT. \end{array}$$

In the spectral sequence, $E_2^{*,*}=H^*(BZ_p; Z_p)\otimes H^*(G; Z_p)$ and the differential $d_r$ is trivial on $\mathrm{Im}\,\pi^*$ since $\mathrm{Im}\,\pi^*\subset\mathrm{Im}\,\pi_p{}^*$. By the naturality of the transgression, the elements $a_i$ of $T_G{}^*$ is also transgressive in this spectral sequence. Let $u$ be a generator of $H^1(BZ_p; Z_p)$ then $H^*(BZ_p; Z_p)=\Delta(u)\otimes Z_p[\beta u]$.

Since $a_i$ is of odd degree, $\tau(a_i)\equiv c_i(\beta u)^r$ for some $c_i\in Z_p$ and $\deg a_i=2r-1$. If $c_i\equiv 0$ for all $i$, then the spectral sequence collapses, which contradicts the finiteness of $H^*(G'; Z_p)$. Thus $c_k\not\equiv 0$ for some $k$, and further we may assume that $c_i\equiv 0$ for $i\neq k$.

By a simple computation we have

$$E_\infty^{*,*}=\Delta(u, a_1, \cdots, \hat{a}_k, \cdots, a_l)\otimes(Z_p[\beta u]/((\beta u)^r)\otimes\mathrm{Im}\,\pi^*),$$

for $2r=\deg a_k+1$. Then a similar equality holds for $H^*(G'; Z_p)$, and the assumptions (i), (ii), (iii) of Lemma 1.2 are easily checked for $G'$, provided that $i_p{}^*(\beta u)\in\mathrm{Im}\,\pi'^*$. Consider the following exact and commutative diagram:

$$\begin{array}{ccccccc} 0 \longrightarrow H^1(BZ_p; Z_p) \longrightarrow H^2(BT'; Z_p) \longrightarrow & H^2(BT; Z_p) & \xrightarrow{\rho^*} & H^2(BZ_p; Z_p) \\ & \downarrow i^* & & \downarrow i_p{}^* \\ & H^2(G/T; Z_p) & \xrightarrow{\pi'^*} & H^2(G'; Z_p). \end{array}$$

Then we see that $\rho^*$ is an epimorphism. Thus $i_p{}^*(\beta u)\in\mathrm{Im}\,\pi'^*$, and (iii) for the case $q=p$ follows from Lemma 1.2.

*Proof of Theorem* 1.1. For any compact connected Lie group $G$ there is a finite covering $\widetilde{G}\to G$ such that $\widetilde{G}$ is the product of simply connected simple Lie groups and a torus. By (i) and (ii) of Lemma 1.3, Theorem 1.1 holds for $\widetilde{G}$. The covering is divided into a sequence of coverings of prime order. Then Theorem 1.1 holds for $G$ by (iii) of Lemma 1.3. Q.E.D.

## § 2. General Arguments

We use the following notations.

$$T_G{}^i=\{x\in T_G{}^*; \deg x=i\},\quad T_G{}^{\mathrm{even}}=\sum_i T_G{}^{2i},\ T_G{}^{\mathrm{odd}}=\sum_i T_G{}^{2i+1}.$$

Thus $T_G{}^*=\sum_i T_G{}^i=T_G{}^{\mathrm{odd}}+T_G{}^{\mathrm{even}}$ and, by (ii) of Theorem 1.1, (2·1) $$T_G^{\text{even}}=\operatorname{Im}\pi^{\tau} \text{ and } T_G^{\text{odd}}=\langle a_1, \cdots, a_l\rangle\cong Z_p^{\,l}.$$

Similarly we denote the submodule of the universally transgressive elements in $H^+(G; Z_p)$ by

$$U_G{}^*=\sum_i U_G{}^i=U_G{}^{\text{odd}}+U_G{}^{\text{even}},$$

and that of the primitive elements by

$$P_G{}^*=\sum_i P_G{}^i=P_G{}^{\text{odd}}+P_G{}^{\text{even}}=\{x\in H^+(G; Z_p);\ \bar{\phi}(x)=0\},$$

where

$$\bar{\phi}(x)=\phi(x)-x\otimes 1-1\otimes x\in H^+(G; Z_p)\otimes H^+(G; Z_p)$$

for the diagonal map (comultiplication)

$$\phi=\mu^*:\ H^*(G; Z_p)\to H^*(G; Z_p)\otimes H^*(G; Z_p)$$

induced by the group multiplication

$$\mu:\ G\times G\to G$$

identifying $H^*(G\times G; Z_p)$ with $H^*(G; Z_p)\otimes H^*(G; Z_p)$ by Künneth formula.

From the naturality of the transgression and the diagonal map we have

(2·2) (i) If $f$: $G\to G'$ is a homomorphism of compact connected Lie groups, $f^*T_{G'}{}^*\subset T_G{}^*$, $f^*U_{G'}{}^*\subset U_G{}^*$ and $f^*P_{G'}{}^*\subset P_G{}^*$.

(ii) For each cohomology operation $\alpha\in\mathcal{A}_p$ (the mod $p$ Steenrod algebra),

$$\alpha T_G{}^*\subset T_G{}^*,\ \alpha U_G{}^*\subset U_G{}^* \text{ and } \alpha P_G{}^*\subset P_G{}^*.$$

As is easily seen

(2·3) $$U_G{}^*\subset T_G{}^* \text{ and } U_G{}^*\subset P_G{}^*.$$

From the associativity of $\mu$ it follows the (co)associativity of $\phi$:

(2·4) $$(\phi\otimes 1)\phi=(1\otimes\phi)\phi \text{ and } (\bar{\phi}\otimes 1)\bar{\phi}=(1\otimes\bar{\phi})\bar{\phi}.$$

Consider a principal $G$-bundle $G\xrightarrow{i}E\xrightarrow{p}B$.

**Lemma 2.1.** *If $x\in H^*(G; Z_p)$ is transgressive with respect to this $G$-bundle, $\phi(x)-x\otimes 1\in\operatorname{Im} i^*\otimes H^*(G; Z_p)$.*

*Proof.* Let $\bar{\mu}: E\times G\to E$ be the action of $G$ and $p_1$: $B\times G\to B$ be the projection to the first factor. Then we have the following commutative diagram:

$$\begin{array}{ccccc} H^*(G;Z_p) & \xrightarrow{\delta} & H^*(E,G;Z_p) & \xleftarrow{p^*} & H^*(B;Z_p) \\ \downarrow \phi=\mu^* & & \downarrow \bar{\mu}^* & & \downarrow p_1{}^* \\ H^*(G;Z_p) & \xrightarrow{\delta\otimes 1} & H^*(E,G;Z_p) & \xleftarrow{p^*\otimes 1} & H^*(B;Z_p) \\ \otimes H^*(G;Z_p) & & \otimes H^*(G;Z_p) & & \otimes H^*(G;Z_p). \end{array}$$

By the assumption, $\delta(x)=p^*(y)$ for some $y\in H^*(B;Z_p)$, and

$$(\delta\otimes 1)(\phi(x)-x\otimes 1)=\bar{\mu}^*(\delta(x))-\delta(x)\otimes 1=\bar{\mu}^*(p^*(y))-p^*(y)\otimes 1$$

$$=(p^*\otimes 1)p_1{}^*(y)-p^*(y)\otimes 1=(p^*\otimes 1)(y\otimes 1)-p^*(y)\otimes 1=0\,.$$

Thus $\phi(x)-x\otimes 1\in \mathrm{Ker}(\delta\otimes 1)=\mathrm{Im}(i^*\otimes 1)=\mathrm{Im}\, i^*\otimes H^*(G;Z_p)$.

Q.E.D.

Remark that the lemma is valid for any associative H-space $G$ and any principal $G$-fibering.

Now we apply the above lemma to the fibering $(1\cdot 1)$ equivalent to the principal $G$-bundle $(1\cdot 2)$.

**Theorem 2. 2.** *For each* $x\in H^+(G;Z_p)$, *the following three conditions are equivalent*:

(i) $x\in T_G{}^*$.

(ii) $\phi(x)-x\otimes 1\in \mathrm{Im}\,\pi^*\otimes H^*(G;Z_p)$.

(iii) $\phi(x)-x\otimes 1\in \mathrm{Im}\,\pi^*\otimes T_G{}^*$.

We shall use the following notations:

(i) $a^I=a_1{}^{\varepsilon_1}\cdots a_l{}^{\varepsilon_l}$ for $I=(\varepsilon_1,\cdots,\varepsilon_l)$ and $\varepsilon_i=0$ or $1$, $I+I'=(\varepsilon_1+\varepsilon_1',\cdots,\varepsilon_l+\varepsilon_l')$ for $I=(\varepsilon_1,\cdots,\varepsilon_l)$ and $I'=(\varepsilon_1',\cdots,\varepsilon_l')$, and $|I|=\varepsilon_1+\cdots+\varepsilon_l$.

(ii) $Q_1$ is the ideal of $H^*(G;Z_p)\otimes H^*(G;Z_p)$ generated by $\mathrm{Im}\,\pi^+\otimes H^*(G;Z_p)$ and $Q_2=H^*(G;Z_p)\otimes \mathrm{Im}\,\pi^*\otimes H^*(G;Z_p)$.

**Lemma 2. 3.** $\phi(h)\equiv 1\otimes h \mod Q_1$ *for* $h\in \mathrm{Im}\,\pi^+=T_G{}^{\mathrm{even}}$,

$$\phi(x) \equiv x \otimes 1 + 1 \otimes x \mod Q_1 \ \textit{for} \ x \in T_G^{\text{odd}}$$

*and*

$$\phi(a^I) \equiv \sum_{I'+I''=I} a^{I'} \otimes a^{I''} \mod Q_1 .$$

This follows easily from Lemma 2.1.

*Proof of Theorem* 2.2. Clearly (iii) implies (ii) and also (i) implies (ii) by Lemma 2.1.

(2·5) (ii) is equivalent to $\bar{\phi}(x) \equiv 0 \mod \operatorname{Im} \pi^* \otimes H^*(G; Z_p)$, thus (ii) implies $\bar{\phi}(x) \equiv 0 \mod Q_1$.

By (i) of Theorem 1.1, such $x$ is written uniquely in the form

$$x = \sum_I a^I h_I + \sum_J \alpha_J a^J \quad (h_I \in \operatorname{Im} \pi^+, \alpha_J \in Z_p).$$

Then by Lemma 2.3,

$$0 \equiv \bar{\phi}(x) \equiv \sum_I \sum_{\substack{I'+I''=I \\ |I'| \neq 0}} a^{I'} \otimes a^{I''} h_I + \sum_J \sum_{\substack{J'+J''=J \\ |J'|, |J''| \neq 0}} \alpha_J a^{J'} \otimes a^{J''} \mod Q_1 .$$

This implies that $h_I = 0$ if $|I| > 0$ and $\alpha_J = 0$ if $|J| > 1$. So, $x$ satisfies (i) proving that (ii) implies (i).

Finally we prove that (ii) implies (iii) by induction on $\deg x$. From the induction hypothesis we have

$$\phi(h') \in \operatorname{Im} \pi^* \otimes \operatorname{Im} \pi^* \ \text{for} \ h' \in \operatorname{Im} \pi^+ \subset T_G^* \ \text{with} \ \deg h' < \deg x.$$

Put $\bar{\phi}(x) = \sum_k h_k' \otimes y_k$ for $h_k' \in \operatorname{Im} \pi^+$, $\deg h_k' < \deg x$, $y_k \in H^+(G; Z_p)$. By the associativity (2·4), we have

$$0 = (1 \otimes \bar{\phi}) \bar{\phi}(x) - (\bar{\phi} \otimes 1) \bar{\phi}(x) = \sum_k (h_k' \otimes \bar{\phi}(y_k) - \bar{\phi}(h_k') \otimes y_k)$$

$$\equiv \sum_k h_k' \otimes \bar{\phi}(y_k) \qquad \mod Q_2 .$$

We may choose $\{h_k'\}$ linearly independent. Then $\bar{\phi}(y_k) \equiv 0 \mod \operatorname{Im} \pi^* \otimes H^*(G; Z_p)$. So, by (2·5) $y_k$ satisfies (ii), and $y_k$ does (i). Thus

$$\phi(x) - x \otimes 1 = 1 \otimes x + \sum_k h_k' \otimes y_k \in \operatorname{Im} \pi^* \otimes T_G^*. \qquad \text{Q.E.D.}$$

**Corollary 2. 4.** $P_G^* \subset T_G^*$. $\operatorname{Im} \pi^* = \langle 1 \rangle + T_G^{\text{even}}$ *is a Hopf sub-algebra of* $H^*(G; Z_p)$.

*Proof.* If $x \in P_G^*$, $\phi(x) - x \otimes 1 = 1 \otimes x \in \operatorname{Im} \pi^* \otimes H^*(G; Z_p)$. It follows $x \in T_G^*$. If $x \in \operatorname{Im} \pi^+ \subset T_G^{\text{even}}$, $\phi(x) - x \otimes 1 \in \operatorname{Im} \pi^* \otimes T_G^{\text{even}} \subset \operatorname{Im} \pi^* \otimes \operatorname{Im} \pi^*$. Thus the subalgebra $\operatorname{Im} \pi^*$ is closed under $\phi$. Q.E.D.

The notation

$$\Delta_p(b_1, \cdots, b_m) \subset H^*(G; Z_p)$$

indicates the subspace having a $p$-simple system of generators $\{b_1, \cdots, b_m\}$, that is, the set $\{b_1^{\varepsilon_1} \cdots b_m^{\varepsilon_m}; 0 \leq \varepsilon_i < p\}$ is a $Z_p$-basis of $\Delta_p(b_1, \cdots, b_m)$. Note that

$$\Delta_2(b_1, \cdots, b_m) = \Delta(b_1, \cdots, b_m).$$

Since $Z_p[b]/(b^{p^r}) = \Delta_p(b, b^p, \cdots, b^{p^{r-1}})$, Hopf-Borel theorem for the Hopf algebra $H^*(G; Z_p)$ has a form:

$$(2\cdot 6) \qquad H^*(G; Z_p) = \Delta(a_1', \cdots, a_l') \otimes \Delta_p(b_1', \cdots, b_m'),$$

$$\deg a_i \colon \text{odd}, \ \deg b_j \colon \text{even}.$$

Also, applying the theorem to the Hopf sub-algebra $\operatorname{Im} \pi^*$, we have

$$(2\cdot 6)' \quad H^*(G; Z_p) = \Delta(a_1, \cdots, a_l) \otimes \Delta_p(b_1, \cdots, b_m), \ a_i \in T_G^{\text{odd}}, \ b_j \in T_G^{\text{even}},$$

$$\operatorname{Im} \pi^* = \Delta_p(b_1, \cdots, b_m) \ \text{and} \ T_G^* = \langle a_1, \cdots, a_l \rangle + \operatorname{Im} \pi^+.$$

**Lemma 2. 5.** *Given any* $(2\cdot 6)$, *there exist elements* $a_i$, $b_j$ *which satisfy* $(2\cdot 6)'$, *such that* $a_i \equiv a_i'$, $b_j \equiv b_j'$ (mod *decomposables*) *and that* $a_i = a_i'$ (*resp.* $b_j = b_j'$) *if* $a_i' \in T_G^{\text{odd}}$ (*resp.* $b_j' \in T_G^{\text{even}}$).

This is proved by changing the generators suitably by induction on the degrees.

**Lemma 2. 6.** $P_G^{\text{odd}} \subset \langle a_1, \cdots, a_l \rangle$ *and* $P_G^{\text{even}} \subset \langle b_1', \cdots, b_m' \rangle$ *for some* $b_j' \in T_G^{\text{even}}$ *such that* $b_j' \equiv b_j$ (mod *decomposables*).

*Proof.* By Corollary 2.4, $P_G^{\text{odd}} \subset T_G^{\text{odd}} = \langle a_1, \cdots, a_l \rangle$. Consider the Hopf algebra $B = \operatorname{Im} \pi^* = \Delta_p(b_1, \cdots, b_m)$. If the elements $b_j$'s are all primitive, we can show that $P(B) = P_G^{\text{even}} = \langle b_1, \cdots, b_m \rangle$ by the same arguments as in the proof of Theorem 2.2. Let $E^0(B)$ be the associated graded Hopf algebra given by the augmentation filtration $\{(B^+)^r\}$, then $E^0(B)$

is primitively generated and $P(E^0(B)) = \langle$the classes of $b_j\rangle$. Since there is a natural injection of $P(B)$ into $P(E^0(B))$, the second assertion follows. Q.E.D.

## § 3. Structure of $H^*(Spin(n); Z_2)$

Consider the following fibering

$$(3\cdot 1) \qquad Spin(n) \xrightarrow{\rho} SO(n) \xrightarrow{\lambda} BZ_2 \quad \text{for } n \geq 3\,,$$

where $\rho$ is the universal covering and $\lambda$ is a map classifying $\rho$.

We use the following notation

$$(3\cdot 2) \qquad s = s(n) \text{ is the integer given by } 2^{s-1} < n \leq 2^s\,,$$

$$N = \{1, 2, 2^2, 2^3, \cdots\}.$$

We quote the following result due to Borel [4], [5], [6].

**Proposition 3. 1.** (i) *$H^*(SO(n); Z_2) = \Delta(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_{n-1})$ for $\bar{x}_i \in U^i_{SO(n)}$.*

(ii) *$Sq^j \bar{x}_i = \binom{i}{j} \bar{x}_{i+j} (=0$ if $i+j \geq n)$, in particular $\bar{x}_i^2 = \bar{x}_{2i}$.*

(iii) *The ideal* Ker $\rho^*$ *is generated by $\bar{x}_1$, and* Im $\rho^* = \Delta(\rho^*\bar{x}_i;$ *$i \notin N$, $i < n)$.*

(iv) *$H^*(Spin(n); Z_2) = \mathrm{Im}\,\rho^* \otimes \Delta(z)$ for an element $z$ of* $\deg z = 2^s - 1$ *which is transgressive with respect to the fibering* $(3\cdot 1)$ *and $\tau(z) \neq 0$ in $H^{2^s}(BZ_2; Z_2)$.*

Since $\mathrm{Im}\,\rho^+$ is transgressive and $\tau(\mathrm{Im}\,\rho^+) = 0$ we have that

$$(3\cdot 3) \qquad \text{the element } z \text{ in (iv) is determined modulo } \mathrm{Im}\,\rho^*.$$

We put

$$(3\cdot 4) \qquad x_i = \rho^* \bar{x}_i \in U^i_{Spin(n)}\,.$$

It follows from $\rho^*\bar{x}_1 = 0$ and $\bar{x}_i^2 = \bar{x}_{2i}$

$$(3\cdot 5) \qquad x_i = 0 \text{ if } i \in N \text{ or if } i \geq n\,.$$

Consider the fibering $(1\cdot 1)$ for $G = Spin(n)$:

$$Spin(n) \xrightarrow{\pi} Spin(n)/T \xrightarrow{i} BT\,.$$

Then the structure of $H^*(Spin(n); Z_2)$ is given by the following

**Theorem 3.2.** *Put* $l=[n/2]$. *There exists an element* $z\in T^{2^s-1}_{Spin(n)}$ *such that* $z\notin \operatorname{Im}\rho^*=\Delta(x_i; i\notin N, i<n)$. *Then we have the following*

(i) $H^*(Spin(n); Z_2)=\Delta(x_3, x_5, \cdots, x_{2l-1}, z)\otimes \operatorname{Im}\pi^*$,

(ii) $T^{\text{odd}}_{Spin(n)}=\langle x_3, x_5, \cdots, x_{2l-1}, z\rangle$,

$T^{\text{even}}_{Spin(n)}+\langle 1\rangle=\operatorname{Im}\pi^*=\Delta(x_{2j}; j\notin N, 2j<n)$,

(iii) $\bar{\phi}(x_i)=0$ *and* $\bar{\phi}(z)=\sum_{\substack{i+j=2^{s-1}\\ i>0}} x_{2i}\otimes x_{2j-1}$

*and*

(iv) $Sq^j x_i=\binom{i}{j}x_{i+j}$, *in particular* $x_i^2=x_{2i}$,

$Sq^1(z)=\sum_{\substack{i+j=2^{s-1}\\ i<j}} x_{2i}x_{2j}$, $Sq^j z=0$ *for* $j>1$, $z^2=0$.

Note that

(3·6) the above element $z$ is unique if $n\neq 2^s$ and unique up to $x_{2l-1}$ if $n=2^s$ $(=2l)$.

By (2·3), $\bar{x}_i\in P^i_{SO(n)}$ implies $x_i\in P^i_{Spin(n)}$, i.e., $\bar{\phi}(x_i)=0$. By Corollary 2.4, $x_i\in T^i_{Spin(n)}$. By Proposition 3.1,

$$H^*(Spin(n); Z_2)=\Delta(x_{2i-1},\ z; 1<i\leq l)\otimes\Delta(x_{2j}; j\notin N, 2j<n).$$

Apply Lemma 2.5, then $z$ can be changed modulo $\operatorname{Im}\rho^*=\Delta(x_i; i\notin N, i<n)$ such that $z\in T^{2^s-1}_{Spin(n)}$. $\tau(z)\neq 0$ shows that $z\notin\operatorname{Im}\rho^*$. Then Lemma 2.5 implies (i) and (ii) of Theorem 3.2. By the naturality of $Sq^i$, (ii) of Proposition 3.1 implies the first assertion of (iv) of Theorem 3.2. Since $Sq^{2i}z\in T^{2^s+2i-1}_{Spin(n)}=0$ for $i>0$, $Sq^j z=0$ for $j>1$. Thus we have obtained

(3·7) Theorem 3.2 holds except the assertion for $\bar{\phi}(z)$ and $Sq^1 z$.

Now we have

**Lemma 3.3.** $P^{\text{even}}_{Spin(n)}=\langle x_{2j}; j\notin N, 2j<n\rangle$.

*Proof.* By Lemma 2.6, $\dim P^{\text{even}}_{Spin(n)}\leq\dim\langle x_{2j}; j\notin N, 2j<n\rangle$. Since $x_{2j}$ is primitive for all $j$, the equality holds.

**Lemma 3.4.** $\bar{\phi}(z)=\sum_{i+j=2^{s-1}} c_i x_{2i}\otimes x_{2j-1}$ *for some* $c_i\in Z_2$.

*Proof.* By Theorem 2.2, $\bar{\phi}(z)=\sum_j h_j\otimes x_{2j-1}$ for some $h_j\in \operatorname{Im}\pi^+$. Then

$$\sum_j \bar{\phi}(h_j)\otimes x_{2j-1}=(\bar{\phi}\otimes 1)\bar{\phi}(z)=(1\otimes\bar{\phi})\bar{\phi}(z)=\sum_j h_j\otimes\bar{\phi}(x_{2j-1})=0\,.$$

If $x_{2j-1}\neq 0$, $\bar{\phi}(h_j)=0$. It follows from Lemma 3.3 that $h_j=a_i x_{2i}$ for some $a_i$ where $2i=2^s-1-(2j-1)=2^s-2j$. Of course, $a_i$ is arbitrary if $x_{2j-1}=0$. Q.E.D.

**Corollary 3.5.** *$\bar{\phi}(z)=0$ and $Sq^1 z=0$ if $n\leq 9$ or if $n=2^{s-1}+1$.*

*Proof.* By dimensional reason $\bar{\phi}(z)=0$ for these cases. By (2·2), (ii) and by Lemma 3.3, $Sq^1 z\in Sq^1 P^{2s-1}_{Spin(n)}\subset P^{2s}_{Spin(n)}=0$.

**Lemma 3.6.** *The coefficients $c_i$ in Lemma 3.4 satisfy $c_i=c_{2^{s-1}-i}$ for $i\notin N$, $2i<n/2$, $2^s-2i<n$, and $Sq^1 z=\sum_{i<2^{s-2}} c_i x_{2i}x_{2^s-2i}$.*

*Proof.* Since $Sq^1 z\in T^{\text{even}}_{Spin(n)}\subset \operatorname{Im}\rho^*$, $Sq^1 z=\sum_I c_I x^I$ for some $c_I\in Z_2$. Since $\phi(x^I)=\sum_{I'+I''=I} x^{I'}\otimes x^{I''}$ is symmetric, so is $\bar{\phi}(Sq^1 z)$. On the other hand, $\bar{\phi}(Sq^1 z)=Sq^1\bar{\phi}(z)=\sum_{i+j=2^{s-1}} c_i x_{2i}\otimes x_{2j}$. Thus $c_i=c_{2^{s-1}-i}$ if $x_{2i}\otimes x_{2^s-2i}\neq 0$, and the first assertion follows. Then we have $\bar{\phi}(Sq^1 z)=\sum_{i<2^{s-2}} c_i(x_{2i}\otimes x_{2^s-2i}+x_{2^s-2i}\otimes x_{2i})$. So, $Sq^1 z-\sum_{i<2^{s-1}} c_i x_{2i}x_{2^s-2i}\in P^{2s}_{Spin(n)}=0$ by Lemma 3.3, and the second assertion follows.

**Lemma 3.7.** *Let $n=2^s$, $s\geq 4$. For some $c\in Z_2$, we have*

$$\bar{\phi}(z)=c\sum_{i+j=2^{s-1}} x_{2i}\otimes x_{2j-1} \quad \textit{and} \quad Sq^1 z=c\sum_{i<2^{s-2}} x_{2i}x_{2^s-2i}\,.$$

*Proof.* By (3·7), $Sq^2 z=Sq^1 x_{2i}=0$, $Sq^2 x_{2i}=i\cdot x_{2i+2}$ and $Sq^2 x_{2j-1}=(j-1)x_{2j+1}$. Thus

$$0=\bar{\phi}(Sq^2 z)=Sq^2\bar{\phi}(z)=\sum_{i+j=2^{s-1}} c_i(i\cdot x_{2i+2}\otimes x_{2j-1}+(j-1)x_{2i}\otimes x_{2j+1})$$

$$=\sum_{1<k<2^{s-2}}(c_{2k-1}+c_{2k})(x_{4k}\otimes x_{2^s-4k+1})\,.$$

Since $4k\notin N$ for $2^{s-2}<2k<2^{s-1}$, we have

$$(3\cdot 8)\qquad c_{2k-1}=c_{2k} \text{ for } 2^{s-2}<2k<2^{s-1}\,.$$

Similarly,

$$0=\bar{\phi}(Sq^4z)=Sq^4\bar{\phi}(z)$$

$$=\sum_{i+j=2^{s-1}} c_i\left(\binom{i}{2}x_{2i+4}\otimes x_{2j-1}+i(j-1)x_{2i+2}\otimes x_{2j+1}+\binom{j-1}{2}x_{2i}\otimes x_{2j+3}\right)$$

$$=\sum_{1<k<2^{s-2}}\{(k-1)(c_{2k-2}+c_{2k})(x_{4k}\otimes x_{2^s-4k+3})$$

$$+k(c_{2k-3}+c_{2k-1})(x_{4k-2}\otimes x_{2^s-4k+5})\},$$

and we have

$$(3\cdot9)\qquad c_{4m-2}=c_{4m} \text{ and } c_{4m-1}=c_{4m+1} \text{ for } 2^{s-2}<4m<2^{s-1}.$$

It follows from $(3\cdot8)$ and $(3\cdot9)$ that $c_{i-1}=c_i$ for $2^{s-2}+1<i<2^{s-1}$. Thus, by Lemma 3.6, $c_i$ is independent of $i<2^{s-1}$, $i\notin N$, proving Lemma 3.7.

Next, consider the homomorphism

$$i^*:\ H^*(Spin(n);Z_2)\longrightarrow H^*(Spin(m);Z_2)$$

induced by the natural inclusion $i$: $Spin(m)\to Spin(n)$, $m\leq n$.

**Lemma 3.8.** $i^*(x_i)=x_i$. *If* $2^{s-1}<m\leq n\leq 2^s$, $i^*(z)=z$.

*Proof.* The first assertion follows from the well-known fact $\bar{i}^*(\bar{x}_i)=\bar{x}_i$ and the commutativity of the following diagram:

$$\begin{array}{ccccc} Spin(m) & \xrightarrow{\rho} & SO(m) & \xrightarrow{\lambda} & BZ_2 \\ \downarrow i & & \downarrow \bar{i} & & \downarrow id \\ Spin(n) & \xrightarrow{\rho} & SO(n) & \xrightarrow{\lambda} & BZ_2. \end{array}$$

If $2^{s-1}<m\leq n\leq 2^s$, by (iv) of Proposition 3.1 and by the naturality of the transgression $\tau$, $\tau(i^*(z))=\tau(z)\neq 0$. By (i) of $(2\cdot2)$ and by $(3\cdot6)$ $i^*(z)=z$ in $H^{2^s-1}(Spin(m);Z_2)$. Q.E.D.

*Proof of Theorem* 3.2. It is known that $H^*(Spin(2^s);Z_2)$ is not primitively generated for $s\geq 4$ (see Kojima [7]). It follows that $c=1$ in Lemma 3.7. Apply the naturality of $\bar{\phi}$ and $Sq^1$ to $i^*$ of Lemma 3.8, then we see that the formulas on $\bar{\phi}(z)$ and $Sq^1(z)$ in Theorem 3.2 holds for $n>8$. Together with $(3\cdot 7)$ and Corollary 3.5, the proof of the theorem has been established.

**Remark 3.9.** In the case $n=2^s$, the element $z$ has not been uniquely determined. In the next §, we shall see that $T^{2s-1}_{Ss(2^s)}=\langle z\rangle$ and this is mapped injectively into $T^{2s-1}_{Spin(2^s)}=\langle z, x_{2s-1}\rangle$ under the homomorphism $\rho_0{}^*$ induced by a double covering $\rho_0:Spin(2^s)\longrightarrow Ss(2^s)$. So, $z$ may be fixed as the image of $\rho_0{}^*$ if we fix $\rho_0$. However, by an automorphism of $Spin(2^s)$, $\rho_0$ is changed to another covering $\rho_1$: $Spin(2^s)\longrightarrow Ss(2^s)$ such that $\rho_1{}^*(z)=\rho_0{}^*(z)+x_{2s-1}$.

## § 4. Structure of $H^*(Ss(n); Z_2)$, $n=4m$

Let $n=4m$ and $l=n/2=2m$. It is well known that the center of $Spin(n)$ is isomorphic to $Z_2\times Z_2$. Let $a$ be the generator of the kernel of $\rho$: $Spin(n)\longrightarrow SO(n)$ and let $b$ be another generator of the center. So, $SO(n)=Spin(n)/\langle a\rangle$ and $PO(n)=Spin(n)/\langle a, b\rangle$. Then the semi-spinor group $Ss(n)$, $n=4m$, is defined by $Ss(n)=Spin(n)/\langle b\rangle$. By an automorphism of $Spin(n)$, $b$ is carried to $a\cdot b$. Thus

$(4\cdot 1)$ $\quad Ss(n)=Spin(n)/\langle b\rangle\cong Spin(n)/\langle a\cdot b\rangle.$

Note that

$(4\cdot 1)'$ $\quad Ss(4)\cong Spin(3)\times SO(3)$ and $Ss(8)\cong SO(8)$.

Let

$$(4\cdot 2)\qquad Ss(n)\xrightarrow{\rho'}PO(n)\xrightarrow{\lambda'}BZ_2$$

be a fibering consists of a double covering $\rho'$ and a map $\lambda'$ classifying $\rho'$. We use the following notations

$$(4\cdot 3)\qquad s=s(n)\qquad \text{for } 2^{s-1}<n\leq 2^s,$$

$$r=r(n)\qquad \text{for } n=2^r\cdot odd\quad (r\geq 2)$$

and $\bar{N}=\bar{N}(n)=N\cup\{2^r-1\}$ where $N=\{1, 2, 2^2, 2^3, \cdots, 2^t, \cdots\}$.

We quote the following result due to Baum-Browder [3].

**Proposition 4.1.** (i) *There are elements* $\bar{v}\in H^1$, $x\in H^{2s-1}$ *and* $w_i\in H^i$ *for* $i\neq 2^r-1$ *such that* $w_i=0$ *for* $i\in N$ *or* $i\geq n$ *and*

$$H^*(Ss(n);Z_2)=\Delta(w_i,x;i\notin\bar{N},0<i<n)\otimes Z_2[\bar{v}]/(\bar{v}^{2r}),\ \bar{v}^{2r}=0.$$

(ii) $$Sq^j(w_i)=\begin{cases}\bar{v}^{2r-1} & \text{if } r\geq 3,\ j=1 \textit{ and } i=2^{r-1}-1,\\ \binom{i}{j}w_{i+j} & \textit{if otherwise.}\end{cases}$$

(iii) $$\bar{\phi}(w_i)=\sum_{1\leq j<i}\binom{i}{j}\bar{v}^j\otimes w_{i-j}.$$

(iv) $w_i,\bar{v}\in\operatorname{Im}\rho'^*$. *$x$ is transgressive with respect to the fibering* $(4\cdot2)$ *and* $\tau(x)\neq 0$.

In order to apply Theorem 1.1, we change the generators:

$$(4\cdot4)\qquad x_{2j-1}=w_{2j-1}+\bar{v}w_{2j-2}\ \text{for}\ 2j-1\neq 2^r-1,\ 1,$$
$$x_{2j}=w_{2j}\ \text{and}\ y=\bar{v}.$$

Here we use the following convention

$$(4\cdot4)'\qquad x_i=0\ \text{for}\ i\in\bar{N}\ \text{and for}\ i\geq n.\quad x_0=1.$$

Obviously $(l=n/2=2m)$,

$$(4\cdot5)\qquad H^*(Ss(n);Z_2)=\Delta(x_i,x;i\notin\bar{N},0<i<n)\otimes Z_2[y]/(y^{2r})$$
$$=\Delta(y,x_{2j-1},x;1<j\leq l,j\neq 2^{r-1})$$
$$\otimes\Delta(x_{2j};2j\notin N,0<2j<n)\otimes Z_2[y^2]/(y^{2r}).$$

From the above proposition it is directly verified

**Lemma 4.2.** (i) $\bar{\phi}(y)=\bar{\phi}(y^{2t})=0,$

$$\bar{\phi}(x_{2j})=\sum_{1\leq k<j}\binom{j}{k}y^{2k}\otimes x_{2j-2k}$$

*and*

$$\bar{\phi}(x_{2j-1})=x_{2j-2}\otimes y+\sum_{1\leq k<j}\binom{j-1}{k}y^{2k}\otimes x_{2j-2k-1}\ (j\neq 2^{r-1}).$$

(ii) $Sq^1x_{2j}=0,$

$$Sq^1x_{2j-1}=\begin{cases}x_{2j}+y^2x_{2j-2} & \textit{for } j\neq 2^{r-2},\ j\neq 2^{r-1}\ \textit{or}\ r=2\\ y^{2r-1}+y^2x_{2r-1-2} & \textit{for } j=2^{r-2},\ r\geq 3,\end{cases}$$

$$Sq^{2j}x_i=\binom{i}{2j}x_{i+2j}\ (i\neq 2^r-1)\ \textit{and}\ Sq^jy^i=\binom{i}{j}y^{i+j}.$$

Consider the fibering $(1\cdot 1)$ for $G=Ss(n)$:

$$Ss(n) \xrightarrow{\pi} Ss(n)/T \xrightarrow{i} BT .$$

**Lemma 4.3.** *$y, y^{2j}$ and $x_i (i \notin \bar{N})$ belong to $T^*_{Ss(n)}$.*

*Proof.* Each element of $H^1(Ss(n); Z_2)$ is universally transgressive, in particular so is $y$. Thus $y \in T^1_{Ss(n)}$. By (ii) of $(2\cdot 2)$ and $(2\cdot 1)$, $y^2 = Sq^1 y \in T^2_{Ss(n)} \subset \operatorname{Im} \pi^*$, and $y^{2j} \in \operatorname{Im} \pi^+ \subset T^*_{Ss(n)}$. The second formula of Lemma 4.2, (i) shows that $x_{2j}$ satisfies (ii) of Theorem 2.2. Thus $x_{2j} \in T^*_{Ss(n)}$. Similarly it follows from the last formula of Lemma 4.2, (i) that $x_{2j-1} \in T^*_{Ss(n)}$. Q.E.D.

Now applying Lemma 2.5 to $(4\cdot 5)$, we see the existence of an element $z$ of $T^{2^s-1}_{Ss(n)}$ such that $z - x \in$ decomposables $\subset \operatorname{Im} \rho'^*$. Thus $\tau(z) = \tau(x) \neq 0$, and we have by Lemma 2.5 that

$(4\cdot 6)$ the following theorem holds except the assertion for $\bar{\phi}(z)$ and $Sq^i(z)$.

**Theorem 4.4.** *There exists an element $z \in T^{2^s-1}_{Ss(n)}$ such that $\tau(z) \neq 0$ with respect to the fibering $(4\cdot 2)$ and that the following holds.*

(i) $H^*(Ss(n); Z_2) = \Delta(x_i, z; i \notin \bar{N}, 0<i<n) \otimes Z_2[y]/(y^{2^r}),\ y^{2^r}=0$.

(ii) $T^{\text{odd}}_{Ss(n)} = \langle y, x_{2j-1}, z; 1<j\leq l, j\neq 2^{r-1}\rangle$,

$\operatorname{Im} \pi^* = \langle 1\rangle + T^{\text{even}}_{Ss(n)} = \Delta(x_{2j}; 2j \notin N, 0<2j<n) \otimes Z_2[y^2]/(y^{2^r})$.

(iii) *$\bar{\phi}$ and $Sq^i$ for $y$ and $x_i$ are given in Lemma 4.2.*

(iv) $$\bar{\phi}(z) = \sum_{\substack{i+j+k=2^{s-1}\\ j>0}} \binom{i+j}{i} y^{2i} x_{2j} \otimes x_{2k-1} + \sum_{\substack{i+j=2^{s-1}-1\\ 0<i<j}} x_{2i}x_{2j} \otimes y ,$$

$$Sq^1(z) = \sum_{\substack{i+j=2^{s-1}\\ 0<i<j}} x_{2i}x_{2j} + \sum_{\substack{i+j=2^{s-1}-1\\ 0<i<j}} y^2 x_{2i} x_{2j}$$

*and* $\quad Sq^j(z) = 0$ *for* $j>1$.

The remaining part of this section is devoted to determine $\bar{\phi}(z)$ and $Sq^1(z)$. Consider the following map between two fiberings $(3\cdot 1)$ and

(4·2):

$$\begin{array}{ccccc} Spin(n) & \xrightarrow{\rho} & SO(n) & \xrightarrow{\lambda} & BZ_2 \\ \downarrow \rho_0 & & \downarrow & & \downarrow id \\ Ss(n) & \xrightarrow{\rho'} & PO(n) & \xrightarrow{\lambda'} & BZ_2, \end{array}$$

where $\rho_0$ is the natural projection (double covering).

**Lemma 4. 5.** $\rho_0{}^*(z)=z$, $\rho_0{}^*(y)=0$ *and* $\rho_0{}^*(x_i)=x_i$ *for* $i\notin\overline{N}$.

*Proof.* By the naturality of the transgression, $\tau(\rho_0{}^*(z))=\tau(z)\neq 0$ in the upper fibering. By (i) of (2·2), $\rho_0{}^*(z)\notin T^{2^s-1}_{Spin(n)}$. Then we can take $z=\rho_0{}^*(z)$ in Theorem 3.2. Next consider the spectral sequence associated with the fibering

$$Spin(n)\xrightarrow{\rho_0}Ss(n)\xrightarrow{\lambda_0}BZ_2.$$

Then the only non-trivial differential is given by the transgression $\tau(x_{2r-1})\neq 0$ in $H^{2r}(BZ_2;Z_2)$. Thus we have that the kernel of $\rho_0{}^*\colon H^*(Ss(n);Z_2)\longrightarrow H^*(Spin(n);Z_2)$ is the ideal generated by $y$. So, $\rho_0{}^*(y)=0$ and $\rho_0{}^*(x_i)\neq 0$ for $i\in\overline{N}$ and $i<n$. If $i$ is odd, $\rho_0{}^*(x_i)\in T^i_{Spin(n)}=\langle x_i\rangle$ by (i) of (2·2) and (ii) of Theorem 3.2. Thus $\rho_0{}^*(x_i)=x_i$ for odd $i\notin\overline{N}$. For even $i\notin\overline{N}$, by (ii) of Lemma 4.2, $x_i=Sq^1x_{i-1}+y\cdot f$ for some $f$. Then $\rho_0{}^*(x_i)=\rho_0{}^*(Sq^1x_{i-1}+y\cdot f)=Sq^1(\rho_0{}^*(x_{i-1}))=Sq^1x_{i-1}=x_i$. Q.E.D.

We use a similar notation as in § 2:

(4·7) (i) $x^J=x_6{}^{\varepsilon_3}\cdots x_{2j}^{\varepsilon_j}\cdots x_{2l-2}^{\varepsilon_{l-1}}$ for $J=(\varepsilon_3,\cdots,\varepsilon_j,\cdots,\varepsilon_{l-1})$, $\varepsilon_j=0$ or $1$, where $x_{2j}$ and $\varepsilon_j$ are omitted if $j\in N$. $|J|=\sum\varepsilon_j$, $d(J)=\sum 2j\varepsilon_j$. $J+J'=(\cdots,\varepsilon_j+\varepsilon_j{}',\cdots)$ for $J'=(\cdots,\varepsilon_j{}',\cdots)$.

(ii) $R_1$ denotes the ideal of $H^*(Ss(n);Z_2)\otimes H^*(Ss(n);Z_2)$ generated by $y\otimes 1$. $R_2=R_1\otimes H^*(Ss(n);Z_2)$.

By Theorem 2.2, $\bar\phi(z)\in\operatorname{Im}\pi^+\otimes T^{\text{odd}}_{Ss(n)}$. Then by (4·6) we have

(4·8) $$\bar\phi(z)=\sum_{2i+d(J)+2k=2^s}a^k_{i,J}y^{2i}x^J\otimes x_{2k-1}+\sum_{2i+d(J)=2^s-2}b_{i,J}y^{2i}x^J\otimes y$$

for some $a^k_{i,J}$, $b_{i,J}\in Z_2$.

**Lemma 4. 6.** *In* (4·8) *we can take* $a^k_{0,J}=1$ *for* $x^J=x_{2^s-2k}$.

In fact, $(\rho_0^*\otimes\rho_0^*)\bar{\phi}(z)=\sum a^k_{0,J}x^J\otimes x_{2k-1}$ coincides with $\bar{\phi}(\rho_0^*(z))=\bar{\phi}(z)=\sum_{i+j=2^{s-1}}x_{2i}\otimes x_{2j-1}$ in Theorem 3.2. The only question might occur to the term $x_{2^s-2^r}\otimes x_{2^r-1}$ which may not be in $(\rho_0^*\otimes\rho_0^*)$-image. But

$$(4\cdot 9)\qquad x_i=0 \text{ if } i\geq 2^s-2^r \text{ and } s\neq r,$$

since $2^s-2^r\geq n=2^r\cdot\text{odd}$ unless $2^s=2^r=n$.

Next, we prove the following lemma by use of the associativity $(\bar{\phi}\otimes 1)\bar{\phi}=(1\otimes\bar{\phi})\bar{\phi}$ of $\bar{\phi}$.

**Lemma 4.7.** 
$$\bar{\phi}(z)=\sum_{\substack{i+j+k=2^{s-1}\\0<j}}\binom{i+k-1}{i}y^{2i}x_{2j}\otimes x_{2k-1}$$
$$+\sum_{\substack{i+j=2^{s-1}-1\\0<i<j}}x_{2i}x_{2j}\otimes y+b(x_{2^s-2}\otimes y+\sum_{\substack{i+j=2^{s-1}\\0<i}}y^{2i}\otimes x_{2j-1}),\ b\in Z_2.$$

*Proof.* By Lemma 4.2,

$\phi(y^{2i})\equiv 1\otimes y^{2i}$ and $\phi(x_{2j})\equiv x_{2j}\otimes 1+1\otimes x_{2j}$ mod $R_1$. Then it follows from $(4\cdot 8)$, modulo $R_2=R_1\otimes H^*(Ss(n);Z_2)$,

$$(\bar{\phi}\otimes 1)\bar{\phi}(z)\equiv\sum a^k_{i,J}\sum_{J'+J''=J}x^{J'}\otimes y^{2i}x^{J''}\otimes x_{2k-1}$$
$$+\sum b_{i,J}\sum_{J'+J''=J}x^{J'}\otimes y^{2i}x^{J''}\otimes y$$
$$\equiv(1\otimes\bar{\phi})\bar{\phi}(z)\equiv\sum_{k\neq 2^{r-1}}a^k_{0,J}x^J\otimes(x_{2k-2}\otimes y+\sum_{1\leq m<k}\binom{k-1}{m}y^{2m}\otimes x_{2k-2m-1}).$$

Comparing the coefficients, we have

$$a^k_{i,J}=0\qquad\text{if } |J|>1 \text{ and } y^{2i}x^J\otimes x_{2k-1}\neq 0,$$
$$b_{i,J}=0\qquad\text{if } |J|>2 \text{ and } y^{2i}x^J\neq 0,$$
$$b_{i,J}=0\qquad\text{if } |J|>0,\ i>0 \text{ and } y^{2i}x^J\neq 0,$$

and using Lemma 4.6,

$$b_{0,J}=a^{j+1}_{0,I}=1 \text{ if } x^I=x_{2i},\ x^J=x_{2i}x_{2j}\neq 0\,(i\neq j) \text{ and } j+1\neq 2^{r-1},$$
$$a^k_{i,J}=\binom{i+k-1}{i}a^{i+k}_{0,J}=\binom{i+k-1}{i}\quad\text{if } x^J=x_{2j},\ y^{2i}x_{2j}\otimes x_{2k-1}\neq 0$$
$$\text{and}\quad i+k\neq 2^{r-1}.$$

Here, $b_{0,J}=1$ for $x^J=x_{2i}x_{2j}\neq 0\,(i\neq j)$ since either $j+1\neq 2^{r-1}$ or $i+1\neq 2^{r-1}$. Also, if $i+k=2^{r-1}$ and $x^J=x_{2j}$, $x_{2j}=0$ by $(4\cdot 9)$ and $a^k_{i,J}y^{2i}x^J$

$\otimes x_{2k-1}=0$.

Now it remains to fix the coefficients of the terms in the following

$$(4\cdot 10) \qquad x_{2^s-2}\otimes y,\ y^{2i}\otimes x_{2^s-2i-1}\ (0<i<2^{s-1}-1) \text{ and } y^{2^s-2}\otimes y\,,$$

which are all trivial or non-trivial for $n\neq 2^s$ or $n=2^s$ respectively.

Let $n=2^s$ and let $b$, $a_i$ and $b'$ be the coefficients of the terms of (4·10) in (4·8) respectively. Compare the coefficients of $y^{2i}\otimes x_{2^s-2i-2}\otimes y$, $y^{2i}\otimes y^{2j}\otimes x_{2^s-2i-2j-1}$ and $y^2\otimes y^{2^s-4}\otimes y$ in the equality $(\bar{\phi}\otimes 1)\bar{\phi}(z)=(1\otimes\bar{\phi})\bar{\phi}(z)$. Then we have

$$b=a_i \text{ for } 2^s-2i-2\notin N,$$

$$\binom{i+j}{j}a_{i+j}=\binom{2^{s-1}-i-1}{j}a_i \quad\text{and}\quad b'=0\,.$$

For even $i<2^{s-1}-2$, $b=a_i=a_{i+1}$. For $i=2^{s-1}-2$, $a_i=\binom{i}{2}a_i=\binom{3}{2}\cdot a_{i-2}=a_{i-2}=b$. Consequently the coefficients of the equality of the lemma are all fixed. Q.E.D.

**Lemma 4. 8.** *Lemma* 4.7 *holds for* $b=0$, *and*

$$Sq^1(z)=\sum_{\substack{i+j=2^{s-1}\\0<i<j}} x_{2i}x_{2j}+\sum_{\substack{i+j=2^{s-1}-1\\0<i<j}} y^2x_{2i}x_{2j}\,.$$

*Proof.* Since $Sq^1y=y^2$, $R_1$ is closed under $Sq^1$. Consider $Sq^1(\bar{\phi}(z))$ modulo $R_1$. For $j+k=2^{s-1}$, $k>1$, the equality $Sq^1(x_{2j}\otimes x_{2k-1})=x_{2j}\otimes(x_{2k}+y^2x_{2k-2})$ holds, even for $k=2^{r-1}$, $2^{r-2}$, since $x_{2j}=0$ for $k\leq 2^{r-1}$ by (4·9). Then it follows from Lemma 4.7

$$Sq^1(\bar{\phi}(z))\equiv\sum x_{2j}\otimes(x_{2k}+y^2x_{2k-2})+\sum_{i<j} x_{2i}x_{2j}\otimes y^2+bx_{2^s-2}\otimes y^2\,.$$

On the other hand, $Sq^1(z)\in Sq^1T^{\text{odd}}_{Ss(n)}\subset T^{\text{even}}_{Ss(n)}=\operatorname{Im}\pi^+$. So, we may put

$$Sq^1(z)=\sum_{2k+d(J)=2^s} c_{k,J}y^{2k}x^J \quad\text{for some } c_{k,J}\in Z_2\,.$$

As in the proof of Lemma 4.7, we have

$$\bar{\phi}(Sq^1(z))\equiv\sum c_{k,J}\sum_{J'+J''=J} x^{J'}\otimes y^{2k}x^{J''} \quad \text{mod } R_1\,.$$

From the naturality $\bar{\phi}(Sq^1(z))=Sq^1(\bar{\phi}(z))$, the coefficients $c_{k,J}$ of $y^{2k}x^J\neq 0$ satisfy the following relations:

$$c_{k,J}=0 \text{ if } |J|>2 \text{ or if } k>1\,,$$

$$c_{0,J}=c_{1,J}=1 \text{ if } |J|=2 \text{ and } c_{1,J}=b \text{ if } |J|=1\,.$$

Thus the lemma is proved except the triviality of the coefficient $b$ of $y^2x_{2^s-2}$. If $n\neq 2^s$, $x_{2^s-2}=0$ and we may take $b=0$ by (4·10).

Let $n=2^s$, and let $Q$ be the ideal of $H^*(Ss(2^s);Z_2)$ generated by the elements $x_i$'s. By (ii) of Lemma 4.2, $x_i^2=Sq^ix_i=x_{2i}$. It follows from (ii) of Theorem 4.4 that $H^*(Ss(2^s);Z_2)/Q\cong\Delta(z)\otimes Z_2[y]/(y^{2^s})$. By (ii) of Lemma 4.2, $x_i\equiv 0$, $Sq^1x_i\equiv 0\,(i\neq 2^{s-1})$ and $Sq^1x_{2^{s-1}-1}\equiv y^{2^{s-1}}$ mod $Q$. Then we have

$$Sq^1(\bar{\phi}(z))\equiv by^{2^{s-1}}\otimes y^{2^{s-1}},\quad \bar{\phi}(Sq^1(z))\equiv 0 \quad \text{mod } Q_1$$

for the ideal $Q_1$ generated by $Q\otimes 1$ and $1\otimes Q$. Thus $b=0$ completing the proof of Lemma 4.8.

*Proof of Theorem* 4.4. The assertions (i), (ii) and (iii) are established by (4·6). For even $j>0$, $Sq^j(z)\in T^{2^s+j-1}_{Ss(n)}=0$ by (ii), and $Sq^{j+1}(z)=Sq^1Sq^j(z)=0$. Thus $Sq^j(z)=0$ for $j>1$. The remaining part of the assertion (iv) follows from Lemma 4.8 and the following (4·11).

Q.E.D.

$$(4\cdot 11)\qquad \binom{i+j}{i}\equiv\binom{i+k}{i} \quad \text{mod } 2 \quad \text{if } i+j+k=2^t-1\,.$$

For,

$$(a+b)^{2^t-1}\equiv(a^{2^t}+b^{2^t})/(a+b)\equiv\sum a^ib^{2^t-i-1}$$

and $(a+b+c)^{2^t-1}\equiv\sum(a+b)^lc^{2^t-l-1}\equiv\sum\binom{i+j}{i}a^ib^jc^{2^t-i-j-1}$.

Similarly

$$(a+b+c)^{2^t-1}\equiv\sum\binom{i+k}{i}a^ib^{2^t-i-k-1}c^k,$$

and (4·11) follows.

## § 5. Structure of $H^*(AdE_7;Z_2)$

Let $E_7$ be the compact simply connected Lie group of type $E_7$. The mod 2 cohomology ring $H^*(E_7;Z_2)$ is determined by Araki [1]. As is seen in [16] or by use of Theorem 1.1 we have

**Proposition 5. 1.** (i) *There are elements* $e_i \in H^i(E_7; Z_2)$ *for* $i=3$, 15 *such that*

$$H^*(E_7; Z_2) = Z_2[e_3, e_5, e_9]/(e_3^4, e_5^4, e_9^4) \otimes \Lambda(e_{15}, e_{17}, e_{23}, e_{27})$$

*where*

$$e_5 = Sq^2 e_3, \ e_9 = Sq^4 e_5, \ e_{17} = Sq^8 e_9, \ e_{23} = Sq^8 e_{15} \text{ and } e_{27} = Sq^4 e_{23}.$$

(ii) $$T^{\text{odd}}_{E_7} = \langle e_3, e_5, e_9, e_{15}, e_{17}, e_{23}, e_{27} \rangle$$

*and* $$\langle 1 \rangle + T^{\text{even}}_{E_7} = \operatorname{Im} \pi^* = \Lambda(e_3^2, e_5^2, e_9^2).$$

Thomas [14] showed that $Sq^2 e_{15} \neq 0$ in $E_6$, and thus in $E_7$. So,

(5·1) $$Sq^2 e_{15} = e_{17}.$$

The following is due to Kono-Mimura-Shimada [9] or Toda [15]:

**Lemma 5. 2.** $P^{15}_{E_7} = 0$.

As is well known the center of $E_7$ is a cyclic group of order 2, and so denoted by $Z_2$. The quotient group of $E_7$ by the center is denoted by $AdE_7$, and the natural projection (double covering) by

$$p: E_7 \longrightarrow AdE_7 = E_7/Z_2.$$

We use the following notations:

(5·2) (i) $$e_6 = e_3^2, \ e_{10} = e_5^2 \text{ and } e_{18} = e_9^2,$$

(ii) $$M = \{1, 2, 5, 6, 9, 10, 15, 17, 18, 23, 27\}$$

and $$\overline{M} = M \cup \{16, 24, 28\}.$$

Then the results on $H^*(AdE_7; Z_2)$ are summarized as follows.

**Theorem 5. 3.** (i) *There exist elements* $x_i \in H^i(AdE_7; Z_2)$ *for* $i \in M$ *such that* $p^*(x_i) = e_i$ *if* $i \neq 1, 2$, $x_i^2 = x_{2i}$ *if* $2i \in M$ *and*

$$\begin{aligned} H^*(AdE_7; Z_2) &= \Delta(x_i; i \in M) \\ &= Z_2[x_1, x_5, x_9]/(x_1^4, x_5^4, x_9^4) \otimes \Lambda(x_6, x_{15}, x_{17}, x_{23}, x_{27}). \end{aligned}$$

(ii) $$T^{\text{odd}}_{AdE_7} = \langle x_1, x_5, x_9, x_{15}, x_{17}, x_{23}, x_{27} \rangle$$

*and* $$\langle 1 \rangle + T^{\text{even}}_{AdE_7} = \operatorname{Im} \pi^* = \Lambda(x_2, x_6, x_{10}, x_{18}).$$

(iii) $$P^*_{AdE_7}=\langle x_1, x_2, x_5, x_6, x_9, x_{10}, x_{17}, x_{18}\rangle$$

$$\bar{\phi}(x_{15})=x_{10}\otimes x_5+x_6\otimes x_9,$$

$$\bar{\phi}(x_{23})=x_{18}\otimes x_5+x_6\otimes x_{17}$$

*and* $$\bar{\phi}(x_{27})=x_{18}\otimes x_9+x_{10}\otimes x_{17}.$$

(iv) *For* $i\in M$, *the relations* $Sq^j x_i=\binom{i}{j}x_{i+j}$ *hold, where* $x_{i+j}=0$ *if* $i+j\notin \overline{M}$, $x_{16}=x_6x_{10}$, $x_{24}=x_6x_{18}$ *and* $x_{28}=x_{10}x_{18}$.

(More precise results will be seen later.)

Let $T$ be a maximal torus of $E_7$. As is well known the center $Z_2$ is a subgroup of $T$. According to Watanabe [17], we take elements $t_1,\cdots,t_7$, $x$ of $H^2(BT)$ such that

(5·3) (i) $H^*(BT)=Z[t_1,\cdots,t_7,x]/(3x-c_1)$, $c_1=t_1+\cdots+t_7$,

(ii) the actions of $\Phi(E_7)$ on $H^*(BT)$ contain the permutations of $t_i$'s.

The inclusions $Z_2\subset T\subset E_7$ induce maps

(5·4) $$\iota=\iota''\circ\iota': BZ_2\xrightarrow{\iota'}BT\xrightarrow{\iota''}BE_7.$$

**Lemma 5.4.** (i) $\iota''^*: H^4(BE_7)\longrightarrow H^4(BT)$ *is injective and its image is generated by* $c_2-4x^2$ *where* $c_2=\sum_{1\le i<j\le 7}t_it_j$.

(ii) $\iota^*: H^4(BE_7;Z_2)\longrightarrow H^4(BZ_2;Z_2)$ *is bijective.*

*Proof.* $\iota''$ is a fibering with a fibre $E_7/T$. Since $BE_7$ is 3-connected, we have a Serre exact sequence

$$H^3(E_7/T)\longrightarrow H^4(BE_7)\xrightarrow{\iota''^*}H^4(BT)\xrightarrow{i^*}H^4(E_7/T).$$

$H^*(E_7/T)$ is given by Theorem 4.1 of [16], in particular $H^3(E_7/T)=0$ and $i^*$ induces an isomorphism $H^4(BT)/\langle c_2-4x^2\rangle\cong H^4(E_7/T)$. Thus (i) follows. By (i) of (5·3), $H^*(BT;Z_2)=Z_2[t_1,\cdots,t_7]$. Obviously $H^*(BZ_2;Z_2)=Z_2[y]$, $y\in H^1$. Put $T'=T/Z_2$. From the fibering $BZ_2\xrightarrow{\iota'}BT\longrightarrow BT'$, we have an exact sequence $0\longrightarrow H^1(BZ_2;Z_2)\longrightarrow H^2(BT';Z_2)\longrightarrow H^2(BT;Z_2)\xrightarrow{\iota'^*}H^2(BZ_2;Z_2)$. Since $T$ and $T'$ are tori of the same dimension, they are isomorphic to each other. It follows easily that $\iota'^*$ is not trivial, i.e., $\iota'^*(t_i)=y^2$ for some $i$. Since $Z_2$ is the center, the action of the Weyl group $\Phi(E_7)$ is trivial on $BZ_2$. It follows from (ii) of (5·3) and from the naturality of the action that $\iota'^*(t_1)=\cdots=\iota'^*(t_7)=y^2$. By (i), a generator $x_4$ of $H^4(BE_7;Z_2)\cong Z_2$ is mapped onto $\iota''^*(x_4)=c_2-4x^2=c_2 \pmod 2$. Thus

$$\iota^*(x_4)=\iota'^*(c_2)=\sum_{1\leq i<j\leq 7}\iota'^*(x_i)\,\iota'^*(x_j)=\binom{7}{2}y^4=y^4,$$

and (ii) follows. Q.E.D.

Consider the following fibering

$$E_7\xrightarrow{p}AdE_7\xrightarrow{f}BZ_2. \tag{5·5}$$

**Lemma 5.5.** *There exist elements $x_i$ of $H^i(AdE_7;Z_2)$ for $i\in M$ such that*

(i) $H^*(AdE_7;Z_2)=\Delta(x_i;i\in M)$,

(ii) $T^{\mathrm{odd}}_{AdE_7}=\langle x_1,x_5,x_9,x_{15},x_{17},x_{23},x_{27}\rangle$,

$\langle 1\rangle+T^{\mathrm{even}}_{AdE_7}=\mathrm{Im}\,\pi^*=\Delta(x_2,x_6,x_{10},x_{18})$,

(iii) $x_i=f^*(y^i)$ *for* $i=1,2$, $p^*(x_i)=e_i$ *for* $i\neq 1,2$,

(iv) $x_2=Sq^1x_1=x_1^2$, $x_6=Sq^1x_5$, $x_9=Sq^4x_5$, $x_{10}=Sq^5x_5=x_5^2$,

$x_{17}=Sq^8x_9$, $x_{18}=Sq^9x_9=x_9^2$, $x_{23}=Sq^8x_{15}$ *and* $x_{27}=Sq^4x_{23}$.

*Proof.* $e_3$ is universally transgressive and its transgression image $x_4$ generates $H^4(BE_7;Z_2)$. By (ii) of Lemma 5.4, we have that the transgression $\tau$ with respect to (5·5) maps $e_3$ to $\tau(e_3)=\iota^*(x_4)=y^4$. The generators $e_i$ belong to $T^*_{E_7}$, i.e., they are transgressive with respect to the fibering

$$E_7\xrightarrow{\pi}E_7/T\longrightarrow BT. \tag{5·5)'$$

By the naturality of the transgression for the natural map of (5·5) to (5·5)′, $e_i$'s are transgressive with respect to (5·5). Since $\tau(e_3)=y^4$, $\tau(e_i)=0$ for $i>3$, that is, $e_i=p^*(x_i')$ for some $x_i'$, $i>3$. In the spectral sequence associated with the fibering (5·5): $E_2=H^*(BZ_2;Z_2)\otimes H^*(E_7;Z_2)=Z_2[y]\otimes\Delta(e_i)$, the only non-trivial differential is $d_4(1\otimes e_3)=y^4\otimes 1$. Then $E_\infty=Z_2[y]/(y^4)\otimes\Delta(e_i;\,i\neq 3)$, and we have $H^*(AdE_7;Z_2)=\Delta(x_1,$

$x_2, x_i'; i>3)$. By Lemma 2.5, we can choose $x_i \in T^i_{AdE_7}$ such that $x_i \equiv x_i'$ (mod decomposables) and that (i) and (ii) of the lemma hold. By (i) of $(2\cdot 2)$, $p^*(x_i) \in T^i_{E_7}$. By (ii) of Proposition 5.1, $T^i_{E_7} = \langle e_i \rangle$ for $i \in M$ and $i>3$. Thus $p^*(x_i) = e_i$, $i>3$, and (iii) of the lemma is proved.

By (ii) of this lemma, $T^i_{AdE_7} = \langle x_i \rangle$ for $i \in M$ and $i \neq 18$ and $T^{18}_{AdE_7} = \langle x_{18}, x_2 x_6 x_{10} \rangle$. We may choose $x_{18}$ as $x_{18} = x_9^2$. Since $Sq^j$ is closed in $T^*_{AdE_7}$, the relation (iv) holds, up to undetermined coefficients. The coefficients are fixed by applying $p^*$ and comparing the coefficients in (i) of Proposition 5.1 with (i) of $(5\cdot 2)$. For example, $p^*(x_6) = e_6 = e_3^2 = Sq^3 e_3 = Sq^1 Sq^2 e_3 = Sq^1 e_5 = p^*(Sq^1 x_5)$, and this implies $Sq^1 x_5 = x_6$. Q.E.D.

**Lemma 5.6.** (i) $x_i^2 = 0$ *for* $i = 2, 6, 10, 15, 17, 18, 23, 27$.

(ii) *The relation* $Sq^j x_i = \binom{i}{j} x_{i+j}$ *in* (iv) *of Theorem* 5.3 *holds for* $i = 1, 2, 5, 6, 9, 10, 17, 18$.

(iii) (iii) *of Theorem* 5.3 *holds.* $Sq^1 x_{15} = x_6 x_{10}$, $Sq^1 x_{23} = x_6 x_{18}$ *and* $Sq^1 x_{27} = x_{10} x_{18}$.

*Proof.* Obviously $x_1 \in P^1_{AdE_7}$. By Theorem 2.2, $\bar{\phi}(x_5) \in \operatorname{Im} \pi^+ \otimes T^{\mathrm{odd}}_{AdE_7}$. It follows from (ii) of Lemma 5.5, $\bar{\phi}(x_5) = 0$, i.e., $x_5 \in P^5_{AdE_7}$. By (ii) of $(2\cdot 2)$, $Sq^j P^i_{AdE_7} \subset P^{i+j}_{AdE_7}$. Then it follows from (iv) of Lemma 5.5

$$(5\cdot 6) \qquad \langle x_1, x_2, x_5, x_6, x_9, x_{10}, x_{17}, x_{18} \rangle \subset P^*_{AdE_7}.$$

Also by Lemma 2.6

$$(5\cdot 6)' \qquad P^*_{AdE_7} \subset \langle x_i; i \in M \rangle.$$

For $i = 2, 6, 10, 17, 18$, $x_i \in P^i_{AdE_7}$ implies $x_i^2 \in P^{2i}_{AdE_7}$ which is trivial by $(5\cdot 6)'$: $x_i^2 = 0$. For $i = 15, 23, 27$, $x_i^2 = Sq^i x_i = Sq^1 Sq^{i-1} x_i$. $x_i \in T^i_{AdE_7}$ implies $Sq^{i-1} x_i \in T^{2i-1}_{AdE_7}$ which is trivial by (ii) of Lemma 5.5. Thus $x_i^2 = 0$ for $i = 15, 23, 27$, and (i) is proved.

Obviously the relation in (ii) holds for $j = 0$, for $j > i$ and also for $j = i$ by (i) of this lemma and (iv) of Lemma 5. 5. Let $0 < j < i$ and consider the cases both that $P^{i+j}_{AdE_7} = 0$ and that $\binom{i}{j} \equiv 0 \pmod 2$ or $i + j \notin \overline{M}$. For such cases $Sq^j x_i = 0 = \binom{i}{j} x_{i+j}$. Then, the remaining cases are the following ones:

($\alpha$) $j = 1$ and $i = 5, 9, 17$; ($\beta$) $j = i - 1$ and $i = 5, 9$;

($\gamma$) $j=i-2$ and $i=6, 10$; ($\delta$) $i+j=15, 23, 27$, and $i=9, 10, 17, 18$.

For the cases ($\alpha$) and ($\beta$) the relation follows from (iv) of Lemma 3.5. and the Adem relation $Sq^1Sq^{2k}=Sq^{2k+1}$ $(i=4k+1)$. For the case ($\gamma$), we have $Sq^4x_6=Sq^4Sq^1x_5=(Sq^5+Sq^2Sq^3)x_5=x_5{}^2=\binom{6}{4}x_{10}$ and $Sq^8x_{10}=Sq^8(x_5{}^2)=(Sq^4x_5)^2=x_9{}^2=\binom{10}{8}x_{18}$. For the case ($\delta$), $\binom{i}{j}\equiv 0 \pmod 2$. For $i=9, 17$, we have $Sq^6x_i=(Sq^2Sq^4+Sq^5Sq^1)x_i=Sq^5x_{i+1}$ and $Sq^5x_{i+1}=Sq^1Sq^4x_{i+1}=0$. We have also, $Sq^{10}x_{17}=(Sq^2Sq^8+Sq^9Sq^1)x_{17}=Sq^9x_{18}$ and $Sq^9x_{18}=Sq^1Sq^8x_{18}=0$.

Consequently the relation $Sq^jx_i=\binom{i}{j}x_{i+j}$ in (ii) is established.

By Theorem 2. 2

$$\bar{\phi}(x_{15})=ax_{10}\otimes x_5+bx_6\otimes x_9 \text{ for } a, b\in Z_2 .$$

Since $Sq^1x_{15}\in T^{16}_{AdE_7}=\langle x_6x_{10}\rangle$, $Sq^1x_{15}=cx_6x_{10}$ for some $c\in Z_2$. Then $c(x_{10}\otimes x_6+x_6\otimes x_{10})=\bar{\phi}Sq^1(x_{15})=Sq^1\bar{\phi}(x_{15})=ax_{10}\otimes x_6+bx_6\otimes x_{10}$. It follows from Lemma 5.2, $a=b=c=1$. That is,

$$\bar{\phi}(x_{15})=x_{10}\otimes x_5+x_6\otimes x_9 \text{ and } Sq^1x_{15}=x_6x_{10}=x_{16} .$$

By use of (ii) and the Cartan formula,

$$\bar{\phi}(x_{23})=\bar{\phi}(Sq^8x_{15})=Sq^8\bar{\phi}(x_{15})=x_{18}\otimes x_5+x_6\otimes x_{17}$$

and $$\bar{\phi}(x_{27})=\bar{\phi}(Sq^4x_{23})=Sq^4\bar{\phi}(x_{23})=x_{18}\otimes x_9+x_{10}\otimes x_{17} .$$

Thus $x_{15}$, $x_{23}$, $x_{27}\notin P^*_{AdE_7}$ and the equality holds in $(5\cdot 6)$.

Finally $Sq^1x_{23}\in T^{24}_{AdE_7}=\langle x_6x_{18}\rangle$, $Sq^1x_{27}\in T^{28}_{AdE_7}=\langle x_{10}x_{18}\rangle$, and the last two formulas of (iii) are proved as above. Q.E.D.

**Lemma 5. 7.** *The relation* $Sq^jx_i=\binom{i}{j}x_{i+j}$ *in* (iv) *of Theorem* 5.3 *holds for* $i=15$, 23, 27.

*Proof.* First consider even $j=2k$. Since $Sq^{2k}x_i\in T^{\text{odd}}_{AdE_7}$, the non-trivial cases are the following ones:

$$Sq^2x_{15}=x_{17},\ Sq^8x_{15}=x_{23},\ Sq^4x_{23}=x_{27} \text{ and } Sq^{12}x_{15}=x_{27}.$$

The first case is reduced to $(5\cdot 1)$ by applying $p^*$. The seconed and the third cases are the definitions. For the last one, $Sq^{12}x_{15}=(Sq^4Sq^8+Sq^{10}Sq^2+Sq^{11}Sq^1)x_{15}=Sq^4x_{23}+Sq^{10}x_{17}+Sq^{11}(x_6x_{10})=x_{27}$.

Together with Lemma 5.6, we see that the formula of the lemma holds for $j=1$ and for even $j=2k$. For odd $j=2k+1$,

$$Sq^j x_i = Sq^1 Sq^{2k} x_i = \binom{i}{2k} Sq^1 x_{i+2k} = \binom{i}{2k}\binom{i+2k}{1} x_{i+j} = \binom{i}{j} x_{i+j}. \quad \text{Q.E.D.}$$

*Proof of Theorem* 5.3. (i) follows from Lemma 5.5 and (i) of Lemma 5.6. Then (ii) of the theorem is (ii) of Lemma 5.5. (iii) of the theorem is (iii) of Lemma 5.6. (iv) follows from (ii) of Lemma 5.6 and Lemma 5.7. Q.E.D.

By quite a similar but a little simpler arguments, we have

**Proposition 5.8.** *Theorem* 5.3 *holds for* $H^*(E_7; Z_2)$ *by omitting* $x_1, x_2$, *by adding* $x_3 \in P^3_{E_7}$ *with* $x_3^2 = x_6$ *and by replacing* $M$ *by* $\{3, 5, 6, 9, 10, 15, 17, 18, 23, 27\}$.

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