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On the Asymptotic Behavior of Solutions of Semi-linear Wave Equations

By

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Introduction

We first consider the following semi-linear wave equation in Part I:

$$u_{tt}-\Delta u+m^2u+g|u|^{p-1}u=0 \tag{1}$$

where $m>0$, $g>0$, $p>1$, $x\in R^n$, $\Delta=$Laplacian. Recently, Glassey [1] showed that if p is small ($1<p\leqq 2$ $n=1$; $1<p\leqq 1+2n^{-1}$ $n\geqq 2$), scattering theory is impossible for complex solutions of (1). We show in 1. 1 Glassey's result is also applicable even to real solutions. Segal showed in [3] and [4] that scattering operator can be constructed for (1) if $p>2+2n^{-1}$. We show in 1. 2 that the solution $u_+(x,t)$ of the free equation [(1) with $g=0$] to which a given solution $u(x,t)$ of (1) is asymptotic in a weak sense as $t\to+\infty$, exists if $p>1+2n^{-1}$, $n\geqq 3$.

In Part II, we consider the following semi-linear wave equation with the first order dissipation:

$$u_{tt}-\Delta u+u_t+f(u,u_t,\nabla u)=0\,. \tag{2}$$

We show the asymptotic properties of the solutions of the linear equation [(2) with $f=0$] in 2. 1 and those of the nonlinear equation (2) in 2. 2 and 2. 3.

Notation. We denote by L^p the space of measurable functions u on R^n whose p-th powers ($1\leqq r\leqq\infty$) are integrable with the norm

$$\|u\|_{L^p}=\left(\int|u(x)|^p\,dx\right)^{1/p}\|u\|_\infty=\mathrm{ess}\cdot\sup_x|u(x)|\,,$$

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by C^∞ the space of infinitely differentiable functions, by C_0^∞ the subspace of C^∞ consisting of functions with compact support in R^n and by H^k the usual Sobolev space on R^n with the norm

$$\|u\|_k^2=\sum_{|\alpha|\leq k}\left\|\left(\frac{\partial}{\partial x}\right)^\alpha u\right\|_{L^2}^2$$

where

$$\left(\frac{\partial}{\partial x}\right)^\alpha=\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}, \quad |\alpha|=\alpha_1+\cdots+\alpha_n .$$

Let X be a Banach space on R^n. Then $\mathcal{E}_t^k(X)\ni u(x, t)$ means that $u(\cdot, t)$ belongs to X for all fixed t and u is k-times continuously differentiable with respect to t in X-topology. We denote grad $u=(\partial u/\partial x_1, \partial u/\partial x_2, \cdots, \partial u/\partial x_n)$ by ∇u. In Part I, all functions are generally complex-valued, but in Part II, all functions are real-valued.

Part I

§ 1. 1. Extention of Glassey's Result

We consider the solutions of the Cauchy problem for the equation (1). We take $g=1$ without loss of generality. In [1], Glassey's result is not valid for real solutions, because he assumes

$$Q(t)=\text{Im}\int\bar{u}u_t dx\neq 0 .$$

Then we define the momentum $P(t)$ by

$$P(t)=\text{Re}\int\bar{u}_t\nabla u\, dx$$

instead of $Q(t)$. We denote the energy norm $\|\cdot\|_e$ by

$$\|u(t)\|_e^2=\int(|u_t|^2+|\nabla u|^2+m^2|u|^2)\,dx .$$

We have the following theorem which is valid for both real and complex solutions.

Theorem 1. *Let $u(x, t)$ be a C^2-solution of equation* (1) *with Cauchy data in C_0^∞ satisfying $P(0)\neq 0$. Suppose that*

$$1<p\leqq 2 \ \textit{if} \ n=1, \ 1<p\leqq 1+2n^{-1} \ \textit{if} \ n\geqq 2.$$

Then, there dose not exist any free solution $v(x, t)$ *in* C_0^∞ *such that*

$$\|u(t)-v(t)\|_e\to 0 \ \textit{as} \ t\to+\infty .$$

Proof. We note the following energy equality holds: if u and v are C_0^2 solution of the nonlinear equation (1) and the free one respectively, it follows that

(3) $$\int\left(|u_t|^2+|\nabla u|^2+m^2|u|^2+\frac{2}{p+1}|u|^{p+1}\right)dx=\text{const.}$$

$$\int(|u_t|^2+|\nabla v|^2+m^2|v|^2)\,dx=\text{const.} \quad \text{for all } t\geqq 0 .$$

We first show that $P(t)$ is a time invariant vector. Differentiating $P(t)$ with respect to t directly,

$$\frac{dP(t)}{dt}=\text{Re}\Big(\int\overline{u}_t\nabla u_t\,dx+\int\overline{u}_{tt}\nabla u\,dx)$$

$$=\frac{1}{2}\int\nabla\left(|u_t|^2-|\nabla u|^2-m^2|u|^2-\frac{2}{p+1}|u|^{p+1}\right)dx$$

$$=0 .$$

Hence, we have

$$P(t)=P(0) \ \text{ for all } t\geqq 0 .$$

Suppose that there exists a free solution $v\in C_0^\infty$ such that

$$\|u(t)-v(t)\|_e\to 0, \ \text{as} \ t\to+\infty .$$

Define

$$P_0(t)=\text{Re}\int\overline{v}_t\nabla v\,dx .$$

We note that $P_0(t)$ is also time invariant vector from the same argument as before. Then

$$P(t)-P_0(t)=\text{Re}\left\{\int(\overline{u}_t-\overline{v}_t)\nabla u+\overline{v}_t(\nabla u-\nabla v)\,dx\right\} ,$$

by using Schwarz's inequality

$$|P(t)-Po(t)|\leqq\|\nabla u\|_{L^2}\|u-v\|_e+\|v_t\|_{L^2}\|u-v\|_e .$$

Since $\|\nabla u\|_{L^2}$ and $\|v_t\|_{L^2}$ are bounded from (3) and

$$\|u(t)-v(t)\|_e \to 0 \text{ as } t\to+\infty,$$

we have

$$|P(t)-P_0(t)| \to 0 \text{ as } t\to+\infty.$$

Hence, from $P(t)-Po(t)=$const., it follows

$$P(t)=P_0(t)=P(0) \text{ for all } t\geqq 0.$$

Then, the assumption $P(0)\neq 0$ gives

$$\begin{aligned} 0<|P(0)| &= |\mathrm{Re}\int \bar{v}_t \nabla v\, dx| = |-\mathrm{Re}\int v\nabla \bar{v}_t dx| \\ &\leqq \|v_t\|_e \|v\|_{L^2}. \end{aligned}$$

On the other hands, since v_t is also free solution, we have from (3),

$$\|v_t(t)\|_e = \|v_t(0)\|_e \text{ for all } t\geqq 0.$$

Therefore, there is a positive constant c_0 such that

$$\int |v(x,t)|^2 dx \geqq c_0 \quad \text{for all } t\geqq 0.$$

Let the data of the free solution v be supported in the ball $|x|\leqq k$. Then, by the support property and Hölder's inequality, we have

$$\begin{aligned} 0<c_0 &\leqq \int |v(x,t)|^2 dx = \int_{|x|\leqq k+t} |v(x,t)|^2 dx \\ &\leqq \left(\int |v(x,t)|^{p+1} dx\right)^{2/(p+1)} \left(\int_{|x|\leqq k+t} 1\, dx\right)^{(p-1)/(p+1)} \\ &\leqq \text{const. } t^{(p-1)/(p+1)n} \left(\int |v(x,t)|^{p+1} dx\right)^{2/(p+1)} \quad \text{for all } t\geqq 1 \end{aligned}$$

where p is as in equation (1). Thus there exists a positive constant c_1 such that

$$\int |v(x,t)|^{p+1} dx \geqq c_1 t^{-n(p-1)/2} \quad \text{for all } t\geqq 1.$$

We define

$$H(t) = \mathrm{Re}\int (v_t \bar{u} - \bar{v} u_t)\, dx.$$

Differentiating $H(t)$ with respect to t,

$$\frac{d}{dt}H(t)=\mathrm{Re}\int \bar{u}\,(\Delta v-m^2v)-\bar{v}\,(\Delta u-m^2u-u|u|^{p-1})\,dx$$

$$=\mathrm{Re}\int \bar{v}u|u|^{p-1}dx$$

$$=\int|v|^{p+1}dx+\mathrm{Re}\int\bar{v}|u|^{p-1}(u-v)\,dx$$

$$+\int|v|^2(|u|^{p-1}-|v|^{p-1})\,dx$$

$$\geqq c_1t^{-n(p-1)/2}-I_1-I_2\,,$$

where

$$I_1=\int|v||u|^{p-1}|u-v|dx\,,\quad I_2=\int|v|^2|\;|u|^{p-1}-|v|^{p-1}|dx\,.$$

Recalling that our free solution v satisfies $\|v(t)\|_\infty=o(t^{-n/2})$ as $t\to+\infty$, first for the special case $p=2$, $n=1$ or 2, we have

$$I_1=\int|v||u||u-v|dx\leqq\text{const.}\;t^{-1/2}\|u\|_{L^2}\|u-v\|_{L^2}$$

$$=o\,(t^{-n(p-1)/2})\,.$$

We now take the general case $1<p\leqq1+2n^{-1}$ for $n\geqq3$ or $1<p<2$ for $n=1$ or 2. Then, using Hölder's inequality, we have

$$I_1=\int|v|^{p-1}|v|^{2-p}|u|^{p-1}|u-v|dx$$

$$\leqq\|v\|_\infty^{p-1}\Big(\int|v|^2dx\Big)^{(2-p)/2}\Big(\int|u|^2dx\Big)^{(p-1)/2}\Big(\int|v-u|^2dx\Big)^{1/2}$$

$$\leqq\text{const.}\;t^{-n(p-1)/2}\|u(t)-v(t)\|_e=o\,(t^{-n(p-1)/2})\,.$$

For I_2, we have

$$I_2=\int|v|^2||u|^{p-1}-|v|^{p-1}|dx=\int|v|^{p-1}|v|^{3-p}||u|^{p-1}-|v|^{p-1}|dx$$

$$\leqq\|v\|_\infty^{p-1}\Big(\int|v|^2dx\Big)^{(3-p)/2}\Big(\int|u-v|^2dx\Big)^{(p-1)/2}$$

$$\leqq\text{const.}\;t^{-n(p-1)/2}\|u(t)-v(t)\|_e=o\,(t^{-n(p-1)/2})\,.$$

Thus both I_1 and I_2 satisfy the same estimate for sufficiently large t, and it follows that there is a positive constant c_2 such that

$$\frac{d}{dt}H(t) \geqq c_2 t^{-n(p-1)/2}$$

for large enough t, say $t \geqq T$. Hence,

$$H(2T) - H(T) \geqq c_2 \int_T^{2T} t^{-n(p-1)/2} dt \geqq c_2 \int_T^{2T} t^{-1} dt \geqq c_2 \log 2 > 0 .$$

However, Schwarz's inequality gives

$$|H(t)| = |\mathrm{Re} \int \bar{u}(v_t - u_t) + u_t(\bar{u} - \bar{v}) dx|$$

$$\leqq \|u(t)\|_e \|u(t) - v(t)\|_e .$$

Thus $|H(t)| \to 0$ as $t \to +\infty$, so that $|H(2T)| + |H(T)| \to 0$ as $T \to +\infty$. A sufficiently large choice of T in the inequality above yields the desired contradiction and completes the proof. Q.E.D.

§ 1. 2. Remarks on Weak Dispersion

We consider the solutions of the Cauchy problem for (1) with initial data $u(x, 0) = \phi(x)$, $u_t(x, 0) = \psi(x)$. If $n = 3$, for example, scattering operator is constructed for $p > 8/3$ and impossible for $1 < p \leqq 5/3$. For $5/3 < p \leqq 8/3$, we don't know if scattering theory can be constructed, but we can get the following weak result. We denote by B the positive selfadjoint operator $(m^2 I - \Delta)^{1/2}$ in L^2, by $\langle\,,\rangle L^2$ inner product, and by D_B the domain of B as a Hilbert space relative to the inner product $\langle x, y\rangle_1 = \langle Bx, By\rangle$. Moreover, we denote by H the Hilbert space direct sum of D_B and L^2 with inner product and norm $\langle\,,\rangle_H$, $|\cdot|_H$ and by $\binom{x}{y}$ the element of H with component x in D_B and component y in L^2. We define

$$U(x,t) = \begin{pmatrix} u(x,t) \\ u_t(x,t) \end{pmatrix}, \quad V(x,t) = \begin{pmatrix} v(x,t) \\ v_t(x,t) \end{pmatrix}, \quad U_0(x) = \begin{pmatrix} \phi(x) \\ \psi(x) \end{pmatrix}$$

$$W(t) = \begin{pmatrix} \cos tB, & \sin tB/B \\ -B \sin tB, & \cos tB \end{pmatrix}, \quad K[U(t)] = \begin{pmatrix} 0 \\ -|u(t)|^{p-1} u(t) \end{pmatrix}.$$

Then, we have the following

Theorem 2. *Let $u(x, t)$ be a_n $\mathcal{E}_t^0(H^1\cap L^{2p})\cap\mathcal{E}_t^1(L^2)$ solution of the equation* (1) *with Cauchy data $\phi(x)\in H^1\cap L^{2p}$, $\psi(x)\in L^2$. Suppose that*

$$p>1+2n^{-1},\ n\geqq 3\,.$$

Then there exists a unique free solution $v(x, t)\in\mathcal{E}_t^0(H^1)\cap\mathcal{E}_t^1(L^2)$ such that

$$\langle U(t)-V(t),\, W(t)X\rangle_H\to 0 \text{ as } t\to+\infty, \text{ for any } X\in H.$$

Proof. This proof is almost similar to the proof of Theorem 4 in [3]. We note that energy equality (3) is valid also for $u\in\mathcal{E}_t^0(H^1\cap L^{2p})\cap\mathcal{E}_t^1(L^2)$ and $v\in\mathcal{E}_t^0(H^1)\cap\mathcal{E}_t^1(L^2)$ by using Friedrichs' mollifier arguments and that $W(t)$ is the unitary operator on H, i.e. $\langle W(t)X, W(t)\cdot Y\rangle_H=\langle X, Y\rangle_H$ for all $X, Y\in H$. Now $U(t)$ satisfies the integral equation

$$U(t)=W(t)U_0+\int_0^t W(t-s)K[U(s)]ds\,. \tag{4}$$

We define $S(t)=W(-t)U(t)$ and represent by Z an arbitrary fixed element in D, where

$$D=\left\{\begin{pmatrix}x\\y\end{pmatrix}\,|\,x\in C_0^\infty, y\in C_0^\infty\right\}.$$

From (4), $S(t)$ satisfies

$$S(t)-S(t')=\int_{t'}^t W(-s)K[U(s)]ds\,.$$

Then we have

$$\langle S(t)-S(t'), Z\rangle_H=\int_{t'}^t\langle K[U(s)],\, W(s)Z\rangle_H ds\,.$$

Noting that $W(t)Z$ is a free solution, say of the form $\binom{z(t)}{z_t(t)}$, it follows that

$$|\langle S(t)-S(t'),\, Z\rangle_H|\leqq\left|\int_{t'}^t\langle -|u(s)|^{p-1}u(s),\, z_t(s)\rangle ds\right|$$
$$\leqq\int_{t'}^t\int|u(s)|^p|z_t(s)|dx\, ds\,.$$

Recalling that $\|z_t(s)\|_\infty \leqq \text{const.}(1+s)^{-n/2}$ and (3), it holds that if $p \geqq 2$

$$|\langle S(t)-S(t'), Z\rangle_H| \leqq \int_{t'}^{t} \|z(s)\|_\infty \int |u(s)|^p dx\, ds$$

$$\leqq \text{const.} \int_{t'}^{t} (1+s)^{-n/2} \int |u(s)|^2 + |u(s)|^{p+1} dx\, ds$$

$$\leqq \text{const.} \int_{t'}^{t} (1+s)^{-n/2} ds\,,$$

and if $2 > p > 1 + 2n^{-1}$

$$|\langle S(t)-S(t'), Z\rangle_H| \leqq \int_{t'}^{t} \left(\int u^2 dx\right)^{p/2} \left(\int |z_t|^{2/(2-p)} dx\right)^{(2-p)/2} ds$$

$$\leqq \text{const.} \int_{t'}^{t} \|z_t\|_\infty^{p-1} \left(\int |z_t|^2 dx\right)^{(2-p)/2} ds$$

$$\leqq \text{const.} \int_{t'}^{t} (1+s)^{-n(p-1)/2} ds\,.$$

Therefore, if $p > 1 + 2n^{-1}$, $n \geqq 3$ right hand sides are integrable, so that

(5) $$\langle S(t)-S(t'), Z\rangle_H \to 0 \quad \text{as} \quad t, t' \to +\infty\,.$$

Next, we can show that $\|S(t)\|_H$ is bounded because (3) gives

(6) $$\|S(t)\|_H = \|W(-t)U(t)\|_H = \|U(t)\|_H \leqq \text{const.}\,.$$

Now, the fact that D is dense in H and (5) and (6) imply that there exists a unique $S_0 \in H$ such that

$$\langle S(t)-S_0, Z\rangle_H \to 0 \quad \text{as} \quad t \to +\infty \quad \text{for any} \quad Z \in H\,.$$

Let $V(t)$ be the free solution given by $V(t) = W(t)S_0$. Then for any Z in H,

$$\langle U(t)-V(t), W(t)Z\rangle_H = \langle W(-t)U(t) - W(-t)V(t), Z\rangle_H$$

$$= \langle S(t)-S_0, Z\rangle \to 0 \quad \text{as} \quad t \to +\infty\,.$$

Finally we show the uniqueness. Suppose that there exist the two different free solutions V_1, V_2 which satisfy the above conditions. From the above arguments, we have

$$\langle V_1(t) - V_2(t), W(t)Z\rangle_H \to 0, \quad t \to +\infty\,.$$

Taking $Z = V_1(0) - V_2(0)$,

$$\|V_1(t)-V_2(t)\|_H \to 0,\ \ t\to+\infty .$$

Since $\|V_1(t)-V_2(t)\|_H$ is constant, this contradiction implies $V_1=V_2$.

Q.E.D.

Part II

§ 2. 1. Solutions of Linear Problem

We consider the linear equation

(7) $$v_{tt}-\Delta v+v_t=0 \quad x\in R^n, \quad t\geqq 0$$

with

$$\begin{cases} v(x,0)=\phi(x)\,, \\ v_t(x,0)=\psi(x)\,. \end{cases}$$

We can represent the solution of (7) as follows:

$$v(x,t)=K_1*\psi+K_2*\phi\,.$$

Let $R_i(\xi,t)$ be the Fourier transform of $K_i(x,t)$ $(i=1,2)$. Then R_i satisfies

(8) $$\frac{d^2}{dt^2}R_i+\frac{d}{dt}R_i+|\xi|^2R_i=0\,,$$

$$\begin{cases} R_1(\xi,0)=0\,, \\ \dfrac{d}{dt}R_1(\xi,0)=1\,, \end{cases} \qquad \begin{cases} R_2(\xi,0)=1\,, \\ \dfrac{d}{dt}R_2(\xi,0)=0\,. \end{cases}$$

We can solve (8) exactly, so that

$$R_1(\xi,t)=\begin{cases} \dfrac{2e^{-(1/2)t}}{\sqrt{1-4|\xi|^2}}\sinh\left(\dfrac{\sqrt{1-4|\xi|^2}}{2}t\right), & |\xi|\leqq\dfrac{1}{2}\,, \\ \dfrac{2e^{-(1/2)t}}{\sqrt{4|\xi|^2-1}}\sin\left(\dfrac{\sqrt{4|\xi|^2-1}}{2}t\right), & |\xi|>\dfrac{1}{2}\,, \end{cases}$$

$$R_2(\xi,t)=R_1(\xi,t)+R_3(\xi,t),$$

$$R_3(\xi,t)=\begin{cases} e^{-(1/2)t}\cosh\left(\dfrac{\sqrt{1-4|\xi|^2}}{2}t\right), & |\xi|\leqq\dfrac{1}{2}\,, \\ e^{-(1/2)t}\cos\left(\dfrac{\sqrt{4|\xi|^2-1}}{2}t\right), & |\xi|>\dfrac{1}{2}\,. \end{cases}$$

Lemma 1. *If* $f\in L^m\cap H^{[n/2]+i+|\alpha|}(1\leqq m\leqq 2)$, *then*

$$\left\|\left(\frac{\partial}{\partial t}\right)^i\left(\frac{\partial}{\partial x}\right)^\alpha(K_1*f)\right\|_\infty\leqq c(1+t)^{-n/(2m)-i-(1/2)|\alpha|}(\|f\|_{L^m}+\|f\|_{[n/2]+i+|\alpha|})$$

$$\left\|\left(\frac{\partial}{\partial t}\right)^i\left(\frac{\partial}{\partial x}\right)^\alpha(K_1*f)\right\|_{L_2}\leqq c(1+t)^{(n/4)-n/(2m)-i-(1/2)|\alpha|}(\|f\|_{L^m}+\|f\|_{i+|\alpha|-1}).$$

If $f\in L^m\cap H^{[n/2]+i+|\alpha|+1}$, *then*

$$\left\|\left(\frac{\partial}{\partial t}\right)^i\left(\frac{\partial}{\partial x}\right)^\alpha(K_2*f)\right\|_\infty\leqq c(1+t)^{-n/(2m)-i-(1/2)|\alpha|}(\|f\|_{L^m}+\|f\|_{[n/2]+i+|\alpha|+1})$$

$$\left\|\left(\frac{\partial}{\partial t}\right)^i\left(\frac{\partial}{\partial x}\right)^\alpha(K_2*f)\right\|_{L^2}\leqq c(1+t)^{(n/4)-n/(2m)-i-(1/2)|\alpha|}(\|f\|_{L^m}+\|f\|_{i+|\alpha|}).$$

Here and hereafter c denotes some constant.

Proof. We note that

$$\|f\|_{L^k}\leqq\|\hat f\|_{L^m},\quad \frac{1}{k}+\frac{1}{m}=1,\quad 1\leqq m\leqq 2, \tag{9}$$

where $\hat f$ denotes Fourier transform of f. We show lemma 1 only for $\|(\partial/\partial t)^i(\partial/\partial x)^\alpha(K_1*f)\|_\infty$. For the other cases, we can give proofs in the same way. We use the following

$$\int_0^\delta|\xi|^k e^{-c|\xi|^2t}d|\xi|\leqq c(1+t)^{-(k+1)/2}, \tag{10}$$

$$\sup_{0\leqq|\xi|\leqq\delta}|\xi|^k e^{-c|\xi|^2t}\leqq c(1+t)^{-k/2}\quad\text{for all } t\geqq 0$$

which are easily verified. From (9), we have

$$\left\|\left(\frac{\partial}{\partial t}\right)^i\left(\frac{\partial}{\partial x}\right)^\alpha(K_1*f)\right\|_\infty\leqq\left\|(i\xi)^\alpha\cdot\frac{d^i}{dt^i}R(\xi,t)\cdot\hat f(\xi)\right\|_{L^1}$$

$$\leqq c\int|\xi|^{|\alpha|}\left|\frac{d^i}{dt^i}R(\xi,t)\right||\hat f(\xi)|d\xi\,.$$

We take a small fixed $\delta>0$ and divide the last integral into four parts:

$$\int=\int_{|\xi|\geqq 1}+\int_{1/2<|\xi|<1}+\int_{\delta\leqq|\xi|\leqq 1/2}+\int_{|\xi|\leqq\delta}\equiv I_1+I_2+I_3+I_4$$

We estimate $I_1\sim I_4$ as follows:

$$|I_1|\leqq ce^{-(1/2)t}\int_{|\xi|\geqq 1}\frac{|\xi|^{|\alpha|}(1+\sqrt{-1+4|\xi|^2})^i|\hat f(\xi)|}{\sqrt{4|\xi|^2-1}}d\xi$$

$$\leqq ce^{-t/2}\sup_{|\xi|\geqq 1}\left(\frac{(1+\sqrt{4|\xi|^2-1})^i}{|\xi|^{i-1}\sqrt{4|\xi|^2-1}}\right)\left(\int_{|\xi|\geqq 1}|\xi|^{-2[n/2]-2}d\xi\right)^{1/2}$$

$$\times\left(\int|\xi|^{2[n/2]+2|\alpha|+2i}|\hat{f}(\xi)|^2d\xi\right)^{1/2}$$

$$\leqq ce^{-t/2}\|f\|_{[n/2]+i+|\alpha|}\,.$$

$$|I_2|\leqq ce^{-t/2}\left\{1+\sup_{1>|\xi|>1/2}\left(\frac{\sin((t/2)\sqrt{-1+4|\xi|^2})}{\sqrt{-1+4|\xi|^2}}\right)\right\}\left(\int|f(\xi)|^2d\xi\right)^{1/2}$$

$$\leqq c(1+t)e^{-t/2}\|f\|_{L^2}\,.$$

$$|I_3|\leqq ce^{-t/2}\left\{1+\sup_{\delta<|\xi|\leqq 1/2}\left(\frac{\sinh((t/2)\sqrt{1-4|\xi|^2})}{\sqrt{1-4|\xi|^2}}\right)\right.$$

$$\left.+\sup_{\delta<|\xi|\leqq 1/2}\cosh\left(\frac{t}{2}\sqrt{1-4|\xi|^2}\right)\right\}\|f\|_{L^2}$$

$$\leqq ce^{-(1/2)(1-\sqrt{1-4\delta^2})t}\|f\|_{L^2}\,.$$

$$|I_4|\leqq c\int_{|\xi|\leqq\delta}\frac{(1-\sqrt{1-4|\xi|^2})^i|\xi|^{|\alpha|}|\hat{f}(\xi)|}{\sqrt{1-4|\xi|^2}}e^{(1/2)t(-1+\sqrt{1-4|\xi|^2})}d\xi$$

$$+c\int_{|\xi|\leqq\delta}\frac{(1+\sqrt{1-4|\xi|^2})^i|\xi|^{|\alpha|}|\hat{f}(\xi)|}{\sqrt{1-4|\xi|^2}}e^{(1/2)t(-1-\sqrt{1-4|\xi|^2})}d\xi\,,$$

since $-4|\xi|^2\leqq -1+\sqrt{1-4|\xi|^2}\leqq -2|\xi|^2$ for $|\xi|<\dfrac{1}{2}$,

$$\leqq c\int_{|\xi|\leqq\delta}|\xi|^{2i+|\alpha|}e^{-t|\xi|^2}|\hat{f}(\xi)|d\xi+ce^{-t/2}\int_{|\xi|\leqq\delta}|\hat{f}(\xi)|d\xi$$

$$\leqq c\left(\int_{|\xi|\leqq\delta}|\xi|^{m(2i+|\alpha|)+n-1}e^{-t|\xi|^2}d|\xi|\right)^{1/m}\left(\int|\hat{f}(\xi)|^kd\xi\right)^{1/k}$$

$$+ce^{-t/2}\left(\int|\hat{f}(\xi)|^kd\xi\right)^{1/k}$$

$$\left(\frac{1}{m}+\frac{1}{k}=1\quad 1\leqq m\leqq 2\right)$$

$$\leqq c\{(1+t)^{-(n+m(2i+|\alpha|))/2}\}^{1/m}\|f\|_{L^m}$$

$$\leqq c(1+t)^{-n/(2m)-i-(1/2)|\alpha|}\|f\|_{L^m}\,.$$

Therefore, from the above estimates, we have

$$\left\|\left(\frac{\partial}{\partial t}\right)^i\left(\frac{\partial}{\partial x}\right)^\alpha(K_1*f)\right\|_\infty\leqq c(1+t)^{-n/(2m)-i-(1/2)|\alpha|}(\|f\|_{L^m}+\|f\|_{[n/2]+i+|\alpha|})\,.$$

Q.E.D.

Lemma 2. *We suppose $\phi(x)\in C_0^\infty$, $\psi(x)\in C_0^\infty$. Then the solution of* (7) *satisfies*

$$\left\|\left(\frac{\partial}{\partial t}\right)^i\left(\frac{\partial}{\partial x}\right)^\alpha v(t)\right\|_\infty \leqq c(1+t)^{-(n/2)-i-(1/2)|\alpha|}(\|\phi\|_{[n/2]+i+|\alpha|+1}+\|\phi\|_{L^1}$$
$$+\|\psi\|_{[n/2]+i+|\alpha|}+\|\psi\|_{L^1}),$$
$$\left\|\left(\frac{\partial}{\partial t}\right)^i\left(\frac{\partial}{\partial x}\right)^\alpha v(t)\right\|_{L^2} \leqq c(1+t)^{-(n/4)-i-(1/2)|\alpha|}(\|\phi\|_{i+|\alpha|}+\|\phi\|_{L^1}$$
$$+\|\psi\|_{i+|\alpha|-1}+\|\psi\|_{L^1}).$$

The proof of this lemma follows immediately from the lemma 1 taking $m=1$.

§ 2. 2. Solutions of Semi-Linear Equations

We consider the following semi-linear wave equation

$$u_{tt}-\Delta u+u_t+f(u,u_t,\nabla u)=0, \tag{2}$$
$$\begin{cases} u(x,0)=\phi(x), \\ u_t(x,0)=\psi(x). \end{cases}$$

We assume that $f(z)=f(z_1,z_2,\cdots,z_{n+2})$ is C^k function on R^{n+2} and satisfies that, for $|z|\leqq 1$,

$$\begin{aligned} &|f(z)|\leqq c|z|^p \\ &\left|\left(\frac{\partial}{\partial z}\right)^\alpha f(z)\right|\leqq c|z|^{p-|\alpha|} \quad \text{if } \min(k,p)>|\alpha|\geqq 1, \\ &\left|\left(\frac{\partial}{\partial z}\right)^\alpha f(z)\right|\leqq c \quad \text{if } k\geqq|\alpha|\geqq p \end{aligned} \tag{11}$$

where $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_{n+2})$, $p>1$.

As to the estimates of composite function $f(v(x,t))$, we have

Lemma 3. *Suppose that $f\in C^{[n/2]+1+s}$ ($s\geqq[(1+[n/2])/2]$) satisfies* (11) *and $v(x,t)\in\mathcal{E}_t^0(H^{[n/2]+1+s})$. Then, $f(v(x,t))\in\mathcal{E}_t^0(H^{[n/2]+1+s})$ and satisfies*

$$\begin{aligned} \|f(v)\|_{[n/2]+1+s} &\leqq c\|v\|_{C^s}^{p-2}\|v\|_{[n/2]+1+s}^2 h(\|v\|_{[n/2]+1+s}) \quad \textit{for } p\geqq 2 \\ &\leqq c\|v\|_{[n/2]+1+s}^p h(\|v\|_{[n/2]+1+s}) \quad \textit{for } p>1 \end{aligned} \tag{12}$$

$$\|f(v)\|_{L^q}\leqq c\|v\|_\infty^{p-(2/q)}\|v\|_{L^2}^{2/q}h(\|v\|_\infty)\quad for\ 1\leqq q\leqq 2,\ pq\geqq 2\,.$$

Here and hereafter, we represent by $h(y)$ some nondecreasing nonnegative and continuous function on $y\geqq 0$, and $\|v\|_{C^s}$ means $\sum_{|\alpha|\leqq s}\sup|(\partial/\partial x)^\alpha v(x)|$. We omit the proof. (See von Wahl [5]). For the nonlinear term f, we consider the three cases

case 1. $f=f(u_t)$,

case 2. $f=f(u_t,\nabla u)$,

case 3. $f=f(u,u_t,\nabla u)$.

If $n=1$, we have the following

Theorem 3. *($n=1$) We suppose $\phi(x)$, $\psi(x)\in C_0^\infty$ and*

$$\|\phi\|_{2+k}+\|\psi\|_{1+k}+\|\phi\|_{L^1}+\|\psi\|_{L^1}\leqq\varepsilon\ \ (k\geqq 0).$$

Furthermore, we suppose that $f(z)\in C^{1+k}$ satisfies (11). Then, there exists a small positive constant ε_0 such that the Cauchy problem for (2) has a unique C^{2+k} solution for $0<^\forall\varepsilon\leqq\varepsilon_0$ and p which satisfies the following conditions, and we have

if $p\geqq 2$ for case 1 and 2 or $p>3$ for case 3

$$\left\|\left(\frac{\partial}{\partial t}\right)^i\left(\frac{\partial}{\partial x}\right)^j u(t)\right\|_{L^2}\leqq(1+t)^{-\min\{(1/4)+i+(1/2)j,\,((1/2)+s)p-1/2\}}\tag{13}$$

for $0\leqq i+j\leqq 2+k$

$$\left\|\left(\frac{\partial}{\partial t}\right)^i\left(\frac{\partial}{\partial x}\right)^j u(t)\right\|_\infty\leqq c(1+t)^{-\min\{(1/2)+i+(1/2)j,\,((1/2)+s)p-1/2\}}\tag{14}$$

for $0\leqq i+j\leqq 1+k$

if $2>p>1$ for case 1, or $2>p>\dfrac{\sqrt{17}-1}{2}$ for case 2

$$\left\|\left(\frac{\partial}{\partial t}\right)^i\left(\frac{\partial}{\partial x}\right)^j u(t)\right\|_{L^2}\leqq c(1+t)^{-\min\{(p-1)/4+i+(1/2)j,\,((p-1)/4+s)p\}}\tag{15}$$

for $0\leqq i+j\leqq 2+k$

$$\left\|\left(\frac{\partial}{\partial t}\right)^i\left(\frac{\partial}{\partial x}\right)^j u(t)\right\|_\infty\leqq c(1+t)^{-\min\{(p/4)+i+(1/2)j,\,((p-1)/4+s)p\}}\tag{16}$$

for $0 \leqq i+j \leqq 1+k$.

Here, s represents

$$s=\begin{cases} 1 & \text{for case } 1, \\ \frac{1}{2} & \text{for case } 2, \\ 0 & \text{for case } 3. \end{cases}$$

If $n \geqq 2$, we treat only $p \geqq 2$. We have the following

Theorem 4. *($n \geqq 2$) We suppose $\phi(x), \psi(x) \in C_0^\infty$ and*

$$\|\phi\|_{[n/2]+2+k}+\|\psi\|_{[n/2]+1+k}+\|\phi\|_{L^1}+\|\psi\|_{L^1} \leqq \varepsilon, \quad k \geqq \left[\frac{[n/2]+1}{2}\right].$$

Furthermore, we suppose that $f(z) \in C^{[n/2]+1+k}$ satisfies (11) and $p>2$ for $n=2$ case 3, or $p \geqq 2$ for the other cases. Then, there exists a small positive constant ε_0 such that the Cauchy problem for (2) has a unique $\mathcal{E}_t^0(H^{[n/2]+3+k}) \cap \mathcal{E}_t^1(H^{[n/2]+2+k})$ solution for $0<{}^\forall\varepsilon \leqq \varepsilon_0$ and we have

$$\left\|\left(\frac{\partial}{\partial t}\right)^i\left(\frac{\partial}{\partial x}\right)^\alpha u(t)\right\|_{L^2} \leqq c(1+t)^{-\min\{(n/4)+i+(1/2)|\alpha|,\, ((n/2)+s)p-(n/2)\{}$$

for $0 \leqq i+|\alpha| \leqq 3+k+\left[\frac{n}{2}\right]$

$$\left\|\left(\frac{\partial}{\partial t}\right)^i\left(\frac{\partial}{\partial x}\right)^\alpha u(t)\right\|_{\infty} \leqq c(1+t)^{-\min\{(n/2)+i+(1/2)|\alpha|,\, ((n/2)+s)p-(n/2)\}}$$

for $0 \leqq i+|\alpha| \leqq 2+k$.

Here, s represents

$$s=\begin{cases} 1 & \text{for case } 1, \\ \frac{1}{2} & \text{for case } 2, \\ 0 & \text{for case } 3. \end{cases}$$

Proof of Theorem 3. The existence and uniqueness for local solution are well known so that we show decay estimates which show *a priori* estimates for global solution. Then we suppose the solution $u(x, t) \in C^{2+k}$ exists. $u(x, t)$ satisfies the integral equation

$$(17)\qquad u(t)=v(t)-\int_0^t K_1(t-\tau)*f(u(\tau),u_t(\tau),u_x(\tau))\,d\tau\,,$$

where $v(t)$ is the solution of linear equation (7) whose Cauchy data are equal to (2). Differentiating (17) with respect to t, we have

$$(18)\qquad u_t(t)=v_t(t)-\int_0^t \frac{\partial}{\partial t}K_1(t-\tau)*f(u(\tau),u_t(\tau),u_x(\tau))\,d\tau\,.$$

We only give a proof for the case 1, $f=f(u_t)$. For the other cases, we can give proofs by almost the same way. We first note that for a, $b>0$,

$$\int_0^t (t-\tau+1)^{-a}(\tau+1)^{-b}d\tau\leqq c(1+t)^{-\min(a,b)},\quad \max(a,b)>1$$

that is shown in Segal [4]. From (18) and Lemma 2,

$$\|u_t(t)\|_{k+1}\leqq c\varepsilon(1+t)^{-5/4}+\int_0^t \left\|\left(\frac{\partial}{\partial t}\right)(K_1*f)\right\|_{k+1}d\tau\,.$$

Now, we suppose $p\geqq 2$. Then Lemma 1 with $m=1$ gives

$$(19)\qquad \|u_t(t)\|_{k+1}\leqq c\varepsilon(1+t)^{-5/4}+c\int_0^t(1+t-\tau)^{-5/4}\{\|f(u_t(\tau))\|_{L^1}+\|f(u_t(\tau))\|_{k+1}\}\,d\tau\,.$$

Lemma 3 gives

$$\|f(u_t)\|_{L^1}+\|f(u_t)\|_{k+1}\leqq c\|u_t\|_{ck}^{p-2}\|u_t\|_{k+1}^2 h(\|u_t\|_{k+1})\,.$$

Substituting the above into (19),

$$\|u_t(t)\|_{k+1}\leqq c\varepsilon(1+t)^{-5/4}+c\int_0^t(1+t-\tau)^{-5/4}\|u_t(\tau)\|_{k+1}^p h(\|u_t(\tau)\|_{k+1})\,d\tau$$

putting $M(t)=\sup_{0\leqq\tau\leqq t}(1+\tau)^{5/4}\|u_t(\tau)\|_{k+1}$,

$$\leqq c\varepsilon(1+t)^{-5/4}+ch(M(t))\int_0^t(1+t-\tau)^{-5/4}(1+\tau)^{-(5/4)p}(M(\tau))^p d\tau$$

$$\leqq c\varepsilon(1+t)^{-5/4}+c(1+t)^{-5/4}(M(t))^p h(M(t))$$

so that

$$M(t)\leqq c\varepsilon+c(M(t))^p h(M(t))\,.$$

Therefore, there exists ε_0 such that $M(t)\leqq c$ for $0<\varepsilon\leqq\varepsilon_0$ (cf [4]) so that

$$(20)\qquad \|u_t(t)\|_{k+1}\leqq c(1+t)^{-5/4}\,.$$

Next, from (18) and Lemmas 1, 2 and 3,

$$\|u_t(t)\|_{C^k} \leqq c(1+t)^{-3/2} + c\int_0^t (1+t-\tau)^{-3/2} \|u_t(\tau)\|_{k+1}^p h(\|u_t(\tau)\|_{k+1})\, d\tau$$

$$\leqq c(1+t)^{-3/2} + c\int_0^t (1+t-\tau)^{-3/2}(1+\tau)^{-(5/4)p} d\tau\,.$$

Hence,

$$\|u_t(t)\|_{C^k} \leqq c(1+t)^{-3/2}\,. \tag{21}$$

Furthermore, for $i=0, 1$,

$$\left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial x}\right)^j u = \left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial x}\right)^j v - \int_0^t \left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial x}\right)^j (K_1 * f)\, d\tau\,. \tag{22}$$

By Lemma 2, we have

$$\left\|\left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial x}\right)^j v(t)\right\|_{L^2} \leqq c(1+t)^{-(1/4)-i-(1/2)j},$$

$$\left\|\left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial x}\right)^j v(t)\right\|_{\infty} \leqq c(1+t)^{-(1/2)-i-(1/2)j}\,.$$

By virtue of Lemmas 1 and 3, we have

$$\int_0^t \left\|\left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial x}\right)^j K_1 * f(u_t(\tau))\right\|_{L^2} d\tau$$

$$\leqq c\int_0^t (1+t-\tau)^{-(1/4)-i-(1/2)j} \|u_t(\tau)\|_{C^k}^{p-2} \|u_t(\tau)\|_{k+1}^2\, d\tau$$

substituting (20) and (21) into the above,

$$\leqq c\int_0^t (1+t-\tau)^{-(1/4)-i-(1/2)j}(1+\tau)^{-(3/2)p+1/2} d\tau$$

$$\leqq c(1+t)^{-\min((1/4)+i+(1/2)j,\,(3/2)p-1/2)}$$

$$\text{for} \quad 0 \leqq i+j \leqq k+2\,.$$

Analogously, we have

$$\int_0^t \left\|\left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial x}\right)^j K_1 * f(u_t(\tau))\right\|_{\infty} d\tau$$

$$\leqq c(1+t)^{-\min((1/2)+i+(1/2)j,\,(3/2)p-1/2)}$$

$$\text{for} \quad 0 \leqq i+j \leqq k+1\,.$$

Thus, we can get the desired estimates (13) and (14) for $i=0, 1$. For

$i \geqq 2$, we have

$$(23) \qquad \left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial x}\right)^j u = \text{right hand side of (22)}$$

$$-\sum_{s=0}^{i-2}\left(\frac{\partial}{\partial t}\right)^s \left(\frac{\partial}{\partial x}\right)^j \left\{\left(\frac{\partial}{\partial t}\right)^{i-1-s} K_1(0)*f(u_t(t))\right\}.$$

Our problem is the last term. For the moment, we suppose that

$$(24) \qquad \sum_{s=1}^{i}\left\|\left(\frac{\partial}{\partial t}\right)^s u\right\|_{k+1-s} \leqq c(1+t)^{-5/4},$$

$$\sum_{s=1}^{i}\left\|\left(\frac{\partial}{\partial t}\right)^s u\right\|_{C^{k+1-s}} \leqq c(1+t)^{-3/2}$$

$$\text{for} \quad 0 \leqq i \leqq i_0 .$$

Then, let's show (13) and (14) for $i=i_0+1$. From (8), it follows

$$\left(\frac{d}{dt}\right)^{s+2} R_1(\xi, 0) = \sum_{n=1}^{s+1} \lambda_1{}^m \lambda_2{}^{s+1-m}, \quad \text{where } \lambda_{1,2} = \frac{-1 \pm \sqrt{1-4|\xi|^2}}{2}.$$

Hence, from (23) with $i=i_0+1$, we have

$$\sum_{s=0}^{i_0-1}\left\|\left(\frac{\partial}{\partial t}\right)^s \left(\frac{\partial}{\partial x}\right)^j \left\{\left(\frac{\partial}{\partial t}\right)^{i-1-s} K_1(0)*f(u_t(t))\right\}\right\|_{L^2}$$

$$\leqq c\sum_{s=0}^{i_0-1}\left\|\left(\frac{\partial}{\partial t}\right)^s f(u_t)\right\|_{i+j-2-s}$$

$$\leqq c\left(\sum_{s=1}^{i_0}\left\|\left(\frac{\partial}{\partial t}\right)^s u\right\|_{C^{k+1-s}}\right)^{p-1}\left(\sum_{s=1}^{i_0}\left\|\left(\frac{\partial}{\partial t}\right)^s u\right\|_{k+2-s}\right)$$

$$\times h\left(\sum_{s=1}^{i_0}\left\|\left(\frac{\partial}{\partial t}\right)^s u\right\|_{C^{k+1-s}}\right)$$

substituting (24) into the above

$$\leqq c(1+t)^{-(3/2)p+1/4} \leqq c(1+t)^{-\min((1/4)+i+(1/2)j,\,(3/2)p-(1/2))}.$$

In the same way, we have

$$\sum_{s=0}^{i-2}\left\|\left(\frac{\partial}{\partial t}\right)^s \left(\frac{\partial}{\partial x}\right)^j \left\{\left(\frac{\partial}{\partial t}\right)^{i-1-s} K_1(0)*f(u_t(t))\right\}\right\|_{\infty}$$

$$\leqq c(1+t)^{-\min((1/2)-i+(1/2)j,\,(3/2)p-(1/2))}.$$

Thus, (13) and (14) hold for $i=i_0+1$ and this clearly shows that (24) holds for $i=i_0+1$. Therefore, we can get the desired decay estimates inductively because (24) is true for $i=0, 1$ as we verified already.

If $1<p<2$, Lemmas 1 and 2 give

$$\|u_t(t)\|_{k+1}\leqq c\varepsilon(1+t)^{-(5/4)}+c\int_0^t(1+t-\tau)^{-(3/4)-(1/2m)}(\|f(u_t)\|_{L^m} + \|f(u_t)\|_{k+1})\,d\tau\,.$$

Then choosing m as $pm=2$ in Lemma 3,

$$\leqq c\varepsilon(1+t)^{-5/4}+c\int_0^t(1+t-\tau)^{-(p+3)/4}\|u_t(\tau)\|_{k+1}^p h(\|u_t(\tau)\|_{k+1})\,d\tau\,.$$

We can get

$$\|u_t(t)\|_{k+1}\leqq c(1+t)^{-(p+3)/4},\quad 0<\varepsilon\leqq\varepsilon_0$$

by the same method as before. The remainder is same as $p\geqq 2$.

Q.E.D.

We omit the proof of the Theorem 4 because we show it by the same way as in the Theorem 3.

§ 2. 3. Special Case $f=|u_t|^{p-1}u_t$, $n=1$

We consider the following equation

$$u_{tt}-u_{xx}+u_t+|u_t|^{p-1}u_t=0\,,\quad x\in R^1\,,\quad p>1 \tag{25}$$

with

$$\begin{cases} u(x, 0)=\phi(x)\in C_0^\infty\,, \\ u_t(x, 0)=\psi(x)\in C_0^\infty\,. \end{cases}$$

For this case, we can get the decay estimates without the smallness conditions of ϕ and ψ. We first prepare the following

Lemma 4. *Let $u(x, t)$ be a C_0^2 solution of the Cauchy problem for* (25). *Then*

$$\|u_t(t)\|_\infty,\ \|u_t(t)\|_{L^2}\to 0\ \text{as}\ t\to+\infty\,.$$

Proof. From (21), we have

$$\int_0^t\int u_t^2(\tau)\,dx\,d\tau\leqq E,\quad \int(u_t^2(t)+u_{tt}^2(t)+u_{tx}^2(t))\,dx\leqq E\ \text{ for }\ t\geqq 0 \tag{26}$$

where E is some constant that depends only on ϕ and ψ. From (26), we have

$$\left|\left(\int u_t^2(t_1)\,dx\right)^2-\left(\int u_t^2(t_2)\,dx\right)^2\right|=\left|4\int_{t_2}^{t_1}\left(\int u_t^2dx\right)\left(\int u_tu_{tt}dx\right)d\tau\right|$$

$$\leqq E\left|\int_{t_2}^{t_1}\int u_t^2(\tau)\,dx\,d\tau\right| .$$

Therefore, $\|u_t(t)\|_{L^2}$ is asymptotic to constant as $t\to+\infty$. If this constant is not zero, it contradicts (26). Thus we have

$$\|u_t(t)\|_{L^2}\to 0 \text{ as } t\to+\infty .$$

Furthermore, from

$$u_t^2(t)=\int_{-\infty}^{t}2u_tu_{tx}dx\leqq 2\|u_t(t)\|_{L^2}\|u_{tx}(t)\|_{L^2} ,$$

we have

$$\|u_t(t)\|_{L^2}\to 0 \text{ as } t\to+\infty .$$

Q.E.D.

Theorem 5. *We suppose that* $\phi(x)\in C_0^\infty$, $\psi(x)\in C_0^\infty$ *and* $p>1$. *Then, the Cauchy problem for* (25) *has a unique* C^2 *solution such that; if* $p\geqq 2$

$$\left\|\left(\frac{\partial}{\partial t}\right)^i\left(\frac{\partial}{\partial x}\right)^j u(t)\right\|_{L^2}\leqq c(1+t)^{-(1/4)-i-(1/2)j} \quad for \quad i+j\leqq 2$$

$$\left\|\left(\frac{\partial}{\partial t}\right)^i\left(\frac{\partial}{\partial x}\right)^j u(t)\right\|_{\infty}\leqq c(1+t)^{-(1/2)-i-(1/2)j} \quad for \quad i+j\leqq 1 ,$$

and if $p<2$

$$\left\|\left(\frac{\partial}{\partial t}\right)^i\left(\frac{\partial}{\partial x}\right)^j u(t)\right\|_{L^2}\leqq c(1+t)^{-\min\{(p-1)/4+i+(1/2)j,\,p(p+3)/4\}} \quad for \quad i+j\leqq 2$$

$$\left\|\left(\frac{\partial}{\partial t}\right)^i\left(\frac{\partial}{\partial x}\right)^j u(t)\right\|_{\infty}\leqq c(1+t)^{-\min\{(p/4)+i+(1/2)j,\,p(p+3)/4\}} \quad for \quad i+j\leqq 1 .$$

Proof. We only give a proof for $p\geqq 2$ because the reasoning for $p<2$ is almost the same. (18) and Lemmas 1 and 2 give

$$\|u_t(t)\|_1\leqq c(1+t)^{-(5/4)}+c\int_0^t(1+t-\tau)^{-(5/4)}\{\|\,|u_t|^{p-1}u_t\|_1$$

$$+\| |u_t|^{p-1}u_t\|_{L^1}\} d\tau .$$

For this case, we use

$$\| |v|^{p-1}v\|_1 \leqq c\|v\|_\infty{}^{p-1}\|v\|_1 , \quad \| |v|^{p-1}v\|_{L^1} \leqq \|v\|_\infty^{p-2}\|v\|_{L^1}^2$$

that are easily verified by the modified lemma 3. Therefore, we have

$$\|u_t(t)\|_1 \leq c(1+t)^{-5/4} + cI(0, t)$$

where

$$I(0, t) = \int_0^t (1+t-\tau)^{-5/4}(\|u_t(\tau)\|_{L^2} + \|u_t(\tau)\|_\infty)\|u_t(\tau)\|_1\|u_t(\tau)\|_\infty^{p-2} d\tau .$$

Now, we divide $I(0, t)$ into $I(0, t/2) + I(t/2, t)$ and define $M(t)$ by

$$M(t) = \sup_{0\leqq t\leqq t} (1+\tau)^{5/4}\|u_t(\tau)\|_1 .$$

It follows that

$$I\left(0, \frac{1}{2}t\right) \leqq c\int_0^{(1/2)t} (1+t-\tau)^{-5/4}\|u_t(\tau)\|_1 d\tau$$

$$\leqq c\left(1+\frac{t}{2}\right)^{-5/4} \int_0^{(1/2)t} (1+\tau)^{-5/4} M(\tau)\, d\tau ,$$

$$I\left(\frac{1}{2}t, t\right) \leqq \sup_{(1/2)t\leqq\tau\leqq t} \{\|u_t\|_\infty^{p-1} + \|u_t\|_\infty^{p-2}\|u_t\|_{L^2}\}$$

$$\times \int_{(1/2)t}^t (1+t-\tau)^{-5/4}(1+\tau)^{-5/4} M(\tau)\, d\tau$$

$$\leqq c\left(1+\frac{t}{2}\right)^{-5/4} M(t) \sup_{(1/2)t\leqq\tau\leqq t} \{\|u_t(\tau)\|_\infty^{p-1} + \|u_t(\tau)\|_\infty^{p-2}\|u_t(\tau)\|_{L^2}\} .$$

From the above estimates, we have

$$M(t) \leqq c + c\int_0^{(1/2)t} (1+\tau)^{-5/4} M(\tau)\, d\tau$$

$$+ c \sup_{(1/2)t\leqq\tau\leqq t} \{\|u_t(\tau)\|_\infty^{p-1} + \|u_t(\tau)\|_\infty^{p-2}\|u_t(\tau)\|_{L^2}\} M(t) .$$

Then, by Lemma 4, the choice of sufficient large t, say $t \geqq T$, gives

$$c \sup_{(1/2)t\leqq\tau\leqq t} \{\|u(\tau)\|_\infty^{p-1} + \|u(\tau)\|_\infty^{p-2}\|u(\tau)\|_{L^2}\} \leqq \frac{1}{4} .$$

Therefore, we have

$$M(t) \leqq c + c \int_0^t (1+\tau)^{-5/4} M(\tau)\, d\tau \quad \text{for all } t \geqq T .$$

This implies

$$\|u_t(t)\|_1 \leqq c(1+t)^{-5/4} ,$$

so that we can get the conclusion by the same way as in Theorem 3.

Q.E.D.

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