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Remarks on the Coefficient Ring of Quaternionic Oriented Cohomology Theories

By

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§ 1. Introduction

Let h^* be a complex (or real) oriented cohomology theory. It has recently been observed that the "formal group" of h^* plays a very important role in such a theory. Above all, D. Quillen [6] showed that the formal group F_U of the complex cobordism theory $MU^*(\ )$ is isomorphic to the Lazard universal formal group, and its coefficients generate the ground ring $MU^*(pt)$.

Unfortunately, in the case of quaternionic oriented cohomology theory (see § 2), there exists no such formal group, since the tensor product of quaternionic line bundles does not yield a quaternionic bundle. This situation makes it difficult, for example, to produce enough generators of $MSp^*(pt)$.

However there are some substitutes for the formal group. In particular, using the total Pontrjagin class of a certain quaternionic vector bundle 2ζ (see § 3), N. Ja. Gozman [5] defined a subring $\tilde{\Lambda}$ of $MU^*(pt)$ which is contained in the image of the forgetful homomorphism $\varphi: MSp^*(pt) \to MU^*(pt)$.

In this paper, we will generalize his approach to arbitrary quaternionic oriented cohomology theory h^*, and define a subring $\tilde{\Lambda}_h$ of $h^*(pt)$ which is generated by the coefficients of certain power series. Then it will be shown that

Proposition 3. $\tilde{\Lambda}_{KO} = \sum_{j\geqq 0} KO^{-4j}(pt)$,

and especially,

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Theorem. (a) *$\tilde{\Lambda}_{MSp}$ contains an element $z_n \in MSp^{-4n}(pt) \cong \Omega_{4n}^{Sp}$ for each $n \geqq 1$, which is represented by an Sp manifold M_n whose Chern number $s_{2n}(M_n)$ is equal to*

$$\begin{array}{lll} 16\lambda_{2n} & \text{if } n+1=2^f & \text{for some } f \geq 1, \\ 8\lambda_{2n} & \text{otherwise} & \end{array}$$

where $\lambda_{2n}=p$ if $2n+1=p^g$ for some prime p and $g \geqq 1$, and $\lambda_{2n}=1$ otherwise.

(b) $\tilde{\Lambda}_{MSp} \otimes Z[\frac{1}{2}] = MSp^*(pt) \otimes Z[\frac{1}{2}] = Z[\frac{1}{2}][z_1, z_2, \cdots, z_n, \cdots]$.

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§ 2. Notations and Preliminaries

Let h^* be a multiplicative cohomology theory defined on the category of CW (or finite CW) pairs, and let $\tilde{h}^*$ be the corresponding reduced theory. We always assume that h^* satisfies the additivity axiom.

Definition 1. *We say that h^* is quaternionic oriented if each quaternionic vector bundle ξ has a Thom class $t_h(\xi) \in \tilde{h}^{4n}(M(\xi)) \cong h^{4n}(M(\xi), \infty)$ $(n = \dim_H \xi)$ such that*

(a) *natural for bundle maps,*

(b) $t_h(\xi \times \eta) = t_h(\xi) \wedge t_h(\eta) \in \tilde{h}^*(M(\xi) \wedge M(\eta))$,

(c) $t_h(pt \times H^n) = \sigma^{4n}1 \in \tilde{h}^{4n}(S^{4n})$

where σ^{4n} denotes the $4n$-fold suspension.

The symplectic cobordism theory $MSp^*(\ )$ is quaternionic oriented. We define $t_{MSp}(\xi)$ as follows: let

$$f: X \to BSp(n), \quad n = \dim_H \xi$$

be the classifying map of ξ, then f induces a map

$$\tilde{f}: M(\xi) \to MSp(n)$$

which defines the desired class $t_{MSp}(\xi) = [\tilde{f}] \in \widetilde{MSp}^{4n}(M(\xi))$. It is easily observed that this $t_{MSp}(\xi)$ satisfies the three conditions of Definition 1.

If there exists a multiplicative natural transformation $\mu: MSp^*(\ ) \to h^*$, then h^* is also quaternionic oriented by defining $t_h(\xi) = \mu(t_{MSp}(\xi))$. Conversely we have

Proposition 1. *(Universality of sympletic cobordism) Let h^* be any quaternionic oriented cohomology theory defined on the category of finite CW pairs. Then there exists a unique multiplicative natural transformation μ_h: $\widetilde{MSp}^*(\ )\to\tilde{h}^*$ such that*

$$\mu_h(t_{MSp}(\xi))=t_h(\xi).$$

Proof. (Compare [4] Theorem 5.1.) Let $x\in\widetilde{MSp}^n(X)$, and let f: $S^{4k-n}X\to MSp(k)$ be the map representing x so that $x=[f]=\sigma^{n-4k}f^*(t_{MSp}(\eta))$ where η is the universal bundle over $BSp(k)$. Put

$$\mu_h(X)=\sigma^{n-4k}f^*(t_h(\eta))\in\tilde{h}^n(X).$$

This defines a unique natural transformation μ_h:$\widetilde{MSp}^*(\ )\to\tilde{h}^*$ such that $\mu_h(t_{MSp}(\xi))=t_h(\xi)$ for any ξ. μ_h is also multiplicative by the property (b) of Thom classes.

Remark. This universality is actually true for any quaternionic oriented h^* defined on the category of arbitrary CW pairs, provided that h^* is additive.

Now let η_n be the canonical line bundle over HP^n. Recall that $M(\eta_n)=HP^{n+1}$. Let i_n: $HP^n\to(HP^{n+1},\infty)$ be the inclusion and let $\rho_n=i_n{}^*t_h(\eta_n)\in h^4(HP^n)$. Then, using the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & MSp^*(S^{4n+4}) & \longrightarrow & MSp^*(HP^{n+1}) & \longrightarrow & MSp^*(HP^n) & \longrightarrow & 0 \\ & & \downarrow \mu_h & & \downarrow \mu_h & & \downarrow \mu_h & & \\ \cdots & \longrightarrow & h^*(S^{4n+4}) & \longrightarrow & h^*(HP^{n+1}) & \longrightarrow & h^*(HP^n) & \longrightarrow & \cdots \end{array}$$

associated with the cofibration $HP^n\to HP^{n+1}\to S^{4n+4}$, and by the similar argument to that of [4] (8·1), we have inductively that:

(2·1) (a) $h^*(HP^n)\cong h^*(pt)[\rho_n]/(\rho_n{}^{n+1})$,

(b) $i_n{}^*\rho_{n+1}=\rho_n$ for $1\leqq n$.

Hence we can apply the Leray-Hirsch theorem to h^* and obtain the uniquely determined (total) Pontrjagin class (cf. [4], [8]):

$$p^h(\xi)=1+p_1{}^h(\xi)+\cdots+p_n{}^h(\xi),\quad p_j{}^h(\xi)\in h^{4j}(X)$$

for $1 \leqq j \leqq n = \dim_H \xi$, such that

PI. $p^h(f^*\xi) = f^*(p^h(\xi))$

PII. $p^h(\xi_1 \oplus \xi_2) = p^h(\xi_1) \cdot p^h(\xi_2)$

PIII. $p^h(\eta_n) = 1 + \rho_n$

From the uniqueness, it follows that if $\mu: \underline{h}^* \to h^*$ is a multiplicative natural transformation of cohomology theories such that $\mu(t_{\underline{h}}(\xi)) = t_h(\xi)$ for any ξ, then

$$(2\cdot2) \qquad \mu(p^{\underline{h}}(\xi)) = p^h(\xi).$$

Now assume that h^* is complex oriented, that is, each complex vector bundle ξ has a Thom class $\mathcal{T}_h(\xi)$ which satisfies the conditions similar to those of Definition 1. Then h^* is also quaternionic oriented, simply by defining $t_h(\xi) = \mathcal{T}_h(\xi)$ for any quaternionic vector bundle ξ. In this case, h^* has two kinds of characteristic classes i.e. the Pontrjagin class $p^h(\xi)$ and the Chern class $c^h(\xi)$ which is (**CI**) natural, (**CII**) multiplicative, and (**CIII**) $c^h(\xi_n) = 1 + j_n^*(\mathcal{T}_h(\xi_n))$ where ξ_n is the canonical complex line bundle over CP^n and $j_n: CP^n \hookrightarrow (CP^{n+1}, \infty)$.

Lemma 1. *Let ξ be an n-dimensional quaternionic vector bundle over X. Then*

$$p_n^h(\xi) = c_{2n}^h(\xi) = i^*(\mathcal{T}_h(\xi)),$$

where $i: X \to (M(\xi), \infty)$ is the natural inclusion.

Proof. Let η_∞ be the canonical quaternionic line bundle over $HP^\infty = BSp(1)$ and let $\varepsilon_n: HP^n \to HP^\infty$ be the inclusion. By $(2\cdot1)$ $h^*(HP^\infty) = \varprojlim_n{}^0 h^*(HP^n) = h^*(pt)[[\rho_\infty]]$ where ρ_∞ is the unique element such that $\varepsilon_n^* \rho_\infty = \rho_n$. Then obviously we have

$$p_1^h(\eta_\infty) = \rho_\infty = i_\infty^*(t_h(\eta_\infty))$$

where $i_\infty: BSp(1) \to (MSp(1), \infty)$. Therefore $p_1^h(\eta) = i^*(t_h(\eta))$, $i: X \to (M(\eta), \infty)$ for any $Sp(1)$ bundle η over X. By the splitting principle and the property PII, we then have

$$p_n^h(\xi) = i^*(t_h(\xi)), \qquad i: X \to (M(\xi), \infty)$$

for any $Sp(n)$-bundle ξ. Similarly we have

$$c_m{}^h(\xi') = i^*(\mathcal{T}_h(\xi')), \quad i: X \to (M(\xi'), \infty)$$

for any $U(m)$-bundle ξ'.

§ 3. Construction of $\tilde{\Lambda}_h$

From now on let h^* be a quaternionic oriented cohomology theory. Consider the complex bundle

$$\zeta = \eta \otimes_C \eta'$$

over $HP^\infty \times HP^\infty$ where η and η' are the canonical quaternionic line bundles in the first and second factors. Then ζ is a self-conjugate complex vector bundle, and hence $2\zeta = \zeta \oplus \bar{\zeta}$ is naturally a 4-dimensional quaternionic bundle, where $\bar{\zeta}$ is the complex conjugate of ζ.

Let $p^h(2\zeta) = \sum_{j=0}^4 p_j{}^h(2\zeta)$ be the total Pontrjagin class of 2ζ. Since $h^*(HP^\infty \times HP^\infty) \cong h^*(pt)[[x_h, y_h]]$ where $x_h = p_1{}^h(\eta)$ and $y_h = p_1{}^h(\eta')$ (compare the proof of Lemma 1), we may consider $p_j{}^h(2\zeta)$, $1 \leqq j \leqq 4$, as a formal power series with coefficients in $h^*(pt)$ (in fact, in $h^{4*}(pt)$).

Definition 2. *$\tilde{\Lambda}_h \subset h^{4*}(pt)$ is the subring generated by the coefficients of the power series $p_j{}^h(2\zeta)$, $1 \leqq j \leqq 4$.*

If $\underline{h}^*$ is another quaternionic oriented cohomology theory and $\mu: \underline{h}^* \to h^*$ is a multiplicative natural transformation such that $\mu(t_{\underline{h}}(\xi)) = t_h(\xi)$ for any quaternionic bundle ξ, then $\mu(p_j{}^{\underline{h}}(2\zeta)) = p_j{}^h(2\zeta)$ from $(2 \cdot 2)$. Hence,

Lemma 2. $\tilde{\Lambda}_h = \mu(\tilde{\Lambda}_{\underline{h}}) \subset \mathrm{Im}(\underline{h}^*(pt) \xrightarrow{\mu} h^*(pt))$. *In particular,* $\tilde{\Lambda}_h \subset \mathrm{Im}(MSp^*(pt) \xrightarrow{\mu_h} h^*(pt))$ *where μ_h is the natural transformation of Proposition* 1.

Now assume that h^* is complex oriented, and let F be a formal group of h^*, so that

$$c_1{}^h(\xi \otimes_C \eta) = F(c_1{}^h(\xi), c_1{}^h(\eta))$$

for any complex line bundles ξ and η. Then the formal power series $p_j{}^h(2\zeta)$ are related to F as follows (see Gozman [5]). Let

$$f: CP^{\infty}\times CP^{\infty}\rightarrow HP^{\infty}\times HP^{\infty}$$

be the standard inclusion. Then we have

$$f^{*}\zeta=(\xi\otimes\bar{\xi})\otimes_{C}(\xi'\oplus\bar{\xi}')$$

where ξ and ξ' are canonical complex line bundles over the first and second factors of $CP^{\infty}\times CP^{\infty}$. Put $\zeta_1=(\xi\otimes_{C}\xi')\oplus(\bar{\xi}\otimes_{C}\bar{\xi}')$ and $\zeta_2=(\xi\otimes_{C}\bar{\xi}')\oplus(\bar{\xi}\otimes_{C}\xi')$ so that

$$(3\cdot 1)\qquad f^{*}\zeta=\zeta_1\oplus\zeta_2\,.$$

Note that ζ_1 and ζ_2 are quaternionic line bundles. Also f induces the homomorphism:

$$\begin{array}{ccc} h^{*}(HP^{\infty}\times HP^{\infty}) & \xrightarrow{f^{*}} & h^{*}({}^{\infty}CP\times CP^{\infty}) \\ \| & & \| \\ h^{*}(pt)[[x,y]] & \longrightarrow & h^{*}(pt)[[u,v]] \end{array}$$

where $x=p_1^h(\eta)$, $y=p_1^h(\eta')\in h^4(HP^{\infty})$ and $u=c_1^h(\xi)$, $v=c_1^h(\xi')\in h^2(CP^{\infty})$. Since

$$\begin{aligned} f^{*}(x) &=p_1^h(\xi\oplus\bar{\xi})=c_2^h(\xi\oplus\bar{\xi}) \qquad \text{(by Lemma 1)}\\ &=c_1^h(\xi)\cdot c_1^h(\bar{\xi})\\ &=u\bar{u} \quad \text{where } F(u,\bar{u})=0 \end{aligned}$$

and

$$f^{*}(y)=v\bar{v} \text{ where } F(v,\bar{v})=0\,,$$

it follows that

(3·2) *f^{*} is an isomorphism onto the subaigeoru*

$h^{*}(pt)[[u\bar{u},v\bar{v}]]$.

Now we have

$$\begin{aligned} f^{*}(p^h(2\zeta)) &=p^h(2(\zeta_1\oplus\zeta_2)) \qquad \text{(by (3·1))}\\ &=(1+p_1^h(\zeta_1\oplus\zeta_2)+p_2^h(\zeta_1\oplus\zeta_2))^2\,. \end{aligned}$$

Here

$$p_1^h(\zeta_1 \oplus \zeta_2) = c_2^h(\zeta_1) + c_2^h(\zeta_2) \qquad \text{(by Lemma 1)}$$
$$= c_1^h(\xi \otimes_C \xi') \, e_1^h(\bar{\xi} \otimes_C \bar{\xi}') + c_1^h(\xi \otimes_C \bar{\xi}') \, c_1^h(\bar{\xi} \otimes_C \xi')$$
$$= F(u, v) F(\bar{u}, \bar{v}) + F(u, \bar{v}) F(\bar{u}, v)$$

and

$$p_2^h(\zeta_1 \oplus \zeta_2) = c_2^h(\zeta_1) c_2^h(\zeta_2)$$
$$= F(u, v) F(\bar{u}, \bar{v}) F(u, \bar{v}) F(\bar{u}, v).$$

Let $\Theta_1(x, y)$ and $\Theta_2(x, y) \in h^*(pt)[[x, y]]$ be the power series such that

$$f^*(\Theta_1(x, y)) = \Theta_1(u\bar{u}, v\bar{v}) = F(u, v) F(\bar{u}, \bar{v}) + F(u, \bar{v}) F(\bar{u}, v)$$

and

$$f^*(\Theta_2(x, y)) = F(u, v) F(\bar{u}, \bar{v}) F(u, \bar{v}) F(\bar{u}, v).$$

Then we have

Proposition 2. *$p^h(2\zeta) = (1 + \Theta_1 + \Theta_2)^2$, that is*

$$p_1^h(2\zeta) = 2\Theta_1, \ p_2^h(2\zeta) = \Theta_1^2 + 2\Theta_2,$$
$$p_3^h(2\zeta) = 2\Theta_1\Theta_2, \ p_4^h(2\zeta) = \Theta_2^2.$$

Thus $\tilde{\Lambda}_h$ is generated by the coefficients of the above four power series.

As an example, consider the ring $\tilde{\Lambda}_K \subset K^*(pt)$ and $\tilde{\Lambda}_{KO} \subset KO^*(pt)$. Let $c: KO^*(\ ) \to K^*(\ )$ be the complexification homomorphism, then $c(t_{KO}(\xi)) = t_K(\xi)$ for any quaternionic vector bundle ξ. (For the definition of the Thom classes in $K^*(\ )$ and $KO^*(\ )$, see Conner-Floyd [4] § 3.) Therefore, by Lemma 2, we have

$$c(\tilde{\Lambda}_{KO}) = \tilde{\Lambda}_K. \tag{3·3}$$

Now the formal group of $K^*(\ )$ is given by

$$F(u, v) = u + v - \sigma uv$$

where σ is the Bott periodicity element: $K^*(pt) = Z[\sigma, \sigma^{-1}]$ (cf. [2]). Hence $u + \bar{u} = \sigma u\bar{u}$, $v + \bar{v} = \sigma v\bar{v}$, and we have

$$F(u, v) F(\bar{u}, \bar{v}) = (u + v - \sigma uv)(\bar{u} + \bar{v} - \sigma \bar{u}\bar{v})$$
$$= u\bar{u} + v\bar{v} - \sigma^2 u\bar{u}v\bar{v} + (u\bar{v} + \bar{u}v),$$
$$F(u, \bar{v}) F(\bar{u}, v) = u\bar{u} + v\bar{v} - \sigma^2 u\bar{u}v\bar{v} + (uv + \bar{u}\bar{v}),$$

so that

$$\Theta_1(x, y) = 2(x+y) - \sigma^2 xy$$

and

$$\Theta_2(x, y) = (x-y)^2 .$$

By Proposition 2 we have

$$p_1^K(2\zeta) = 4(x+y) - 2\sigma^2 xy ,$$
$$p_2^K(2\zeta) = 6x^2 + 4xy + 6y^2 - 4\sigma^2(x^2y + xy^2) + \sigma^4 x^2y^2 ,$$
$$p_3^K(2\zeta) = 4(x^3 - 2x^2y - 2xy^2 + y^3) - 2\sigma^2(x^3y - 2x^2y^2 + xy^3),$$
$$p_4^K(2\zeta) = (x-y)^4 .$$

Thus $\tilde{\Lambda}_K$ is generated by 1, $2\sigma^2$, σ^4 and coincides with $\mathrm{Im}(c\colon KO^*(pt) \to K^*(pt))$ in negative dimensions (see [8] 13.93). Since c is injective for $*=4k$ for any k, we have the following

Proposition 3. $\tilde{\Lambda}_{KO} = \sum_{j\geqq 0} KO^{-4j}(pt)$.

§ 4. $\tilde{\Lambda}_{MSp}$ and $\tilde{\Lambda}_{MU}$

Consider now the subring $\tilde{\Lambda}_{MSp} \subset MSp^*(pt)$ and $\tilde{\Lambda}_{MU} \subset MU^*(pt)$. Let $\varphi\colon MSp^*(\ ) \to MU^*(\ )$ be the forgetful transformation. By definition, we have $\varphi(t_{MSp}(\xi)) = t_{MU}(\xi)$ for any quaternionic vector bundle ξ. Hence it follows from Lemma 2 that

$$(4\cdot 1) \qquad \varphi(\tilde{\Lambda}_{MSp}) = \tilde{\Lambda}_{MU} \subset \mathrm{Im}(MSp^*(pt) \xrightarrow{\varphi} MU^*(pt)).$$

Recall that $MU^*(pt)$ is identified with the Lazard ring L ([1], [7]). Hereafter we fix a polynomial basis $\{x_j; j=1, 2, 3, \cdots\}$, $|x_j| = -2j$ for $MU^*(pt) \cong L$, and denote the universal formal group over L by

$$F_U(u, v) = u + v + \sum \alpha_{ij} u^i v^j , \qquad \alpha_{ij} \in MU^{2(1-i-j)}(pt).$$

As is well-known, the coefficients α_{ij} generate the ground ring $MU^*(pt)$.

Let Θ_1 and Θ_2 be the power series defined in the preceding section, so that

$$(4\cdot 2) \qquad \Theta_1(u\bar{u}, v\bar{v}) = F_U(u, v) F_U(\bar{u}, \bar{v}) + F_U(u, \bar{v}) F_U(\bar{u}, v),$$
$$\Theta_2(u\bar{u}, v\bar{v}) = F_U(u, v) F_U(\bar{u}, \bar{v}) F_U(u, \bar{v}) F_U(\bar{u}, v).$$

Then we have

$$\Theta_1(x,y)=2(x+y)+\sum_{i+j\geqq 2}\beta_{ij}x^iy^j, \qquad \beta_{ij}\in MU^{4(1-i-j)}(pt),$$

$$\Theta_2(x,y)=(x-y)^2+\sum_{i+j\geqq 3}\gamma_{ij}x^iy^j, \qquad \gamma_{ij}\in MU^{4(2-i-j)}(pt)$$

where $\beta_{ij}=\beta_{ji}$, $\gamma_{ij}=\gamma_{ji}$ and $\beta_{0j}=\gamma_{0j}=0$.

Proposition 4.

$$\beta_{j\ n+1-j}=(-1)^n 4(\alpha_{2j\ 2n+1-2j}+\alpha_{2j-1\ 2n+2-2j})+\text{decomposables},$$
$$\textit{for}\ 1\leqq j\leqq n,$$

$$\gamma_{j\ n+2-j}=(-1)^n 4(\alpha_{2j\ 2n+1-2j}-\alpha_{2j-1\ 2n+2-2j}-\alpha_{2j-2\ 2n+3-2j}+\alpha_{2j-3\ 2n+4-2j})$$
$$+\text{decomposables},\quad \textit{for}\ 2\leqq j\leqq n,$$

$$\gamma_{1\ n+1}=\gamma_{n+1\ 1}=(-1)^n 4(\alpha_{2\ 2n-1}-\alpha_{1\ 2n})+\text{decomposables}.$$

For the proof of this proposition, we require a lemma. let R and R' be commutative rings with unit, and let $\mu:R\to R'$ be a ring homomorphism. Let F be a formal group over R and $F'=\mu_*F$ an induced formal group over R'. We denote the formal inverse of F (resp. F') by ι_F (resp. $\iota_{F'}$) i.e. $F(T,\iota_F(T))=0$ (resp. $F'(T,\iota_{F'}(T))=0$). Then,

Lemma 3. *The following diagram is commutative:*

$$\begin{array}{ccc} R[[x,y]] & \xrightarrow{\mu_*} & R'[[x,y]] \\ \rho_F\downarrow & & \rho_{F'}\downarrow \\ R[[u,v]] & \xrightarrow{\mu_*} & R'[[u,v]] \end{array}$$

where ρ_F (resp. $\rho_{F'}$) *is a homomorphism given by*

$$\rho_F(x)=u\cdot\iota_F(u),\ \rho_F(y)=v\cdot\iota_F(v)$$

$$(\text{resp. } \rho_{F'}(x)=u\cdot\iota_{F'}(u),\ \rho_{F'}(y)=v\cdot\iota_{F'}(v)).$$

Proof. It suffices to show that

$$\iota_{F'}(T)=\mu_*\iota_F(T).$$

Since $F'(T,\mu_*\iota_F(T))=\mu_*(F(T,\iota_F(T)))=\mu_*(0)=0$, we certainly have $\iota_F'(T)=\mu_*\iota_F(T)$.

In particular, we have

$$(4\cdot 3) \qquad \mu_* \Theta_1 = \Theta_1', \quad \mu_* \Theta_2 = \Theta_2'$$

where Θ_1 and Θ_2 (resp. Θ_1' and Θ_2') are the power series of $R[[x, y]]$ (resp. $R'[[x, y]]$) satisfying the equation (4·2) with respect to F (resp. F').

Proof of Proposition 4. In the above lemma, put $R=L=MU^*(pt)$ and $R'=Z\oplus Q^{-4n}(n>0)$ where $Q^*=I/I^2$, $I=\sum_{j>0} MU^{-j}(pt)$, is the indecomposable quotient of $MU^*(pt)$ and $Q^{-4n}\approx\mathbf{Z}$ is a free abelian group generated by $[x_{2n}]$. We make R' into a graded algebra (Adams [1]):

$$R'^0=\mathbf{Z},$$

$$R'^{-4n}=Q^{-4n},$$

$$R'^j=0, \qquad j\neq 0, -4n.$$

Also let $\mu=\phi_n: L\to R'$ be an obvious map i.e.

$$\phi_n(x_{2n})=[x_{2n}], \qquad \phi_n(x_j)=0 \qquad \text{for } j\neq 2n.$$

Then $F'(u, v)=\phi_{n*}F_U(u, v)=u+v+\sum_{i+j=2n+1}\alpha'_{ij}u^iv^j$,

$$\alpha'_{ij}=\phi_n(\alpha_{ij})\in Q^{-4n}(L),$$

and we have only to prove that,

$$(4\cdot 4) \qquad \text{a)} \quad \phi_n(\beta_{j\ n+1-j})=\beta'_{j\ n+1-j}=4(\alpha'_{2j\ 2n+1-2j}+\alpha'_{2j-1\ 2n+2-2j}),$$

$$\text{b)} \quad \phi_n(\gamma_{j\ n+2-j})=\gamma'_{j\ n+2-j}=4(\delta_j'-\delta'_{j-1})$$

where

$$\delta_j'=\alpha'_{2j\ 2n+1-2j}-\alpha'_{2j-1\ 2n+2-2j} \qquad \text{for } 1\leqq j\leqq n,$$

and

$$\delta_0'=\delta'_{n+1}=0.$$

Since $\iota_{F'}(T)=-T$, we have $\bar{u}=\iota_{F'}(u)=-u$ and $\bar{v}=-v$.
Hence

$$F'(u, v)F'(\bar{u}, \bar{v})=-(u+v+\sum_{i+j=2n+1}\alpha'_{ij}u^iv^j)^2$$

$$=-(u+v)^2-\sum_{i+j=2n+1}2\alpha'_{ij}u^iv^j(u+v),$$

$$F'(u,\bar{v})F'(\bar{u},v)=-(u-v+\sum_{i+j=2n+1}\alpha'_{ij}u^i(-v)^j)^2$$

$$=-(u-v)^2-\sum_{i+j=2n+1}2\alpha'_{ij}u^i(-v)^j(u-v),$$

so that

$$F'(u,v)F'(\bar{u},\bar{v})+F'(u,\bar{v})F'(\bar{u},v)$$

$$=-2(u^2+v^2)-\sum_{j=1}^{n}4(\alpha'_{2j\ 2n+1-2j}+\alpha'_{2j-1\ 2n+2-2j})u^{2j}v^{2(n+1-j)}.$$

Therefore we have

$$\Theta_1'(x,y)=2(x+y)+(-1)^n4\sum_{j=1}^{n}(\alpha'_{2j\ 2n+1-2j}+\alpha'_{2j-1\ 2n+1-2j})x^jy^{n+1-j}.$$

Similarly we have

$$\Theta_2'(x,y)=(x-y)^2+(-1)^n4\sum_{j=1}^{n+1}(\delta_j'-\delta'_{j-1})x^jy^{n+2-j}.$$

This proves (4·4), and the proposition follows.

Now we obtain

Proposition 5. *$\tilde{\Lambda}_{MSp}$ contains an element $z_n\in MSp^{-4n}(pt)$ such that*

$$\varphi(z_n)=\begin{cases}16x_{2n}+\text{decomposables}, & \textit{if } n+1=2^f,\ f\geqq 1\\ 8x_{2n}+\text{decomposables}, & \textit{otherwise}\end{cases}$$

where $\varphi:MSp^*(pt)\to MU^*(pt)$ is the forgetful homomolphism and x_1, $x_2,\cdots,x_n,\cdots$ is a polynomial basis for $MU^*(pt)$.

Proof. Since $\tilde{\Lambda}_{MU}$ is generated by the cofficients of the power series $2\Theta_1$, $\Theta_1^2+2\Theta_2$, $2\Theta_1\cdot\Theta_2$ and Θ_2^2, it follows from Proposition 4 (or rather, (4·4)) that $\phi_n(\tilde{\Lambda}_{MU})\subset\mathbf{Z}+Q^{-4n}$ is generated by

$$2\beta_j'{}_{\ n+1-j}=8(\alpha'_{2j\ 2n+1-2j}+\alpha'_{2j-1\ 2n+2-2j})$$

and

$$\sum_{r=1}^{j}2\gamma_r'{}_{\ n+2-r}=8(\alpha'_{2j\ 2n+1-2j}-\alpha'_{2j-1\ 2n+2-2j}).$$

Now we use the fact that

$$\alpha'_{j\ 2n+1-j}=\frac{1}{\lambda_{2n}}\binom{2n+1}{j}[x_{2n}]$$

where $\lambda_{2n}=p$ if $2n+1=p^f$ for some prime p, and $\lambda_{2n}=1$ otherwise ([1], § 7). The greatest common divisor of the numbers

$$\binom{2n+1}{2j}+\binom{2n+1}{2j-1}=\binom{2n+2}{2j}$$

and

$$\binom{2n+1}{2j}-\binom{2n+1}{2j-1}=\binom{2n+2}{2j}-2\binom{2n+1}{2j-1}$$

for $1\leqq j\leqq n$ is $2\lambda_{2n}$ if $2n+2=2^{f+1}$, $f\geqq 1$, and λ_{2n} otherwise. Hence $\phi_n(\tilde{\Lambda}_{MU})$ is generated by

$$16[x_{2n}] \text{ if } n+1=2^f,\ 8[x_{2n}] \text{ otherwise}.$$

This completes the proof of Proposition 5.

Now we identify $MSp^*(pt)$ with Ω_*^{Sp}, the bordism ring of Sp manifolds. Then we have

Theorem. (a) *$\tilde{\Lambda}_{MSp}$ contains the bordism class $z_n=[M_n]$ of a $4n$-dimensional Sp manifold M_n whose Chern number $s_{2n}(M_n)$ is equal to*

$$16\lambda_{2n} \quad \textit{if } n+1=2^f \quad \textit{for some } f\geqq 1,$$
$$8\lambda_{2n} \quad \textit{otherwise}$$

where $\lambda_{2n}=p$ if $2n+1=p^g$ for some prime p and $g\geqq 1$, and $\lambda_{2n}=1$ otherwise.

(b) $\tilde{\Lambda}_{MSp}\otimes Z[\frac{1}{2}]=MSp^*(pt)\otimes Z[\frac{1}{2}]=Z[\frac{1}{2}][z_1, z_2, \cdots, z_n, \cdots]$.

Proof. (a) follows immediately from Proposition 5, since the Chern number of a manifold representing $x_{2n}\in MU^{-4n}(pt)\cong\Omega_{4n}^U$ is precisely λ_{2n}, and

$$s_{2n}(M\times M')=0$$

for any U manifolds M and M' of positive dimensions. Also (b) is now obvious, for the elements $\varphi(z_j)$ for $j=1, 2, \cdots$ form a polynomial basis for

$$\operatorname{Im}(MSp^*(pt)\otimes Z[\tfrac{1}{2}] \overset{\varphi\otimes 1}{\hookrightarrow} MU^*(pt)\otimes Z[\tfrac{1}{2}]).$$

Remark. (1) (Compare [3].) Let $\Lambda\subset MU^*(pt)=\Omega_U$ be the subring generated by the coefficients of Θ_1 and Θ_2. Buhštaber-Novikov [3] studied this ring. They showed that Λ is contained in (the image of) $\mathrm{Hom}_{A^U}(U^*(MSp);\Omega_U)$ and that

$$(*)\ \mathrm{Hom}_{A^U}(U^*(MSp);\Omega_U)\otimes Z[\tfrac{1}{2}]\cong\Lambda\otimes Z[\tfrac{1}{2}]$$

(Theorem 2.22 of [3]), using the Chern-Dold character. Note that Λ is not contained in the image of $MSp^*(pt)\to MU^*(pt)$ (see Gozman [5] Corollary 1).

Does Λ contain the image of $MSp^*(pt)\xrightarrow{\varphi}MU^*(pt)$?

(2) (Compare [5].) Let $MSC^*(\ )$ be the self-conjugate cobordism and let $\phi:MSC^*(\ )\to MU^*(\ )$ be the natural transformation. Using the Euler class of the self-conjugate bundle $\zeta=\eta\otimes_C\eta'$, Gozman showed that $\mathrm{Im}(MSC^*(pt)\xrightarrow{\phi}MU^*(pt))$ contains the subring generated by the coefficients of $2\Theta_1$, Θ_1^2 and Θ_2 ([5] Proposition 2, Corollary 4). Thus by the calculation similar to that of the proof of Theorem (a). We have

Assertion. *The image of ϕ contains the elements*

$$8x_{2n}+\text{decomposables},\qquad \text{if } n+1=2^f,\quad f\geqq 1,$$

$$4x_{2n}+\text{decomposables},\qquad \textit{otherwise.}$$

(3) The ring structure of $MSp^*(pt)/\mathrm{Tors}$ as well as $MSp^*(pt)$ is of course unknown, and it is very interesting to study the divisibility relations between elements of $\tilde{\Lambda}_{MU}=\varphi(\tilde{\Lambda}_{MSp})$. For this purpose, we can use the various formal groups by Lemma 3. For example, (a) using the formal group

$$h_*F_U(u,v)=\exp(\log u+\log v)$$

over $H_*(MU)\cong\pi_*(H\wedge MU)$ where $h:\pi_*(MU)\to H_*(MU)$ is the Hurewicz homomorphism, and the corresponding Θ_1 and Θ_2, we get the ring $\tilde{\Lambda}_{H\wedge MU}\approx\tilde{\Lambda}_{MU}$, and (b) using the formal group F_{SO} of $MSO^*(\ )$, we get $\tilde{\Lambda}_{MSO}\approx\tilde{\Lambda}_{MU}$.

The latter has an advantage that the formal inverse of F_{SO} is given by

$$\iota_{SO}(T)=-T\qquad\text{i.e. } F_{SO}(T,-T)=0\qquad(\text{see [2]})$$

so that $u\bar{u}=-u^2$ and $v\bar{v}=-v^2$; hence the computation of Θ_1 and Θ_2 becomes easier.

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