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Toward Resolution of Singularities over a Field of Positive Characteristic (The Idealistic Filtration Program)

Dedicated to Professor Heisuke Hironaka

Part II. Basic invariants associated to the idealistic filtration and their properties

by

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Abstract

This is the second of a series of four papers entitled “Toward resolution of singularities over a field of positive characteristic (the Idealistic Filtration Program)”. The goal is to present the IFP, and to ultimately construct an explicit algorithm guided by the program.

In the classical setting in characteristic zero, resolution of singularities was carried out by induction on dimension. We take a so-called “hypersurface of maximal contact” to reduce the dimension by one. In the algorithm, we construct the strand of invariants “$\mathrm{inv}_{\mathrm{classic}}$” of the following form:

$$\mathrm{inv}_{\mathrm{classic}} = (w, s)(w, s)(w, s) \cdots ,$$

where the unit (w, s) consists of the weak order w and the number s of the “old” components in the boundary. Going from one unit to the next, the dimension of the object which we use to extract the information to compute the invariants drops by one, manifesting the induction on dimension. We run the algorithm with the center of blowup determined as the maximal locus of “$\mathrm{inv}_{\mathrm{classic}}$”.

In our new setting in positive characteristic, we no longer have a hypersurface of maximal contact. However, we try to carry out the induction on “invariant σ”, which indicates the behavior of “a Leading Generator System”. The notion of an LGS plays the role of a collective substitute for a hypersurface of maximal contact in the IFP.

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Accordingly, in our new algorithm, we construct the strand of invariants "inv" of the following form:

$$\mathrm{inv} = (\sigma, \widetilde{\mu}, s)(\sigma, \widetilde{\mu}, s)(\sigma, \widetilde{\mu}, s) \cdots ,$$

where the unit $(\sigma, \widetilde{\mu}, s)$ consists of the above mentioned σ, followed by $\widetilde{\mu}$ and s, which correspond to w and s in the classical setting, respectively. Going from one unit to the next, the invariant σ of the LGS of the object, namely an idealistic filtration, strictly drops, manifesting the induction on the invariant σ. We run the new algorithm with the center of blowup determined as the maximal locus of "inv".

The main purpose of this paper, Part II of the series, is to study the basic properties of the invariants that appear in the strand of invariants "inv", establishing the upper semi-continuity of the pair $(\sigma, \widetilde{\mu})$ among others.

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Contents

Chapter 0. Introduction to Part II 361
§0.1. Overview of the series 361
§0.2. Outline of Part II 361
0.2.1. Invariant σ 362
0.2.2. Power series expansion 365
0.2.3. Invariant $\widetilde{\mu}$ 367
0.2.4. Appendix 369
§0.3. Current status of the Idealistic Filtration Program 369
0.3.1. Current status 369
0.3.2. Roles of σ and $\widetilde{\mu}$ 370
0.3.3. Note to Part I 370

Chapter 1. Invariant σ 370
§1.1. Definition and computation of σ 370
1.1.1. Definition of σ 370
1.1.2. Computation of σ 372
§1.2. Upper semi-continuity 375
1.2.1. Basic facts surrounding the definition of the upper semi-continuity 375
1.2.2. Upper semi-continuity of the invariant σ 376
§1.3. Local behavior of a leading generator system 378
1.3.1. Definition of a leading generator system and a remark about the subscripts 378
1.3.2. A basic question 379
1.3.3. Modification of a given leading generator system into one which is uniformly pure 379

Chapter 2. Power series expansion 387
§2.1. Existence and uniqueness 388
2.1.1. Setting for the power series expansion 388
2.1.2. Existence and uniqueness of the power series expansion 388
§2.2. Formal coefficient lemma 391

2.2.1. Setting for the formal coefficient lemma 392
2.2.2. Statement of the formal coefficient lemma and its proof 392
Chapter 3. Invariant $\widetilde{\mu}$ 397
§3.1. Definition of $\widetilde{\mu}$ 398
3.1.1. Definition of $\widetilde{\mu}$ as $\mu_{\mathcal{H}}$ 398
3.1.2. Invariant $\mu_{\mathcal{H}}$ is independent of $\mathcal{H}$ 399
§3.2. Interpretation of $\widetilde{\mu}$ in terms of the power series expansion 402
3.2.1. The order $\mathrm{ord}_{\mathcal{H}}(f)$ of f modulo $(\mathcal{H})$ is equal to the order $\mathrm{ord}(c_{\mathbb{O}})$ of the constant term of the power series expansion for f 402
3.2.2. Alternative proof of the Coefficient Lemma 403
3.2.3. Alternative proof of Proposition 3.1.2.1 405
§3.3. Upper semi-continuity of $(\sigma, \widetilde{\mu})$ 407
3.3.1. Statement of the upper semi-continuity of $(\sigma, \widetilde{\mu})$ and its proof 407
3.3.2. Alternative proof of the upper semi-continuity of $(\sigma, \widetilde{\mu})$ 412
Appendix 413
§A.1. Nonsingularity principle using only $\mathfrak{D}$-saturation but not $\mathfrak{R}$-saturation 414
A.1.1. Statement 414
A.1.2. Proof 415
References 421

Chapter 0. Introduction to Part II

§0.1. Overview of the series

This is the second of a series of papers under the title "Toward resolution of singularities over a field of positive characteristic (the Idealistic Filtration Program)":

Part I.	Foundation; the language of the idealistic filtration
Part II.	Basic invariants associated to the idealistic filtration and their properties
Part III.	Transformations and modifications of the idealistic filtration
Part IV.	Algorithm in the framework of the idealistic filtration.

For a brief summary of the entire series, including its goal and the overview of the Idealistic Filtration Program (called the IFP for short), we refer the reader to the introduction in Part I [Kaw07]. The outline of Part II is presented below.

§0.2. Outline of Part II

As described in the overview of the IFP (cf. §0.2 in Part I), we construct a strand of invariants, whose maximum locus determines each center of blowup of our algorithm for resolution of singularities. The strand of invariants consists of units (cf. 0.2.3.2.2 in Part I), each of which is a triplet of numbers $(\sigma, \widetilde{\mu}, s)$ associated to a certain idealistic filtration (cf. Chapter 2 in Part I) and a simple normal crossing divisor E called a boundary. (To be precise, the invariant σ is a sequence of numbers indexed by $\mathbb{Z}_{\geq 0}$ as described in Definition 3.2.1.1 in Part I.) The purpose of Part II is to establish the fundamental properties of the invariants σ and $\widetilde{\mu}$. They are the main constituents of the unit, while the remaining factor s can easily be computed as the number of (certain specified) components in the boundary passing through a given point, and needs no further mathematical discussion. Our goal is to study the intrinsic nature of these invariants associated to a given idealistic filtration. The discussion in Part II does not involve the analysis regarding the exceptional divisors created by blowups, and hence could only be directly applied to the situation *in year 0* of our algorithm. The systematic discussion on how some subtle adjustments should be made in the presence of the exceptional divisors *after year 0* and on how the strand of invariants functions in the algorithm, built upon the analysis in Part III of the modifications and transformations of an idealistic filtration, will have to wait for Part IV.

In the appendix, we report a new development, unexpected at the time of writing Part I, which suggests a possibility of constructing an algorithm using only the $\mathfrak{D}$-saturation (or $\mathfrak{D}_E$-saturation) but not the $\mathfrak{R}$-saturation, still within the framework of the IFP. This would avoid the problem of termination, which we specified in the introduction to Part I as the only missing piece toward completing our algorithm in positive characteristic. (See §0.3 for the further developments and "evolution" of the IFP up to date.)

The following is a rough description of the content of each chapter and the appendix in Part II. Throughout the description, let R be the coordinate ring of an affine open subset of a nonsingular variety W of dimension $d = \dim W$ over an algebraically closed field k of characteristic $\mathrm{char}(k) = p > 0$ or $\mathrm{char}(k) = 0$, where in the latter case we set $p = \infty$ formally (cf. 0.2.3.2.1 and Definition 3.1.1.1(2) in Part I).

0.2.1. Invariant σ. Chapter 1 is devoted to the discussion of the invariant σ, which is defined for a $\mathfrak{D}$-saturated idealistic filtration $\mathbb{I}$ over R (cf. 2.1.2 in Part I). The subtle adjustment of the invariant σ, in the presence of the exceptional divisor E, which is defined for a $\mathfrak{D}_E$-saturated idealistic filtration (cf. 1.2.2 in Part I), will be postponed to Parts III and IV.

0.2.1.1. **Leading algebra and its structure.** We fix a closed point $P \in \mathrm{Spec}\, R \subset W$, with $\mathfrak{m}_P$ denoting the maximal ideal of the local ring R_P. The leading algebra $L(\mathbb{I}_P)$ of the localization $\mathbb{I}_P$ of the idealistic filtration $\mathbb{I}$ at P is defined to be the graded k-subalgebra of $G_P = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} (G_P)_n = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathfrak{m}_P^n/\mathfrak{m}_P^{n+1}$ (cf. 3.1.1 in Part I)

$$L(\mathbb{I}_P) = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} L(\mathbb{I}_P)_n \subset G_P,$$

where

$$L(\mathbb{I}_P)_n = \{\overline{f} = (f \bmod \mathfrak{m}_P^{n+1})\,;\, (f,n) \in \mathbb{I}_P,\, f \in \mathfrak{m}_P^n\}.$$

For $e \in \mathbb{Z}_{\geq 0}$ with $p^e \in \mathbb{Z}_{>0}$, we define the pure part $L(\mathbb{I}_P)^{\text{pure}}_{p^e}$ of $L(\mathbb{I}_P)_{p^e}$ by the formula

$$L(\mathbb{I}_P)^{\text{pure}}_{p^e} = L(\mathbb{I}_P)_{p^e} \cap F^e((G_P)_1) \subset L(\mathbb{I}_P)_{p^e}$$

where F^e is the e-th power of the Frobenius map of G_P.

The most remarkable structure of the leading algebra $L(\mathbb{I}_P)$ is that it is generated by its pure part (cf. Lemma 3.1.2.1 in Part I), i.e.,

$$L(\mathbb{I}_P) = k[L(\mathbb{I}_P)^{\text{pure}}] \quad \text{where} \quad L(\mathbb{I}_P)^{\text{pure}} = \bigsqcup_{e \in \mathbb{Z}_{\geq 0}} L(\mathbb{I}_P)^{\text{pure}}_{p^e}.$$

This follows from the fact that $\mathbb{I}_P$ is $\mathfrak{D}$-saturated, since so is $\mathbb{I}$ (cf. compatibility of $\mathfrak{D}$-saturation with localization, discussed in §2.4 in Part I).

0.2.1.2. **Definition of the invariant σ and its computation.** We define the invariant $\sigma(P)$ by the formula

$$\sigma(P) = (d - l^{\text{pure}}_{p^e}(P))_{e \in \mathbb{Z}_{\geq 0}} \in \prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0} \quad \text{where} \quad l^{\text{pure}}_{p^e}(P) = \dim_k L(\mathbb{I}_P)^{\text{pure}}_{p^e},$$

which reflects the behavior of the pure part of the leading algebra $L(\mathbb{I}_P)$. Varying P among all the closed points, $\mathfrak{m}$-Spec R (i.e., all the maximal ideals of R), we obtain the invariant

$$\sigma : \mathfrak{m}\text{-Spec}\, R \to \prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}.$$

Recall that Lemma 3.1.2.1 in Part I gives the description of a specific set of generators for the leading algebra $L(\mathbb{I}_P)$ taken from its pure part. Using this lemma, we can compute the dimension of the pure part, $l^{\text{pure}}_{p^e}(P) = \dim_k L(\mathbb{I}_P)^{\text{pure}}_{p^e}$, in terms of the dimension of the entire degree p^e component, $l_{p^e}(P) = \dim_k L(\mathbb{I}_P)_{p^e}$, and in terms of the dimensions of the pure parts, $l^{\text{pure}}_{p^\alpha}(P)$ for $\alpha = 0, \ldots, e-1$. That is, $l^{\text{pure}}_{p^e}(P)$ can be computed inductively from $l_{p^e}(P)$ and the dimensions of the pure parts of lower degree.

0.2.1.3. **Upper semi-continuity of the invariant σ.** We observe that $l_{p^e}(P)$ can be computed as the rank of a certain "Jacobian-like" matrix, and hence is easily seen to be lower semi-continuous as a function of P. The upper semi-continuity of the invariant $\sigma = (d - l^{\text{pure}}_{p^e})_{e \in \mathbb{Z}_{\geq 0}}$ then follows immediately from the inductive computation of the pure part $l^{\text{pure}}_{p^e}$ described in 0.2.1.2. The upper semi-continuity of the invariant σ as a function over $\mathfrak{m}$-Spec R also allows us to extend its domain to Spec R. That is, we have the invariant σ defined over the extended domain

$$\sigma : \operatorname{Spec} R \to \prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0},$$

which is automatically upper semi-continuous as a function over Spec R.

0.2.1.4. **Clarification of the meaning of the upper semi-continuity.** We say by definition that a function $f : X \to T$, from a topological space X to a totally ordered set T, is *upper semi-continuous* if the set $X_{\geq t} = \{x \in X\,;\, f(x) \geq t\}$ is closed for any $t \in T$. When the target space T is not well-ordered, however, we have to be extra careful if we try to see the equivalence of this definition to the other "well-known" conditions for the upper semi-continuity. The target space of the invariant $\sigma : \mathfrak{m}\text{-}\operatorname{Spec} R \to \prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}$ is a priori not well-ordered. Nevertheless, using the fact that $l^{\text{pure}}_{p^e}(P)$ is nondecreasing as a function of $e \in \mathbb{Z}_{\geq 0}$ for a fixed $P \in \mathfrak{m}\text{-}\operatorname{Spec} R$, we observe that the target space for the invariant σ can be replaced by some well-ordered subset. Then it can be easily seen that the upper semi-continuity of the invariant σ in the above sense is actually equivalent to the condition that, given a point $P \in \mathfrak{m}\text{-}\operatorname{Spec} R$, there exists a neighborhood U_P of P such that $\sigma(P) \geq \sigma(Q)$ for any point $Q \in U_P$. From this upper semi-continuity, interpreted in the equivalent condition, it follows that the domain of the invariant σ can be extended from $\mathfrak{m}$-Spec R to Spec R, as mentioned at the end of 0.2.1.3. We summarize the basic facts surrounding the definition of the upper semi-continuity in Chapter 1 for the sake of clarification.

0.2.1.5. **Local behavior of a leading generator system[1], and its modification into one which is uniformly pure.** Recall that a subset $\mathbb{H} = \{(h_l, p^{e_l})\}_{l=1}^N \subset \mathbb{I}_P$ with associated nonnegative integers $0 \leq e_1 \leq \cdots \leq e_N$ is said to be a *leading generator system* (LGS) of the idealistic filtration $\mathbb{I}_P$ if the leading terms of its elements provide a specific set of generators for the leading algebra $L(\mathbb{I}_P)$, as described in Lemma 3.1.2.1 in Part I. More precisely, it satisfies the following conditions (cf. 3.1.3 in Part I):

(i) $h_l \in \mathfrak{m}_P^{p^{e_l}}$ and $\overline{h_l} = (h_l \bmod \mathfrak{m}_P^{p^{e_l}+1}) \in L(\mathbb{I}_P)^{\text{pure}}_{p^{e_l}}$ for $l = 1, \ldots, N$,

(ii) $\{\overline{h_l}^{p^{e-e_l}}\,;\, e_l \leq e\}$ consists of $\#\{l\,;\, e_l \leq e\}$ distinct elements, and forms a k-basis of $L(\mathbb{I}_P)^{\text{pure}}_{p^e}$ for any $e \in \mathbb{Z}_{\geq 0}$.

[1]We use the abbreviation "LGS" for "Leading Generator System". Prof. Cossart kindly suggested to us that "LGS" could be read "Leading *Giraud* System" in honor of J. Giraud, whose contribution (cf. [Gir74], [Gir75]) is profound in search of the right notion of "a hypersurface of maximal contact" in positive characteristic.

Since the leading algebra $L(\mathbb{I}_P)$ is generated by its pure part

$$L(\mathbb{I}_P)^{\text{pure}} = \bigsqcup_{e \in \mathbb{Z}_{\geq 0}} L(\mathbb{I}_P)^{\text{pure}}_{p^e} \qquad \text{(cf. 0.2.1.1)},$$

we conclude from condition (ii) that the leading terms $\{\overline{h_l} = (h_l \bmod \mathfrak{m}_P^{p^{e_l}+1})\}_{l=1}^N$ of $\mathbb{H}$ provide a set of generators for $L(\mathbb{I}_P)$, i.e., $L(\mathbb{I}_P) = k[\{\overline{h_l}\}_{l=1}^N]$.

A basic question then about the local behavior of an LGS is:

Does $\mathbb{H}$ remain an LGS of $\mathbb{I}_Q$ for any closed point Q in a neighborhood U_P of P (if we take U_P small enough)?

Even though the answer is *no* in general, we show that we can modify a given LGS $\mathbb{H}$ into a new one $\mathbb{H}'$ such that $\mathbb{H}'$ is an LGS of $\mathbb{I}_Q$ for any closed point Q in a neighborhood U_P, as long as Q is in the local maximum locus of the invariant σ (and Q is in the support of $\mathbb{I}$). The last extra condition is equivalent to $\sigma(Q) = \sigma(P)$ by the upper semi-continuity of the invariant σ. We then say $\mathbb{H}'$ is *uniformly pure*. We will use this modification as the main tool to derive the upper semi-continuity of the pair of invariants $(\sigma, \widetilde{\mu})$ in Chapter 3.

0.2.2. Power series expansion. Chapter 2 is devoted to the discussion of the power series expansion with respect to an LGS and its (weakly-)associated regular system of parameters.

0.2.2.1. **Similarities between a regular system of parameters and a leading generator system.** If we have an LGS $\mathbb{H} = \{(h_l, p^{e_l})\}_{l=1}^N$ *in characteristic zero* (for a $\mathfrak{D}$-saturated idealistic filtration $\mathbb{I}_P$ over R_P at a closed point $P \in \operatorname{Spec} R \subset W$), then the elements in the LGS are all concentrated at level 1, i.e., $e_l = 0$ and $p^{e_l} = 1$ for $l = 1, \ldots, N$ (cf. Chapter 3 in Part I). This implies by the definition of an LGS that the set of the elements $H = (h_1, \ldots, h_l)$ forms (a part of) a regular system of parameters $(x_1, \ldots, x_d)$. (Say, $h_l = x_l$ for $l = 1, \ldots, N$.) *In positive characteristic*, this is no longer the case. However, we can still regard the notion of an LGS as a generalization of the notion of a regular system of parameters, and we may expect some similarities between the two notions. One of such expected similarities is the power series expansion, which we discuss next.

0.2.2.2. **Power series expansion with respect to a leading generator system.** In characteristic zero, any element $f \in R_P$ has a power series expansion (with respect to the regular system of parameters $X = (x_1, \ldots, x_d)$, where $h_l = x_l$ for $l = 1, \ldots, N$, with $H = (h_1, \ldots, h_N)$ consisting of the elements of an LGS as described in 0.2.2.1)

$$f = \sum_{I \in (\mathbb{Z}_{\geq 0})^d} a_I X^I = \sum_{B \in (\mathbb{Z}_{\geq 0})^N} c_B H^B$$

where $a_I \in k$ and where c_B is a power series in terms of the remainder $(x_{N+1}, \ldots, x_d)$ of the regular system of parameters.

In positive characteristic, we expect to have a power series expansion with respect to an LGS. More specifically and more generally, the setting for Chapter 2 is given as follows. We have a subset $\mathcal{H} = \{h_1, \ldots, h_N\} \subset R_P$ consisting of N elements, and nonnegative integers $0 \leq e_1 \leq \cdots \leq e_N$ attached to these elements, satisfying the following conditions (cf. 4.1.1 in Part I):

(i) $h_l \in \mathfrak{m}_P^{p^{e_l}}$ and $\overline{h_l} = (h_l \bmod \mathfrak{m}_P^{p^{e_l}+1}) = v_l^{p^{e_l}}$ with $v_l \in \mathfrak{m}_P/\mathfrak{m}_P^2$ for $l = 1, \ldots, N$,

(ii) $\{v_l\,;\, l = 1, \ldots, N\} \subset \mathfrak{m}_P/\mathfrak{m}_P^2$ consists of N distinct and k-linearly independent elements in the k-vector space $\mathfrak{m}_P/\mathfrak{m}_P^2$.

We also take a regular system of parameters $(x_1, \ldots, x_d)$ such that

$$v_l = \overline{x_l} = (x_l \bmod \mathfrak{m}_P^2) \quad \text{for } l = 1, \ldots, N.$$

(We say that a regular system of parameters $(x_1, \ldots, x_d)$ is *associated* to $H = (h_1, \ldots, h_N)$ if the above condition is satisfied. For the description of the condition of $(x_1, \ldots, x_d)$ being *weakly-associated* to H, we refer the reader to Chapter 2.)

Now we claim that any element $f \in R_P$ has a power series expansion of the form

$$(\star) \qquad f = \sum_{B \in (\mathbb{Z}_{\geq 0})^N} c_B H^B \quad \text{where} \quad c_B = \sum_{K \in (\mathbb{Z}_{\geq 0})^d} b_{B,K} X^K,$$

with $b_{B,K}$ being a power series in terms of the remainder $(x_{N+1}, \ldots, x_d)$ of the regular system of parameters, and with $K = (k_1, \ldots, k_d)$ varying in the range satisfying the condition

$$0 \leq k_l \leq p^{e_l} - 1 \quad \text{for } l = 1, \ldots, N \quad \text{and} \quad k_l = 0 \quad \text{for } l = N+1, \ldots, d.$$

The existence of a power series expansion of the form $(\star)$ and its uniqueness (with respect to a fixed subset $\mathcal{H}$ and its chosen (weakly-)associated regular system of parameters $(x_1, \ldots, x_d)$) follow immediately, and are the results stated independently of the notion of an idealistic filtration.

0.2.2.3. **Formal coefficient lemma.** In the general setting as described in 0.2.2.2, the discussion of the power series expansion of the form $(\star)$ does not involve the notion of an idealistic filtration. The most interesting and important result regarding the power series expansion of the form $(\star)$, however, is obtained when we introduce and require the following condition for H to satisfy, involving a $\mathfrak{D}$-saturated idealistic filtration $\mathbb{I}_P$ over R_P:

(iii) $(h_l, p^{e_l}) \in \mathbb{I}_P$ for $l = 1, \ldots, N$.

Now the formal coefficient lemma claims

$$(f,a) \in \widehat{\mathbb{I}}_P, f = \sum_{B \in (\mathbb{Z}_{\geq 0})^N} c_B H^B \ \Rightarrow\ (c_B, a - |[B]|) \in \widehat{\mathbb{I}_P} \text{ for any } B \in (\mathbb{Z}_{\geq 0})^N.$$

(We recall that, for $B = (b_1, \ldots, b_N) \in (\mathbb{Z}_{\geq 0})^N$, we denote $(p^{e_1}b_1, \ldots, p^{e_N}b_N)$ by $[B]$ and $\sum_{l=1}^N p^{e_l}b_l$ by $|[B]|$. For the definition of the completion $\widehat{\mathbb{I}_P}$ of the idealistic filtration $\mathbb{I}_P$, we refer the reader to §2.4 in Part I.) The statement of the formal coefficient lemma turns out to be quite useful and powerful. In fact, Lemma 4.1.4.1 (Coefficient Lemma) in Part I can be obtained as a corollary to this formal version in Part II. We will see some applications of the formal coefficient lemma not only in Chapter 3 when we study the invariant $\widetilde{\mu}$, but also in Part III when we analyze the modifications and transformations of an idealistic filtration, and in Part IV when we give the description of our algorithm.

0.2.3. Invariant $\widetilde{\mu}$. Chapter 3 is devoted to the discussion of the invariant $\widetilde{\mu}$, which is a counterpart in the new setting of the IFP to the notion of the "weak order" in the classical setting, whose definition involves the exceptional divisors. Naturally, when we carry out our algorithm, the definition of the invariant $\widetilde{\mu}$ in the middle of its process involves the exceptional divisors created by blowups. It also involves the subtle adjustments we have to make to the notion of an LGS for a $\mathfrak{D}_E$-saturated idealistic filtration in the presence of the exceptional divisor E (cf. 0.2.1). However, we restrict the discussion of the invariant $\widetilde{\mu}$ in Part II to the one with no exceptional divisors involved, and hence to the discussion which could only be directly applied to the situation *in year 0* of the algorithm. The discussion with the exceptional divisors taken into consideration, i.e., the discussion which can then be applied to the situation *after year 0* of the algorithm, will be postponed until it finds an appropriate place in Part III or Part IV, where we will show how we should adjust the arguments in Part II in the presence of the exceptional divisors.

0.2.3.1. **Definition of $\widetilde{\mu}$.** Let $\mathbb{I}$ be a $\mathfrak{D}$-saturated idealistic filtration over R as before. Let $P \in \operatorname{Spec} R \subset W$ be a closed point. Take an LGS $\mathbb{H}$ for $\mathbb{I}_P$, and let $\mathcal{H}$ be the set consisting of its elements. Recall that in 3.2.2 in Part I we set

$$\mu_{\mathcal{H}}(\mathbb{I}_P) = \inf\left\{\mu_{\mathcal{H}}(f,a) := \frac{\operatorname{ord}_{\mathcal{H}}(f)}{a}\ ;\ (f,a) \in \mathbb{I}_P, a > 0\right\}$$

where

$$\operatorname{ord}_{\mathcal{H}}(f) = \sup\{n \in \mathbb{Z}_{\geq 0}\ ;\ f \in \mathfrak{m}_P^n + (\mathcal{H})\},$$

and that we define the invariant $\widetilde{\mu}(P)$ by the formula

$$\widetilde{\mu}(P) = \mu_{\mathcal{H}}(\mathbb{I}_P).$$

There are two main issues concerning the invariant $\widetilde{\mu}(P)$.

Issue 1: Is $\widetilde{\mu}(P)$ independent of the choice of $\mathbb{H}$ and hence of $\mathcal{H}$?

Issue 2: Is $\widetilde{\mu}$ an upper semi-continuous function of the closed point $P \in \operatorname{Spec} R \subset W$?

0.2.3.2. $\widetilde{\mu}(P)$ **is independent of the choice of** $\mathcal{H}$**.** We settled Issue 1 affirmatively via the Coefficient Lemma in Part I. We would like to emphasize, on one hand, that we carried out the entire argument in Part I at the algebraic level of a local ring. This argument, showing that the invariant $\widetilde{\mu}(P)$ is determined independently of the choice of an LGS, seems to be in contrast to the argument by Włodarczyk [Wło05], who uses some (analytic) automorphism of the completion of the local ring, showing that certain invariants are determined independently of the choice of a hypersurface of maximal contact via the notion of homogenization. Note that the notion of an LGS is a collective substitute for the notion of a hypersurface of maximal contact (cf. 0.2.3.2.1 in Part I).

We remark, on the other hand, that we can give an analytic interpretation of the invariant $\widetilde{\mu}(P)$ using the power series expansion discussed in Chapter 2. In fact, we see that $\operatorname{ord}_{\mathcal{H}}(f) = \operatorname{ord}(c_{\mathbb{O}})$ where $c_{\mathbb{O}}$ with $\mathbb{O} = (0, \ldots, 0) \in (\mathbb{Z}_{\geq 0})^N$ is the "constant term" of the power series expansion of the form $(\star)$. This explicit interpretation leads to an alternative way to settle Issue 1, though quite similar in spirit to the proof at the algebraic level, via the formal coefficient lemma. Note that $\widetilde{\mu}(P)$ is rational, i.e., $\widetilde{\mu}(P) \in \mathbb{Q}$, if we assume that $\mathbb{I}$ is of r.f.g. type (and hence that so is $\mathbb{I}_P$). We recall that "of r.f.g. type" is an abbreviation for "of rationally and finitely generated type" (cf. Definition 2.1.1.1(4) in Part I).

0.2.3.3. **Upper semi-continuity of** $(\sigma, \widetilde{\mu})$**.** Regarding Issue 2, the more precise and correct formulation of the question is to ask if the pair $(\sigma, \widetilde{\mu})$ is upper semi-continuous with respect to the lexicographical order. Since the invariant σ is upper semi-continuous, this is equivalent to asking if the invariant $\widetilde{\mu}$ is upper semi-continuous along the local maximum locus of the invariant σ. We settle Issue 2 affirmatively in this precise form.

The difficulty in studying the behavior of the invariant $\widetilde{\mu}(P) = \mu_{\mathcal{H}}(\mathbb{I}_P)$, as we let P vary along the local maximum locus of the invariant σ, lies in the fact that we also have to change the LGS $\mathbb{H}$ and hence $\mathcal{H}$ simultaneously. This is caused by the fact that our definition of an LGS is a priori "pointwise" in nature and hence we do not know, even if $\mathbb{H}$ is an LGS for $\mathbb{I}_P$ at a point P, whether $\mathbb{H}$ remains an LGS for $\mathbb{I}_Q$ at a point Q in a neighborhood of P. In general, it does not. There arises the need to modify a given LGS into one which is *uniformly pure* as discussed in 0.2.1.5. With the modified and uniformly pure LGS, the upper semi-continuity at issue is reduced to that of the multiplicity of a function in the usual setting. The upper semi-continuity can also be verified if we look at the power series expansion with respect to a uniformly pure LGS, and study the behavior of its coefficients.

0.2.4. Appendix. In the appendix, we report a new development, which establishes the nonsingularity principle using only the $\mathfrak{D}$-saturation but not the $\mathfrak{R}$-saturation. Recall that in Part I we established the nonsingularity principle using both the $\mathfrak{D}$-saturation and $\mathfrak{R}$-saturation (cf. 0.2.3.2.4 and Chapter 4 in Part I). This opens up a possibility of constructing an algorithm, still in the framework of the IFP, using only the $\mathfrak{D}$-saturation but not the $\mathfrak{R}$-saturation. Note that the $\mathfrak{R}$-saturation invites the problem of termination, which we specified in the introduction to Part I as the only missing piece toward completing our algorithm in positive characteristic (cf. 0.3 below). Therefore, we believe that this new development is a substantial step forward in our quest for establishing an algorithm for resolution of singularities in positive characteristic.

This finishes the outline of Part II.

§0.3. Current status of the Idealistic Filtration Program

It has been more than a year since we posted the original version of Part II on the electronic archive in August 2007. We would like to report on the current status of the IFP, and make a note to Part I.

0.3.1. Current status. Since the advent of the new nonsingularity principle as described in 0.2.4, we have been pursuing the scheme of constructing an algorithm using only the $\mathfrak{D}$-saturation (or $\mathfrak{D}_E$-saturation in the presence of an exceptional divisor E). In fact, in characteristic zero, the scheme works almost perfectly, providing an algorithm for the local uniformization. We construct the strand of invariants weaving the units, where each unit is the triplet of the form $(\sigma, \widetilde{\mu}, s)$. (In order to obtain the global resolution of singularities, we have to work a little more to overcome an anomaly: we observe the gap between the maximum locus of the strand and the support of the "last" modification of an idealistic filtration. We have to fill in this gap in order to globalize the choice of a center.) In positive characteristic, as we do not use the $\mathfrak{R}$-saturation any more, the denominators of the invariant $\widetilde{\mu}$ are well-controlled, being no obstruction to showing the termination of the algorithm. Recently, however, some "bad" examples surfaced; if we try to naively follow the analogy to the case in characteristic zero, even when we blow up a "$(\sigma, \widetilde{\mu}, s)$-permissible" center, we observe the strict increase of the invariant $\widetilde{\mu}$ in the examples. This would violate the principle that the strand of invariants we construct should never increase after blowup. A few of these examples also indicate that the so-called monomial case needs a more careful treatment in positive characteristic than in characteristic zero. In order to overcome these pathologies observed in the "bad" examples, we introduce and insert a new invariant $\widetilde{\nu}$, making the quadruple $(\sigma, \widetilde{\mu}, \widetilde{\nu}, s)$ the new unit to constitute the strand of invariants. The invariant $\widetilde{\nu}$ is closely related to the invariant "ν" used in [CP08] and [CP09]. We are now testing if our algorithm, which constructs the "$(\sigma, \widetilde{\mu}, \widetilde{\nu}, s)$-permissible" center in a quite explicit way, will provide a solution to the problem of local uniformization (and global resolution) in positive characteristic. We want to emphasize that we consider these new developments as the events in the process of "evolution" of the IFP, rather than mutation, since the basic strategy of the IFP remains intact as envisioned in Part I throughout our project. We reported the current status of the evolution of the IFP at the workshop held at RIMS in December 2008, and we refer the reader to [RIMS08] for the precise content of the report. More details will be published in our subsequent papers in the near future.

0.3.2. Roles of σ and $\widetilde{\mu}$. Despite all the changes in the evolution process of the IFP discussed above, the fundamental roles of the invariants σ and $\widetilde{\mu}$, as the first two factors of the unit constituting the strand of invariants, remain unchanged. Therefore, the main portion of Part II, discussing these fundamental roles, remains unchanged.

0.3.3. Note to Part I. After Part I was published, we learned that the result stated as Proposition 2.3.2.4 in Part I had already appeared in [LT74]. The arguments both in Part I and [LT74] are closely related to the classical results of Nagata [Nag57]. Due to our negligence, this fact was never mentioned in Part I, even though [LT74] was included in the references for Part I.

Chapter 1. Invariant σ

The purpose of this chapter is to investigate the basic properties of the invariant σ.

In this chapter, R represents the coordinate ring of an affine open subset $\operatorname{Spec} R$ of a nonsingular variety W of $\dim W = d$ over an algebraically closed field k of characteristic $\operatorname{char}(k) = p > 0$ or $\operatorname{char}(k) = 0$, where in the latter case we formally set $p = \infty$ (cf. 0.2.3.2.1 and Definition 3.1.1.1(2) in Part I).

Let $\mathbb{I}$ be a $\mathfrak{D}$-saturated idealistic filtration over R, and $\mathbb{I}_P$ its localization at a closed point $P \in \operatorname{Spec} R \subset W$ (cf. Chapter 2 in Part I).

§1.1. Definition and computation of σ

1.1.1. Definition of σ. First we recall the definition, given in 3.2.1 in Part I, of the invariant σ at a closed point $P \in \text{Spec } R \subset W$.

Definition 1.1.1.1. The invariant σ at P, which we denote by $\sigma(P)$, is defined to be the following infinite sequence indexed by $e \in \mathbb{Z}_{\geq 0}$:

$$\sigma(P) = (d - l_{p^0}^{\mathrm{pure}}(P), d - l_{p^1}^{\mathrm{pure}}(P), \dots, d - l_{p^e}^{\mathrm{pure}}(P), \dots) = (d - l_{p^e}^{\mathrm{pure}}(P))_{e \in \mathbb{Z}_{\geq 0}}$$

where

$$d = \dim W, \quad l_{p^e}^{\mathrm{pure}}(P) = \dim_k L(\mathbb{I}_P)_{p^e}^{\mathrm{pure}}.$$

(We refer the reader to Chapter 3 in Part I or 0.2.1.1 in the introduction to Part II for the definitions of the leading algebra $L(\mathbb{I}_P)$ of the idealistic filtration $\mathbb{I}_P$, its degree p^e component $L(\mathbb{I}_P)_{p^e}$, and its pure part $L(\mathbb{I}_P)_{p^e}^{\mathrm{pure}}$.)

The invariant σ obviously depends on the idealistic filtration $\mathbb{I}$ of concern. However, since in Part II we mostly deal with a situation where the idealistic filtration $\mathbb{I}$ is fixed, we suppress this dependence and omit $\mathbb{I}$ from the notation for simplicity.

Remark 1.1.1.2. (1) The reason why we take the infinite sequence $(d - l_{p^e}^{\mathrm{pure}}(P))_{e \in \mathbb{Z}_{\geq 0}}$ instead of the infinite sequence $(l_{p^e}^{\mathrm{pure}}(P))_{e \in \mathbb{Z}_{\geq 0}}$ is two-fold:

(i) If we consider the infinite sequence $(l_{p^e}^{\mathrm{pure}}(P))_{e \in \mathbb{Z}_{\geq 0}}$, it is lower semi-continuous as a function of P. Taking the negative of each factor $(+d)$ of the sequence, we have our invariant upper semi-continuous, as we will see below. (We consider that the bigger $(l_{p^e}^{\mathrm{pure}}(P))_{e \in \mathbb{Z}_{\geq 0}}$, the better the singularity. Therefore, as the measure of how bad the singularity is, it is also natural to define our invariant using its negative $(-l_{p^e}^{\mathrm{pure}}(P))_{e \in \mathbb{Z}_{\geq 0}}$.)

(ii) We reduce the problem of resolution of singularities of an abstract variety X to that of embedded resolution. Therefore, it would be desirable or even necessary to come up with an algorithm which would induce the "same" process of resolution of singularities, no matter what ambient variety W we choose for an embedding $X \hookrightarrow W$ (locally).
While the infinite sequence $(l_{p^e}^{\mathrm{pure}}(P))_{e \in \mathbb{Z}_{\geq 0}}$ (or its negative $(-l_{p^e}^{\mathrm{pure}}(P))_{e \in \mathbb{Z}_{\geq 0}}$) depends on the choice of W, the infinite sequence $(\dim W - l_{p^e}^{\mathrm{pure}}(P))_{e \in \mathbb{Z}_{\geq 0}}$ does not. Therefore, the latter is more appropriate as an invariant toward constructing such an algorithm.

(2) The dimension of the pure part is nondecreasing as a function of $e \in \mathbb{Z}_{\geq 0}$, and is uniformly bounded from above by $d = \dim W$, i.e.,

$$0 \leq l_{p^0}^{\mathrm{pure}}(P) \leq l_{p^1}^{\mathrm{pure}}(P) \leq \cdots \leq l_{p^{e-1}}^{\mathrm{pure}}(P) \leq l_{p^e}^{\mathrm{pure}}(P) \leq \cdots \leq d = \dim W$$

and hence stabilizes after some point, i.e., there exists $e_M \in \mathbb{Z}_{\geq 0}$ such that

$$l_{p^e}^{\mathrm{pure}}(P) = l_{p^{e_M}}^{\mathrm{pure}}(P) \quad \text{for } e \geq e_M.$$

That is, after some point, the dimension of the pure part stabilizes to a constant N $(= l^{\mathrm{pure}}_{p^{e_M}}(P)) \in \mathbb{Z}_{\geq 0}$. Therefore, although $\sigma(P)$ is an infinite sequence by definition, essentially we are only looking at some finite part of it.

We remark that the constant N above is the number of elements in an LGS (cf. 1.3.1), and that N is uniformly bounded by the dimension of the ambient space, i.e., $N \leq \dim W = d$.

(3) In characteristic zero, the invariant $\sigma(P)$ consists of only one term $d - l^{\mathrm{pure}}_{p^0}$, while the remaining terms $d - l^{\mathrm{pure}}_{p^e}$ are not defined for $e > 0$, as we set $p = \infty$ in characteristic zero. (However, we may still say $\sigma(P)$ is an infinite sequence and write $\sigma(P) \in \prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}$, for the sake of simplicity and uniformity of presentation, intentionally ignoring the particular situation in characteristic zero.) Note that $l^{\mathrm{pure}}_{p^0} = l_{p^0} = \dim_k L(\mathbb{I}_P)_1$ can be regarded as the number indicating "how many linearly independent hypersurfaces of maximal contact we can take" for $\mathbb{I}_P$ (cf. Chapter 3 in Part I).

(4) The so-called "Hironaka's invariant τ", according to the original definition (cf. [Oda87]), is associated to a standard basis. Given a fixed ideal I, we count the minimum number of (additive) elements which generate a k-algebra containing the initial ideal of I. On the other hand, our invariant σ is associated to an LGS. Given a $\mathfrak{D}$-saturated idealistic filtration $\mathbb{I}$, we determine σ by looking at its leading algebra (cf. 1.3.1). Therefore, these two invariants generally arise in two distinct settings, and hence they are different (cf. 1.3.1.1), especially in the sense that the notion of a standard basis does not appear in the definition of an LGS or of an idealistic filtration. We note, however, that, given a fixed ideal, it is possible to create an idealistic filtration, using the degrees of the initial terms as the levels, so that we recover the information on Hironaka's invariants τ and ν^* from it. For example, the minimum number of (additive) generators mentioned above coincides with the number of elements in an LGS of its $\mathfrak{D}$-saturation, and the information on the invariant ν^* is encoded in the leading algebra of the original idealistic filtration before taking its $\mathfrak{D}$-saturation. We would like to mention that neither investigating the leading algebra of a non-$\mathfrak{D}$-saturated idealistic filtration nor focusing only on the total number of elements in an LGS is at the center of the construction of our algorithm according to the IFP.

1.1.2. Computation of σ. The next lemma computes $l^{\mathrm{pure}}_{p^e}(P)$ in terms of $l_{p^e}(P)$ and in terms of $l^{\mathrm{pure}}_{p^\alpha}(P)$ for $\alpha = 0, \ldots, e-1$, which we can assume inductively have already been computed. We also see that $l_{p^e}(P)$ can be computed as the rank of a certain "Jacobian-like" matrix, and hence that it is lower semi-continuous as a function of P. This immediately leads to the lower semi-continuity of the sequence $(l^{\mathrm{pure}}_{p^e}(P))_{e \in \mathbb{Z}_{\geq 0}}$ and hence to the upper semi-continuity of $\sigma(P) = (d - l^{\mathrm{pure}}_{p^e}(P))_{e \in \mathbb{Z}_{\geq 0}}$ as a function of P. We will discuss the upper semi-continuity of σ in detail in the next section.

Lemma 1.1.2.1. *The following assertions concerning the computation of $l^{\mathrm{pure}}_{p^e}(P)$ hold.*

(1) *Assume $P \notin \mathrm{Supp}(\mathbb{I})$. Then $\mathbb{I}_P = R_P \times \mathbb{R}$, and hence $l^{\mathrm{pure}}_{p^e}(P) = d$ for any $e \in \mathbb{Z}_{\geq 0}$. Accordingly, the invariant $\sigma(P)$ takes the absolute minimum value, i.e.,*

$$\sigma(P) = (0, 0, \ldots, 0, \ldots) = (0)_{e \in \mathbb{Z}_{\geq 0}}.$$

(2) *Assume $P \in \mathrm{Supp}(\mathbb{I})$. Then $l^{\mathrm{pure}}_{p^e}(P)$ can be computed inductively as follows: Suppose we have already computed $l^{\mathrm{pure}}_{p^\alpha}(P)$ for $\alpha = 0, \ldots, e-1$. We define $l^{\mathrm{mixed}}_{p^e}(P)$ as the coefficient of the term z^{p^e} in the expansion of the polynomial $\prod_{\alpha=0}^{e-1} (\sum_{j=1}^{p-1} z^{jp^\alpha})^{l^{\mathrm{pure}}_{p^\alpha}(P)} \in \mathbb{Z}_{\geq 0}[z]$. Then*

$$l^{\mathrm{pure}}_{p^e}(P) = l_{p^e}(P) - l^{\mathrm{mixed}}_{p^e}(P).$$

Moreover, in case (2), we can compute $l_{p^e}(P)$ as follows: Let $\{s_1, \ldots, s_r\}$ be a set of generators for the ideal $\mathbb{I}_{p^e}$ of the idealistic filtration at level p^e, i.e., $(s_1, \ldots, s_r) = \mathbb{I}_{p^e} \subset R$, and $(x_1, \ldots, x_d)$ a regular system of parameters at P. Then

$$l_{p^e}(P) = \mathrm{rank}\, [\partial_{X^I}(s_t)]^{t=1,\ldots,r}_{|I|=p^e}.$$

Proof. (1) In this case, by the definition of $\mathrm{Supp}(\mathbb{I})$ (cf. Definition 2.1.1.1(6) in Part I), there exists an element $(f, a) \in \mathbb{I}_P$ with $a > 0$ such that $\mathrm{ord}_P(f) < a$. There also exists an appropriate differential operator δ of degree $\mathrm{ord}_P(f)$ such that $\delta(f) = u$ is a unit of R_P. Then we have, by the (differential) condition in Definition 2.1.2.1 in Part I,

$$(\delta(f), a - \mathrm{ord}_P(f)) = (u, a - \mathrm{ord}_P(f)) \in \mathbb{I}_P$$

and hence by condition (i) in Definition 2.1.1.1 in Part I,

$$(\mathbb{I}_P)_{a - \mathrm{ord}_P(f)} = R_P.$$

This implies by condition (ii) in Definition 2.1.1.1 in Part I that

$$(\mathbb{I}_P)_{n(a - \mathrm{ord}_P(f))} = R_P \quad \forall n \in \mathbb{Z}_{>0}.$$

We then conclude by condition (iii) in Definition 2.1.1.1 in Part I that

$$\mathbb{I}_P = R_P \times \mathbb{R}.$$

From this the assertions on $l^{\mathrm{pure}}_{p^e}(P)$ and $\sigma(P)$ easily follow, since $L(\mathbb{I}_P) = G_P$.

(2) We abbreviate $l^{\mathrm{pure}}_{p^w}(P)$ as γ_w for $w \in \mathbb{Z}_{\geq 0}$. We see by Lemma 3.1.2.1 in Part I that there exists a subset $\{\nu_\beta\,;\, 1 \leq \beta \leq \gamma_e\} \subset \mathfrak{m}/\mathfrak{m}^2$ such that $\{F^\alpha(\nu_\beta)\,;$

$1 \le \beta \le \gamma_\alpha\}$ is a k-basis of $L(\mathbb{I}_P)^{\mathrm{pure}}_{p^\alpha}$ for $0 \le \alpha \le e$. Note that $L(\mathbb{I}_P)$ is generated by $L(\mathbb{I}_P)^{\mathrm{pure}} = \bigsqcup_{e\in\mathbb{Z}_{\ge 0}} L(\mathbb{I}_P)^{\mathrm{pure}}_{p^e}$. Accordingly, the set of monomials $\Phi = \{V^{[S]}\,;\, |[S]| = p^e\}$ is a k-basis of $L(\mathbb{I}_P)_{p^e}$, where $S = (s_{\alpha\beta}\,;\, 0 \le \alpha \le e,$ $1 \le \beta \le \gamma_\alpha)$ is a multi-index, $[S] = (s_{\alpha\beta}p^\alpha\,;\, \alpha, \beta)$, and $V^T = \prod_{\alpha,\beta} \nu_\beta^{t_{\alpha\beta}}$. Since $V^{[S+p\mathbf{e}_{\alpha,\beta}]} = V^{[S+\mathbf{e}_{\alpha+1,\beta}]}$ for $\alpha < e$, we may assume that $0 \le s_{\alpha,\beta} < p$ for any $\alpha < e$. Note also that $s_{e,\beta} \le 1$ for any $0 \le \beta \le \gamma_e$, since $s_{e,\beta}p^e \le |[S]| = p^e$. Therefore, the monomials in the set Φ are the ones appearing in the homogeneous part of degree p^e of the polynomial below:

$$\Big[\Big(\prod_{0\le\alpha<e}\prod_{1\le\beta\le\gamma_\alpha}\sum_{s_{\alpha,\beta}=0}^{p-1} v^{s_{\alpha,\beta}p^\alpha}\Big)\prod_{1\le\beta\le\gamma_e}(1+v_\beta^{p^e})\Big]_{p^e}$$
$$= \underbrace{\Big[\prod_{0\le\alpha<e}\prod_{1\le\beta\le\gamma_\alpha}\sum_{s_{\alpha,\beta}=0}^{p-1} v_\beta^{s_{\alpha,\beta}p^\alpha}\Big]_{p^e}}_{\heartsuit} + \sum_{1\le\beta\le\gamma_e} v_\beta^{p^e}.$$

Observe that each monomial in the set Φ appears with coefficient 1 in the polynomial above, a fact which follows immediately when we consider the p-adic expansion of the exponent of $V^{[S]}$. Thus, in order to count the number of monomials in $\heartsuit$, we have only to set $\nu_\beta = z$ for all β's, and look at the coefficient of z^{p^e}, which is exactly $l^{\mathrm{mixed}}_{p^e}(P)$. Therefore, we conclude

$$l_{p^e}(P) = \#\Phi = l^{\mathrm{mixed}}_{p^e}(P) + l^{\mathrm{pure}}_{p^e}(P).$$

In order to prove the "moreover" part, we have only to recall that $L(\mathbb{I}_P)_{p^e}$ is generated as a k-vector space by the degree p^e terms of the power series expansions of $\{s_t\}_{t=1,\dots,r}$ with respect to a regular system of parameters $(x_1,\dots,x_d)$, i.e.,

$$L(\mathbb{I}_P)_{p^e} = \langle s_t \bmod \mathfrak{m}_P^{p^e+1}\,;\, t = 1,\dots,r\rangle = \langle s_t \bmod (x_1,\dots,x_d)^{p^e+1}\,;\, t = 1,\dots,r\rangle$$

and that their coefficients appear as the entries of the matrix given in the statement, i.e.,

$$s_t = \sum_{|I|=p^e} \partial_{X^I}(s_t)X^I \bmod (x_1,\dots,x_d)^{p^e+1}.$$

This completes the proof of Lemma 1.1.2.1.

Remark 1.1.2.2. (1) The description of $L(\mathbb{I}_P)_{p^e}$, using a specific set of generators for the leading algebra $L(\mathbb{I}_P)$ given by Lemma 3.1.2.1 in Part I, and its decomposition into the pure and mixed parts, will be discussed again in relation to the proof of Proposition 1.3.3.3.

(2) Let us consider $\zeta(P) = (l_{p^e}(P))_{e\in\mathbb{Z}_{\ge 0}}$. Then noting $l_{p^0}(P) = l^{\mathrm{pure}}_{p^0}(P)$, we conclude by Lemma 1.1.2.1 that $\sigma(P)$ determines $\zeta(P)$ and vice versa.

In particular, for $P, Q \in \mathfrak{m}\text{-Spec}\, R$, we have

$$\sigma(P) = \sigma(Q) \;\Leftrightarrow\; \zeta(P) = \zeta(Q), \qquad \sigma(P) \geq \sigma(Q) \;\Leftrightarrow\; \zeta(P) \leq \zeta(Q).$$

Therefore, the upper semi-continuity of the invariant σ, which we will show in the next section, is equivalent to the lower semi-continuity of the invariant ζ.

§1.2. Upper semi-continuity

1.2.1. Basic facts surrounding the definition of the upper semi-continuity. In this subsection, we clarify some basic facts surrounding the definition of the upper semi-continuity. We denote by $f : X \to T$ a function from a topological space X to a totally ordered set T.

Definition 1.2.1.1. We say f is *upper semi-continuous* if the set

$$X_{\geq t} := \{x \in X\, ;\, f(x) \geq t\}$$

is closed for any $t \in T$.

Lemma 1.2.1.2. *Consider the conditions below:*

(i) *For any $x \in X$, there exists an open neighborhood U_x such that $f(x) \geq f(y)$ for any $y \in U_x$.*
(ii) *The set $X_{>t} = \{x \in X\, ;\, f(x) > t\}$ is closed for any $t \in T$.*
(iii) *f is upper semi-continuous.*

Then we have the following implications:

$$\text{(i)} \Leftrightarrow \text{(ii)} \Rightarrow \text{(iii)}.$$

Moreover, if $f(X) \subset \mathcal{W} \subset T$ where $\mathcal{W}$ is well-ordered (in the sense that every nonempty subset has the least element), then conditions (ii) *and* (iii) *are equivalent.*

Proof. The proof is elementary, and left to the reader as an exercise.

Corollary 1.2.1.3. *For the invariant $\sigma : \mathfrak{m}\text{-Spec}\, R \to \prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}$, where the target space $\prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}$ is totally ordered with respect to the lexicographical order, conditions* (i), (ii), (iii) *in Lemma* 1.2.1.2 *are all equivalent.*

Proof. As mentioned in Remark 1.1.1.2(2), the dimension of the pure part, $l^{\text{pure}}_{p^e}(P)$, is nondecreasing as a function of $e \in \mathbb{Z}_{\geq 0}$. Accordingly, $\sigma(P)(e) = d - l^{\text{pure}}_{p^e}(P)$ is nonincreasing as a function of $e \in \mathbb{Z}_{\geq 0}$. Therefore, setting $f = \sigma$, $X = \mathfrak{m}\text{-Spec}\, R$ and $T = \prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}$, we see $f(X) \subset S \subset T$ where $S = \{(t_e)_{e \in \mathbb{Z}_{\geq 0}} \in T\, ;\, t_{e_1} \geq t_{e_2} \text{ if } e_1 < e_2\}$. Observe that S is well-ordered (with respect to the total order induced by the one on T). In fact, for a nonempty subset $\mathcal{U} \subset S$, we can construct its least element $u_{\min} = (u_{\min,e})_{e \in \mathbb{Z}_{\geq 0}}$ inductively by the following formula:

$$u_{\min,e} = \min\{u_e \in \mathbb{Z}_{\geq 0}\,;\, u = (u_i)_{i \in \mathbb{Z}_{\geq 0}} \in \mathcal{U} \text{ such that } u_i = u_{\min,i} \text{ for } i < e\}.$$

Now we see that the statement of the corollary follows from Lemma 1.2.1.2.

The following basic description of the stratification into level sets, in the case where the ambient space X is noetherian, can be easily seen, and its proof is left to the reader.

Corollary 1.2.1.4. *Let $f : X \to T$ be an upper semi-continuous function. Suppose that X is noetherian, and that $f(X) \subset \mathcal{W} \subset T$ where $\mathcal{W}$ is well-ordered. Then f takes only finitely many values over X, i.e.,*

$$\{f(x)\,;\, x \in X\} = \{t_1 < \cdots < t_n\} \subset T.$$

Accordingly, we have a strictly decreasing finite sequence of closed subsets

$$X = X_{\geq t_1} \supsetneq \cdots \supsetneq X_{\geq t_n} \supsetneq \emptyset,$$

which provides the stratification of X into the level sets

$$\{x \in X\,;\, f(x) = t_i\} = X_{\geq t_i} \setminus X_{\geq t_{i+1}} \quad \text{for } i = 1, \ldots, n.$$

1.2.2. Upper semi-continuity of the invariant σ

Proposition 1.2.2.1. *The invariant $\sigma : \mathfrak{m}\text{-Spec}\, R \to \prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}$ is upper semi-continuous.*

Proof. Set $X = \mathfrak{m}\text{-Spec}\, R$ and $T = \prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}$ for notational simplicity. By the definition of the upper semi-continuity (cf. 1.2.1.1), we have only to show that

$$X_{<t} = \{x \in X\,;\, \sigma(x) < t\}$$

is open for any $t \in T$. Fix $t = (t_e)_{e \in \mathbb{Z}_{\geq 0}} \in T$, and take $x \in X_{<t}$. We define

$$\alpha = \min\{e \in \mathbb{Z}_{\geq 0}\,;\, \sigma(x)(e) < t_e\},$$
$$U_e = \{y \in X\,;\, l_{p^e}(y) > l_{p^e}(x) - 1\} \quad (0 \leq e \leq \alpha).$$

Since the function l_{p^e} is lower semi-continuous (cf. Lemma 1.1.2.1), each U_e is open. Set

$$U = \bigcap_{0 \leq e \leq \alpha} U_e = \{y \in X\,;\, l_{p^e}(y) \geq l_{p^e}(x),\, 0 \leq e \leq \alpha\}.$$

Since U is obviously an open neighborhood of x, we have only to show $U \subset X_{<t}$. Assuming $U \not\subset X_{<t}$, we deduce a contradiction. Take $y \in U \setminus X_{<t}$. Observe that $\sigma(y) \geq t$. Observe also that, by the definition of α, we have $\sigma(x)(e) = t_e$ for $0 \leq e < \alpha$ and $\sigma(x)(\alpha) < t_\alpha$. Set

$$\beta = \min\{0 \leq e \leq \alpha\,;\, \sigma(y)(e) > \sigma(x)(e)\}.$$

We have $\sigma(y)(e) = \sigma(x)(e)$ for $0 \leq e < \beta$ and $\sigma(y)(\beta) > \sigma(x)(\beta)$ by definition. It then follows from the definition of σ that $l^{\text{pure}}_{p^e}(y) = l^{\text{pure}}_{p^e}(x)$ for $0 \leq e < \beta$. This implies by Lemma 1.1.2.1 that $l^{\text{mixed}}_{p^\beta}(y) = l^{\text{mixed}}_{p^\beta}(x)$. Since $y \in U \subset U_\beta$, we conclude that

$$l^{\text{pure}}_{p^\beta}(y) = l_{p^\beta}(y) - l^{\text{mixed}}_{p^\beta}(y) \geq l_{p^\beta}(x) - l^{\text{mixed}}_{p^\beta}(x) = l^{\text{pure}}_{p^\beta}(x),$$

and hence that

$$\sigma(y)(\beta) = d - l^{\text{pure}}_{p^\beta}(y) \leq d - l^{\text{pure}}_{p^\beta}(x) = \sigma(x)(\beta),$$

which is a contradiction.

This completes the proof of Proposition 1.2.2.1.

Corollary 1.2.2.2. *We can extend the domain from* $\mathfrak{m}$-Spec R *to* Spec R *to have the invariant* $\sigma : \operatorname{Spec} R \to \prod_{e\in\mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}$, *by defining*

$$\sigma(Q) = \min\{\sigma(P)\,;\, P \in \mathfrak{m}\text{-}\operatorname{Spec} R,\ P \in \overline{Q}\} \qquad \textit{for } Q \in \operatorname{Spec} R.$$

The formula is equivalent to saying that $\sigma(Q)$ *is equal to* $\sigma(P)$ *with* P *being a general closed point on* $\overline{Q}$. *The invariant* σ *with the extended domain is also upper semi-continuous.*

Moreover, since Spec R *is noetherian and since* $\sigma(\operatorname{Spec} R) \subset S$ *where* S *is the well-ordered subset of* $T = \prod_{e\in\mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}$ *as described in the proof of Corollary* 1.2.1.3, *conditions* (i) *and* (ii) *in Lemma* 1.2.1.2, *as well as the assertions of Corollary* 1.2.1.4, *hold for the upper semi-continuous function* $\sigma : \operatorname{Spec} R \to T$.

Proof. Observe that, given $Q \in \operatorname{Spec} R$, $\sigma(Q)$ is well-defined, since the existence of the minimum (i.e., the least element) on the right hand side is guaranteed by the fact that the value set of the invariant σ is well-ordered (cf. the proof of Corollary 1.2.1.3). Note that there exists a nonempty dense open subset U of $\overline{Q} \cap \mathfrak{m}$-Spec R such that $\sigma(Q) = \sigma(P)$ for $P \in U$, a fact implied by condition (i) of the upper semi-continuity of the invariant σ (cf. Corollary 1.2.1.3). The upper semi-continuity of the invariant σ with the extended domain Spec R is immediate from the upper semi-continuity of the invariant σ with the original domain $\mathfrak{m}$-Spec R.

The "moreover" part follows immediately from the statements of Lemma 1.2.1.2, Corollary 1.2.1.3 and Corollary 1.2.1.4.

This completes the proof of Corollary 1.2.2.2.

§1.3. Local behavior of a leading generator system

1.3.1. Definition of a leading generator system and a remark about the subscripts. We say that a subset $\mathbb{H} = \{(h_l, p^{e_l})\}_{l=1}^N \subset \mathbb{I}_P$, with nonnegative integers $0 \leq e_1 \leq \cdots \leq e_N$ attached, is an LGS (of the localization $\mathbb{I}_P$ of the $\mathfrak{D}$-saturated idealistic filtration $\mathbb{I}$ over R at a closed point $P \in \mathfrak{m}\text{-Spec}\, R$) if the leading terms of its elements provide a specific set of generators for the leading algebra $L(\mathbb{I}_P)$ as described in Lemma 3.1.2.1 in Part I (cf. 0.2.1.2). More precisely, it satisfies the following conditions:

(i) $h_l \in \mathfrak{m}_P^{p^{e_l}}$ and $\overline{h_l} = (h_l \bmod \mathfrak{m}_P^{p^{e_l}+1}) \in L(\mathbb{I}_P)_{p^{e_l}}^{\text{pure}}$ for $l = 1, \ldots, N$,

(ii) $\{\overline{h_l}^{p^{e-e_l}} ; e_l \leq e\}$ consists of $\#\{l ; e_l \leq e\}$ distinct elements, and forms a k-basis of $L(\mathbb{I}_P)_{p^e}^{\text{pure}}$ for any $e \in \mathbb{Z}_{\geq 0}$.

(We refer the reader to Definition 3.1.3.1 and Proposition 3.1.3.2 in Part I for the definition and existence of an LGS, respectively.)

Since the leading algebra $L(\mathbb{I}_P)$ is generated by its pure part $L(\mathbb{I}_P)^{\text{pure}} = \bigsqcup_{e \in \mathbb{Z}_{\geq 0}} L(\mathbb{I}_P)_{p^e}^{\text{pure}}$ (cf. 0.2.1.1), we conclude from condition (ii) that the leading terms of $\mathbb{H}$

$$\{\overline{h_l} = (h_l \bmod \mathfrak{m}_P^{p^{e_l}+1})\}_{l=1}^N$$

provide a set of generators for $L(\mathbb{I}_P)$, i.e., $L(\mathbb{I}_P) = k[\overline{h_1}, \ldots, \overline{h_N}]$.

We remark that, for the subscripts of an LGS $\mathbb{H}$, we sometimes use the letter "l" as above, writing $\mathbb{H} = \{(h_l, p^{e_l})\}_{l=1}^N$ with nonnegative integers $0 \leq e_1 \leq \cdots \leq e_N$ attached, and sometimes we use the letters i and j, writing $\mathbb{H} = \{(h_{ij}, p^{e_i})\}_{i=1,\ldots,M}^{j}$ with nonnegative integers $0 \leq e_1 < \cdots < e_M$ attached. In the latter use of the subscripts, conditions (i) and (ii) are written as in 3.1.3 of Part I:

(i) $h_{ij} \in \mathfrak{m}_P^{p^{e_i}}$ and $\overline{h_{ij}} = (h_{ij} \bmod \mathfrak{m}_P^{p^{e_i}+1}) \in L(\mathbb{I}_P)_{p^{e_i}}^{\text{pure}}$ for any i, j,

(ii) $\{\overline{h_{ij}}^{p^{e-e_i}} ; e_i \leq e\}$ consists of $\#\{(i,j) ; e_i \leq e\}$ distinct elements, and forms a k-basis of $L(\mathbb{I}_P)_{p^e}^{\text{pure}}$ for any $e \in \mathbb{Z}_{\geq 0}$.

In the future, we use the subscripts in both ways, while choosing one at a time, depending upon the situation and convenience.

Remark 1.3.1.1. The definition of an LGS may remind some reader of that of a standard basis: A standard basis of an ideal I consists of those elements

$\{h_\lambda\}_{\lambda\in\Lambda} \subset I$ whose initial terms generate the initial ideal $\mathrm{in}(I)$. An LGS of an idealistic filtration $\mathbb{I}$ consists of those elements $\{(h_l, p^{e_l})\}_{l=1}^N \subset \mathbb{I}$ whose leading terms generate the leading algebra $L(\mathbb{I})$. While they look similar (and are closely related in some aspects), they are different objects: a standard basis deals with a fixed ideal, while an LGS deals with a collection of ideals $\{\mathbb{I}_a\}_{a\in\mathbb{R}}$ indexed by the levels $a \in \mathbb{R}$. Accordingly, we find most of the arguments on the properties of an LGS, including the one on Proposition 1.3.3.3 below, unique to our situation and not directly derived as consequences of the classical results about a standard basis.

1.3.2. A basic question. Let $\mathbb{H}$ be an LGS of $\mathbb{I}_P$. If we take a neighborhood U_P of P small enough, then $\mathbb{H}$ is a subset of $\mathbb{I}_Q$ for any closed point $Q \in U_P \cap \mathfrak{m}\text{-Spec}\, R$. We may then ask the following question regarding the local behavior of the LGS:

> Is $\mathbb{H}$ an LGS of $\mathbb{I}_Q$?

A moment's thought reveals that the answer to this question is in general no. In fact, due to the upper semi-continuity of the invariant σ, by shrinking U_P if necessary, we may assume $\sigma(P) \geq \sigma(Q)$ for any closed point $Q \in U_P \cap \mathfrak{m}\text{-Spec}\, R$. If $\sigma(P) > \sigma(Q)$, then there is no way that $\mathbb{H}$ could be an LGS of $\mathbb{I}_Q$. (Note that the invariant σ is completely determined by the LGS.)

We refine our question to avoid the obvious calamity as above:

> Is $\mathbb{H}$ an LGS of $\mathbb{I}_Q$ for any closed point $Q \in C \cap \mathfrak{m}\text{-Spec}\, R \subset U_P \cap \mathfrak{m}\text{-Spec}\, R$ where $C = \{Q \in U_P\, ;\, \sigma(P) = \sigma(Q)\}$?

The answer to this question, for an arbitrary LGS $\mathbb{H}$ of $\mathbb{I}_P$, is still no. One of the conditions for $\mathbb{H}$ to be an LGS of $\mathbb{I}_P$ requires any element $(h_{ij}, p^{e_i}) \in \mathbb{H}$ to be pure at P, i.e., $(h_{ij} \bmod \mathfrak{m}_P^{p^{e_i}+1}) \in L(\mathbb{I}_P)^{\mathrm{pure}}_{p^{e_i}}$. However, even when a closed point $Q \in U_P \cap \mathfrak{m}\text{-Spec}\, R$ satisfies the condition $Q \in C \cap \mathfrak{m}\text{-Spec}\, R$, some element (h_{ij}, p^{e_i}) may fail to be pure at Q, i.e., $(h_{ij} \bmod \mathfrak{m}_Q^{p^{e_i}+1}) \notin L(\mathbb{I}_Q)^{\mathrm{pure}}_{p^{e_i}}$, and hence $\mathbb{H}$ fails to be an LGS at Q.

Now we refine our question further:

> Can we modify a given LGS $\mathbb{H}$ of $\mathbb{I}_P$ into $\mathbb{H}'$ so that $\mathbb{H}'$ remains an LGS of $\mathbb{I}_Q$ for any closed point $Q \in C \cap \mathfrak{m}\text{-Spec}\, R \subset U_P \cap \mathfrak{m}\text{-Spec}\, R$ where $C = \{Q \in U_P\, ;\, \sigma(P) = \sigma(Q)\}$?

The main goal of the next subsection is to give an affirmative answer to this last question (adding one extra condition of the point Q being in the support $\mathrm{Supp}(\mathbb{I})$ of the idealistic filtration), and also to give an explicit description of how we make the modification. We say we modify the given LGS into one which is "uniformly pure" (along C intersected with $\mathrm{Supp}(\mathbb{I})$).

1.3.3. Modification of a given leading generator system into one which is uniformly pure

Definition 1.3.3.1. Let $\mathbb{H}$ be an LGS of the localization $\mathbb{I}_P$ of the $\mathfrak{D}$-saturated idealistic filtration $\mathbb{I}$ over R at a closed point $P \in \mathfrak{m}\text{-Spec}\, R$. We say $\mathbb{H}$ is *uniformly pure* (in a neighborhood U_P of P along the local maximum locus C of the invariant σ intersected with the support $\mathrm{Supp}(\mathbb{I})$ of the idealistic filtration) if there exists an open neighborhood U_P of P such that the following conditions are satisfied:

(1) $\mathbb{H} \subset \mathbb{I}_Q$ for all $Q \in U_P$,

(2) $\sigma(P)$ is the maximum of the invariant σ over U_P, i.e., $\sigma(P) \geq \sigma(Q)$ for all $Q \in U_P$,

(3) $C = \{Q \in U_P\,;\, \sigma(P) = \sigma(Q)\}$ is a closed subset of U_P,

(4) $\mathbb{H}$ is an LGS of $\mathbb{I}_Q$ for any $Q \in C \cap \mathrm{Supp}(\mathbb{I}) \cap \mathfrak{m}\text{-Spec}\, R$.

(For the definition of $\mathrm{Supp}(\mathbb{I})$, we refer the reader to Definition 2.1.1.1(6) in Part I.)

Remark 1.3.3.2. We remark that in condition (4) of Definition 1.3.3.1, in order for $\mathbb{H}$ to be uniformly pure, we require $\mathbb{H}$ is an LGS of $\mathbb{I}_Q$ for any closed point "$Q \in C \cap \mathrm{Supp}(\mathbb{I}) \cap \mathfrak{m}\text{-Spec}\, R$" (i.e., we only consider those closed points in the support $\mathrm{Supp}(\mathbb{I})$ of the idealistic filtration $\mathbb{I}$), whereas in the last form of the basic question in 1.3.2 we merely wrote "$Q \in C \cap \mathfrak{m}\text{-Spec}\, R$". The reason for adding this extra condition on Q to be in $\mathrm{Supp}(\mathbb{I})$ (as mentioned in the last paragraph of 1.3.2) is as follows:

(i) Consider the case where $\sigma(P) = (0, 0, \ldots, 0, \ldots) = (0)_{e \in \mathbb{Z}_{\geq 0}}$. (Recall that $(0)_{e \in \mathbb{Z}_{\geq 0}}$ is the absolute minimum in the value set of the invariant σ (cf. Lemma 1.1.2.1(1)). By the upper semi-continuity of the invariant σ, for a sufficiently small open neighborhood U_P of P, we have $\sigma(Q) = \sigma(P) = (0)_{e \in \mathbb{Z}_{\geq 0}}$ for any closed point $Q \in U_P \cap \mathfrak{m}\text{-Spec}\, R$ and hence $C \cap \mathfrak{m}\text{-Spec}\, R = U_P \cap \mathfrak{m}\text{-Spec}\, R$. On the other hand, the condition $\sigma(P) = (0)_{e \in \mathbb{Z}_{\geq 0}}$ implies that, given any LGS $\mathbb{H}$ of $\mathbb{I}_P$, the elements $\{h_{ij}\}$ are generators of the maximal ideal $\mathfrak{m}_P$ with $\#\{(i,j)\} = d$. (Note that, in this case, all the elements of a leading generator system are concentrated at level 1, i.e., $1 = i = M$ and $0 = e_1 = e_i = e_M$.) Therefore, $\mathbb{H}$ cannot be an LGS of $\mathbb{I}_Q$ for a closed point $Q \in U_P \cap \mathfrak{m}\text{-Spec}\, R$ if $Q \neq P$. That is, it would not satisfy the condition described in the last form of the basic question. However, in this case, we have either $U_P \cap \mathrm{Supp}(\mathbb{I}) = \emptyset$ or $U_P \cap \mathrm{Supp}(\mathbb{I}) = \{P\}$ (if we take U_P sufficiently small). Therefore, condition (4) in Definition 1.3.3.1 is automatically satisfied.

(ii) Consider the case where $\sigma(P) \neq (0)_{e \in \mathbb{Z}_{\geq 0}}$. In this case, $C \cap \mathfrak{m}\text{-Spec}\, R = C \cap \mathrm{Supp}(\mathbb{I}) \cap \mathfrak{m}\text{-Spec}\, R$, since any closed point $Q \in C \cap \mathfrak{m}\text{-Spec}\, R$ (i.e., $\sigma(Q) = \sigma(P) \neq (0)_{e \in \mathbb{Z}_{\geq 0}}$) is necessarily in $\mathrm{Supp}(\mathbb{I})$ (cf. Lemma 1.1.2.1(1)). Therefore, there is no difference between the condition in the last form of the basic question and condition (4) in Definition 1.3.3.1.

In other words, the extra condition on Q to be in $\mathrm{Supp}(\mathbb{I})$ is introduced so that we can avoid the "obvious" counterexample to an affirmative answer to the last form of the basic question in the special case $\sigma(P) = (0)_{e \in \mathbb{Z}_{\geq 0}}$.

Before giving the statement of Proposition 1.3.3.3 and its proof, we recall that Lemma 3.1.2.1 in Part I gives the description of a specific set of generators for the leading algebra $L(\mathbb{I}_P)$:

We can choose $\{e_1 < \cdots < e_M\} \subset \mathbb{Z}_{\geq 0}$ and $V_1 \sqcup \cdots \sqcup V_M \subset G_1 = \mathfrak{m}_P^1/\mathfrak{m}_P^2$ with $V_i = \{v_{ij}\}_j$ satisfying the following conditions:

(i) $F^{e_i}(V_i) \subset L(\mathbb{I}_P)^{\mathrm{pure}}_{p^{e_i}}$ for $1 \leq i \leq M$,

(ii) $\bigsqcup_{e_i \leq e} F^e(V_i)$ is a k-basis of $L(\mathbb{I}_P)^{\mathrm{pure}}_{p^e}$ for any $e \in \mathbb{Z}_{\geq 0}$.

Since $L(\mathbb{I}_P)^{\mathrm{pure}}$ generates $L(\mathbb{I}_P)$, we have $L(\mathbb{I}_P) = k[\bigsqcup_{i=1}^M F^{e_i}(V_i)]$.

Using these notations, we can take the following k-bases of $L_{p^e}(P)$ and $L^{\mathrm{pure}}_{p^e}(P)$ consisting of monomials:

$$\Big\{ \prod_{e_\alpha \leq e} (v_{\alpha\beta}^{p^{e_\alpha}})^{b_{\alpha\beta}} \,;\, \sum_{\alpha,\beta} p^{e_\alpha} b_{\alpha\beta} = p^e \Big\} : \quad \text{a } k\text{-basis of } L_{p^e}(\mathbb{I}_P),$$

$$\{ (v_{\alpha\beta}^{p^{e_\alpha}})^{b_{\alpha\beta}} \,;\, e_\alpha \leq e,\, p^{e_\alpha} b_{\alpha\beta} = p^e \} : \quad \text{a } k\text{-basis of } L^{\mathrm{pure}}_{p^e}(\mathbb{I}_P).$$

Though not canonical, we define $L^{\mathrm{mixed}}_{p^e}(\mathbb{I}_P)$ to be the k-vector space spanned by the monomials which are not pure:

$$\Big\{ \prod_{e_\alpha \leq e} (v_{\alpha\beta}^{p^{e_\alpha}})^{b_{\alpha\beta}} \,;\, \sum_{\alpha,\beta} p^{e_\alpha} b_{\alpha\beta} = p^e, \text{ and } p^{e_\alpha} b_{\alpha\beta} \neq p^e \ \forall \alpha, \beta \Big\} : \qquad \text{a } k\text{-basis of } L^{\mathrm{mixed}}_{p^e}(\mathbb{I}_P).$$

Let $\mathbb{H} = \{(h_{ij}, p^{e_i})\}^j_{i=1,\ldots,M}$ be an LGS such that the leading terms of its elements correspond to the specific generators of $L(\mathbb{I}_P)$ mentioned above, i.e.,

$$\overline{h_{ij}} = F^{e_i}(v_{ij}) \qquad \forall i, j.$$

We define $\mathrm{Mix}_{\mathbb{H},i}$ to be the set of indices B that give rise to the monomials of the leading terms of $\mathbb{H}$ in the mixed part $L(\mathbb{I}_P)^{\mathrm{mixed}}_{p^{e_i}}$, i.e.,

$$\mathrm{Mix}_{\mathbb{H},i} = \Big\{ B = (b_{\alpha\beta}) \in (\mathbb{Z}_{\geq 0})^{\#\mathbb{H}} \,;\, |[B]| = \sum_{\alpha,\beta} p^{e_\alpha} b_{\alpha\beta} = p^{e_i}, \text{ and } p^{e_\alpha} b_{\alpha\beta} \neq p^{e_i} \ \forall \alpha, \beta \Big\}.$$

Now we are ready to state and prove Proposition 1.3.3.3.

Proposition 1.3.3.3. *Let $\mathbb{H} = \{(h_{ij}, p^{e_i})\}^j_{i=1,\dots,M}$ be an LGS of the localization $\mathbb{I}_P$ of the $\mathfrak{D}$-saturated idealistic filtration $\mathbb{I}$ over R at a closed point $P \in \mathfrak{m}\text{-Spec}\, R$, with nonnegative integers $0 \le e_1 < \cdots < e_M$ attached. Then $\mathbb{H}$ can be modified into another LGS $\mathbb{H}'$ which is uniformly pure.*

More precisely, there exists $\{g_{ijB}\} \subset \mathfrak{m}_P$ where the subscript B ranges over the set $\mathrm{Mix}_{\mathbb{H},i}$ such that, setting $h'_{ij} = h_{ij} - \sum g_{ijB} H^B$, the modified set $\mathbb{H}' = \{(h'_{ij}, p^{e_i})\}^j_{i=1,\dots,M}$ is an LGS of $\mathbb{I}_P$ which is uniformly pure.

Proof. It suffices to prove that there exists an affine open neighborhood $U_P = \mathrm{Spec}\, R_f$ of P, where R_f is the localization of R at an element $f \in R$, such that the following conditions are satisfied:

(1) $\mathbb{H} \subset \mathbb{I}_f$ (and hence $\mathbb{H}' \subset \mathbb{I}_f$ where $\mathbb{H}'$ is described in condition (4) below),

(2) $\sigma(P)$ is the maximum of σ over U_P, i.e., $\sigma(P) \ge \sigma(Q)$ for all $Q \in U_P$,

(3) $C = \{Q \in U_P\,;\, \sigma(P) = \sigma(Q)\}$ is a closed subset of U_P,

(4) there exists $\{g_{ijB}\} \subset R_f$ where the subscript B ranges over the set $\mathrm{Mix}_{\mathbb{H},i}$ such that $\{g_{ijB}\} \subset \mathfrak{m}_P$ and that, setting $h'_{ij} = h_{ij} - \sum g_{ijB} H^B$, the modified set $\mathbb{H}' = \{(h'_{ij}, p^{e_i})\}^j_{i=1,\dots,M}$ is an LGS of $\mathbb{I}_Q$ for any $Q \in C \cap \mathrm{Supp}(\mathbb{I}) \cap \mathfrak{m}\text{-Spec}\, R$.

Step 1. *Check of conditions* (1), (2) *and* (3).

It is easy to choose an affine open neighborhood $U_P = \mathrm{Spec}\, R_f$ of P satisfying condition (1). By the upper semi-continuity of the invariant σ, we may also assume condition (2) is satisfied (cf. condition (i) in Lemma 1.2.1.2 and Corollary 1.2.1.3). Then condition (3) automatically follows, since $C = U_P \cap (\mathrm{Spec}\, R)_{\ge \sigma(P)}$ is closed (cf. Definition 1.2.1.1).

We remark that in terms of the invariant ζ (cf. Remark 1.1.2.2(2)) conditions (2) and (3) are equivalent to the following:

$(2)_\zeta$ $\zeta(P)$ is the minimum of ζ over U_P, i.e., $\zeta(P) \le \zeta(Q)$ for all $Q \in U_P$,

$(3)_\zeta$ $C = \{Q \in U_P\,;\, \zeta(P) = \zeta(Q)\}$.

Now we have only to check, by shrinking U_P if necessary, that condition (4) is also satisfied.

Step 2. *Preliminary analysis to check condition* (4).

First consider the idealistic filtration $\mathbb{J} = G_{R_f}(\mathbb{H})$ generated by $\mathbb{H}$ over R_f. Note that $\mathbb{J} \subset \mathbb{I}_f$ but that $\mathbb{J}$ may not be $\mathfrak{D}$-saturated. In order to distinguish the invariant ζ for $\mathbb{I}$ (or equivalently for $\mathbb{I}_f$ over U_P) from the invariant ζ for $\mathbb{J}$, we denote them by $\zeta_{\mathbb{I}}$ and $\zeta_{\mathbb{J}}$, respectively.

Since $\zeta_{\mathbb{J}}$ is lower semi-continuous, by shrinking U_P if necessary we may assume

$(2)_{\zeta_{\mathbb{J}}}$ $\zeta_{\mathbb{J}}(P)$ is the minimum of $\zeta_{\mathbb{J}}$ over U_P, i.e., $\zeta_{\mathbb{J}}(P) \le \zeta_{\mathbb{J}}(Q)$ for all $Q \in U_P$.

For any closed point $Q \in C \cap \mathrm{Supp}(\mathbb{I}) \cap \mathfrak{m}\text{-}\mathrm{Spec}\, R$, we compute

$$\zeta_{\mathbb{I}}(P) = \zeta_{\mathbb{J}}(P) \le \zeta_{\mathbb{J}}(Q) \le \zeta_{\mathbb{I}}(Q) = \zeta_{\mathbb{I}}(P).$$

We remark that the first equality is a consequence of the fact that the set $\{\overline{h_{ij,P}} = (h_{ij} \bmod \mathfrak{m}_P^{p^{e_i}+1})\}_{i=1,\dots,M}^{j}$ generates both $L(\mathbb{I}_P)$ and $L(\mathbb{J}_P)$ as k-algebras, the second inequality is a consequence of $(2)_{\zeta_{\mathbb{J}}}$, the third inequality is a consequence of the inclusion $\mathbb{J} \subset \mathbb{I}_f$, and the last equality follows from the definition of the closed subset C.

Therefore, we see that

$$\zeta_{\mathbb{I}}(P) = \zeta_{\mathbb{J}}(P) = \zeta_{\mathbb{J}}(Q) = \zeta_{\mathbb{I}}(Q) \qquad \forall Q \in C \cap \mathrm{Supp}(\mathbb{I}) \cap \mathfrak{m}\text{-}\mathrm{Spec}\, R.$$

Step 3. *Some consequences of the equality $\zeta_{\mathbb{I}}(P) = \zeta_{\mathbb{J}}(P) = \zeta_{\mathbb{J}}(Q) = \zeta_{\mathbb{I}}(Q)$ for any $Q \in C \cap \mathrm{Supp}(\mathbb{I}) \cap \mathfrak{m}\text{-}\mathrm{Spec}\, R$.*

The equality obtained at the end of Step 2 leads to a few conclusions that we list below:

(a) The set $\{\overline{h_{ij,Q}} = (h_{ij} \bmod \mathfrak{m}_Q^{p^{e_i}+1})\}_{i=1,\dots,M}^{j}$ generates $L(\mathbb{I}_Q)$ as a k-algebra for any $Q \in C \cap \mathrm{Supp}(\mathbb{I}) \cap \mathfrak{m}\text{-}\mathrm{Spec}\, R$. Moreover

$$\{\overline{H_Q}^B\,;\, B = (b_{ij}),\ |[B]| = p^e,\ \text{and } b_{ij} = 0 \text{ if } e_i > e\}$$

forms a basis of $L(\mathbb{I}_Q)_{p^e}$ as a k-vector space, since it obviously generates $L(\mathbb{I}_Q)_{p^e}$ and since

$$\begin{aligned} \#\{\overline{H_Q}^B\,;\, B = (b_{ij}),\ |[B]| = p^e, \text{ and } b_{ij} = 0 \text{ if } e_i > e\} \\ = \#\{\overline{H_P}^B\,;\, B = (b_{ij}),\ |[B]| = p^e, \text{ and } b_{ij} = 0 \text{ if } e_i > e\} \\ = l_{p^e}(P) = l_{p^e}(Q) = \dim_k L(\mathbb{I}_Q)_{p^e}. \end{aligned}$$

(b) There exist nonnegative integers $0 \le e_1 < \cdots < e_M$, independent of $Q \in C \cap \mathrm{Supp}(\mathbb{I}) \cap \mathfrak{m}\text{-}\mathrm{Spec}\, R$, such that a jump of the dimension of the pure part only occurs at these numbers, i.e.,

$$\begin{aligned} 0 = l_{p^0}^{\mathrm{pure}}(Q) &= \cdots = l_{p^{e_1-1}}^{\mathrm{pure}}(Q) \\ < l_{p^{e_1}}^{\mathrm{pure}}(Q) &= \cdots = l_{p^{e_2-1}}^{\mathrm{pure}}(Q) \\ &\vdots \\ < l_{p^{e_M}}^{\mathrm{pure}}(Q) &= \cdots, \end{aligned}$$

as $l_{p^e}^{\mathrm{pure}}(Q) = l_{p^e}^{\mathrm{pure}}(P)$ for any $Q \in C \cap \mathrm{Supp}(\mathbb{I}) \cap \mathfrak{m}\text{-}\mathrm{Spec}\, R$ and $e \in \mathbb{Z}_{\ge 0}$ (cf. Lemma 1.1.2.1).

Applying Lemma 3.1.2.1 in Part I to $L(\mathbb{I}_Q)$ for $Q \in C \cap \mathrm{Supp}(\mathbb{I}) \cap \mathfrak{m}\text{-Spec}\, R$, we see that we can take $V_{1,Q} \sqcup \cdots \sqcup V_{M,Q} \subset G_{1,Q} = \mathfrak{m}_Q^1/\mathfrak{m}_Q^2$ with $V_{i,Q} = \{v_{ij,Q}\}_j$, where $1 \le j \le l_{p^{e_i}}^{\mathrm{pure}}(Q) - l_{p^{e_{i-1}}}^{\mathrm{pure}}(Q)$, satisfying the following conditions:

(i) $F^{e_i}(V_{i,Q}) \subset L(\mathbb{I}_Q)_{p^{e_i}}^{\mathrm{pure}}$ for $1 \le i \le M$,

(ii) $\bigsqcup_{e_i \le e} F^e(V_{i,Q})$ is a k-basis of $L(\mathbb{I}_Q)_{p^e}^{\mathrm{pure}}$ for any $e \in \mathbb{Z}_{\ge 0}$.

Since $L(\mathbb{I}_Q)^{\mathrm{pure}}$ generates $L(\mathbb{I}_Q)$, we have $L(\mathbb{I}_Q) = k[\bigsqcup_{i=1}^M F^{e_i}(V_{i,Q})]$.

Using this information, we also deduce the following.

(c) The $\overline{h_{ij,Q}}$ are all pure when $i = 1$, i.e., $\overline{h_{1j,Q}} \in L(\mathbb{I}_Q)_{p^{e_1}}^{\mathrm{pure}}$, and we take $v'_{1j,Q} \in G_{1,Q}$ so that $F^{e_1}(v'_{1j,Q}) = \overline{h_{1j,Q}}$ for $j = 1, \ldots, l_{p^{e_1}}^{\mathrm{pure}}(Q)$. As can be seen by induction on $i = 1, \ldots, M$, for each (i,j) there exists a *unique* set $\{\kappa_{ijB,Q}\}_{B \in \mathrm{Mix}_{\mathbb{H},i}} \subset k$ such that $\overline{h_{ij,Q}} - \sum_{B \in \mathrm{Mix}_{\mathbb{H},i}} \kappa_{ijB,Q} \overline{H_Q}^B$ is pure, i.e., $\overline{h_{ij,Q}} - \sum_{B \in \mathrm{Mix}_{\mathbb{H},i}} \kappa_{ijB,Q} \overline{H_Q}^B \in L(\mathbb{I}_Q)_{p^{e_i}}^{\mathrm{pure}}$. We take $v'_{ij,Q} \in G_{1,Q}$ such that $F^{e_i}(v'_{ij,Q}) = \overline{h_{ij,Q}} - \sum_{B \in \mathrm{Mix}_{\mathbb{H},i}} \kappa_{ijB,Q} \overline{H_Q}^B$. Setting $V'_{i,Q} = \{v'_{ij,Q}\}_j$, we see that we can replace $V_{1,Q} \sqcup \cdots \sqcup V_{M,Q}$ with $V'_{1,Q} \sqcup \cdots \sqcup V'_{M,Q}$, i.e.,

- $F^{e_i}(V'_{i,Q}) \subset L(\mathbb{I}_Q)_{p^{e_i}}^{\mathrm{pure}}$ for $1 \le i \le M$,
- $\bigsqcup_{e_i \le e} F^e(V'_{i,Q})$ is a k-basis of $L(\mathbb{I}_Q)_{p^e}^{\mathrm{pure}}$ for any $e \in \mathbb{Z}_{\ge 0}$.

We also have $L(\mathbb{I}_Q) = k[\bigsqcup_{i=1}^M F^{e_i}(V'_{i,Q})]$.

In fact, we prove conclusion (c) below, claiming the existence and uniqueness of $\{\kappa_{ijB,Q}\} \subset k$ as described above, showing simultaneously by induction on i that we can replace $V_{1,Q} \sqcup \cdots \sqcup V_{M,Q}$ with $V'_{1,Q} \sqcup \cdots \sqcup V'_{i,Q} \sqcup V_{i+1,Q} \sqcup \cdots \sqcup V_{M,Q}$ in the assertions of Lemma 3.1.2.1 in Part I, and hence ultimately with $V'_{1,Q} \sqcup \cdots \sqcup V'_{M,Q}$.

(Existence) By induction hypothesis, we may replace $V_{1,Q} \sqcup \cdots \sqcup V_{M,Q}$ with $V'_{1,Q} \sqcup \cdots \sqcup V'_{i-1,Q} \sqcup V_{i,Q} \sqcup \cdots \sqcup V_{M,Q}$ in the assertions of Lemma 3.1.2.1 in Part I. Expressing $\overline{h_{ij,Q}}$ as a degree p^{e_i} homogeneous polynomial in terms of $F^{e_1}(V'_{1,Q}) \sqcup \cdots \sqcup F^{e_{i-1}}(V'_{i-1,Q}) \sqcup F^{e_i}(V_{i,Q})$, we see that there exists $\{\tau_{ijB,Q}\}_{B \in \mathrm{Mix}_{\mathbb{H},i}} \subset k$ such that $\overline{h_{ij,Q}} - \sum_{B \in \mathrm{Mix}_{\mathbb{H},i}} \tau_{ijB,Q} F^*(V'_Q)^B \in L(\mathbb{I}_Q)_{p^{e_i}}^{\mathrm{pure}}$, where $F^*(V'_Q) = (F^{e_\alpha}(v'_{\alpha\beta,Q}))$. Note that, although $v'_{\alpha\beta,Q}$ has yet to be defined if $\alpha \ge i$, since $b_{\alpha\beta} = 0$ if $\alpha \ge i$ for $B = (b_{\alpha\beta}) \in \mathrm{Mix}_{\mathbb{H},i}$, the expression $\overline{h_{ij,Q}} - \sum_{B \in \mathrm{Mix}_{\mathbb{H},i}} \tau_{ijB,Q} F^*(V'_Q)^B$ is well-defined. By substituting

$$F^{e_\alpha}(v'_{\alpha\beta,Q}) = \overline{h_{\alpha\beta,Q}} - \sum_{B \in \mathrm{Mix}_{\mathbb{H},\alpha}} \kappa_{\alpha\beta B,Q} \overline{H_Q}^B \quad \text{for } \alpha < i,$$

we see that there exists $\{\kappa_{ijB,Q}\}_{B \in \mathrm{Mix}_{\mathbb{H},i}} \subset k$ such that

$$\overline{h_{ij,Q}} - \sum_{B \in \mathrm{Mix}_{\mathbb{H},i}} \kappa_{ijB,Q} \overline{H_Q}^B \in L(\mathbb{I}_Q)_{p^{e_i}}^{\mathrm{pure}}.$$

We remark that the set

$$\Big\{F^{e_i-e_\alpha}\Big(\overline{h_{\alpha\beta,Q}}-\sum_{B\in\mathrm{Mix}_{\mathbb{H},\alpha}}\kappa_{\alpha\beta B,Q}\overline{H_Q}^B\Big)\Big\}_{\alpha=1,\dots,i,\,\beta=1,\dots,l^{\mathrm{pure}}_{p^{e_\alpha}}(Q)-l^{\mathrm{pure}}_{p^{e_{\alpha-1}}}(Q)} \subset L(\mathbb{I}_Q)^{\mathrm{pure}}_{p^{e_i}}$$

is linearly independent, since

$$\{\overline{H_Q}^B\,;\, B=(b_{\alpha\beta}),\ |[B]|=p^{e_i},\text{ and } b_{\alpha\beta}=0 \text{ if } e_\alpha>e_i\}$$

is linearly independent (cf. conclusion (a) above), and that its cardinality $\sum_{\alpha=1}^{i}(l^{\mathrm{pure}}_{p^{e_\alpha}}(Q)-l^{\mathrm{pure}}_{p^{e_{\alpha-1}}}(Q))$ is equal to $l^{\mathrm{pure}}_{p^{e_i}}(Q)$. Therefore, the above set forms a basis of $L(\mathbb{I}_Q)^{\mathrm{pure}}_{p^{e_i}}$.

(Uniqueness) Suppose there exists another set $\{\kappa'_{ijB,Q}\}_{B\in\mathrm{Mix}_{\mathbb{H},i}}\subset k$ such that $\overline{h_{ij,Q}}-\sum_{B\in\mathrm{Mix}_{\mathbb{H},i}}\kappa'_{ijB,Q}\overline{H_Q}^B\in L(\mathbb{I}_Q)^{\mathrm{pure}}_{p^{e_i}}$. Then

$$\sum_{B\in\mathrm{Mix}_{\mathbb{H},i}}\kappa_{ijB,Q}\overline{H_Q}^B-\sum_{B\in\mathrm{Mix}_{\mathbb{H},i}}\kappa'_{ijB,Q}\overline{H_Q}^B = \sum_{B\in\mathrm{Mix}_{\mathbb{H},i}}(\kappa_{ijB,Q}-\kappa'_{ijB,Q})\overline{H_Q}^B\in L(\mathbb{I}_Q)^{\mathrm{pure}}_{p^{e_i}}.$$

From the conclusion at the end of the argument for "Existence" it follows that there exists

$$\{\gamma_{\alpha\beta}\}_{\alpha=1,\dots,i,\,\beta=1,\dots,l^{\mathrm{pure}}_{p^{e_\alpha}}(Q)-l^{\mathrm{pure}}_{p^{e_{\alpha-1}}}(Q)}\subset k$$

such that

$$\sum_{B\in\mathrm{Mix}_{\mathbb{H},i}}(\kappa_{ijB,Q}-\kappa'_{ijB,Q})\overline{H_Q}^B = \sum_{\substack{1\le\alpha\le i\\ 1\le\beta\le l^{\mathrm{pure}}_{p^{e_\alpha}}(Q)-l^{\mathrm{pure}}_{p^{e_{\alpha-1}}}(Q)}}\gamma_{\alpha\beta}F^{e_i-e_\alpha}\Big(\overline{h_{\alpha\beta,Q}}-\sum_{B\in\mathrm{Mix}_{\mathbb{H},\alpha}}\kappa_{\alpha\beta B,Q}\overline{H_Q}^B\Big).$$

Again since $\{\overline{H_Q}^B\,;\,B=(b_{\alpha\beta}),\ |[B]|=p^{e_i},\text{ and } b_{\alpha\beta}=0\text{ if } e_\alpha>e_i\}$ is linearly independent, we conclude that $\gamma_{\alpha\beta}=0$ for all α,β and hence that

$$\kappa_{ijB,Q}-\kappa'_{ijB,Q}=0\quad \forall B\in\mathrm{Mix}_{\mathbb{H},i}\,.$$

This finishes the proof of the uniqueness.

Now take $v'_{ij,Q}\in G_{1,Q}$ such that $F^{e_i}(v'_{ij,Q})=\overline{h_{ij,Q}}-\sum_{B\in\mathrm{Mix}_{\mathbb{H},i}}\kappa_{ijB,Q}\overline{H_Q}^B$. Setting $V'_{i,Q}=\{v'_{ij,Q}\}_j$, we see that we can replace $V_{1,Q}\sqcup\cdots\sqcup V_{M,Q}$ with $V'_{1,Q}\sqcup\cdots\sqcup V'_{i,Q}\sqcup V_{i+1,Q}\sqcup\cdots\sqcup V_{M,Q}$ in the assertions of Lemma 3.1.2.1 in Part I.

This completes the proof for conclusion (c) by induction on i.

Step 4. *Finishing argument to check condition* (4).

In order to check condition (4), it suffices to show that there exists $\{g_{ijB}\}_{B\in\mathrm{Mix}_{\mathbb{H},i}} \subset R_f$ such that

$$g_{ijB}(Q) = \kappa_{ijB,Q} \quad \forall Q \in C \cap \mathrm{Supp}(\mathbb{I}) \cap \mathfrak{m}\text{-}\mathrm{Spec}\, R.$$

Fix a regular system of parameters $(x_1, \ldots, x_d)$ at P. By shrinking U_P if necessary, we may assume that $(x_1, \ldots, x_d)$ is a regular system of parameters over U_P, i.e., $(x_1 - x_1(Q), \ldots, x_d - x_d(Q))$ is a regular system of parameters at Q for any $Q \in U_P \cap \mathfrak{m}\text{-}\mathrm{Spec}\, R$.

Now we analyze the condition of $\overline{h_{ij,Q}} - \sum_{B\in\mathrm{Mix}_{\mathbb{H},i}} \kappa_{ijB,Q}\overline{H_Q}^B$ being pure, i.e.,

$$(\heartsuit) \qquad \overline{h_{ij,Q}} - \sum_{B\in\mathrm{Mix}_{\mathbb{H},i}} \kappa_{ijB,Q}\overline{H_Q}^B \in L(\mathbb{I}_Q)^{\mathrm{pure}}_{p^{e_i}}.$$

This happens if and only if, when we compute the power series expansions of h_{ij} and $\sum_{B\in\mathrm{Mix}_{\mathbb{H},i}} \kappa_{ijB,Q}H_Q^B$ with respect to the regular system of parameters $(x_1 - x_1(Q), \ldots, x_d - x_d(Q))$ and when we compare the degree p^{e_i} terms, their mixed parts coincide (even though their pure parts may well not coincide). Since the coefficients of (the mixed parts of) the power series can be computed using the partial derivatives with respect to $X = (x_1, \ldots, x_d)$, we conclude that condition ($\heartsuit$) is equivalent to the linear equation

$$(\heartsuit\heartsuit) \qquad [\partial_{X^I} H^B(Q)]^{B\in\mathrm{Mix}_{\mathbb{H},i}}_{I\in\mathrm{Mix}_{X,i}}[\kappa_{ijB,Q}]_{B\in\mathrm{Mix}_{\mathbb{H},i}} = [\partial_{X^I} h_{ij}(Q)]_{I\in\mathrm{Mix}_{X,i}},$$

where

$$\mathrm{Mix}_{X,i} = \{I = (i_1, \ldots, i_d)\,;\, |I| = p^{e_i},\, i_l \neq p^{e_i}\ \forall l = 1, \ldots, d\}$$

and where

$$\begin{array}{rl} [\partial_{X^I} H^B(Q)]^{B\in\mathrm{Mix}_{\mathbb{H},i}}_{I\in\mathrm{Mix}_{X,i}} & \text{is a matrix of size } (\#\,\mathrm{Mix}_{X,i}) \times (\#\,\mathrm{Mix}_{\mathbb{H},i}), \\ [\kappa_{ijB,Q}]_{B\in\mathrm{Mix}_{\mathbb{H},i}} & \text{is a matrix of size } (\#\,\mathrm{Mix}_{\mathbb{H},i}) \times 1, \\ [\partial_{X^I} h_{ij}(Q)]_{I\in\mathrm{Mix}_{X,i}} & \text{is a matrix of size } (\#\,\mathrm{Mix}_{X,i}) \times 1. \end{array}$$

In particular, at the closed point P, we have the linear equation

$$[\partial_{X^I} H^B(P)]^{B\in\mathrm{Mix}_{\mathbb{H},i}}_{I\in\mathrm{Mix}_{X,i}}[\kappa_{ijB,P}]_{B\in\mathrm{Mix}_{\mathbb{H},i}} = [\partial_{X^I} h_{ij}(P)]_{I\in\mathrm{Mix}_{X,i}}.$$

Since the *unique* solution $[\kappa_{ijB,P}]_{B\in\mathrm{Mix}_{\mathbb{H},i}}$ exists (cf. conclusion (c)), we conclude that the coefficient matrix of the linear equation has full rank, i.e.,

$$\mathrm{rank}\, [\partial_{X^I} H^B(P)]^{B\in\mathrm{Mix}_{\mathbb{H},i}}_{I\in\mathrm{Mix}_{X,i}} = \#\,\mathrm{Mix}_{\mathbb{H},i}\,.$$

Therefore, there exists a subset $S \subset \mathrm{Mix}_{X,i}$ with $\#S = \#\,\mathrm{Mix}_{\mathbb{H},i}$ such that the corresponding minor has a nonzero determinant, i.e.,

$$\det\, [\partial_{X^I} H^B(P)]^{B\in\mathrm{Mix}_{\mathbb{H},i}}_{I\in S} \in k^\times.$$

Then the solution $[\kappa_{ijB,P}]_{B\in \mathrm{Mix}_{\mathbb{H},i}}$ can be expressed as follows:

$$[\kappa_{ijB,P}]_{B\in \mathrm{Mix}_{\mathbb{H},i}} = ([\partial_{X^I} H^B(P)]_{I\in S}^{B\in \mathrm{Mix}_{\mathbb{H},i}})^{-1}[\partial_{X^I} h_{ij}(P)]_{I\in S}.$$

(Note that actually the matrix $[\kappa_{ijB,P}]_{B\in \mathrm{Mix}_{\mathbb{H},i}}$ as well as the matrix $[\partial_{X^I} h_{ij}(P)]_{I\in S}$ is a zero matrix.) By shrinking U_P if necessary, we may assume

$$\det [\partial_{X^I} H^B]_{I\in S}^{B\in \mathrm{Mix}_{\mathbb{H},i}} \in (R_f)^\times$$

and hence that

$$\det [\partial_{X^I} H^B(Q)]_{I\in S}^{B\in \mathrm{Mix}_{\mathbb{H},i}} \in k^\times \qquad \forall Q \in C \cap \mathrm{Supp}(\mathbb{I}) \cap \mathfrak{m}\text{-}\mathrm{Spec}\, R.$$

Then the solution $[\kappa_{ijB,Q}]_{B\in \mathrm{Mix}_{\mathbb{H},i}}$ for ($\heartsuit\heartsuit$) can be expressed as follows:

$$[\kappa_{ijB,Q}]_{B\in \mathrm{Mix}_{\mathbb{H},i}} = ([\partial_{X^I} H^B(Q)]_{I\in S}^{B\in \mathrm{Mix}_{\mathbb{H},i}})^{-1}[\partial_{X^I} h_{ij}(Q)]_{I\in S}.$$

It follows immediately that if we define the set $\{g_{ijB}\}_{B\in \mathrm{Mix}_{\mathbb{H},i}}$ by the formula

$$[g_{ijB}]_{B\in \mathrm{Mix}_{\mathbb{H},i}} = ([\partial_{X^I} H^B]_{I\in S}^{B\in \mathrm{Mix}_{\mathbb{H},i}})^{-1}[\partial_{X^I} h_{ij}]_{I\in S},$$

then it satisfies the desired condition

$$g_{ijB}(Q) = \kappa_{ijB,Q} \qquad \forall Q \in C \cap \mathrm{Supp}(\mathbb{I}) \cap \mathfrak{m}\text{-}\mathrm{Spec}\, R.$$

Finally, by shrinking U_P if necessary so that the above argument is valid for any element h_{ij} taken from the given LGS $\mathbb{H}$, we see that condition (4) is satisfied.

This completes the proof of Proposition 1.3.3.3.

Chapter 2. Power series expansion

As in Chapter 1, we denote by R the coordinate ring of an affine open subset $\mathrm{Spec}\, R$ of a nonsingular variety W of $\dim W = d$ over an algebraically closed field k with $\mathrm{char}(k) = p > 0$ or $\mathrm{char}(k) = 0$, where in the latter case we formally set $p = \infty$ (cf. 0.2.3.2.1 and Definition 3.1.1.1(2) in Part I).

We fix a closed point $P \in W$. Let $\mathbb{I}_P$ be a $\mathfrak{D}$-saturated idealistic filtration over $R_P = \mathcal{O}_{W,P}$, the local ring at the closed point, with $\mathfrak{m}_P$ being its maximal ideal. Let $\mathbb{H} = \{(h_l, p^{e_l})\}_{l=1}^N$ be an LGS of $\mathbb{I}_P$.

In characteristic zero, the elements in the LGS are all concentrated at level 1, i.e., $e_l = 0$ and $p^{e_l} = 1$ for $l = 1, \ldots, N$ (cf. Chapter 3 in Part I). This implies by the definition of an LGS that the set of the elements $H = (h_l\,;\, l = 1, \ldots, N)$ forms (a part of) a regular system of parameters $(x_1, \ldots, x_d)$. (Say $h_l = x_l$ for $l = 1, \ldots, N$.) In positive characteristic, this is no longer the case. However, we can still regard the notion of an LGS as a generalization of the notion of a regular system of parameters, and we may expect some common properties shared by the two notions.

Now any element $f \in R_P$ (or more generally any element $f \in \widehat{R_P}$) can be expressed as a power series with respect to the regular system of parameters and hence with respect to the LGS as above in characteristic zero. That is, we can write

$$f = \sum_{I \in (\mathbb{Z}_{\geq 0})^d} a_I X^I = \sum_{B \in (\mathbb{Z}_{\geq 0})^N} c_B H^B$$

where $a_I \in k$ and c_B is a power series in terms of the remainder $(x_{N+1}, \ldots, x_d)$ of the regular system of parameters.

Chapter 2 is devoted to the study of the power series expansion with respect to the elements in an LGS (and its (weakly-)associated regular system of parameters), one of the expected common properties mentioned above, which is valid both in characteristic zero and in positive characteristic.

§2.1. Existence and uniqueness

2.1.1. Setting for the power series expansion. First we describe the setting for Chapter 2, which is slightly more general than just dealing with an LGS. Actually, until we reach §2.2, our argument does *not* involve the notion of an idealistic filtration.

Let $\mathcal{H} = \{h_1, \ldots, h_N\} \subset R_P$ be a subset consisting of N elements, and $0 \leq e_1 \leq \cdots \leq e_N$ nonnegative integers attached to these elements, satisfying the following conditions (cf. 4.1.1 in Part I):

(i) $h_l \in \mathfrak{m}_P^{p^{e_l}}$ and $\overline{h_l} = (h_l \bmod \mathfrak{m}_P^{p^{e_l}+1}) = v_l^{p^{e_l}}$ with $v_l \in \mathfrak{m}_P/\mathfrak{m}_P^2$ for $l = 1, \ldots, N$,

(ii) $\{v_l\,;\, l = 1, \ldots, N\} \subset \mathfrak{m}_P/\mathfrak{m}_P^2$ consists of N distinct and k-linearly independent elements in the k-vector space $\mathfrak{m}_P/\mathfrak{m}_P^2$.

We also take a regular system of parameters $(x_1, \ldots, x_d)$ such that

$$v_l = \overline{x_l} = (x_l \bmod \mathfrak{m}_P^2) \quad \text{for } l = 1, \ldots, N.$$

We say $(x_1, \ldots, x_d)$ is *associated* to $H = (h_1, \ldots, h_N)$ if the above condition is satisfied.

2.1.2. Existence and uniqueness of the power series expansion

Lemma 2.1.2.1. *Let the setting be as described in* 2.1.1. *Then any element* $f \in \widehat{R_P}$ *has a power series expansion, with respect to* $H = (h_1, \ldots, h_N)$ *and its associated regular system of parameters* $(x_1, \ldots, x_d)$*, of the form*

$$(\star) \qquad f = \sum_{B \in (\mathbb{Z}_{\geq 0})^N} c_B H^B \quad \textit{where} \quad c_B = \sum_{K \in (\mathbb{Z}_{\geq 0})^d} b_{B,K} X^K,$$

with $b_{B,K}$ being a power series in terms of the remainder $(x_{N+1},\ldots,x_d)$ of the regular system of parameters, and with $K=(k_1,\ldots,k_d)$ varying in the range satisfying the condition

$$0 \leq k_l \leq p^{e_l}-1 \quad \textit{for } l=1,\ldots,N \quad \textit{and} \quad k_l=0 \quad \textit{for } l=N+1,\ldots,d.$$

Moreover, the power series expansion of the form $(\star)$ is unique.

Note that, when we want to make explicit the dependence of the coefficient c_B on f, we write $c_B(f)$ instead of c_B.

Proof. (Existence) We construct a sequence $\{f_r\}_{r\in\mathbb{Z}_{\geq 0}} \subset R_P$ inductively, satisfying the following conditions:

(i) $f - f_r \in \widehat{\mathfrak{m}_P}^{r+1}$,

(ii) $f_r = \sum_{|[B]|\leq r} c_{B,r}H^B$ where

- $c_{B,r} = \sum b_{B,K,r}X^K$,
- $b_{B,K,r} = \sum_{|[B]|+|K|+|J|\leq r} a_{B,K,J}X^J$ is a polynomial in $(x_{N+1},\ldots,x_d)$ with
  - $a_{B,K,J} \in k$,
  - $J=(j_1,\ldots,j_d)$ varying in the range $j_l=0$ for $l=1,\ldots,N$,
  - K varying in the range specified above, satisfying the condition $|[B]| + |K|+|J| \leq r$.

Case 1: *Construction of f_0.* In this case, we set $f_0 = c_{\mathbb{O},0} = b_{\mathbb{O},\mathbb{O},0} = a_{\mathbb{O},\mathbb{O},\mathbb{O}} = \overline{f} \in k$. Then conditions (i) and (ii) are obviously satisfied.

Case 2: *Construction of f_{r+1} assuming that of f_r.* Suppose inductively that we have constructed f_r satisfying conditions (i) and (ii) above. Now express $f - f_r = \sum a_{I,r}X^I$ with $a_{I,r} \in k$ as a power series expansion in terms of the regular system of parameters $X=(x_1,\ldots,x_d)$.

Given $I=(i_1,\ldots,i_d)$ with $|I|=r+1$, determine

$$\begin{cases} B=(b_1,\ldots,b_N), \\ K=(k_1,\ldots,k_N,0,\ldots,0) \in (\mathbb{Z}_{\geq 0})^d, \\ J=(0,\ldots,0,j_{N+1},\ldots,j_d) \in (\mathbb{Z}_{\geq 0})^d \end{cases}$$

by the formulas

$$i_l = \begin{cases} b_l p^{e_l} + k_l & \text{with } b_l \in \mathbb{Z}_{\geq 0} \text{ and } 0 \leq k_l \leq p^{e_l}-1 \text{ for } l=1,\ldots,N, \\ j_l & \text{for } l=N+1,\ldots,d. \end{cases}$$

Then it is straightforward to see, after renaming $a_{I,r}$ as $a_{B,K,J}$, that

$$\sum_{|I|=r+1} a_{I,r}X^I = \sum_{|[B]|+|K|+|J|=r+1} a_{B,K,J}X^J X^K H^B \bmod \mathfrak{m}_P^{r+2}.$$

Set

$$b_{B,K,r+1} = \sum_{|[B]|+|K|+|J| \le r+1} a_{B,K,J} X^J, \quad c_{B,r+1} = \sum b_{B,K,r+1} X^K,$$
$$f_{r+1} = \sum_{|[B]| \le r+1} c_{B,r+1} H^B.$$

Then f_{r+1} clearly satisfies conditions (i) and (ii).

This finishes the inductive construction of the sequence $\{f_r\}_{r \in \mathbb{Z}_{\ge 0}} \subset R_P$.

Now set

$$b_{B,K} = \lim_{r \to \infty} b_{B,K,r} = \sum a_{B,K,J} X^J, \qquad c_B = \lim_{r \to \infty} c_{B,r} = \sum b_{B,K} X^K,$$

where each of the above limits exists by condition (ii). Then condition (i) implies

$$f = \lim_{r \to \infty} f_r = \lim_{r \to \infty} \sum_{|[B]| \le r} c_{B,r} H^B = \sum c_B H^B,$$

proving the existence of a power series expansion of the form $(\star)$.

(Uniqueness) In order to show the uniqueness of the power series expansion of the form $(\star)$, we have only to verify

$$0 = \sum_{B \in (\mathbb{Z}_{\ge 0})^N} c_B H^B \text{ of the form } (\star) \iff c_B = 0 \ \forall B \in (\mathbb{Z}_{\ge 0})^N.$$

As the implication $(\Leftarrow)$ is obvious, we show $(\Rightarrow)$. Suppose $0 = \sum_{B \in (\mathbb{Z}_{\ge 0})^N} c_B H^B$, and assume that there exists $B \in (\mathbb{Z}_{\ge 0})^N$ such that $c_B \neq 0$. Set $s = \min\{\mathrm{ord}(c_B H^B)\,;\, c_B \neq 0\}$. Write

$$c_B = \sum_{K \in (\mathbb{Z}_{\ge 0})^d} b_{B,K} X^K \quad \text{and} \quad b_{B,K} = \sum_{J \in (\mathbb{Z}_{\ge 0})^d} a_{B,K,J} X^J \quad \text{with} \quad a_{B,K,J} \in k,$$

where $K = (k_1, \ldots, k_d)$ varies in the range

$$0 \le k_l \le p^{e_l} - 1 \quad \text{for } l = 1, \ldots, N \quad \text{and} \quad k_l = 0 \quad \text{for } l = N+1, \ldots, d,$$

and where $J = (j_1, \ldots, j_d)$ varies in the range

$$j_l = 0 \quad \text{for } l = 1, \ldots, N.$$

Then we have

$$\begin{aligned} 0 = \sum_B c_B H^B &= \sum_B \sum_K \Big(\sum_J a_{B,K,J} X^J X^K \Big) H^B \\ &= \sum_{|[B]|+|K|+|J|=s} a_{B,K,J} X^J X^K \Big(\prod_{l=1}^N x_l^{p^{e_l} b_l} \Big) \bmod \widehat{\mathfrak{m}_P}^{s+1}. \end{aligned}$$

On the other hand, we observe that the set $\{X^J X^K (\prod_{l=1}^N x_l^{p^{e_l} b_l})\}_{|[B]|+|K|+|J|=s} = \{X^I\}_{|I|=s}$ of all monomials of degree s obviously forms a basis of the vector space $\widehat{\mathfrak{m}_P}^s/\widehat{\mathfrak{m}_P}^{s+1}$, and $a_{B,K,J} \neq 0$ for some B, K, J with $|[B]| + |K| + |J| = s$ by the assumption and by the choice of s. This is a contradiction! Therefore, we conclude that $c_B = 0$ for any $B \in (\mathbb{Z}_{\geq 0})^N$.

This finishes the proof of the implication $(\Rightarrow)$, and hence the proof of the uniqueness of the power series expansion of the form $(\star)$.

This completes the proof of Lemma 2.1.2.1.

Remark 2.1.2.2. (1) It follows immediately from the argument showing the existence and uniqueness of the power series expansion $f = \sum c_B H^B$ of the form $(\star)$ that

$$\mathrm{ord}(f) = \min\{\mathrm{ord}(c_B H^B)\} = \min\{\mathrm{ord}(c_B) + |[B]|\}$$

and hence that

$$\mathrm{ord}(c_B) \geq \mathrm{ord}(f) - |[B]| \quad \forall B \in (\mathbb{Z}_{\geq 0})^N.$$

(2) In the setting 2.1.1, we defined the notion of a regular system of parameters associated to $H = (h_1, \ldots, h_N)$. We say that a regular system of parameters $(x_1, \ldots, x_d)$ is *weakly-associated* to $H = (h_1, \ldots, h_N)$ if

$$\det [\partial_{x_i^{p^e}} (h_l^{p^{e-e_l}})]_{i=1,\ldots,L_e}^{l=1,\ldots,L_e} \in R_P^\times \quad \text{for } e = e_1, \ldots, e_N \quad \text{where} \quad L_e = \#\{l\,;\, e_l \leq e\}.$$

All the assertions of Lemma 2.1.2.1 hold if we only require a regular system of parameters $(x_1, \ldots, x_d)$ to be weakly-associated to H, instead of associated to H.

Example 2.1.2.3. The existence part of the above proof provides an actual algorithm to compute the power series expansion of the form $(\star)$ for any given $f \in \widehat{R}$. The reader is encouraged to carry out the computation for himself. Here we only mention one example: Assume $\mathrm{char}(k) = 2$. Let $f = x^2$. Then, with respect to $\mathcal{H}' = \{h' = x^2\}$ (with $e_1 = 2$ attached), the power series expansion of the form $(\star)$ is trivial, i.e., $f = h'$. However, with respect to $\mathcal{H} = \{h = uh'\}$ where $u = 1 + xy$ and hence where h has the same leading term as h', the power series expansion of the form $(\star)$ becomes an infinite series

$$\begin{aligned} x^2 &= uh + y^2h^2 + uy^4h^3 + uy^8h^5 + y^{10}h^6 + uy^{16}h^9 + y^{18}h^{10} + \cdots \\ &= \sum_{k=0}^{\infty} \eta(k) y^{2k} h^{k+1} \quad \text{where} \quad \eta(k) = \begin{cases} 0 & \text{if } \lfloor k/2^j \rfloor \equiv 3 \bmod 4 \ (\exists j \in \mathbb{Z}) \\ u & \text{if } \lfloor k/2^j \rfloor \not\equiv 3 \bmod 4 \ (\forall j \in \mathbb{Z}),\ k \in 2\mathbb{Z}, \\ 1 & \text{if } \lfloor k/2^j \rfloor \not\equiv 3 \bmod 4 \ (\forall j \in \mathbb{Z}),\ k \notin 2\mathbb{Z}. \end{cases} \end{aligned}$$

§2.2. Formal coefficient lemma

2.2.1. Setting for the formal coefficient lemma. As we can see from the description of the setting 2.1.1, our discussion on the power series expansion of the form $(\star)$ (cf. Lemma 2.1.2.1) so far does not involve the notion of an idealistic filtration. However, the most interesting and important result of Chapter 2 is obtained in Lemma 2.2.2.1 below, called the formal coefficient lemma, when we get the notion of an idealistic filtration involved and impose an extra condition related to it as follows:

Let $\mathcal{H} = \{h_1, \dots, h_N\} \subset R_P$ be a subset consisting of N elements, and $0 \le e_1 \le \cdots \le e_N$ nonnegative integers attached to these elements, satisfying conditions (i) and (ii) of 2.1.1. Let $X = (x_1, \dots, x_d)$ be a regular system of parameters associated to $H = (h_1, \dots, h_N)$ with $h_l = x_l^{p^{e_l}} \bmod \mathfrak{m}_P^{p^{e_l}+1}$ for $l = 1, \dots, N$. Let $\mathbb{I}_P$ be a $\mathfrak{D}$-saturated idealistic filtration over R_P. We impose the following extra condition:

(iii) $(h_l, p^{e_l}) \in \mathbb{I}_P$ for $l = 1, \dots, N$.

2.2.2. Statement of the formal coefficient lemma and its proof. Now our assertion is that, under the setting of 2.2.1 and given an element in (the completion of) the idealistic filtration, the coefficients of the power series expansion of the form $(\star)$, with "appropriate" levels attached, belong to the completion of the idealistic filtration. We formulate this assertion as the following *formal coefficient lemma.*

Lemma 2.2.2.1. *Let the setting be as described in* 2.2.1. *Let* $\widehat{\mathbb{I}_P}$ *be the completion of the idealistic filtration* $\mathbb{I}_P$ (*cf.* §2.4 *in Part* I)*. Take an element* $(f, a) \in \widehat{\mathbb{I}_P}$*. Let* $f = \sum_{B \in (\mathbb{Z}_{\ge 0})^N} c_B H^B$ *be the power series expansion of the form* $(\star)$ (*cf. Lemma* 2.1.2.1)*. Then*

$$(c_B, a - |[B]|) \in \widehat{\mathbb{I}_P} \quad \forall B \in (\mathbb{Z}_{\ge 0})^N.$$

Proof. We will derive a contradiction assuming

$$(c_B, a - |[B]|) \notin \widehat{\mathbb{I}_P} \quad \text{for some } B \in (\mathbb{Z}_{\ge 0})^N.$$

Note that, under this assumption, there should exist $B \in (\mathbb{Z}_{\ge 0})^N$ with $B \ne \mathbb{O}$ such that $(c_B, a - |[B]|) \notin \widehat{\mathbb{I}_P}$. (In fact, suppose $(c_B, a - |[B]|) \in \widehat{\mathbb{I}_P}$ for all $B \ne \mathbb{O}$. Then the equality $c_{\mathbb{O}} = f - \sum_{B \ne \mathbb{O}} c_B H^B$ and the inclusions $(f, a) \in \widehat{\mathbb{I}_P}$ and $(c_B H^B, a) \in \widehat{\mathbb{I}_P}$ for all $B \ne \mathbb{O}$ would imply $(c_{\mathbb{O}}, a) = (c_{\mathbb{O}}, a - |[\mathbb{O}]|) \in \widehat{\mathbb{I}_P}$, contrary to the assumption.)

We introduce the following notations:

$$l_B = |[B]| + \sup\{n \in \mathbb{Z}_{\ge 0} \,;\, c_B \in (\widehat{\mathbb{I}_P})_{a - |[B]|} + \widehat{\mathfrak{m}_P}^n\} \quad \text{for } B \in (\mathbb{Z}_{\ge 0})^N \setminus \{\mathbb{O}\},$$

$$
\begin{aligned}
l &= \min_{B \in (\mathbb{Z}_{\geq 0})^N,\, B \neq \mathbb{O}} \{l_B\}, \\
\Gamma_B &= (\widehat{\mathbb{I}_P})_{a-|[B]|} + \widehat{\mathfrak{m}_P}^{\,l-|[B]|+1} \qquad \text{for } B \in (\mathbb{Z}_{\geq 0})^N, \\
L_B &= \max\Big\{ B + K\,;\, c_B \in \Gamma_B + \sum_{K \leq M} \widehat{\mathfrak{m}_P}^{\,l-|[B+M]|} H^M \Big\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{for } B \in (\mathbb{Z}_{\geq 0})^N \setminus \{\mathbb{O}\},\, l_B = l, \\
L &= \min_{B \in (\mathbb{Z}_{\geq 0})^N,\, B \neq \mathbb{O},\, l_B = l} \{L_B\}, \\
B_0 &= \max_{B \in (\mathbb{Z}_{\geq 0})^N,\, B \neq \mathbb{O},\, l_B = l, L_B = L} \{B\} \\
\Lambda_B &= \Gamma_B + \sum_{L < B+M} \widehat{\mathfrak{m}_P}^{\,l-|[B+M]|} H^M \qquad \text{for } B \in (\mathbb{Z}_{\geq 0})^N.
\end{aligned}
$$

Note that $l < \infty$ by the assumption $c_B \notin (\widehat{\mathbb{I}_P})_{a-|[B]|}$ for some $B \neq \mathbb{O}$. Note that the maximum of $B + K$, the minimum of L_B, and the maximum of B are all taken with respect to the lexicographical order on $(\mathbb{Z}_{\geq 0})^N$. Observe that if $l_B = l$ with $B \neq \mathbb{O}$, then $[B] < a$. Observe also that we have only to consider K with $|[K]| \leq l - |[B]|$ in order to compute the maximum for $B + K$. These observations guarantee the existence of the maximum of $B + K$ for $B \in (\mathbb{Z}_{\geq 0})^N, B \neq \{\mathbb{O}\}$, $l_B = l$, and the maximum of B for $B \in (\mathbb{Z}_{\geq 0})^N$, $B \neq \mathbb{O}$, $l_B = l$, $L_B = L$. We remark that when $r \leq 0$, we understand that by convention $\widehat{\mathfrak{m}_P}^{\,r}$ represents $\widehat{R_P}$.

We claim, for $B, K \in (\mathbb{Z}_{\geq 0})^N$,

(i) $H^K \Lambda_{B+K} \subset \Lambda_B$,

(ii) $\partial_{[K]}(\Lambda_B) \subset \Lambda_{B+K}$.

(We identify $[K]$, for $K = (k_1, \ldots, k_N) \in (\mathbb{Z}_{\geq 0})^N$, with $(p^{e_1}k_1, \ldots, p^{e_N}k_N, 0, \ldots, 0) \in (\mathbb{Z}_{\geq 0})^d$, and hence $\partial_{[K]}$ denotes $\partial_{X^{[K]}} = \partial_{x_1^{p^{e_1}k_1} \cdots x_N^{p^{e_N}k_N}}$ in claim (ii).)

In fact, since $(H^K, |[K]|) \in \widehat{\mathbb{I}_P}$ and $H^K \in \widehat{\mathfrak{m}_P}^{\,|[K]|}$, we see that

$$
\begin{aligned}
H^K \Lambda_{B+K} &= H^K \Big(\Gamma_{B+K} + \sum_{L < B+K+M} \widehat{\mathfrak{m}_P}^{\,l-|[B+K+M]|} H^M \Big) \\
&= H^K \Big((\widehat{\mathbb{I}_P})_{a-|[B+K]|} + \widehat{\mathfrak{m}_P}^{\,l-|[B+K]|+1} + \sum_{L < B+K+M} \widehat{\mathfrak{m}_P}^{\,l-|[B+K+M]|} H^M \Big) \\
&\subset (\widehat{\mathbb{I}_P})_{a-|[B]|} + \widehat{\mathfrak{m}_P}^{\,l-|[B]|+1} + \sum_{L < B+M} \widehat{\mathfrak{m}_P}^{\,l-|[B+M]|} H^M \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by replacing old } M + K \text{ with new } M) \\
&= \Gamma_B + \sum_{L < B+M} \widehat{\mathfrak{m}_P}^{\,l-|[B+M]|} H^M = \Lambda_B,
\end{aligned}
$$

proving claim (i).

In order to show claim (ii), observe

- $\partial_{[K]}((\widehat{\mathbb{I}_P})_{a-|[B]|}) \subset (\widehat{\mathbb{I}_P})_{a-|[B+K]|}$, since $\widehat{\mathbb{I}_P}$ is $\mathfrak{D}$-saturated,
- $\partial_{[K]}(\widehat{\mathfrak{m}_P}^{l-|[B]|+1}) \subset \widehat{\mathfrak{m}_P}^{l-|[B+K]|+1}$,
  $\partial_{[K]-I}(\widehat{\mathfrak{m}_P}^{l-|[B+M]|}) \subset \widehat{\mathfrak{m}_P}^{l-|[B+K+M]|+|I|}$ for I with $I \le [K]$,
- $\partial_I(H^M) \subset \binom{[M]}{I} H^{M-I} + \widehat{\mathfrak{m}_P}^{|[M]|-|I|+1}$, and $\binom{[M]}{I} = 0$ unless $I = [J]$ for some $J \in (\mathbb{Z}_{\ge 0})^N$.

Using these observations, we compute

$$
\begin{aligned}
\partial_{[K]}(\Lambda_B) &= \partial_{[K]}\Big(\Gamma_B + \sum_{L<B+M} \widehat{\mathfrak{m}_P}^{l-|[B+M]|} H^M\Big) \\
&= \partial_{[K]}\Big((\widehat{\mathbb{I}_P})_{a-|[B]|} + \widehat{\mathfrak{m}_P}^{l-|[B]|+1} + \sum_{L<B+M} \widehat{\mathfrak{m}_P}^{l-|[B+M]|} H^M\Big) \\
&= \partial_{[K]}((\widehat{\mathbb{I}_P})_{a-|[B]|}) + \partial_{[K]}(\widehat{\mathfrak{m}_P}^{l-|[B]|+1}) + \sum_{L<B+M} \partial_{[K]}(\widehat{\mathfrak{m}_P}^{l-|[B+M]|} H^M) \\
&= \partial_{[K]}((\widehat{\mathbb{I}_P})_{a-|[B]|}) + \partial_{[K]}(\widehat{\mathfrak{m}_P}^{l-|[B]|+1}) \\
&\quad + \sum_{L<B+M}\Big[\sum_{I\le[K]} \partial_{[K]-I}(\widehat{\mathfrak{m}_P}^{l-|[B+M]|})\partial_I(H^M)\Big] \\
&\qquad\qquad \text{(by the generalized product rule; cf. Lemma 1.2.1.2(3) in Part I)} \\
&= \partial_{[K]}((\widehat{\mathbb{I}_P})_{a-|[B]|}) + \partial_{[K]}(\widehat{\mathfrak{m}_P}^{l-|[B]|+1}) \\
&\quad + \sum_{L<B+M}\Big[\sum_{I=[J],\, I\le[K]} \partial_{[K]-I}(\widehat{\mathfrak{m}_P}^{l-|[B+M]|})\partial_I(H^M) \\
&\qquad\qquad + \sum_{I\ne[J],\, I\le[K]} \partial_{[K]-I}(\widehat{\mathfrak{m}_P}^{l-|[B+M]|})\partial_I(H^M)\Big] \\
&\subset (\widehat{\mathbb{I}_P})_{a-|[B+K]|} + \widehat{\mathfrak{m}_P}^{l-|[B+K]|+1} \\
&\quad + \sum_{L<B+M}\Big[\sum_{I=[J],\, J\le K,\, J\le M} \widehat{\mathfrak{m}_P}^{l-|[B+M+K-J]|} H^{M-J}\Big] \\
&\subset \Gamma_{B+K} + \sum_{L<B+M+(K-J)=B+K+(M-J),\, J\le K,\, J\le M} \widehat{\mathfrak{m}_P}^{l-|[B+K+M-J]|} H^{M-J} \\
&\subset \Gamma_{B+K} + \sum_{L<B+K+M} \widehat{\mathfrak{m}_P}^{l-|[B+K+M]|} H^M = \Lambda_{B+K} \\
&\qquad\qquad \text{(by replacing old } M-J \text{ with new } M\text{)},
\end{aligned}
$$

checking claim (ii).

Now by definition, for each $B \in (\mathbb{Z}_{\ge 0})^N$ with $B \ne \mathbb{O}$, $l_B = l$, $L_B = L$, we can choose $b_B \in \widehat{\mathfrak{m}_P}^{l-|[L]|}$ such that $c_B - b_B H^{L-B} \in \Lambda_B$. For each $B \in (\mathbb{Z}_{\ge 0})^N$ with $B \ne \mathbb{O}$ but $l_B \ne l$ or $L_B \ne L$, we set $b_B = 0$ and have $c_B - b_B H^{L-B} \in \Lambda_B$.

Therefore, for each $B \in (\mathbb{Z}_{\geq 0})^N$ with $B \neq \mathbb{O}$, we have

$$c_B - b_B H^{L-B} \in \Lambda_B$$

and hence by claim (i) (with $B, K \in (\mathbb{Z}_{\geq 0})^N$ there being equal to $\mathbb{O}, B$ below, respectively)

$$(c_B - b_B H^{L-B})H^B \in \Lambda_{\mathbb{O}}.$$

Now we compute (with "$\equiv$" denoting equality modulo Λ_{B_0}):

$$\begin{aligned}
\partial_{[B_0]} f &= \partial_{[B_0]}\Big(\sum c_B H^B\Big) = \partial_{[B_0]}\Big(\sum_{B\neq\mathbb{O}} c_B H^B\Big) \equiv \partial_{[B_0]}\Big(\sum_{B\neq\mathbb{O}} b_B H^L\Big) \\
&\qquad \Big(\text{since } B_0 \neq \mathbb{O},\ \sum_{B\neq\mathbb{O}} c_B H^B - \sum_{B\neq\mathbb{O}} b_B H^L \in \Lambda_{\mathbb{O}} \text{ and by claim (ii)}\Big) \\
&= \sum_{B\neq\mathbb{O}, l_B=l, L_B=L} \partial_{[B_0]}(b_B H^{L-B} H^B) \\
&= \sum_{B\neq\mathbb{O},\, l_B=l,\, L_B=L} \Big[\sum_{I\leq[B_0]} \partial_I(b_B H^{L-B})\partial_{[B_0]-I}(H^B)\Big] \\
&\qquad \text{(by the generalized product rule; cf. Lemma 1.2.1.2(3) in Part I)} \\
&\equiv \sum_{B\neq\mathbb{O},\, l_B=l,\, L_B=L} \Big[\sum_{I\leq[B_0],\, I=[K]} \partial_{[K]}(b_B H^{L-B})\partial_{[B_0-K]}(H^B)\Big] \\
&\qquad \text{(refer to the last observation used to prove claim (ii)).}
\end{aligned}$$

Therefore we obtain the equation

$$\partial_{[B_0]} f \equiv \sum_{B\neq\mathbb{O},\, l_B=l,\, L_B=L} b_B H^{L-B} \partial_{[B_0]}(H^B)$$

by using the following observation:

> For $K \neq \mathbb{O}$, we have $\partial_{[K]}(b_B H^{L-B}) = \partial_{[K]}(-(c_B - b_B H^{L-B})) \in \Lambda_{B+K}$ and hence $\partial_{[K]}(b_B H^{L-B})\partial_{[B_0-K]}(H^B) \in \Lambda_{B+K}\partial_{[B_0-K]}(H^B) \subset \Lambda_{B_0}$.

Note that the last inclusion in the observation above is verified as follows:

$$\begin{aligned}
&\Lambda_{B+K}\partial_{[B_0-K]}(H^B) \\
&= \Big(\Gamma_{B+K} + \sum_{L<B+K+M} \widehat{\mathfrak{m}_P}^{\,l-|[B+K+M]|} H^M\Big)\partial_{[B_0-K]}(H^B) \\
&= \Big((\widehat{\mathbb{I}_P})_{a-|[B+K]|} + \widehat{\mathfrak{m}_P}^{\,l-|[B+K]|+1} + \sum_{L<B+K+M} \widehat{\mathfrak{m}_P}^{\,l-|[B+K+M]|} H^M\Big)\partial_{[B_0-K]}(H^B)
\end{aligned}$$

$$
\begin{aligned}
&\subset (\widehat{\mathbb{I}_P})_{a-|[B_0]|} + \widehat{\mathfrak{m}_P}^{l-|[B_0]|+1} + \Big(\sum_{L<B+K+M} \widehat{\mathfrak{m}_P}^{l-|[B+K+M]|} H^M\Big) \\
&\quad \cdot \left(\binom{[B]}{[B_0-K]} H^{B-(B_0-K)} + \widehat{\mathfrak{m}_P}^{[B]-[B_0-K]+1}\right) \\
&\subset (\widehat{\mathbb{I}_P})_{a-|[B_0]|} + \widehat{\mathfrak{m}_P}^{l-|[B_0]|+1} \\
&\quad + \sum_{L<B_0+(M+B+K-B_0)} \widehat{\mathfrak{m}_P}^{l-|[B_0+(M+B+K-B_0)]|} H^{M+B+K-B_0} \\
&\subset (\widehat{\mathbb{I}_P})_{a-|[B_0]|} + \widehat{\mathfrak{m}_P}^{l-|[B_0]|+1} + \sum_{L<B_0+M} \widehat{\mathfrak{m}_P}^{l-|[B_0+M]|} H^M \\
&= \Gamma_{B_0} + \sum_{L<B_0+M} \widehat{\mathfrak{m}_P}^{l-|[B_0+M]|} H^M = \Lambda_{B_0}.
\end{aligned}
$$

In carrying out the above computation, we make a couple of notes. Note that, in order to obtain the first inclusion, we used the observation $\partial_{[B_0-K]}(H^B) \in (\widehat{\mathbb{I}_P})_{|[B]|-|[B_0-K]|}$ and $\partial_{[B_0-K]}(H^B) \in \widehat{\mathfrak{m}_P}^{|[B]|-|[B_0-K]|}$, as well as the last observation used to prove claim (ii). Note also that, when $[B] \not\geq [B_0-K]$, we have $\binom{[B]}{[B_0-K]} = 0$, and hence we have the second inclusion valid ignoring the third term starting with $\sum_{L<B_0+(M+B+K-B_0)}$.

On the other hand, continuing with the last term of the equations $(*)$, we have, by the maximality of B_0,

$$
\begin{aligned}
&\sum_{B\neq \mathbb{O},\, l_B=l,\, L_B=L} b_B H^{L-B} \partial_{[B_0]}(H^B) \\
&\qquad\qquad \equiv \sum_{B\neq \mathbb{O},\, l_B=l,\, L_B=L} \binom{B}{B_0} b_B H^{L-B_0} = b_{B_0} H^{L-B_0}.
\end{aligned}
$$

Therefore, we conclude $\partial_{[B_0]} f \equiv b_{B_0} H^{L-B_0}$. However, since $\partial_{[B_0]} f \in (\widehat{\mathbb{I}_P})_{a-|[B_0]|} \subset \Lambda_{B_0}$, we conclude that $b_{B_0} H^{L-B_0} \in \Lambda_{B_0}$ and hence $c_{B_0} \in \Lambda_{B_0}$, which contradicts the choice of B_0 with $L_{B_0} = L$.

This finishes the proof of Lemma 2.2.2.1.

Remark 2.2.2.2. The essential idea of the proof by contradiction above actually leads to an explicit and concrete construction of the coefficients using the differential operators and taking limits. We present this construction below. Given $f \in \widehat{R}$, let $f = \sum_{B\in(\mathbb{Z}_{\geq 0})^N} c_B H^B$ be the power series expansion of the form $(\star)$.

∘ *Construction of the "constant" term* $c_{\mathbb{O}}$. We construct a sequence $\{g_n\}_{n\in\mathbb{Z}_{\geq 0}} \subset \widehat{R}$ inductively as follows:

(1) $g_0 = f$.

(2) $g_n = (1 - H^{B_{n-1}}\partial_{[B_{n-1}]})g_{n-1}$, where the multi-index B_{n-1} is characterized and chosen in the following manner: with $g_{n-1} = \sum_{B\in(\mathbb{Z}_{\geq 0})^N} c_B(g_{n-1})H^B$ being the power series expansion of the form $(\star)$ for g_{n-1}, we have

$$\begin{cases} \operatorname{ord}(c_{B_{n-1}}(g_{n-1})H^{B_{n-1}}) = \min\{\operatorname{ord}(c_B(g_{n-1})H^B)\,;\, B \neq \mathbb{O}\} = \nu_{n-1}, \\ B_{n-1} = \min\{B\,;\, B \neq \mathbb{O},\, \operatorname{ord}(c_B(g_{n-1})H^B) = \nu_{n-1}\}. \end{cases}$$

Then we realize the constant term $c_{\mathbb{O}}$ as the limit of the above sequence, i.e.,

$$c_{\mathbb{O}} = \lim_{n\to\infty} g_n.$$

∘ *Construction of the coefficient c_B for $B \in (\mathbb{Z}_{\geq 0})^N$ in general.* We construct a sequence $\{f_n\}_{n\in\mathbb{Z}_{\geq 0}} \subset \widehat{R}$ inductively as follows:

(1) $f_0 = 0$.

(2) $f_n = f_{n-1} + H^{B_{n-1}}c_{\mathbb{O}}(\partial_{[B_{n-1}]}(f - f_{n-1}))$, where $c_{\mathbb{O}}$ is the operator taking the "constant" term and where the multi-index B_{n-1} is characterized and chosen in the following manner: with $f - f_{n-1} = \sum_{B\in(\mathbb{Z}_{\geq 0})^N} c_B(f - f_{n-1})H^B$ being the power series expansion of the form $(\star)$ for $f - f_{n-1}$, we have

$$\begin{cases} \operatorname{ord}(c_{B_{n-1}}(f - f_{n-1})H^{B_{n-1}}) = \operatorname{ord}(f - f_{n-1}), \\ B_{n-1} = \min\{B\,;\, \operatorname{ord}(c_B(f - f_{n-1})H^B) = \operatorname{ord}(f - f_{n-1})\}. \end{cases}$$

Then we realize f and c_B as the limits of the above sequence and $\{c_B(f_n)\}_{n\in\mathbb{Z}_{\geq 0}}$, respectively, i.e.,

$$f = \lim_{n\to\infty} f_n, \quad c_B = \lim_{n\to\infty} c_B(f_n).$$

Starting from $(f, a) \in \widehat{\mathbb{I}_P}$, we see inductively that $(g_n, a) \in \widehat{\mathbb{I}_P}$. This implies $(c_{\mathbb{O}}, a) \in \widehat{\mathbb{I}_P}$, since $(\widehat{\mathbb{I}_P})_a$ is complete. Using this information, we also prove inductively $(c_B(f_n), a - |[B]|) \in \widehat{\mathbb{I}_P}$. This implies $(c_B, a - |[B]|) \in \widehat{\mathbb{I}_P}$, since $(\widehat{\mathbb{I}_P})_{a-|[B]|}$ is complete. In this way, the construction gives an "alternative" proof (but essentially the same as above) to Lemma 2.2.2.1.

Chapter 3. Invariant $\widetilde{\mu}$

The purpose of this chapter is to study the basic properties of the invariant $\widetilde{\mu}$. Since the unit for the strand of invariants in our algorithm is a triplet of numbers $(\sigma, \widetilde{\mu}, s)$ (or a quadruplet $(\sigma, \widetilde{\mu}, \widetilde{\nu}, s)$ (cf. 0.3.1)), we also study the behavior the pair $(\sigma, \widetilde{\mu})$ endowed with the lexicographical order. The discussion of the invariant $\widetilde{\mu}$ or of the pair $(\sigma, \widetilde{\mu})$ in this chapter is restricted to and concentrated on the case where there are no exceptional divisors involved, and hence can only be applied directly to the process *in year 0* of our algorithm. We will postpone the general discussion, involving the exceptional divisors and hence applicable to the process *after year 0* of our algorithm, until Part III and Part IV (cf. 0.2.3).

The setting for this chapter is identical to that of Chapter 1.

Namely, R represents the coordinate ring of an affine open subset $\operatorname{Spec} R$ of a nonsingular variety W of $\dim W = d$ over an algebraically closed field k with $\operatorname{char}(k) = p > 0$ or $\operatorname{char}(k) = 0$, where in the latter case we formally set $p = \infty$ (cf. 0.2.3.2.1 and Definition 3.1.1.1(2) in Part I).

Let $\mathbb{I}$ be an idealistic filtration over R. We assume that $\mathbb{I}$ is $\mathfrak{D}$-saturated. We remark that then, by compatibility of localization with $\mathfrak{D}$-saturation (cf. Proposition 2.4.2.1(2) in Part I), the localization $\mathbb{I}_P$ is also $\mathfrak{D}$-saturated for any closed point $P \in \operatorname{Spec} R$.

§3.1. Definition of $\widetilde{\mu}$

3.1.1. Definition of $\widetilde{\mu}$ as $\mu_{\mathcal{H}}$. We fix a closed point $P \in \operatorname{Spec} R \subset W$. Take an LGS $\mathbb{H} = \{(h_l, p^{e_l})\}_{l=1,\dots,N}$ with nonnegative integers $0 \leq e_1 \leq \cdots \leq e_N$ attached for the $\mathfrak{D}$-saturated idealistic filtration $\mathbb{I}_P$. Let $\mathcal{H} = \{h_l\}_{l=1,\dots,N}$ be the set of its elements in $\mathbb{H}$, and $(\mathcal{H}) \subset R_P$ the ideal generated by $\mathcal{H}$.

Definition 3.1.1.1. First we recall a few definitions given in 3.2.2 in Part I. For $f \in R_P$ (or more generally for $f \in \widehat{R_P}$), we define its *multiplicity* (or *order*) modulo $(\mathcal{H})$, denoted by $\operatorname{ord}_{\mathcal{H}}(f)$, to be

$$\operatorname{ord}_{\mathcal{H}}(f) = \sup\{n \in \mathbb{Z}_{\geq 0}\,;\, f \in \mathfrak{m}_P^n + (\mathcal{H})\} \quad (\text{or } \sup\{n \in \mathbb{Z}_{\geq 0}\,;\, f \in \widehat{\mathfrak{m}_P}^n + (\mathcal{H})\}).$$

Note that we set $\operatorname{ord}_{\mathcal{H}}(0) = \infty$ by definition. We also define

$$\mu_{\mathcal{H}}(\mathbb{I}_P) := \inf\left\{\mu_{\mathcal{H}}(f,a) = \frac{\operatorname{ord}_{\mathcal{H}}(f)}{a}\,;\, (f,a) \in \mathbb{I}_P,\, a > 0\right\}.$$

(We remark that $\mu_{\mathcal{H}}(\widehat{\mathbb{I}_P})$ is defined in a similar manner.)

Finally the invariant $\widetilde{\mu}$ at P, which we denote by $\widetilde{\mu}(P)$, is defined by the formula

$$\widetilde{\mu}(P) = \mu_{\mathcal{H}}(\mathbb{I}_P).$$

In order to justify the definition, we should show that $\mu_{\mathcal{H}}(\mathbb{I}_P)$ is independent of the choice of $\mathcal{H}$, i.e., independent of the choice of an LGS $\mathbb{H}$ for $\mathbb{I}_P$. We will show this independence in the next subsection.

Remark 3.1.1.2. (1) The usual order is multiplicative, i.e., we have an equality

$$\operatorname{ord}(fg) = \operatorname{ord}(f) + \operatorname{ord}(g) \quad \forall f, g \in R_P.$$

The order modulo $(\mathcal{H})$ is also multiplicative if $e_1 = \cdots = e_N = 0$. However, in general, we can only expect that the order modulo $(\mathcal{H})$ is weakly multiplicative, i.e., we only have an inequality

$$\mathrm{ord}_{\mathcal{H}}(fg) \geq \mathrm{ord}_{\mathcal{H}}(f) + \mathrm{ord}_{\mathcal{H}}(g) \quad \forall f, g \in R_P.$$

In fact, if $e_l > 0$ for some $l = 1, \ldots, N$, then it is easy to see (cf. Remark 3.2.1.2(1)) that we indeed have a strict inequality for some $f, g \in R_P$, i.e.,

$$\mathrm{ord}_{\mathcal{H}}(fg) > \mathrm{ord}_{\mathcal{H}}(f) + \mathrm{ord}_{\mathcal{H}}(g) \quad \text{for some } f, g \in R_P.$$

(2) Assume further that the idealistic filtration $\mathbb{I}$ is of r.f.g. type (cf. Definition 2.1.1.1(4) and §2.3 in Part I). Then the invariant $\widetilde{\mu}$ takes rational values with some bounded denominator δ (independent of P).

In fact, take a finite set of generators T for $\mathbb{I} = G_R(T)$ of the form

$$T = \{(f_\lambda, a_\lambda)\}_{\lambda \in \Lambda} \subset R \times \mathbb{Q}_{>0},\ \#\Lambda < \infty, \text{ with } a_\lambda = p_\lambda / q_\lambda \text{ where } p_\lambda, q_\lambda \in \mathbb{Z}_{>0}.$$

Set $\delta = \prod_{\lambda \in \Lambda} p_\lambda$. Then

$$\begin{aligned}\widetilde{\mu}(P) &= \mu_{\mathcal{H}}(\mathbb{I}_P) = \inf\left\{\mu_{\mathcal{H}}(f,a) = \frac{\mathrm{ord}_{\mathcal{H}}(f)}{a}\ ;\ (f,a) \in \mathbb{I}_P,\ a > 0\right\} \\ &= \min\left\{\mu_{\mathcal{H}}(f_\lambda, a_\lambda) = \frac{\mathrm{ord}_{\mathcal{H}}(f_\lambda)}{a_\lambda} = \frac{\mathrm{ord}_{\mathcal{H}}(f_\lambda) \cdot q_\lambda}{p_\lambda}\right\} \\ &\qquad \text{(cf. Lemma 2.2.1.2(1) in Part I and Remark 3.1.1.2(1) above)} \\ &\in \frac{1}{\delta}\mathbb{Z}_{\geq 0} \cup \{\infty\}.\end{aligned}$$

(3) Our invariant $\widetilde{\mu}$, the order modulo an LGS, is quite different from the so-called "residual order", the order modulo a p-th power element (or modulo a p^e-th power element for some appropriate $e \in \mathbb{Z}_{\geq 0}$, depending upon the situation) used in some other approaches toward resolution of singularities in positive characteristic. In fact, they form quite a contrast. Firstly, even though the leading terms of the elements in an LGS are p^e-th power elements for some $e \in \mathbb{Z}_{\geq 0}$, their higher terms are not necessarily p^e-th power elements. Therefore, we consider two different modulo's in order to compute the invariant $\widetilde{\mu}$ and the residual order. Secondly, they behave differently under blowup. The residual order may strictly increase under blowup, as Hauser's example of a "Kangaroo point" or Moh's example exhibits (cf. [RIMS08]). On the other hand, when we compute the pair $(\sigma, \widetilde{\mu})$ for these examples, we see that the pair never increases. A further comparison of the invariant $\widetilde{\mu}$ and the residual order will be given in our subsequent papers.

3.1.2. Invariant $\mu_{\mathcal{H}}$ is independent of $\mathcal{H}$. We show that $\mu_{\mathcal{H}}(\mathbb{I}_P)$ is independent of the choice of $\mathcal{H}$.

Proposition 3.1.2.1. *Let the setting be as described in* 3.1.1. *Then $\mu_{\mathcal{H}}(\mathbb{I}_P)$ is independent of the choice of $\mathcal{H}$, i.e., independent of the choice of an LGS $\mathbb{H}$ for $\mathbb{I}_P$.*

Proof. Suppose

$$\mu_P(\mathbb{I}) = \inf\left\{\mu_P(f,a) = \frac{\mathrm{ord}_P(f)}{a}\,;\,(f,a)\in\mathbb{I}_P,\ a>0\right\} < 1.$$

Then $P \notin \mathrm{Supp}(\mathbb{I})$. By Lemma 1.1.2.1(1), we have $\mathbb{I}_P = R_P \times \mathbb{R}$. We conclude that the set of elements $\mathcal{H}$ in any LGS $\mathbb{H}$ for $\mathbb{I}_P$ is a regular system of parameters $\{x_1,\ldots,x_d\}$ for R_P, where $d = \dim W$. Accordingly, we have $\mu_{\mathcal{H}}(\mathbb{I}_P) = 0$, independently of the choice of $\mathcal{H}$.

Therefore, in the following, we may assume $1 \le \mu_P(\mathbb{I})$ and hence that $1 \le \mu_P(\mathbb{I}) \le \mu_{\mathcal{H}}(\mathbb{I}_P)$ for any choice of $\mathcal{H}$.

Case 1: $\mu_{\mathcal{H}}(\mathbb{I}_P) = 1$ *for any choice of $\mathcal{H}$.* In this case, $\mu_{\mathcal{H}}(\mathbb{I}_P) = 1$ is obviously independent of the choice of $\mathcal{H}$ by the case assumption.

Case 2: $\mu_{\mathcal{H}}(\mathbb{I}_P) > 1$ *for some choice of $\mathcal{H}$.* In this case, fixing the set of the elements $\mathcal{H}$ in an LGS $\mathbb{H}$ for $\mathbb{I}_P$ with $\mu_{\mathcal{H}}(\mathbb{I}_P) > 1$, we show

$$(*) \qquad \mu_{\mathcal{H}'}(\mathbb{I}_P) \ge \mu_{\mathcal{H}}(\mathbb{I}_P) \quad (>1)$$

where $\mathcal{H}'$ is the set of the elements in another LGS $\mathbb{H}'$ for $\mathbb{I}_P$.

This is actually sufficient to show the required independence, since by switching the roles of $\mathcal{H}$ and $\mathcal{H}'$, we conclude $\mu_{\mathcal{H}}(\mathbb{I}_P) \ge \mu_{\mathcal{H}'}(\mathbb{I}_P)$ and hence $\mu_{\mathcal{H}}(\mathbb{I}_P) = \mu_{\mathcal{H}'}(\mathbb{I}_P)$.

First we make the following two easy observations:

(A) Let $\mathcal{H}'' = \{h''_l\}_{l=1,\ldots,N}$ be another set of elements in R_P obtained from $\mathcal{H}'$ by a linear transformation, i.e., for each $e \in \mathbb{Z}_{\ge 0}$ we have

$$[h_l''^{p^{e-e_l}}\,;\,e_l \le e] = [h_l'^{p^{e-e_l}}\,;\,e_l \le e]g_e \quad \text{for some } g_e \in \mathrm{GL}(\#\{e_l\,;\,e_l \le e\}, k).$$

Then $\mathcal{H}''$ is the set of the elements in an LGS $\mathbb{H}'' = \{(h''_l, p^{e_l})\}_{l=1}^N$ for $\mathbb{I}_P$, and we have $\mu_{\mathcal{H}'}(\mathbb{I}_P) = \mu_{\mathcal{H}''}(\mathbb{I}_P)$.

Going back to our situation, we see that there is $\mathcal{H}''$, obtained from $\mathcal{H}'$ by a linear transformation, such that $\mathcal{H}''$ and $\mathcal{H}$ share the same leading terms. Therefore, in order to show the inequality $(*)$, by replacing $\mathcal{H}'$ with $\mathcal{H}''$ we may assume that $\mathcal{H}$ and $\mathcal{H}'$ share the same leading terms, i.e.,

$$h_l \equiv h'_l \bmod \mathfrak{m}_P^{p^{e_l}+1} \quad \text{for } l = 1,\ldots,N.$$

(B) Assume that $\mathcal{H}$ and $\mathcal{H}'$ share the same leading terms. Then we have a sequence of the sets of the elements of LGS's for $\mathbb{I}_P$

$$\mathcal{H} = \mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_N = \mathcal{H}'$$

where the adjacent sets share all but one element. We have only to show

$$\mu_{\mathcal{H}_l}(\mathbb{I}_P) \geq \mu_{\mathcal{H}_{l-1}}(\mathbb{I}_P) \quad \text{for } l = 1, \ldots, N$$

in order to verify the inequality $(*)$.

According to the observations above, therefore, we have only to show the inequality $(*)$ under the following extra assumptions:

(1) $\mathcal{H}$ and $\mathcal{H}'$ share the same leading terms, i.e.,

$$h_l \equiv h'_l \bmod \mathfrak{m}_P^{p^{e_l}+1} \quad \text{for } l = 1, \ldots, N.$$

(2) $\mathcal{H}$ and $\mathcal{H}'$ share all but one element, i.e.,

$$h_l = h'_l \quad \text{for } l = 1, \ldots, N \text{ except } l = l_0.$$

In order to ease the notation, we set

$$h = h_{l_0}, \quad h' = h'_{l_0}, \quad \mathcal{G} = \mathcal{H} \setminus \{h_{l_0}\} = \mathcal{H}' \setminus \{h'_{l_0}\}.$$

Let ν be any positive number such that $1 < \nu < \mu_{\mathcal{H}}(\mathbb{I}_P)$.

Since $(h, p^{e_{l_0}}), (h', p^{e_{l_0}}) \in \mathbb{I}_P$, we have $(h - h', p^{e_{l_0}}) \in \mathbb{I}_P$. Therefore, by the definition of $\mu_{\mathcal{H}}(\mathbb{I}_P)$ and by the inequality $1 < \nu < \mu_{\mathcal{H}}(\mathbb{I}_P)$, we have

$$h - h' \in \mathfrak{m}_P^{\lceil \nu p^{e_{l_0}} \rceil} + (\mathcal{H}), \quad \text{i.e.,} \quad h - h' = f_1 + f_2 \quad \text{with} \quad f_1 \in \mathfrak{m}_P^{\lceil \nu p^{e_{l_0}} \rceil},\ f_2 \in (\mathcal{H}).$$

On the other hand, by the extra assumption (1), we have

$$h - h' \in \mathfrak{m}_P^{p^{e_{l_0}}+1}.$$

Observing $\lceil \nu p^{e_{l_0}} \rceil \geq p^{e_{l_0}} + 1$ (recall $\nu > 1$), we thus conclude that

$$f_2 = (h - h') - f_1 \in (\mathcal{H}) \cap \mathfrak{m}_P^{p^{e_{l_0}}+1}$$

and hence that

$$h - h' = f_1 + f_2 \in \mathfrak{m}_P^{\lceil \nu p^{e_{l_0}} \rceil} + (\mathcal{H}) \cap \mathfrak{m}_P^{p^{e_{l_0}}+1} \subset \mathfrak{m}_P^{\lceil \nu p^{e_{l_0}} \rceil} + h\mathfrak{m}_P + (\mathcal{G}) \cap \mathfrak{m}_P^{p^{e_{l_0}}+1},$$

where the last inclusion follows from Lemma 4.1.2.3 in Part I. That is, we have

$$h - h' = g_1 + hr + g_2 \quad \text{with} \quad g_1 \in \mathfrak{m}_P^{\lceil \nu p^{e_{l_0}} \rceil},\ r \in \mathfrak{m}_P,\ g_2 \in (\mathcal{G}) \cap \mathfrak{m}_P^{p^{e_{l_0}}+1}.$$

Therefore, $(1-r)h = g_1 + h' + g_2$. Since $u = 1 - r$ is a unit in R_P, we conclude

$$h = u^{-1}g_1 + u^{-1}h' + u^{-1}g_2 \in \mathfrak{m}_P^{\lceil \nu p^{e_{l_0}} \rceil} + (\mathcal{H}').$$

Given an element $(f, a) \in \mathbb{I}_P$ $(a > 0)$, we hence have

$$\begin{aligned} f &\in \sum_B (\mathbb{I}_P)'_{a-|[B]|} H^B \\ &\qquad \text{(by the Coefficient Lemma in Part I, where } (\mathbb{I}_P)'_t = (\mathbb{I}_P)_t \cap \mathfrak{m}_P^{\lceil \nu t \rceil}) \\ &= \sum_{b=b_{l_0},\, C=(b_1,\dots,b_{l_0-1},0,b_{l_0+1},\dots,b_N)} (\mathbb{I}_P)'_{a-|[C]|-bp^{e_{l_0}}} h^b H^C \subset \sum_b (\mathbb{I}_P)'_{a-bp^{e_{l_0}}} h^b + (\mathcal{G}) \\ &\subset \sum_b (\mathbb{I}_P)'_{a-bp^{e_{l_0}}} \mathfrak{m}_P^{b\lceil \nu p^{e_{l_0}} \rceil} + (\mathcal{H}') \quad \text{(since } h \in \mathfrak{m}_P^{\lceil \nu p^{e_{l_0}} \rceil} + (\mathcal{H}')). \end{aligned}$$

Therefore,

$$\begin{aligned} \operatorname{ord}_{\mathcal{H}'}(f) &\geq \min_b \{\operatorname{ord}_P((\mathbb{I}_P)'_{a-bp^{e_{l_0}}} \mathfrak{m}_P^{b\lceil \nu p^{e_{l_0}} \rceil})\} \geq \min_b \{\lceil \nu(a - bp^{e_{l_0}}) \rceil + b\lceil \nu p^{e_{l_0}} \rceil\} \\ &\geq \nu a. \end{aligned}$$

This implies

$$\mu_{\mathcal{H}'}(f, a) = \frac{\operatorname{ord}_{\mathcal{H}'}(f)}{a} \geq \nu.$$

Since this inequality holds for any positive number with $1 < \nu < \mu_{\mathcal{H}}(\mathbb{I}_P)$, we conclude

$$\mu_{\mathcal{H}'}(f, a) \geq \mu_{\mathcal{H}}(\mathbb{I}_P).$$

Since $(f, a) \in \mathbb{I}_P$ $(a > 0)$ is arbitrary, we finally conclude

$$\mu_{\mathcal{H}'}(\mathbb{I}_P) \geq \mu_{\mathcal{H}}(\mathbb{I}_P).$$

This completes the proof of the inequality $(*)$, and hence the proof in Case 2.

This completes the proof of Proposition 3.1.2.1.

§3.2. Interpretation of $\widetilde{\mu}$ in terms of the power series expansion

The purpose of this section is to give an interpretation of the invariant $\widetilde{\mu} = \mu_{\mathcal{H}}$ in terms of the power series expansion of the form $(\star)$ discussed in Chapter 2.

3.2.1. The order $\operatorname{ord}_{\mathcal{H}}(f)$ of f modulo $(\mathcal{H})$ is equal to the order $\operatorname{ord}(c_{\mathbb{O}})$ of the constant term of the power series expansion for f

Lemma 3.2.1.1. *Let the setting be as described in* 3.1.1. *Then, given $f \in \widehat{R_P}$, we have*

$$\operatorname{ord}_{\mathcal{H}}(f) = \operatorname{ord}(c_{\mathbb{O}}),$$

where $c_{\mathbb{O}}$ is the "constant" term of the power series expansion $f = \sum c_B H^B$ of the form $(\star)$ as described in Lemma 2.1.2.1.

Proof. Since $f \equiv c_{\mathbb{O}} \bmod (\mathcal{H})$, we obviously have

$$\mathrm{ord}_{\mathcal{H}}(f) = \mathrm{ord}_{\mathcal{H}}(c_{\mathbb{O}}) \geq \mathrm{ord}(c_{\mathbb{O}}).$$

Suppose $\mathrm{ord}_{\mathcal{H}}(f) > \mathrm{ord}(c_{\mathbb{O}}) = r$. Then by definition we can write

$$f = f_1 + f_2 \quad \text{with} \quad f_1 \in \widehat{\mathfrak{m}_P}^{r+1},\ f_2 \in (\mathcal{H}).$$

Therefore, we have

$$f_1 = f - f_2 = \sum c_B H^B - f_2.$$

Since $f_2 \in (\mathcal{H})$, we conclude by the uniqueness of the power series expansion of the form $(\star)$ that the constant term $c_{\mathbb{O}} = c_{\mathbb{O}}(f)$ for f is also the constant term $c_{\mathbb{O}}(f_1)$ for f_1, i.e., $c_{\mathbb{O}} = c_{\mathbb{O}}(f_1)$. On the other hand, by Remark 2.1.2.2(1),

$$r = \mathrm{ord}(c_{\mathbb{O}}) = \mathrm{ord}(c_{\mathbb{O}}(f_1)) \geq \mathrm{ord}(f_1) \geq r + 1,$$

a contradiction! Therefore, $\mathrm{ord}_{\mathcal{H}}(f) = \mathrm{ord}(c_{\mathbb{O}})$. This completes the proof of Lemma 3.2.1.1.

Remark 3.2.1.2. (1) We justify Remark 3.1.1.2(1), using Lemma 3.2.1.1. Suppose $e_l > 0$ for some $l = 1, \ldots, N$. Take $a, b \in \mathbb{Z}_{>0}$ such that $a+b = p^{e_l}$. Set $f = x_l^a$ and $g = x_l^b$. Then $c_{\mathbb{O}}(f) = x_l^a$ and $c_{\mathbb{O}}(g) = x_l^b$. Therefore, by Lemma 3.2.1.1 we have

$$\mathrm{ord}_{\mathcal{H}}(f) + \mathrm{ord}_{\mathcal{H}}(g) = \mathrm{ord}(c_{\mathbb{O}}(f)) + \mathrm{ord}(c_{\mathbb{O}}(g)) = a + b.$$

On the other hand, we observe that

$$fg = x_l^a x_l^b = x_l^{p^{e_l}} \in \mathfrak{m}_P^{p^{e_l}+1} + (h_l) \subset \mathfrak{m}_P^{p^{e_l}+1} + (\mathcal{H}),$$

which implies

$$\mathrm{ord}_{\mathcal{H}}(fg) \geq p^{e_l} + 1 > p^{e_l} = a + b = \mathrm{ord}_{\mathcal{H}}(f) + \mathrm{ord}_{\mathcal{H}}(g).$$

(2) We remark that the above interpretation of $\mathrm{ord}_{\mathcal{H}}(f)$ is still valid even if we consider the power series expansion of the form $(\star)$ with respect to $H = (h_1, \ldots, h_N)$ and a regular system of parameters only weakly-associated to H (cf. Remark 2.1.2.2(2)), instead of the power series expansion of the form $(\star)$ with respect to H and a regular system of parameters associated to H as described in Lemma 2.1.2.1.

3.2.2. Alternative proof of the Coefficient Lemma. The interpretation given in 3.2.1 allows us to derive the Coefficient Lemma (Lemma 4.1.4.1 in Part I) as a corollary to the formal coefficient lemma (Lemma 2.2.2.1 in Part II).

Corollary 3.2.2.1 (= Coefficient Lemma). *Let $\nu \in \mathbb{R}_{\geq 0}$ be such that $\nu < \mu_{\mathcal{H}}(\mathbb{I}_p)$. Set*

$$(\mathbb{I}_P)'_t = (\mathbb{I}_P)_t \cap \mathfrak{m}_P^{\lceil \nu t \rceil},$$

where we use the convention that $\mathfrak{m}_P^n = R_P$ for $n \leq 0$. Then for any $a \in \mathbb{R}$, we have

$$(\mathbb{I}_P)_a = \sum_B (\mathbb{I}_P)'_{a-|[B]|} H^B.$$

Proof. Note that we already gave a proof to the Coefficient Lemma in Part I. Here we present an alternative proof based upon the formal coefficient lemma, although both proofs share some common spirit.

Since $H^B \in (\mathbb{I}_P)_{|[B]|}$, we clearly have the inclusion

$$(\mathbb{I}_P)_a \supset \sum_B (\mathbb{I}_P)'_{a-|[B]|} H^B.$$

Therefore, we have only to show the opposite inclusion

$$(\mathbb{I}_P)_a \subset \sum_B (\mathbb{I}_P)'_{a-|[B]|} H^B.$$

Now, as observed in Remark 4.1.4.2(2) in Part I, we have

$$\sum_B (\mathbb{I}_P)'_{a-|[B]|} H^B = \sum_{|[B]| < a + p^{e_N}} (\mathbb{I}_P)'_{a-|[B]|} H^B.$$

Therefore, actually we have only to show

$$(\mathbb{I}_P)_a \subset \sum_{|[B]| < a + p^{e_N}} (\mathbb{I}_P)'_{a-|[B]|} H^B.$$

Since $\widehat{R_P}$ is faithfully flat over R_P, we have only to prove this inclusion at the level of completion. That is, we have only to show

$$(\widehat{\mathbb{I}_P})_a \subset \sum_{|[B]| < a + p^{e_N}} (\widehat{\mathbb{I}_P})'_{a-|[B]|} H^B,$$

noting

$$\begin{cases} (\mathbb{I}_P)_t \otimes_{R_P} \widehat{R_P} = (\widehat{\mathbb{I}_P})_t, \\ (\mathbb{I}_P)'_t \otimes_{R_P} \widehat{R_P} = ((\mathbb{I}_P)_t \cap \mathfrak{m}_P^{\lceil \nu t \rceil}) \otimes_R \widehat{R_P} = (\widehat{\mathbb{I}_P})_t \cap \widehat{\mathfrak{m}_P}^{\lceil \nu t \rceil} = (\widehat{\mathbb{I}_P})'_t. \end{cases}$$

Take $f \in (\widehat{\mathbb{I}_P})_a$. Let $f = \sum_B c_B H^B$ be the power series expansion of the form $(\star)$ as described in Lemma 2.1.2.1.

Observe that, for each $C \in (\mathbb{Z}_{\geq 0})^N$ with $|[C]| \geq a + p^{e_N}$, there exists $B_C \in (\mathbb{Z}_{\geq 0})^N$ with $a \leq |[B_C]| < a + p^{e_N}$ such that $B_C < C$ (cf. Remark 4.1.4.2(2) in Part I). We choose one such B_C and call it $\phi(C)$.

For each $B \in (\mathbb{Z}_{\geq 0})^N$ with $a \leq |[B]| < a + p^{e_N}$, we set

$$c'_B = c_B + \sum_{C \text{ with } \phi(C)=B} c_C H^{C-B}.$$

Then $c'_B \in \widehat{R_P} = (\widehat{\mathbb{I}_P})'_{a-|[B]|}$, since $a - |[B]| \leq 0$.

On the other hand, for each $B \in (\mathbb{Z}_{\geq 0})^N$ with $|[B]| < a$, we have by the formal coefficient lemma

$$c_B \in (\widehat{\mathbb{I}_P})_{a-|[B]|}.$$

We also have by Lemma 3.2.1.1

$$\mathrm{ord}(c_B) = \mathrm{ord}_{\mathcal{H}}(c_B) \geq \lceil \mu_{\mathcal{H}}(\mathbb{I}_P)(a - |[B]|) \rceil \geq \lceil \nu(a - |[B]|) \rceil.$$

Therefore,

$$c_B \in (\widehat{\mathbb{I}_P})_{a-|[B]|} \cap \widehat{\mathfrak{m}_P}^{\lceil \nu(a-|[B]|) \rceil} = (\widehat{\mathbb{I}_P})'_{a-|[B]|}.$$

We conclude

$$f = \sum_B c_B H^B = \sum_{|[B]|<a} c_B H^B + \sum_{a \leq |[B]| < a+p^{e_N}} c'_B H^B \in \sum_{|[B]|<a+p^{e_N}} (\widehat{\mathbb{I}_P})'_{a-|[B]|} H^B.$$

Since $f \in (\widehat{\mathbb{I}_P})_a$ is arbitrary, we finally conclude

$$(\widehat{\mathbb{I}_P})_a \subset \sum_{|[B]|<a+p^{e_N}} (\widehat{\mathbb{I}_P})'_{a-|[B]|} H^B.$$

This completes the "alternative" proof of the Coefficient Lemma.

3.2.3. Alternative proof of Proposition 3.1.2.1. The interpretation given in 3.2.1 also allows us to provide an alternative proof of Proposition 3.1.2.1 via the formal coefficient lemma (cf. Lemma 2.2.2.1).

Corollary 3.2.3.1 (= Proposition 3.1.2.1)**.** *Let the setting be as described in* 3.1.1. *Then $\mu_{\mathcal{H}}(\mathbb{I}_P)$ is independent of the choice of $\mathcal{H}$, i.e., independent of the choice of a leading generator system $\mathbb{H}$ for $\mathbb{I}_P$.*

Alternative proof. Let $\mathcal{H}'$ be the set of the elements in another LGS $\mathbb{H}'$. We want to show $\mu_{\mathcal{H}'}(\mathbb{I}_P) = \mu_{\mathcal{H}}(\mathbb{I}_P)$. By the same argument as in the proof of Proposition 3.1.2.1, we may assume that $\mathcal{H}$ and $\mathcal{H}'$ share the same leading terms, i.e.,

$$h_l \equiv h'_l \bmod \mathfrak{m}_P^{p^{e_l}+1} \quad \text{for } l = 1, \ldots, N.$$

Since $\mathcal{H}$ and $\mathcal{H}'$ share the same leading terms, we can take a regular system of parameters $(x_1, \ldots, x_d)$ associated to both H and H' *simultaneously*. In the following, when we consider the power series expansion of the form $(\star)$, we understand that it is with respect to H and $(x_1, \ldots, x_d)$ or with respect to H' and $(x_1, \ldots, x_d)$.

Now since $\mu_{\mathcal{H}'}(\mathbb{I}_P) = \mu_{\mathcal{H}'}(\widehat{\mathbb{I}_P})$ and $\mu_{\mathcal{H}}(\mathbb{I}_P) = \mu_{\mathcal{H}}(\widehat{\mathbb{I}_P})$, we have only to show

$$\mu_{\mathcal{H}'}(\widehat{\mathbb{I}_P}) = \mu_{\mathcal{H}}(\widehat{\mathbb{I}_P}).$$

We observe that

$$\begin{aligned}
\mu_{\mathcal{H}'}(\widehat{\mathbb{I}_P}) &= \inf\left\{\mu_{\mathcal{H}'}(f,a) = \frac{\operatorname{ord}_{\mathcal{H}'}(f)}{a}\,;\,(f,a) \in \widehat{\mathbb{I}_P},\, a > 0\right\} \\
&= \inf\left\{\frac{\operatorname{ord}(c'_{\mathbb{O}}(f))}{a}\,;\,(f,a) \in \widehat{\mathbb{I}_P},\, a > 0,\, f = \sum c'_B(f){H'}^B\right\} \\
&\qquad\qquad\text{(by the interpretation given in 3.2.1)} \\
&= \inf\left\{\frac{\operatorname{ord}(f)}{a}\,;\,(f,a) \in \widehat{\mathbb{I}_P},\, a > 0,\, f = c'_{\mathbb{O}}(f)\right\} \\
&\qquad\qquad\text{(by the formal coefficient lemma)}
\end{aligned}$$

and similarly that

$$\begin{aligned}
\mu_{\mathcal{H}}(\widehat{\mathbb{I}_P}) &= \inf\left\{\mu_{\mathcal{H}}(f,a) = \frac{\operatorname{ord}_{\mathcal{H}}(f)}{a}\,;\,(f,a) \in \widehat{\mathbb{I}_P},\, a > 0\right\} \\
&= \inf\left\{\frac{\operatorname{ord}(c_{\mathbb{O}}(f))}{a}\,;\,(f,a) \in \widehat{\mathbb{I}_P},\, a > 0,\, f = \sum c_B(f)H^B\right\} \\
&\qquad\qquad\text{(by the interpretation given in 3.2.1)} \\
&= \inf\left\{\frac{\operatorname{ord}(f)}{a}\,;\,(f,a) \in \widehat{\mathbb{I}_P},\, a > 0,\, f = c_{\mathbb{O}}(f)\right\} \\
&\qquad\qquad\text{(by the formal coefficient lemma).}
\end{aligned}$$

On the other hand, the condition $f = c'_{\mathbb{O}}(f)$ is equivalent to saying that f, as a power series in terms of $(x_1, \ldots, x_N, x_{N+1}, \ldots, x_d)$, is of the form $f = \sum b_K X^K$, with b_K being a power series in terms of the remainder $(x_{N+1}, \ldots, x_d)$ of the regular system of parameters, and with $K = (k_1, \ldots, k_d)$ varying in the range

$$0 \leq k_l \leq p^{e_l} - 1 \quad \text{for } l = 1, \ldots, N \quad \text{and} \quad k_l = 0 \quad \text{for } l = N+1, \ldots, d.$$

Since the regular system of parameters $(x_1, \ldots, x_N, x_{N+1}, \ldots, x_d)$ is associated to both H and H' simultaneously, this condition is no different from the condition $f = c_{\mathbb{O}}(f)$. That is, we have

$$f = c'_{\mathbb{O}}(f) \;\Leftrightarrow\; f = c_{\mathbb{O}}(f).$$

Therefore, by looking at the last expressions for $\mu_{\mathcal{H}'}(\widehat{\mathbb{I}_P})$ and $\mu_{\mathcal{H}}(\widehat{\mathbb{I}_P})$ above, we conclude

$$\mu_{\mathcal{H}'}(\widehat{\mathbb{I}_P}) = \mu_{\mathcal{H}}(\widehat{\mathbb{I}_P}).$$

This completes the alternative proof.

§3.3. Upper semi-continuity of $(\sigma, \widetilde{\mu})$

The purpose of this section is to establish the upper semi-continuity of $(\sigma, \widetilde{\mu})$, where the pair is endowed with the lexicographical order.

Recall that we have a $\mathfrak{D}$-saturated idealistic filtration $\mathbb{I}$ over R.

3.3.1. Statement of the upper semi-continuity of $(\sigma, \widetilde{\mu})$ and its proof

Proposition 3.3.1.1. *The function*

$$(\sigma, \widetilde{\mu}) : X = \mathfrak{m}\text{-Spec}\, R \to \Big(\prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}\Big) \times (\mathbb{R}_{\geq 0} \cup \{\infty\})$$

is upper semi-continuous with respect to the lexicographical order on $(\prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}) \times (\mathbb{R}_{\geq 0} \cup \{\infty\})$*. That is, for any* $(\alpha, \beta) \in (\prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}) \times (\mathbb{R}_{\geq 0} \cup \{\infty\})$*, the locus* $X_{\geq(\alpha,\beta)}$ *is closed* (*cf. Definition* 1.2.1.1).

Assume further that the idealistic filtration $\mathbb{I}$ *is of r.f.g. type* (*cf. Definition* 2.1.1.1(4) *and* §2.3 *in Part I*). *Then the invariant* $\widetilde{\mu}$ *takes the rational values with some bounded denominator* δ*, and the upper semi-continuity allows us to extend the domain to define the function*

$$(\sigma, \widetilde{\mu}) : \operatorname{Spec} R \to \Big(\prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}\Big) \times (\mathbb{R}_{\geq 0} \cup \{\infty\}),$$

where for $Q \in \operatorname{Spec} R$ *we have by definition*

$$(\sigma, \widetilde{\mu})(Q) = \min\{(\sigma, \widetilde{\mu})(P) = (\sigma(P), \widetilde{\mu}(P))\,;\, P \in \mathfrak{m}\text{-Spec}\, R,\, P \in \overline{Q}\},$$

or equivalently $(\sigma, \widetilde{\mu})(Q)$ *is equal to* $(\sigma, \widetilde{\mu})(P)$ *with* P *being a general closed point in* $\overline{Q}$*. The function* $(\sigma, \widetilde{\mu})$ *with the extended domain is upper semi-continuous.*

Moreover, since $\operatorname{Spec} R$ *is noetherian and since* $(\sigma, \widetilde{\mu})(\operatorname{Spec} R)$ *is contained in a well-ordered subset* $\mathcal{W}$ *of* $T = (\prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}) \times (\mathbb{R}_{\geq 0} \cup \{\infty\})$ (*e.g., we can set* $\mathcal{W} = S \times (\frac{1}{\delta}\mathbb{Z}_{\geq 0} \cup \{\infty\})$ *where* S *is the well-ordered subset of* $\prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}$ *as described in the proof of Corollary* 1.2.1.3)*, conditions* (i) *and* (ii) *in Lemma* 1.2.1.2*, as well as the assertions in Corollary* 1.2.1.4*, hold for the upper semi-continuous function* $(\sigma, \widetilde{\mu}) : \operatorname{Spec} R \to T$.

Proof. First we show the upper semi-continuity of the function

$$(\sigma, \widetilde{\mu}) : X = \mathfrak{m}\text{-Spec}\, R \to \Big(\prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}\Big) \times (\mathbb{R}_{\geq 0} \cup \{\infty\}).$$

We have only to show that, for any $(\alpha, \beta) \in (\prod_{e \in \mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}) \times (\mathbb{R}_{\geq 0} \cup \{\infty\})$, the locus $X_{\geq(\alpha,\beta)}$ is closed.

Step 1. *Reduction to the (local) situation where $X = \mathfrak{m}\text{-}\mathrm{Spec}\, R$ is an affine open neighborhood of a fixed point P, $\alpha = \sigma(P) \neq (0)_{e \in \mathbb{Z}_{\geq 0}}$ is the maximum of the invariant σ, and where an LGS $\mathbb{H}$ of $\mathbb{I}_P$ is uniformly pure along the (local) maximum locus C of the invariant σ.*

Observe that $X_{\geq(\alpha,\beta)} = X_{>\alpha} \cup (X_{\leq\alpha} \cap X_{\geq(\alpha,\beta)})$. Since $X_{>\alpha}$ is a closed subset by the upper semi-continuity of the invariant σ (cf. Corollary 1.2.1.3 and Proposition 1.2.2.1), we have only to show $X_{\geq(\alpha,\beta)}$ is closed in the open subset $X_{\leq\alpha}$, or equivalently in any affine open subset U contained in $X_{\leq\alpha}$. By replacing X with U, we may assume that the invariant σ never exceeds α in X. Then again by the upper semi-continuity of the invariant σ, the maximum locus $C = \{Q \in X\,; \sigma(Q) = \alpha\}$ $(= X_{\geq\alpha})$ of the invariant σ is a closed subset. Since $X_{\geq(\alpha,\beta)} \subset C$, we have only to show that, for any point $P \in C$, there exists an affine open neighborhood U_P of P such that $U_P \cap X_{\geq(\alpha,\beta)}$ is closed.

Suppose $\alpha = \sigma(P) = (0)_{e \in \mathbb{Z}_{\geq 0}}$. Then, taking U_P sufficiently small, we have $U_P \cap \mathrm{Supp}(\mathbb{I}) = \emptyset$ or $\{P\}$ (cf. Remark 1.3.3.2(i)). Therefore, we conclude that $U_P \cap X_{\geq(\alpha,\beta)} = U_P, \{P\}$ or $\emptyset$, and hence is closed. (Note that, for a point $Q \in U_P$, we have $(\sigma, \widetilde{\mu})(Q) = ((0)_{e \in \mathbb{Z}_{\geq 0}}, 0)$ if $Q \notin \mathrm{Supp}(\mathbb{I})$, and $(\sigma, \widetilde{\mu})(Q) = ((0)_{e \in \mathbb{Z}_{\geq 0}}, \infty)$ if $Q \in \mathrm{Supp}(\mathbb{I})$.)

Therefore, in the following, we may concentrate on the case where $\alpha = \sigma(P) \neq (0)_{e \in \mathbb{Z}_{\geq 0}}$. We take a leading generator system $\mathbb{H}$ of $\mathbb{I}_P$. By Proposition 1.3.3.3 and by shrinking U_P if necessary, we may assume that $\mathbb{H}$ is uniformly pure along C. Note that $C = C \cap \mathrm{Supp}(\mathbb{I})$, due to the condition $\sigma(P) \neq (0)_{e \in \mathbb{Z}_{\geq 0}}$ (cf. Remark 1.3.3.2). Finally by replacing X with U_P, we are reduced to the (local) situation as described in Step 1.

We may also assume by shrinking U_P if necessary, after taking a regular system of parameters $(x_1, \ldots, x_d)$ associated to $H = (h_1, \ldots, h_N)$ at P, that we have a regular system of parameters $(x_1, \ldots, x_d)$ over $\mathrm{Spec}\, R$ such that the matrices

$$[\partial_{x_i^{p^e}}(h_j^{p^{e-e_j}})]_{i=1,\ldots,L_e}^{j=1,\ldots,L_e} \quad \text{for } e = e_1, \ldots, e_N \quad \text{where} \quad L_e = \#\{l\,;\, e_l \leq e\}$$

are all invertible, and hence that the conditions described in the setting 4.1.1 of Part I for the supporting lemmas to hold are satisfied (at any point in C).

(We would like to bring to the reader's attention the difference in notation from 4.1.1 of Part I. The symbol "R" here denotes the coordinate ring of an affine open subset $\mathrm{Spec}\, R$ in W (cf. the beginning of Chapter 3), while the "R" there denotes the local ring at a closed point.)

Step 2. *Reduction to statement* (♠), *which is further reduced to statement* (♡).

We observe that, in order to provide an argument for the upper semi-continuity, it suffices to prove the following slightly more general statement (♠) (which does not involve any idealistic filtration):

(♠) Let $C \subset \mathfrak{m}\text{-Spec}\, R$ be a closed subset. Let $\mathcal{H} = \{h_1, \ldots, h_N\} \subset R$ be a subset consisting of N elements, and $0 \leq e_1 \leq \cdots \leq e_N$ nonnegative integers attached to these elements, satisfying the following conditions at each point $P \in C$ (cf. 4.1.1 in Part I):

(i) $h_l \in \mathfrak{m}_P^{p^{e_l}}$ and $\overline{h_l} = (h_l \bmod \mathfrak{m}_P^{p^{e_l}+1}) = v_l^{p^{e_l}}$ with $v_l \in \mathfrak{m}_P/\mathfrak{m}_P^2$ for $l = 1, \ldots, N$,

(ii) $\{v_l\,;\, l = 1, \ldots, N\} \subset \mathfrak{m}_P/\mathfrak{m}_P^2$ consists of N distinct and k-linearly independent elements in the k-vector space $\mathfrak{m}_P/\mathfrak{m}_P^2$.

We also have a regular system of parameters $(x_1, \ldots, x_d)$ over $\operatorname{Spec} R$ such that the matrices

$$[\partial_{x_i^{p^e}}(h_j^{p^{e-e_j}})]_{i=1,\ldots,L_e}^{j=1,\ldots,L_e} \quad \text{for } e = e_1, \ldots, e_N \quad \text{where} \quad L_e = \#\{l\,;\, e_l \leq e\}$$

are all invertible. Then for any $f \in R$ and $r \in \mathbb{Z}_{\geq 0}$ the locus

$$V_r(f, \mathcal{H}) := \{P \in C\,;\, f \in \mathfrak{m}_P^r R_P + (\mathcal{H})R_P\} = \{P \in C\,;\, \operatorname{ord}_{\mathcal{H}}(f)(P) \geq r\}$$

is a closed subset.

In fact, if we prove statement (♠), then $X_{(\alpha,\beta)} = \bigcap_{(f,a)\in\mathbb{I}, a>0} V_{\lceil \beta a \rceil}(f, \mathcal{H})$ is closed for any $(\alpha, \beta) \in (\prod_{e\in\mathbb{Z}_{\geq 0}} \mathbb{Z}_{\geq 0}) \times (\mathbb{R}_{\geq 0} \cup \{\infty\})$, and hence we have the required upper semi-continuity of the function $(\sigma, \widetilde{\mu})$.

Furthermore, in order to prove statement (♠), we have only to show the following statement (♡) for any $f \in R$ and $r \in \mathbb{Z}_{\geq 0}$:

(♡) There exist $\omega_l \in R$ $(l = 1, \ldots, N)$ such that

$$V_r(f, \mathcal{H}) = \left\{P \in C\,;\, f - \sum_{l=1}^{N} \omega_l h_l \in \mathfrak{m}_P^r R_P\right\}.$$

In fact, if we show statement (♡), then $V_r(f, \mathcal{H})$ being a closed set follows from the usual upper semi-continuity of the order function for $f - \sum_{l=1}^{N} \omega_l h_l$, and hence we have statement (♠).

Step 3. *Proof of statement* (♡) *by induction on* r.

We use induction on r. We set

$$\begin{aligned}
e &:= e_1 = \min\{e_l\,;\, l = 1, \ldots, N\}, \\
L &:= \max\{l\,;\, l = 1, \ldots, N,\, e_l = e\} = \#\{l\,;\, l = 1, \ldots, N,\, e_l = e\}, \\
e' &:= e_{L+1} \text{ (if } L = N, \text{ then we set } e' := \infty), \\
\chi &:= \#\{e_1, \ldots, e_N\}.
\end{aligned}$$

Case 1: $r \leq p^e$. In this case, we have only to set $\omega_l = 0 \ \ (l = 1, \ldots, l)$ in order to show statement (♡).

Case 2: $r > p^e$. Observing $V_r(f, \mathcal{H}) \subset V_{r-1}(f, \mathcal{H})$ and replacing f with $f - \sum_{l=1}^{N} \omega_l h_l$ via the application of statement (♡) for $r - 1$ by induction, we may assume

$$f \in \mathfrak{m}_P^{r-1} R_P \quad \forall P \in V_r(f, \mathcal{H}).$$

We also observe, by Supporting Lemma 3 in Part I (cf. Lemma 4.1.2.3 in Part I), that, at each point $P \in V_r(f, \mathcal{H})$, there exist $\beta_{l,P} \in \mathfrak{m}_P^{r-1-p^{e_l}} R_P$ such that

$$f - \sum_{l=1} \beta_{l,P} h_l \in \mathfrak{m}_P^r R_P.$$

Now we use induction on the pair (χ, L).

Case $\chi = 1$ $(L = N,\, e' = \infty)$

In this case, by applying Supporting Lemma 2 in Part I (cf. Lemma 4.1.2.2 in Part I) with $v = r$, $s = r - 1$ and $\alpha = -f$, we see that

$$(*) \qquad \begin{aligned}
\beta_L &\in F_v(-f) + \sum_{l=1}^{N-1} (F_v \beta_{l,P}) h_l + (h_L^r) + \mathfrak{m}_P^{r-p^e} R_P \\
&\subset F_v(-f) + \sum_{l=1}^{N-1} (F_v \beta_{l,P}) h_l + \mathfrak{m}_P^{r-p^e} R_P.
\end{aligned}$$

See Supporting Lemmas 1 and 2 in Part I (cf. Lemma 4.1.2.1 and Lemma 4.1.2.2 in Part I) for the definition of the differential operator F_v. We would like to emphasize that, even though Supporting Lemma 2 is a local statement at P, the differential operator F_v is defined globally over $\operatorname{Spec} R$ and hence $F_v(-f) \in R$.

From $(*)$, we deduce the following.

When $N = 1$, we have only to set $\omega_1 = F_v(-f)$ in order to obtain statement (♡).

When $N > 1$, we observe that

$$V_r(f, \mathcal{H}) = V_r(f, \{h_1, \ldots, h_{N-1}, h_N\}) = V_r(f - F_v(-f) h_N, \{h_1, \ldots, h_{N-1}\}).$$

Now statement ($\heartsuit$) for f and r with respect to $\mathcal{H} = \{h_1, \ldots, h_{N-1}, h_N\}$ follows from statement ($\heartsuit$) for $f - F_v(-f)h_N$ and r with respect to $\{h_1, \ldots, h_{N-1}\}$, which holds by induction on $(\chi, L) = (1, N-1)$.

Case $\chi > 1$

In this case, by applying Supporting Lemma 2 in Part I (cf. Lemma 4.1.2.2 in Part I) with $v = p^{e'-e} - 1$, $s = r - 1$ and $\alpha = -f$, we see that

$$\begin{aligned}\beta_L &\in F_v(-f) + \sum_{1 \le l \le N, l \ne L} (F_v \beta_{l,P}) h_l + (h_L^v) + \mathfrak{m}_P^{r-p^e} R_P \\ &\subset F_v(-f) + \sum_{1 \le l \le N, l \ne L} h_l R_P + (h_L^v) + \mathfrak{m}_P^{r-p^e} R_P.\end{aligned}$$

Hence, we conclude that

$$f - F_v(-f)h_L \in \sum_{1 \le l \le N, l \ne L} h_l R_P + h_L^{p^{e'-e}} R_P + \mathfrak{m}_P^r R_P$$

and so

$$\begin{aligned}V_r(f, \mathcal{H}) &= V_r(f, \{h_1, \ldots, h_{L-1}, h_L, h_{L+1}, \ldots, h_N\}) \\ &= V_r(f - F_v(-f)h_L, \{h_1, \ldots, h_{L-1}, h_L^{p^{e'-e}}, h_{L+1}, \ldots, h_N\}).\end{aligned}$$

Now statement ($\heartsuit$) for f and r with respect to $\mathcal{H} = \{h_1, \ldots, h_{L-1}, h_L, h_{L+1}, \ldots, h_N\}$ follows from statement ($\heartsuit$) for $f - F_v(-f)h_L$ and r with respect to $\{h_1, \ldots, h_{L-1}, h_L^{p^{e'-e}}, h_{L+1}, \ldots, h_N\}$, which holds by induction on (χ, L). (In fact, if originally $L = 1$, then the invariant χ drops by 1, and if originally $L > 1$, then the invariant χ remains the same but the invariant L drops by 1.)

This completes the proof of statement ($\heartsuit$).

This completes the proof of the upper semi-continuity of the function

$$(\sigma, \widetilde{\mu}) : X = \mathfrak{m}\text{-Spec}\, R \to \Big(\prod_{e \in \mathbb{Z}_{\ge 0}} \mathbb{Z}_{\ge 0} \Big) \times (\mathbb{R}_{\ge 0} \cup \{\infty\}).$$

If we assume further that the idealistic filtration is of r.f.g. type, then, as shown in Remark 3.1.1.2(2), the invariant $\widetilde{\mu}$ takes rational values with some bounded denominator δ. Then $(\sigma, \widetilde{\mu})(\mathfrak{m}\text{-Spec}\, R)$ is contained in a well-ordered set $\mathcal{W} = S \times (\frac{1}{\delta}\mathbb{Z}_{\ge 0} \cup \{\infty\})$ where S is the well-ordered subset of $\prod_{e \in \mathbb{Z}_{\ge 0}} \mathbb{Z}_{\ge 0}$ as described in the proof of Corollary 1.2.1.3. Now the assertion regarding the extension of the domain of the function $(\sigma, \widetilde{\mu})$ from $\mathfrak{m}$-Spec R to Spec R and the rest of the assertions in Proposition 3.3.1.1 follow from the same argument as in the proof of Corollary 1.2.2.2, where we discussed the extension of the domain of the invariant σ from $\mathfrak{m}$-Spec R to Spec R.

This completes the proof of Proposition 3.3.1.1.

Remark 3.3.1.2. Hironaka discusses the phenomenon of "generic up and down", some pathological behavior of the residual order at nonclosed points, before he deals with its upper semi-continuity over closed points. In contrast, the pair of our invariants $(\sigma, \widetilde{\mu})$, in the framework of the IFP, is defined originally only at closed points. The values of the pair at the nonclosed points are defined through the upper semi-continuity over closed points, extending the domain from $\mathfrak{m}$-Spec R to Spec R as in Proposition 3.3.1.1. Therefore, in the framework of the IFP, we do not face the issue of specialization or generization, which is inherent to the consideration of the residual order.

3.3.2. Alternative proof of the upper semi-continuity of $(\sigma, \widetilde{\mu})$. We can give an alternative proof of the upper semi-continuity of $(\sigma, \widetilde{\mu})$, using the interpretation of $\widetilde{\mu}$ in terms of the power series expansion of the form $(\star)$ as presented in 3.2.

Alternative proof of the upper semi-continuity of $(\sigma, \widetilde{\mu})$. By the same argument as before, we are reduced to the (local) situation as described in Step 1 of the original proof.

Take a regular system of parameters $X_P = (x_1, \ldots, x_d) = (x_{1,P}, \ldots, x_{d,P})$ at P, which is associated to $H = (h_1, \ldots, h_N)$. By shrinking Spec R if necessary, we may assume that $X_Q = (x_{1,Q}, \ldots, x_{d,Q}) = (x_1 - x_1(Q), \ldots, x_d - x_d(Q))$ with $x_i(Q) \in k$ is a regular system of parameters, which is weakly-associated to H at any point $Q \in C$.

By the same argument as before, we have only to show that, given $f \in R$ and $r \in \mathbb{Z}_{\geq 0}$, the locus $V_r(f, \mathcal{H}) = \{Q \in C\,; \operatorname{ord}_{\mathcal{H}}(f)(Q) \geq r\}$ is a closed subset as in Step 2 of the original proof.

This is where the alternative argument using the interpretation of $\widetilde{\mu}$ presented in 3.2 begins: Let $f = \sum c_{B,Q} H^B$ be the power series expansion of f at $Q \in C$ with respect to H and the regular system of parameters X_Q, which is weakly-associated to H at Q. By Lemma 3.2.1.1 and Remark 3.2.1.2(2), we have

$$\operatorname{ord}_{\mathcal{H}}(f)(Q) = \operatorname{ord}(c_{\mathbb{O},Q}).$$

Let

$$c_{\mathbb{O},Q} = \sum \gamma_{I,Q} X_Q^I, \qquad \gamma_{I,Q} \in k,$$

be the power series expansion of $c_{\mathbb{O},Q}$ with respect to X_Q. Then we have

$$\operatorname{ord}(c_{\mathbb{O},Q}) \geq r \ \Leftrightarrow \ \gamma_{I,Q} = 0 \ \forall I \text{ with } |I| < r.$$

On the other hand, since the coefficients $\gamma_{I,Q}$ can be computed from the coefficients of the power series expansions

$$f = \sum a_{I,f,Q} X_Q^I \quad \text{where} \quad a_{I,f,Q} = \partial_{X_Q^I}(f)(Q) = \partial_{X_P^I}(f)(Q) \in k,$$
$$h_l = \sum a_{I,h_l,Q} X_Q^I \text{ where } a_{I,h_l,Q} = \partial_{X_Q^I}(h_l)(Q) = \partial_{X_P^I}(h_l)(Q) \in k \text{ for } l = 1, \dots, N,$$

via the invertible matrices appearing in the condition of X_Q being weakly-associated to H,

$$[\partial_{x_{i,Q}^{p^e}}(h_j^{p^{e-e_j}})]_{i=1,\dots,L_e}^{j=1,\dots,L_e}(Q) = [\partial_{x_{i,P}^{p^e}}(h_j^{p^{e-e_j}})]_{i=1,\dots,L_e}^{j=1,\dots,L_e}(Q) \quad \text{for } e = e_1, \dots, e_N$$

where $L_e = \#\{l\,;\, e_l \leq e\}$, we conclude that, for each I, there exists $\gamma_I \in R$ such that $\gamma_I(Q) = \gamma_{I,Q}$ for all $Q \in C$.

Finally we conclude that

$$V_r(f, \mathcal{H}) = \{Q \in C\,;\, \mathrm{ord}_{\mathcal{H}}(f)(Q) \geq r\} = \{Q \in C\,;\, \gamma_I(Q) = 0 \ \forall I \text{ with } |I| < r\}$$

is a closed subset.

This completes the alternative proof of the upper semi-continuity of $(\sigma, \widetilde{\mu})$.

Appendix

The purpose of this appendix is to present the new nonsingularity principle using only the $\mathfrak{D}$-saturation, as opposed to the old nonsingularity principle using both the $\mathfrak{D}$-saturation and $\mathfrak{R}$-saturation. (The combination of the $\mathfrak{D}$-saturation and $\mathfrak{R}$-saturation was called the $\mathfrak{B}$-saturation in 2.1.5 and 2.2.3 in Part I.)

In Part I, we emphasized the importance of the $\mathfrak{R}$-saturation (and of the $\mathfrak{B}$-saturation) in carrying out the IFP. In fact, the $\mathfrak{R}$-saturation was crucial in establishing the nonsingularity principle, as formulated in Chapter 4 of Part I, which was supposed to lie at the heart of constructing our algorithm. However, the $\mathfrak{R}$-saturation has also been the main culprit in our quest to complete the algorithm, causing the following problems:

- By taking the $\mathfrak{R}$-saturation, we may increase the denominator of the invariant $\widetilde{\mu}$ indefinitely, and hence may not have the descending chain condition on the value set of the strand of invariants consisting of the units of the form $(\sigma, \widetilde{\mu}, s)$. This brings about the problem of termination, as mentioned in the introduction to Part I.
- If we take the $\mathfrak{R}$-saturation, the value of the invariant $\widetilde{\mu}$ may strictly increase under blowup, even when the value of the invariant σ stays the same. This violates the principle that our strand of invariants, consisting of the units of the form $(\sigma, \widetilde{\mu}, s)$, should never increase under blowup.

While writing Part II, we came to realize that we can establish the nonsingularity principle, as formulated below, using only the $\mathfrak{D}$-saturation but not the $\mathfrak{R}$-saturation. This indicates that we can construct an algorithm, still in the framework of the IFP, without using the $\mathfrak{R}$-saturation, and hence that we may avoid the problem of termination, as well as the other technical problems, that the use of the $\mathfrak{R}$-saturation brings about.

Even though we are still in the evolution process of the IFP (see 0.3.1 for the current status of the IFP), we consider this new nonsingularity principle a substantial step forward in our quest to construct an algorithm for the local or global resolution of singularities in positive characteristic.

In this appendix, R represents the coordinate ring of an affine open subset $\operatorname{Spec} R$ of a nonsingular variety W of $\dim W = d$ over an algebraically closed field k with $\operatorname{char}(k) = p > 0$ or $\operatorname{char}(k) = 0$, where in the latter case we formally set $p = \infty$ (cf. 0.2.3.2.1 and Definition 3.1.1.1(2) in Part I).

§A.1. Nonsingularity principle using only $\mathfrak{D}$-saturation but not $\mathfrak{R}$-saturation

A.1.1. Statement

Theorem A.1.1.1. *Let $\mathbb{I}$ be an idealistic filtration over R. Let $P \in \operatorname{Spec} R \subset W$ be a closed point.*

(1) *Assume that $\mathbb{I}$ is $\mathfrak{D}$-saturated and that $\widetilde{\mu}(P) = \infty$. Then there exist a regular system of parameters $(x_1, \dots, x_N, y_{N+1}, \dots, y_d)$ at P and nonnegative integers $e_1 \leq \cdots \leq e_N$ such that $\mathbb{H} = \{(x_l^{p^{e_l}}, p^{e_l})\}_{l=1}^N$ is an LGS of $\mathbb{I}_P$ and $\mathbb{I}_P = G_{R_P}(\mathbb{H})$.*

(2) *Assume further that $\mathbb{I}$ is of r.f.g. type. Then there exists an affine open neighborhood $P \in U_P = \operatorname{Spec} R_r$ of P such that $(x_1, \dots, x_N, y_{N+1}, \dots, y_d)$ is a regular system of parameters over U_P, and*

$$\mathbb{H} = \{(x_l^{p^{e_l}}, p^{e_l})\}_{l=1}^N \subset \mathbb{I}_r, \quad \mathbb{I}_r = G_{R_r}(\mathbb{H}).$$

(Note that R_r and $\mathbb{I}_r$ represent the localizations of R and $\mathbb{I}$ at $r \in R$, respectively.) In particular,

- $\operatorname{Supp}(\mathbb{I}) \cap U_P = V(x_1, \dots, x_N)$*, which is hence nonsingular,*
- $(\sigma(Q), \widetilde{\mu}(Q)) = (\sigma(P), \infty)$ *for any closed point $Q \in \operatorname{Supp}(\mathbb{I}) \cap U_P$.*

Remark A.1.1.2. (1) It is straightforward to see that assertion (1) actually gives the following characterization: An idealistic filtration $\mathbb{I}_P$ over R_P is $\mathfrak{D}$-saturated and $\widetilde{\mu}(P) = \infty$ if and only if there exist a regular system of parameters

$(x_1, \ldots, x_N, y_{N+1}, \ldots, y_d)$ and a subset of the form $\mathbb{H} = \{(x_l^{p^{e_l}}, p^{e_l})\}_{l=1}^{N} \subset \mathbb{I}_P$ such that $\mathbb{I}_P = G_{R_P}(\mathbb{H})$. (The subset $\mathbb{H}$ is then automatically an LGS of $\mathbb{I}_P$.)

(2) We construct the strand of invariants in our algorithm (cf. 0.2.3.2.2 in the introduction to Part I), and in year 0 it takes the form

$$\mathrm{inv}_{\mathrm{new}}(P) = (\sigma_0^1, \widetilde{\mu}_0^1, s_0^1)(\sigma_0^2, \widetilde{\mu}_0^2, s_0^2) \cdots (\sigma_0^{n-1}, \widetilde{\mu}_0^{n-1}, s_0^{n-1})(\sigma_0^n, \widetilde{\mu}_0^n, s_0^n),$$

with the last n-th unit $(\sigma_0^n, \widetilde{\mu}_0^n, s_0^n)$ being equal to $(\sigma_0^n, \infty, 0)$. The subscript "$_0$" refers to year "0", while the superscript "^{j}" refers to stage "j". (Note that, if we insert the new invariant $\widetilde{\nu}$ so that the unit changes from the triplet $(\sigma, \widetilde{\mu}, s)$ to the quadruplet $(\sigma, \widetilde{\mu}, \widetilde{\nu}, s)$, then the strand of invariants also changes accordingly (cf. 0.3.1).)

The (local) maximum locus of the strand of invariants coincides with the support $\mathrm{Supp}(\mathbb{I}_0^n)$ of the last n-th modification $\mathbb{I}_0^n$. (Note that in year 0 we always have $\widetilde{\mu} > 1$ and hence there is no gap between the (local) maximum locus and the support of the modification, an anomaly observed when $\widetilde{\mu} = 1$.) The idealistic filtration $\mathbb{I}_0^n$ is $\mathfrak{D}$-saturated with $\widetilde{\mu}(P) = \infty$. Therefore, applying Theorem A.1.1.1, we conclude that $\mathrm{Supp}(\mathbb{I}_0^n)$ is nonsingular (in a neighborhood of P). (Note that all the idealistic filtrations we deal with in our algorithm are of r.f.g. type.) Therefore, we conclude that the center of blowup, which is chosen to be the maximum locus of the strand, is nonsingular. This is why Theorem A.1.1.1 is called the (new) nonsingularity principle of the center. (After year 0, we have to make several technical adjustments, including an adjustment to overcome the gap between the (local) maximum locus and the support of the last modification and another adjustment to introduce the $\mathfrak{D}_E$-saturation in the presence of the exceptional divisor E instead of the usual $\mathfrak{D}$-saturation. The basic tool for us to guarantee the nonsingularity of the center, however, is still Theorem A.1.1.1.)

(3) If we assume further that $\mathbb{I}_P$ is $\mathfrak{R}$-saturated, then having assertion (1), we immediately come to the conclusion that $e_1 = \cdots = e_N = 0$, i.e., all the elements in the LGS (and hence of any LGS) are concentrated at level 1. That is, we obtain the old nonsingularity principle Theorem 4.2.1.1 in Part I as a corollary to the new nonsingularity principle Theorem A.1.1.1 of this appendix.

A.1.2. Proof

Proof for assertion (1).

Step 1. *Show* $\mathbb{I}_P = G_{R_P}(\mathbb{H})$ *for any LGS* $\mathbb{H}$ *of* $\mathbb{I}_P$.

First, note that, if $P \notin \mathrm{Supp}(\mathbb{I})$, then we would have $\mathbb{I}_P = R_P \times \mathbb{R}$ (cf. Lemma 1.1.2.1(1)) and hence $\widetilde{\mu}(P) = 0$. Thus our assumption $\widetilde{\mu}(P) = \infty$ implies

$P \in \mathrm{Supp}(\mathbb{I})$. Second, we claim $\mathbb{I}_P = G_{R_P}(\mathbb{H})$ for *any* LGS $\mathbb{H}$ of $\mathbb{I}_P$. To prove this, we can use the same argument as in the proof of the nonsingularity principle in Chapter 4 of Part I. Note that this part of the proof did not use the assumption that $\mathbb{I}_P$ is $\mathfrak{R}$-saturated.

Alternatively, we can give a proof of the claim using the formal coefficient lemma (cf. Lemma 2.2.2.1), without referring to the arguments in Part I:

Take an element $f \in (\mathbb{I}_P)_a \subset (\widehat{\mathbb{I}_P})_a$, and let $f = \sum_{B \in (\mathbb{Z}_{\geq 0})^N} c_B H^B$ be the power series expansion of the form $(\star)$ as described in Lemma 2.1.2.1. From the formal coefficient lemma it follows that

$$(c_B, a - |[B]|) \in \widehat{\mathbb{I}_P} \quad \forall B \in (\mathbb{Z}_{\geq 0})^N.$$

Suppose there exists $B \in (\mathbb{Z}_{\geq 0})^N$ with $|[B]| < a$ such that $c_B \neq 0$. Then we would have

$$\widetilde{\mu}(P) = \mu_{\mathcal{H}}(\mathbb{I}_P) = \mu_{\mathcal{H}}(\widehat{\mathbb{I}_P}) \leq \frac{\mathrm{ord}_{\mathcal{H}}(c_B)}{a - |[B]|} = \frac{\mathrm{ord}(c_B)}{a - |[B]|} < \infty \quad \text{(cf. Lemma 3.2.1.1)},$$

a contradiction! Therefore, we conclude

$$c_B = 0 \quad \forall B \in (\mathbb{Z}_{\geq 0})^N \text{ with } |[B]| < a.$$

This implies $f \in (H^B\,;\,|[B]| \geq a)$. Since $f \in (\mathbb{I}_P)_a$ is arbitrary, we conclude $(\mathbb{I}_P)_a \subset (H^B\,;\,|[B]| \geq a)$, while the opposite inclusion is obvious. Therefore, we finally conclude

$$(\mathbb{I}_P)_a = (H^B\,;\,|[B]| \geq a) \quad \forall a \in \mathbb{R},$$

which is equivalent to $\mathbb{I}_P = G_{R_P}(\mathbb{H})$ (cf. Lemma 2.2.1.2 in Part I).

Step 2. *Inductive construction of an LGS and a regular system of parameters of the desired form via claim* $(\diamondsuit)$.

Now, by induction, we assume that we have found an LGS $\mathbb{H} = \{(h_{ij}, p^{e_i})\}$ of $\mathbb{I}_P$ and a regular system of parameters $(\{x_{ij}\}, y_{N+1}, \ldots, y_d)$ at P such that

$$h_{ij} = \begin{cases} x_{ij}^{p^{e_i}} & \text{if } e_i < e_u, \\ x_{ij}^{p^{e_i}} \bmod \mathfrak{m}_P{}^{p^{e_i}+1} & \text{if } e_i \geq e_u. \end{cases}$$

Note that we use double subscripts for elements in the LGS, where the first subscript indicates the level p^{e_i} with $e_1 < \cdots < e_M$. So we have the total of $N = \#\{(i,j)\}$ elements at M distinct levels in the LGS. (See 1.3.1.) The inductive assumption means that we have found an LGS and a regular system of parameters of the desired form up to $i = u - 1$.

We want to show, by replacing h_{ij} and x_{ij} for $i = u$ via the use of claim ($\diamondsuit$), which we state next, that we can also have

$$h_{ij} = \begin{cases} x_{ij}^{p^{e_i}} & \text{if } e_i < e_{u+1}, \\ x_{ij}^{p^{e_i}} \bmod \mathfrak{m}_P{}^{p^{e_i}+1} & \text{if } e_i \geq e_{u+1}. \end{cases}$$

Step 3. *Statement and proof of claim* ($\diamondsuit$).

For $l \in \mathbb{Z}_{\geq 0}$, we set

$$J_l = F^{e_u}(\mathfrak{m}_P) + \Big(X^{[C]} = \prod_{e_i < e_u} x_{ij}^{p^{e_i} c_{ij}} \,;\, C = (c_{ij} \,;\, e_i < e_u),\ |[C]| \geq p^{e_u}\Big) \\ + (\mathbb{I}_P)_{p^{e_u}} \cap \mathfrak{m}_P{}^{p^{e_u}+1} + \mathfrak{m}_P{}^l,$$

where F represents the Frobenius map. This step is devoted to proving the following claim:

$$(\diamondsuit) \qquad (\mathbb{I}_P)_{p^{e_u}} \subset J_l \quad \forall l \in \mathbb{Z}_{\geq 0}.$$

Observe that

$$\begin{aligned} (\mathbb{I}_P)_{p^{e_u}} &= (H^B \,;\, |[B]| \geq p^{e_u}) \quad (\text{since } \mathbb{I}_P = G_{R_P}(\mathbb{H})) \\ &= (X^{[C]} \,;\, C = (c_{ij} \,;\, e_i < e_u),\ |[C]| \geq p^{e_u}) \\ &\quad + (h_{ij} \,;\, e_i = e_u) + (h_{ij} \,;\, e_i > e_u) \\ &\subset (X^{[C]} \,;\, C = (c_{ij} \,;\, e_i < e_u),\ |[C]| \geq p^{e_u}) + F^{e_u}(\mathfrak{m}_P) + \mathfrak{m}_P{}^{p^{e_u}+1} \\ &\subset J_{p^{e_u}+1}. \end{aligned}$$

Therefore, the required inclusion holds for $l \leq p^{e_u} + 1$.

Now assume, by induction, that $(\mathbb{I}_P)_{p^{e_u}} \subset J_l$ holds for a fixed $l \geq p^{e_u} + 1$. We want to show $(\mathbb{I}_P)_{p^{e_u}} \subset J_{l+1}$. Take an arbitrary element $f \in (\mathbb{I}_P)_{p^{e_u}} \subset J_l$. We may choose $\{\alpha_{ST} \,;\, S, T\} \subset k$ such that

$$f - \sum_{|(S,T)|=l} \alpha_{ST} X^S Y^T \in J_{l+1}.$$

Note that then there exists $w \in \mathfrak{m}_P$ such that

$$(\heartsuit) \qquad w^{p^{e_u}} + \sum_{|(S,T)|=l} \alpha_{ST} X^S Y^T \in (\mathbb{I}_P)_{p^{e_u}} + \mathfrak{m}_P{}^{l+1}.$$

Set

$$s_{ij} = p^{e_i} s_{ij,q} + s_{ij,r} \quad \text{with } 0 \leq s_{ij,r} < p^{e_i}, \quad S_q = (s_{ij,q}) \quad \text{and} \quad S_r = (s_{ij,r}).$$

Then we have $S = [S_q] + S_r$ and $X^S Y^T = X^{[S_q]} X^{S_r} Y^T$. We analyze the terms in

$$\sum_{|(S,T)|=l} \alpha_{ST} X^S Y^T = \sum_{|(S,T)|=l} \alpha_{ST} X^{[S_q]} X^{S_r} Y^T.$$

Case 1: $S_q = 0$. In this case, we write for simplicity

$$X^S Y^T = X^{S_r} Y^T = Z^V$$

by setting

$$\begin{cases} (X,Y) = (\{x_{ij}\}, y_{N+1}, \ldots, y_d) = (z_1, \ldots, z_d) = Z, \\ (S,T) \ = (S_r, T) = (\{s_{ij,r}\}, t_{N+1}, \ldots, t_d) = (v_1, \ldots, v_d). \end{cases}$$

Subcase 1.1: $p^{e_u} \mid V$. In this subcase, we conclude

$$\alpha_{ST} X^S Y^T = \alpha_{ST} Z^V \in F^{e_u}(\mathfrak{m}_P) \subset J_{l+1}.$$

Subcase 1.2: $p^{e_u} \nmid V$. In this subcase, let v_ω be a factor, not divisible by p^{e_u}, of V. Set

$$v_\omega = p^{e_u} v_{\omega,q} + v_{\omega,r} \qquad \text{with } 0 < v_{\omega,r} < p^{e_u}.$$

Apply $\partial_{z_\omega^{v_{\omega,r}}}$ to ($\heartsuit$) and obtain

$$\begin{aligned} \partial_{z_\omega^{v_{\omega,r}}} \Big(w^{p^{e_u}} + \sum_{|(S,T)|=l} \alpha_{ST} X^S Y^T \Big) &= \partial_{z_\omega^{v_{\omega,r}}} \Big(\sum_{|(S,T)|=l} \alpha_{ST} X^S Y^T \Big) \\ &= \alpha_{ST} Z^{V - v_{\omega,r} \mathbf{e}_\omega} + (\text{other monomials of degree } l - v_{\omega,r}) \\ &\in ((\mathbb{I}_P)_{p^{e_u} - v_{\omega,r}} + \mathfrak{m}_P{}^{l - v_{\omega,r}+1}) \cap \mathfrak{m}_P{}^{l - v_{\omega,r}} \\ &= (\mathbb{I}_P)_{p^{e_u} - v_{\omega,r}} \cap \mathfrak{m}_P{}^{l - v_{\omega,r}} + \mathfrak{m}_P{}^{l - v_{\omega,r}+1}. \end{aligned}$$

On the other hand, we observe that

$$(\mathbb{I}_P)_{p^{e_u} - v_{\omega,r}} \cap \mathfrak{m}_P{}^{l - v_{\omega,r}} \subset \sum_{1 \le i \le M} h_{ij} \mathfrak{m}_P{}^{l - v_{\omega,r} - p^{e_i}}.$$

(We use the convention that $\mathfrak{m}_P{}^{l - v_{\omega,r} - p^{e_i}} = R_P$ if $l - v_{\omega,r} - p^{e_i} \le 0$.)

In fact, let $g \in (\mathbb{I}_P)_{p^{e_u} - v_{\omega,r}} \cap \mathfrak{m}_P{}^{l - v_{\omega,r}}$ be an arbitrary element, and $g = \sum c_B(g) H^B$ the power series expansion of the form ($\star$) as described in Lemma 2.1.2.1. Then it follows from the condition $\widetilde{\mu}(P) = \infty$ and $0 < p^{e_u} - v_{\omega,r}$ that $c_{\mathbb{O}}(g) = 0$ (cf. Lemma 3.2.1.1), and from the construction that $\mathrm{ord}_P(c_B(g)) \ge (l - v_{\omega,r}) - |[B]|$ for any $B \in (\mathbb{Z}_{\ge 0})^N$ (cf. Remark 2.1.2.2(1)). Therefore, $f \in \sum_{1 \le i \le M} h_{ij} \mathfrak{m}_P{}^{l - v_{\omega,r} - p^{e_i}}$. This proves the inclusion above. (Note that the inclusion can also be derived using Lemma 4.1.2.3 in Part I via the fact that $\mathbb{I}_P = G_{R_P}(\mathbb{H})$.)

However, this inclusion implies that any monomial of degree $l - v_{\omega,r}$ in the power series expansion of an element in $(\mathbb{I}_P)_{p^{e_u} - v_{\omega,r}} \cap \mathfrak{m}_P{}^{l - v_{\omega,r}}$ with respect to the regular system of parameters $(x_1, \ldots, x_N, y_{N+1}, \ldots, y_d)$ should be divisible by some element in the set $\{x_{ij}^{p^{e_i}}\,;\, 1 \leq i \leq M\}$, and hence that the monomial $Z^{V - v_{\omega,r}\mathbf{e}_\omega}$ cannot appear as $S_q = 0$.

Therefore, in this subcase, we conclude $\alpha_{ST} = 0$.

Case 2: $S_q \neq 0$

Subcase 2.1: $s_{ij,q} > 0$ for some $i \geq u$. In this subcase,

$$X^S Y^T \in x_{ij}^{p^{e_i}} \mathfrak{m}_P{}^{l-p^{e_i}} \subset (h_{ij} + \mathfrak{m}_P{}^{p^{e_i}+1})\mathfrak{m}_P{}^{l-p^{e_i}} \subset h_{ij}\mathfrak{m}_P{}^{l-p^{e_i}} + \mathfrak{m}_P{}^{l+1} \subset (\mathbb{I}_P)_{p^{e_u}} \cap \mathfrak{m}_P{}^{p^{e_u}+1} + \mathfrak{m}_P{}^{l+1} \subset J_{l+1}.$$

(Note that, in order to obtain the second last inclusion above, we use the fact that $h_{ij} \in \mathfrak{m}_P{}^{p^{e_u}+1}$ if $i > u$, and the condition $l \geq p^{e_u} + 1$ if $i = u$.) Therefore, $\alpha_{ST} X^S Y^T \in J_{l+1}$.

Subcase 2.2: $s_{ij,q} = 0$ for any $i \geq u$ and $|[S_q]| \geq p^{e_u}$. In this subcase, we conclude

$$\alpha_{ST} = \alpha_{ST} X^{[S_q]} X^{S_r} Y^T \in (X^{[C]}\,;\, C = (c_{ij}\,;\, e_i < e_u),\, |[C]| \geq p^{e_u}) \subset J_{l+1}.$$

Subcase 2.3: $s_{ij,q} = 0$ for any $i \geq u$ and $|[S_q]| < p^{e_u}$. Note that $0 < |[S_q]|$ by the case assumption. In this subcase, apply $\partial_{X^{[S_q]}}$ to $(\heartsuit)$ and obtain

$$\begin{aligned}\partial_{X^{[S_q]}}\Big(w^{p^{e_u}} + \sum_{|(S,T)|=l} \alpha_{ST} X^S Y^T\Big) &= \partial_{X^{[S_q]}}\Big(\sum_{|(S,T)|=l} \alpha_{ST} X^S Y^T\Big) \\ &= \alpha_{ST} X^{S_r} Y^T + (\text{other monomials of degree } l - |[S_q]|) \\ &\in ((\mathbb{I}_P)_{p^{e_u} - |[S_q]|} + \mathfrak{m}_P{}^{l-|[S_q]|+1}) \cap \mathfrak{m}_P{}^{l-|[S_q]|} \\ &= (\mathbb{I}_P)_{p^{e_u} - |[S_q]|} \cap \mathfrak{m}_P{}^{l-|[S_q]|} + \mathfrak{m}_P{}^{l-|[S_q]|+1}.\end{aligned}$$

On the other hand, we observe

$$(\mathbb{I}_P)_{p^{e_u} - |[S_q]|} \cap \mathfrak{m}_P{}^{l-|[S_q]|} \subset \sum_{1 \leq i \leq M} h_{ij} \mathfrak{m}_P{}^{l-|[S_q]|-p^{e_i}}.$$

(We use the convention that $\mathfrak{m}_P{}^{l-|[S_q]|-p^{e_i}} = R_P$ if $l - |[S_q]| - p^{e_i} \leq 0$. The inclusion follows from the same argument as in Subcase 1.2.)

However, this inclusion implies that any monomial of degree $l - |[S_q]|$ in the power series expansion of an element in $(\mathbb{I}_P)_{p^{e_u} - |[S_q]|} \cap \mathfrak{m}_P{}^{l-|[S_q]|}$ with respect to the regular system of parameters $(x_1, \ldots, x_N, y_{N+1}, \ldots, y_d)$ should be divisible by some element in the set $\{x_{ij}^{p^{e_i}}\,;\, 1 \leq i \leq M\}$, and hence that the monomial $X^{S_r} Y^T$ cannot appear.

Therefore, in this subcase, we conclude $\alpha_{ST} = 0$.

From the above analysis of the terms in $\sum_{|(S,T)|=l} \alpha_{ST} X^S Y^T$, it follows that

$$f \in \sum_{|(S,T)|=l} \alpha_{ST} X^S Y^T + J_{l+1} = J_{l+1}.$$

Since $f \in (\mathbb{I}_P)_{p^{e_u}}$ is arbitrary, we conclude $(\mathbb{I}_P)_{p^{e_u}} \subset J_{l+1}$, completing the inductive argument for claim $(\diamondsuit)$.

Step 4. *Finishing argument for the inductive construction.*

Claim $(\diamondsuit)$ states

$$\begin{aligned}(\mathbb{I}_P)_{p^{e_u}} \subset J_l = F^{e_u}(\mathfrak{m}_P) + (X^{[C]}\,;\, C = (c_{ij}\,;\, e_i < e_u),\, |[C]| \geq p^{e_u}) \\ + (\mathbb{I}_P)_{p^{e_u}} \cap \mathfrak{m}_P{}^{p^{e_u}+1} + \mathfrak{m}_P{}^l \quad \forall l \in \mathbb{Z}_{\geq 0}.\end{aligned}$$

This implies

$$\begin{aligned}(\mathbb{I}_P)_{p^{e_u}} \subset F^{e_u}(\mathfrak{m}_P) + (X^{[C]}\,;\, C = (c_{ij}\,;\, e_i < e_u),\, |[C]| \geq p^{e_u}) \\ + (\mathbb{I}_P)_{p^{e_u}} \cap \mathfrak{m}_P{}^{p^{e_u}+1} + F^{e_u}(\mathfrak{m}_P{}^l) R_P \quad \forall l \in \mathbb{Z}_{\geq 0}.\end{aligned}$$

Since R_P is a finite $F^{e_u}(R_P)$-module, including

$$F^{e_u}(\mathfrak{m}_P) + (X^{[C]}\,;\, C = (c_{ij}\,;\, e_i < e_u),\, |[C]| \geq p^{e_u}) + (\mathbb{I}_P)_{p^{e_u}} \cap \mathfrak{m}_P{}^{p^{e_u}+1}$$

as an $F^{e_u}(R_P)$-submodule, we conclude (cf. [Mat86, page 62, last line]) that

$$\begin{aligned}(\mathbb{I}_P)_{p^{e_u}} \subset \bigcap_{l \in \mathbb{Z}_{\geq 0}} [F^{e_u}(\mathfrak{m}_P) + (X^{[C]}\,;\, C = (c_{ij}\,;\, e_i < e_u),\, |[C]| \geq p^{e_u}) \\ + (\mathbb{I}_P)_{p^{e_u}} \cap \mathfrak{m}_P{}^{p^{e_u}+1} + F^{e_u}(\mathfrak{m}_P{}^l) R_P] \\ = F^{e_u}(\mathfrak{m}_P) + (X^{[C]}\,;\, C = (c_{ij}\,;\, e_i < e_u),\, |[C]| \geq p^{e_u}) + (\mathbb{I}_P)_{p^{e_u}} \cap \mathfrak{m}_P{}^{p^{e_u}+1}.\end{aligned}$$

Since $(X^{[C]}\,;\, C = (c_{ij}\,;\, e_i < e_u),\, |[C]| \geq p^{e_u}) \subset (\mathbb{I}_P)_{p^{e_u}}$, we also conclude

$$\begin{aligned}(\mathbb{I}_P)_{p^{e_u}} = F^{e_u}(\mathfrak{m}_P) \cap (\mathbb{I}_P)_{p^{e_u}} + (X^{[C]}\,;\, C = (c_{ij}\,;\, e_i < e_u),\, |[C]| \geq p^{e_u}) \\ + (\mathbb{I}_P)_{p^{e_u}} \cap \mathfrak{m}_P{}^{p^{e_u}+1}.\end{aligned}$$

Now choose $\{h'_{uj} = x'_{uj}{}^{p^{e_u}}\} \subset F^{e_u}(\mathfrak{m}_P) \cap (\mathbb{I}_P)_{p^{e_u}}$ such that

$$\{h'_{uj} \bmod \mathfrak{m}_P^{p^{e_u}+1}\} \cup \{X^{[C]} \bmod \mathfrak{m}_P^{p^{e_u}+1}\,;\, C = (c_{ij}\,;\, e_i < e_u),\, |[C]| = p^{e_u}\}$$

forms a k-basis of $L(\mathbb{I}_P)_{p^{e_u}}$.

In order to finish the inductive argument (cf. Step 2) to complete the proof for assertion (1), we have only to replace $\{h_{ij}\}$ and $\{x_{ij}\}$ with $\{h'_{ij}\}$ and $\{x'_{ij}\}$.

Proof for assertion (2). Take a regular system of parameters $(x_1, \ldots, x_N, y_{N+1}, \ldots, y_d)$ and an LGS $\mathbb{H}$ of $\mathbb{I}_P$ as described in assertion (1).

By choosing an affine neighborhood $P \in U_P = \operatorname{Spec} R_r$ of P sufficiently small, we may assume that $(x_1, \ldots, x_N, y_{N+1}, \ldots, y_d)$ is a regular system of parameters over U_P and that $\mathbb{H} = \{(x_l^{p^{e_l}}, p^{e_l})\}_{l=1}^N \subset \mathbb{I}_r$.

Now we know by assumption that $\mathbb{I}$ is of r.f.g. type, i.e., $\mathbb{I} = G_R(\{(f_\lambda, a_\lambda)\}_{\lambda \in \Lambda})$ for some $\{(f_\lambda, a_\lambda)\}_{\lambda\in\Lambda} \subset R \times \mathbb{Q}_{\geq 0}$ with $\#\Lambda < \infty$.

Since $\mathbb{I}_P = G_{R_P}(\mathbb{H})$, we can write each f_λ as a finite sum of the form $\sum g_{B,\lambda} H^B$ with $g_{B,\lambda} \in R_P$ and $|[B]| \geq a_\lambda$. By shrinking $U_P = \operatorname{Spec} R_r$ if necessary, we may assume that the coefficients $g_{B,\lambda}$ are in R_r for all B and $\lambda \in \Lambda$. Then we have

$$\mathbb{I}_r = G_{R_r}(\{(f_\lambda, a_\lambda)\}_{\lambda\in\Lambda}) \subset G_{R_r}(\mathbb{H}).$$

Since the opposite inclusion $\mathbb{I}_r \supset G_{R_r}(\mathbb{H})$ is obvious, we conclude $\mathbb{I}_r = G_{R_r}(\mathbb{H})$. It follows immediately from the above conclusions that

$$\begin{aligned}\operatorname{Supp}(\mathbb{I}) \cap U_P &= \operatorname{Supp}(\mathbb{I}_r) = \operatorname{Supp}(G_{R_r}(\mathbb{H}))\\ &= \{Q \in U_P\,;\, \mu_Q(x_l^{p^{e_l}}, p^{e_l}) \geq 1 \text{ for } l = 1, \ldots, N\}\\ &= V(x_1, \ldots, x_N),\end{aligned}$$

which is nonsingular.

Given any closed point $Q \in \operatorname{Supp}(\mathbb{I}) \cap U_P = V(x_1, \ldots, x_N)$, it also follows from the above conclusions that $(x_1, \ldots, x_N)$ is a part of a regular system of parameters at Q with a subset $\mathbb{H} = \{(x_l^{p^{e_l}}, p^{e_l})\}_{l=1}^N \subset \mathbb{I}_Q$ such that $\mathbb{I}_Q = G_{R_Q}(\mathbb{H})$. This implies that $\mathbb{H}$ is an LGS of $\mathbb{I}_Q$ and that $\widetilde{\mu}(Q) = \infty$. Therefore, $(\sigma(Q), \widetilde{\mu}(Q)) = (\sigma(P), \infty)$. This concludes the proof of Theorem A.1.1.1.

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