

Pseudoconvex Domains on a Kähler Manifold with Positive Holomorphic Bisectional Curvature¹⁾

By

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§ 1. Introduction

Let M be an n -dimensional complex manifold with a *real analytic* kähler metric. Throughout this paper, a kähler manifold is assumed to have a real analytic kähler metric without mentioning it. A relatively compact domain D in M is called a pseudoconvex (resp. strongly pseudoconvex) domain if there exist a neighborhood U of p and a pseudoconvex (resp. strongly pseudoconvex) function φ on U satisfying $D \cap U = \{\varphi < 0\}$ for each boundary point $p \in \partial D$. We write simply s -pseudoconvex domains (resp. functions) for strongly pseudoconvex domains (resp. functions). Note that pseudoconvex domains are not always Stein manifolds.

The purpose of the present paper is to show the following theorem:

Main Theorem. *If M admits a kähler metric with positive holomorphic bisectional curvature (see, Definition(2.7)), then pseudoconvex domains are always holomorphically convex.*

For the definition of holomorphic bisectional curvature and its basic properties, see S. I. Goldberg and S. Kobayashi [2].

Since the holomorphic bisectional curvature of the Fubini-Study metric of the complex projective space \mathbf{P}^n is positive and \mathbf{P}^n admits no exceptional sets in the sense of H. Grauert [5], we find that pseudoconvex domains in \mathbf{P}^n are always Stein manifolds, which implies the result of

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R. Fujita [1] and A. Takeuchi [8].

We shall prove our Main Theorem by using the variations of arc lengths which are well known in Riemannian Geometry (see, Lemma (2, 3) in § 2) and by using a nice property of real analytic kähler metrics which is essentially due to A. Takeuchi (see, Lemma(2, 10) in § 2).

For a point $p \in D$, we set

$$d(p) = \inf_{q \in \partial D} d(p, q) \quad \text{and} \quad \varphi(p) = -\log d(p).$$

The essential step in the proof of Main Theorem is the following Lemma (for the proof, see § 4):

Key Lemma. *Let D be an s -pseudoconvex domain whose boundary is a real submanifold in M . Suppose that the holomorphic bisectional curvature is bounded below by a positive constant ρ . Then there exists a positive constant δ^* such that*

$$W(\varphi) \geq \rho/8 \quad \text{for} \quad p \in D_{\delta^*},$$

where $W(\varphi)$ means the infimum of the eigenvalues of the Hessian of φ on D_{δ^*} and $D_{\delta} = \{p \in D: d(p) < \delta\}$.

By this Lemma we prove our Main Theorem in § 5.

The author understands from Professor K. Saito that Professor G. Elencwajg has announced a similar result at A. M. S. Summer Institute 1975 (Williamstown). The present work was done independently of this.

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§ 2. Variations of Arc Lengths and Real Analytic Kähler Metrics

First we consider a Riemannian manifold M . We follow notations in S. Kobayashi and K. Nomizu [7]. For vectors X, Y we denote their symmetric inner product by $g(X, Y)$. Connections, covariant differentiations and Riemannian curvature tensor are assumed to be induced from the inner product g in a canonical manner. To state the theorem of Synge,

we prepare more notations: Let N and P be two submanifolds in M and let p and q be two points in N and P respectively. Also suppose that a geodesic curve σ between p and q is given. A C^∞ -mapping ψ from a rectangle $[-\varepsilon, \varepsilon] \times [\alpha, \beta]$ to M is called a variation of σ between N and P , if the following condition is satisfied: For any fixed $\xi \in [-\varepsilon, \varepsilon]$, $\psi(\xi, t)$ gives a parametrization of an arc C_ξ between N and P and especially when $\xi=0$, $\psi(0, t)$ gives a parametrization of σ . We want to know the values of the first and the second derivatives of arc length of C_ξ at $\xi=0$. By $D(\xi)$ we denote the arc length C_ξ :

$$D(\xi) = \int_\alpha^\beta \|\dot{\psi}(\xi, t)\| dt,$$

where $\|\dot{\psi}(\xi, t)\|$ is the length of the tangent vector $\dot{\psi}(\xi, t)$ of C_ξ at (ξ, t) . We set

$$X = \psi_*(D_1) \text{ and } Y = \psi_*(D_2),$$

where $D_1 = \partial/\partial t$ and $D_2 = \partial/\partial \xi$. Let $\{U\}$ be a local coordinate covering on some neighborhood of σ whose local coordinates are denoted by x^1, x^2, \dots, x^m . Suppose that Y is expressed as

$$Y = \sum_{i=1}^m Y^i \frac{\partial}{\partial x^i} \circ \psi \text{ on } U.$$

We define $\nabla_{D_i} Y$ ($i=1, 2$) by

$$(2.1) \quad \nabla_{D_1} Y = \sum_{k=1}^m \frac{\partial Y^k}{\partial t} \frac{\partial}{\partial x^k} \circ \psi + \sum_{i,j,k=1}^m \frac{\partial(x^i \circ \psi)}{\partial t} Y^j \Gamma_{ij}^k \circ \psi \frac{\partial}{\partial x^k} \circ \psi,$$

$$(2.2) \quad \nabla_{D_2} Y = \sum_{k=1}^m \frac{\partial Y^k}{\partial \xi} \frac{\partial}{\partial x^k} \circ \psi + \sum_{i,j,k=1}^m \frac{\partial(x^i \circ \psi)}{\partial \xi} Y^j \Gamma_{ij}^k \circ \psi \frac{\partial}{\partial x^k} \circ \psi,$$

where Γ_{ij}^k is defined by

$$\nabla_{(\partial/\partial x^i)} \left(\frac{\partial}{\partial x^j} \right) = \sum \Gamma_{ij}^k \frac{\partial}{\partial x^k}.$$

Now Synge's formula is stated as follows:

Lemma (2.3). *(The first and the second variations of arc lengths) Suppose that $\|\dot{\psi}(0, t)\| = 1$ for each t . Then we obtain*

(i) $D'(0) = g(X, Y) \circ \psi(0, t)|_\alpha^beta,$

$$(ii) \quad D''(0) = \int_{\alpha}^{\beta} g(\nabla_{D_1} \tilde{Y}, \nabla_{D_1} \tilde{Y}) \circ \psi(0, t) dt \\ - \int_{\alpha}^{\beta} g(R(Y, X)X, Y) \circ \psi(0, t) dt + g(\nabla_{D_2} Y, X) \circ \psi(0, t) \Big|_{\alpha}^{\beta},$$

where $\tilde{Y} = Y - g(Y, X)X$.

For the proof of this Lemma, see D. Gromoll, W. Klingenberg and W. Meyer [6, Satz, p. 122].

In what follows we restrict ourselves to kähler manifolds. Let M be a kähler manifold whose metric is denoted by

$$(2.4) \quad ds^2 = 2 \sum g_{\alpha\bar{\beta}} dz^{\alpha} \cdot d\bar{z}^{\beta}.$$

As stated in Introduction, we assume that $g_{\alpha\bar{\beta}}$ ($\alpha, \beta = 1, 2, \dots, n$) are real analytic functions. In a canonical manner, M may be considered as a Riemannian manifold by the following metric:

$$g_{\alpha\bar{\beta}} = g\left(\frac{\partial}{\partial z^{\alpha}}, \frac{\partial}{\partial \bar{z}^{\beta}}\right).$$

By definition, we see that

$$g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0, \quad \overline{g_{\alpha\bar{\beta}}} = g_{\bar{\alpha}\beta} \quad \text{and} \quad g_{\alpha\bar{\beta}} = g_{\bar{\beta}\alpha}.$$

When we say connections or Riemannian curvature tensors, we mean ones which are induced from the above metric by the canonical way. By J we denote the complex structure of M .

We set

$$z^{\alpha} = x^{\alpha} + \sqrt{-1}y^{\alpha}, \quad \bar{z}^{\alpha} = x^{\alpha} - \sqrt{-1}y^{\alpha}$$

and

$$\frac{\partial}{\partial z^{\alpha}} = \frac{1}{2} \left(\frac{\partial}{\partial x^{\alpha}} - \sqrt{-1} \frac{\partial}{\partial y^{\alpha}} \right), \quad \frac{\partial}{\partial \bar{z}^{\alpha}} = \frac{1}{2} \left(\frac{\partial}{\partial x^{\alpha}} + \sqrt{-1} \frac{\partial}{\partial y^{\alpha}} \right).$$

We infer that

$$J\left(\frac{\partial}{\partial x^{\alpha}}\right) = \frac{\partial}{\partial y^{\alpha}} \quad \text{and} \quad J\left(\frac{\partial}{\partial y^{\alpha}}\right) = -\frac{\partial}{\partial x^{\alpha}}.$$

In terms of holomorphic coordinates z^1, z^2, \dots, z^n , connections and some of Riemannian curvature tensors can be expressed as follows (see, S. Kobayashi and K. Nomizu [7, II, P. 157]):

$$(2.5) \quad \Gamma_{\beta\bar{\gamma}}^{*\alpha} = \sum_{\varepsilon=1}^n g^{\alpha\bar{\varepsilon}} \frac{\partial g_{\varepsilon\beta}}{\partial z^{\gamma}},$$

$$K_{\alpha\beta\bar{\gamma}\bar{\delta}} = \frac{\partial^2 g_{\alpha\beta}}{\partial z^{\gamma} \partial \bar{z}^{\delta}} - \sum_{\tau, \varepsilon=1}^n g^{\bar{\varepsilon}\tau} \frac{\partial g_{\alpha\bar{\varepsilon}}}{\partial z^{\gamma}} \cdot \frac{\partial g_{\beta\bar{\tau}}}{\partial \bar{z}^{\delta}},$$

where $|g^{\alpha\bar{\beta}}|$ denotes the inverse matrix of $|g_{\alpha\bar{\beta}}|$. We see that

$$(2.6) \quad K_{\alpha\beta\bar{\gamma}\bar{\delta}} = K_{\gamma\bar{\delta}\alpha\bar{\beta}} = -K_{\alpha\beta\bar{\delta}\bar{\gamma}} = -K_{\beta\alpha\bar{\gamma}\bar{\delta}}, \quad \bar{K}_{\alpha\beta\bar{\gamma}\bar{\delta}} = K_{\bar{\alpha}\bar{\beta}\bar{\gamma}\bar{\delta}}.$$

Definition (2.7). (1) The metric (2.4) is said to have positive holomorphic bisectional (resp. positive sectional) curvature if there exists a positive constant ρ for every point p in M such that the following inequality holds for every pair of vectors $X, Y \in T_p(M)$:

$$(2.8) \quad g(R(Y, X)X, Y) + g(R(Z, X)X, Z) \\ \geq \rho[\{g(X, X) \cdot g(Y, Y) - g(X, Y)^2\} + \{g(X, X) \cdot g(Z, Z) - g(Z, X)^2\}],$$

where $Z = J(Y)$,

$$\text{(resp. } g(R(Y, X)X, Y) \geq \rho\{g(X, X) \cdot g(Y, Y) - g(X, Y)^2\}.$$

We prepare two lemmas on real analytic kähler metrics. The first lemma is a slight generalization of Lemma 2 in A. Takeuchi [9, p. 333]. By using this lemma we prove the second lemma, which will be used in the proof of Key Lemma.

Lemma (2.9). *For any point $p \in M$ and for any positive constant C_0 with $C_0 > 1$, there exist a neighborhood $V(p)$ and a positive constant δ^* such that for any point $q \in V(p)$ and for any geodesic σ through q , there exists a system of local coordinates z^1, z^2, \dots, z^n on $\bar{B}_{\delta^*}(q)$, where $B_{\delta}(q) = \{p \in M: d(p, q) < \delta\}$ and $\bar{B}_{\delta^*}(q)$ denotes the closure of $B_{\delta^*}(q)$, satisfying the following conditions:*

- (1) σ is expressed as follows: $Im z^1 = 0, z^2 = z^3 = \dots = z^n = 0,$
- (2) $2g_{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}} + \sum 2K_{\alpha\beta\bar{\gamma}\bar{\delta}}(0)z^{\gamma}\bar{z}^{\delta} + \dots$ on $\bar{B}_{\delta^*}(q),$
- (3) $2g_{1,1} = 1$ on $\sigma,$
- (4) $\|2g_{\alpha\bar{\beta}}\| < C_0$ and $\|\frac{1}{2}g^{\alpha\bar{\beta}}\| < C_0$ on $\bar{B}_{\delta^*}(q),$

where $\|f\|$ is defined as follows: For a real analytic function f with

$$f = \sum a_{i_1, i_2, \dots, i_n, j_1, j_2, \dots, j_n} (z^1)^{i_1} (z^2)^{i_2} \dots (\bar{z}^n)^{j_n},$$

we define

$$\|f\| = \sum |a_{i_1, i_2, \dots, i_n, j_1, j_2, \dots, j_n} (z^1)^{i_1} (z^2)^{i_2} \dots (\bar{z}^n)^{j_n}|.$$

The proof is almost the same as that of Lemma 2 in A. Takeuchi [9].
By this Lemma, we prove the following Lemma:

Lemma (2·10). (i) *For any point p and for any positive constant C_0 with $C_0 > 1$, there exist positive numbers δ , δ_0 and a neighborhood $V(p)$ such that for any point $q \in V(p)$ and for any geodesic σ through q , there exist a local coordinate neighborhood $U(p)$ and a system of local coordinates z^1, z^2, \dots, z^n on $U(q)$ with following properties:*

$$(0) \quad B_\delta(q) \subset U_{\delta_0/2}(z:q) \subset \bar{U}_{\delta_0}(z:q) \subset U(q),$$

where $U_\sigma(z:q) = \{(z^1, z^2, \dots, z^n) : |z^1| < \sigma, |z^2| < \sigma, \dots, |z^n| < \sigma\}$,

$$(1) \quad \sigma \text{ is expressed as } \text{Im } z^1 = 0, z^2 = z^3 = \dots = z^n = 0,$$

$$(2) \quad 2g_{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}} + \sum 2K_{\alpha\bar{\beta}\gamma\bar{\delta}}(0) z^\gamma \bar{z}^\delta + \dots \text{ on } \bar{U}_{\delta_0}(z:q),$$

$$(3) \quad 2g_{1, \bar{1}} = 1 \text{ on } \sigma,$$

$$(4) \quad \|2g_{\alpha\bar{\beta}}\| < C_0 \text{ and } \|\frac{1}{2}g^{\alpha\bar{\beta}}\| < C_0 \text{ on } \bar{U}_{\delta_0}(z:q).$$

(ii) *For any relatively compact domain D in M and for any positive constant C_0 with $C_0 > 1$, there exist positive constants δ , $\delta_0^{(1)}$, $\delta_0^{(2)}$, \dots , $\delta_0^{(N)}$ such that for any point $p \in \bar{D}$ and for any geodesic σ through p , there exist $\delta_0^{(1)}$ and a local coordinate neighborhood $U_{\delta_0^{(1)}}(z:q)$ satisfying (0)~(4) in (i) with respect to a certain system of local coordinates z^1, z^2, \dots, z^n .*

Proof of (i). We choose a system of local coordinates w_1, w_2, \dots, w_n at p and by $g_{\alpha\bar{\beta}}$ we denote the metric tensor with respect to these local coordinates. By A. Takeuchi [7, p. 329] we can find a neighborhood $V(p)$ and a system of real analytic functions $\lambda_{\gamma\bar{\delta}}(w)$ ($\gamma, \delta = 1, 2, \dots, n$) on $V(p)$ such that

$$\sum_{\alpha, \beta=1}^n g_{\alpha\bar{\beta}} \lambda_{\alpha\gamma} \overline{\lambda_{\beta\delta}} = \delta_{\gamma\delta} \quad \text{on } V(p).$$

Take an arbitrary point q in $V(p)$. We set $w_i(q) = \zeta_i$ for $i=1, 2, \dots, n$. We define a system of local coordinates z_1', z_2', \dots, z_n' by

$$w_\alpha - \zeta_\alpha = \sum_{\beta=1}^n \lambda_{\alpha\beta} z_\beta' \quad \text{for } \alpha=1, 2, \dots, n.$$

Now setting

$$u_\alpha = z_\alpha' + \sum_{k=2}^{\infty} \sum_{i_1+i_2+\dots+i_n=k} \beta_{i_1, i_2, \dots, i_n}^{(\alpha)} z_1'^{i_1} z_2'^{i_2} \dots z_n'^{i_n}. \quad (\alpha=1, 2, \dots, n),$$

we can determine $\{\beta_{i_1, i_2, \dots, i_n}^{(\alpha)}\}$ uniquely so that they satisfy the condition (2). Moreover, after A. Takeuchi, we may assume that u_1, u_2, \dots, u_n give local coordinates on $\overline{B_{\delta^*}}(q)$ (see, Lemma 1 in A. Takeuchi [9]).

Now we choose a constant δ_0' so that $U_{\delta_0'}(u; q) \subset B_{\delta^*}(q)$. We denote the supremum of such δ_0' by $\delta_0''(q)$. Then letting $\delta_0'' = \inf_{q \in V(p)} \delta_0''(q)$, we see that $\delta_0'' > 0$. Next we choose an arbitrary geodesic σ through q . We choose a unitary matrix $|\alpha_{i,j}|$ and a system of local coordinates u_1', u_2', \dots, u_n' with

$$u_i' = \sum_{j=1}^n \alpha_{i,j} u_j \quad (i=1, 2, \dots, n)$$

such that the tangent vector of σ at $u_1'=0, u_2'=0, \dots, u_n'=0$ is normalized as follows:

$$\frac{du_1'}{ds}(0) = 1 \quad \text{and} \quad \frac{du_i'}{ds}(0) = 0 \quad (i=2, \dots, n),$$

where s denotes the length of σ . We infer that $U_{\delta_0''/n}(u'; q) \subset \overline{B_{\delta^*}}(q)$. Owing to the real analyticity of the metric tensor, σ can be expressed as

$$u_i' = \sum_{k=1}^{\infty} a_{i,k} s^k \quad (i=1, 2, \dots, n).$$

Replacing s by $z^1 = s + \sqrt{-1}t$ in u_i' and using the implicit function theorem, we obtain

$$(2 \cdot 11) \quad z^1 = \varphi(u_1').$$

Set

$$z^1 = \varphi(u_1'), \quad z^2 = u_2', \quad \dots, \quad z^n = u_n'.$$

Then we get again a new system of local coordinates on $\overline{B_{\delta^*}}(q)$ satisfying

(1)~(3) (see, section 5 in A. Takeuchi [9]). With reference to the univalence of (2.11) and $du_1'/dz^1(0)=1$, we see that $U_{\delta_0}(z:q) \subset \overline{B}_{\delta_0^*}(q)$, where $\delta_0 = \delta_0''/4n$. Then there exists a constant $\underline{\delta}$ such that $B_{\underline{\delta}}(q) \subset U_{\delta_0/2}(z:q)$. We denote the supremum of such $\underline{\delta}$ by $\delta(q)$. Then we see that $\delta = \inf_{q \in V(p)} \delta(q)$ is positive. So we see that z^1, z^2, \dots, z^n satisfy (0)~(3). Finally making $V(p)$ smaller, we can staisfy (0)~(4). This completes the proof.

Proof of (ii). For each $p \in \overline{D}$ we choose $V(p)$ and $\delta(p)$, $\delta_0(p)$ by using (i). We can cover \overline{D} by $\{V(p_i)\}$ ($i=1, 2, \dots, N$). Letting $\delta_0^{(i)} = \delta_0(p_i)$ ($i=1, 2, \dots, N$) and $\delta = \min \delta(p_i)$, we see that $\delta, \delta_0^{(1)}, \delta_0^{(2)}, \dots, \delta_0^{(N)}$ satisfy the condition in (ii).

In terms of the real coordinates $x^1, y^1, \dots, x^n, y^n$, we denote connection coefficient $\Gamma_{j,k}^i$ by

$$\begin{aligned} \mathcal{V}_{(\partial/\partial x^i)}\left(\frac{\partial}{\partial x^j}\right) &= \sum_{k=1}^n \Gamma_{i,j}^k \frac{\partial}{\partial x^k} + \sum_{k=1}^n \Gamma_{i,j}^{n+k} \frac{\partial}{\partial y^k}, \\ \mathcal{V}_{(\partial/\partial x^i)}\left(\frac{\partial}{\partial y^j}\right) &= \sum_{k=1}^n \Gamma_{i,n+j}^k \frac{\partial}{\partial x^k} + \sum_{k=1}^n \Gamma_{i,n+j}^{n+k} \frac{\partial}{\partial y^k}, \\ \mathcal{V}_{(\partial/\partial y^i)}\left(\frac{\partial}{\partial x^j}\right) &= \sum_{k=1}^n \Gamma_{n+i,j}^k \frac{\partial}{\partial x^k} + \sum_{k=1}^n \Gamma_{n+i,j}^{n+k} \frac{\partial}{\partial y^k}, \\ \mathcal{V}_{(\partial/\partial y^i)}\left(\frac{\partial}{\partial y^j}\right) &= \sum_{k=1}^n \Gamma_{n+i,n+j}^k \frac{\partial}{\partial x^k} + \sum_{k=1}^n \Gamma_{n+i,n+j}^{n+k} \frac{\partial}{\partial y^k}. \end{aligned}$$

Because the connection is induced by the metric, we see that $\Gamma_{j,i}^k = \Gamma_{i,j}^k$ for $i, j, k=1, 2, \dots, 2n$. Also we have

$$\mathcal{V}_{(\partial/\partial x^i)}\left(\frac{\partial}{\partial y^j}\right) = J\left(\mathcal{V}_{(\partial/\partial x^i)}\left(\frac{\partial}{\partial x^j}\right)\right)$$

and

$$\mathcal{V}_{(\partial/\partial x^i)}\left(\frac{\partial}{\partial x^j}\right) = \mathcal{V}_{(\partial/\partial x^j)}\left(\frac{\partial}{\partial x^i}\right), \quad \mathcal{V}_{(\partial/\partial x^i)}\left(\frac{\partial}{\partial y^j}\right) = \mathcal{V}_{(\partial/\partial y^i)}\left(\frac{\partial}{\partial x^j}\right).$$

By these relations, we obtain

$$(2.12) \quad \begin{cases} \Gamma_{i,n+j}^k = -\Gamma_{i,j}^{n+k}, & \Gamma_{n+i,n+j}^k = -\Gamma_{n+i,j}^{n+k}, & \Gamma_{n+i,n+j}^k = -\Gamma_{i,j}^k \\ \Gamma_{i,n+j}^{n+k} = \Gamma_{i,j}^k, & \Gamma_{n+i,n+j}^{n+k} = \Gamma_{n+i,j}^k, & \Gamma_{n+i,n+j}^{n+k} = -\Gamma_{i,j}^{n+k}. \end{cases}$$

Recall that σ is a geodesic. Then with reference to (1) in (i),

Lemma (2·10), we see that

$$(2·13) \quad \Gamma'_{i,\bar{i}} = 0 \text{ and } \Gamma'_{i,n+1} = 0 \text{ on } \sigma \quad (j=1, 2, \dots, 2n),$$

and that

$$(2·14) \quad g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i}\right) = 1 \quad \text{and} \quad g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^i}\right) = 0 \quad \text{on } \sigma.$$

We prepare two propositions which will be used in the proof of Key Lemma (see, § 4):

Proposition (2·15). *Let D be a relatively compact domain and let C_0 be a positive constant with $C_0 > 1$. Suppose that for a point $p \in \bar{D}$, a geodesic σ through p is given. We choose local coordinates as in (ii), Lemma (2·10). Moreover, suppose that $\phi: \mathbf{R} \rightarrow M$ is a mapping defined by $z^1 = t, z^2 = 0, \dots, z^n = 0$. Then we have*

$$(1) \quad g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right) \circ \phi = \delta_{i,j} + 2 \operatorname{Re} K_{i\bar{j}\bar{i}\bar{i}}(0) \cdot t^2 + O(t^3),$$

$$(2) \quad g\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial x^i}\right) \circ \phi = 2 \operatorname{Im} K_{i\bar{j}\bar{i}\bar{i}}(0) \cdot t^2 + O(t^3),$$

$$(3) \quad \Gamma_{i,j}^k \circ \phi = 2 \operatorname{Re} K_{\bar{k}\bar{i}\bar{i}j}(0) \cdot t + O(t^2),$$

$$\Gamma_{i,j}^{n+k} \circ \phi = 2 \operatorname{Im} K_{\bar{k}\bar{i}\bar{i}j}(0) \cdot t + O(t^2),$$

(4) by (2·13)

$$\operatorname{Re} K_{j\bar{i}\bar{i}}(0) = 0 \quad \text{and} \quad \operatorname{Im} K_{j\bar{i}\bar{i}}(0) = 0.$$

These follow from

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \left\{ g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right) + \overline{g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right)} \right\},$$

$$g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j}\right) = \frac{1}{\sqrt{-1}} \left\{ g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right) - \overline{g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}\right)} \right\},$$

and

$$\mathcal{V}_{(a/\partial x^i)}\left(\frac{\partial}{\partial x^j}\right) = \sum_{k=1}^n \operatorname{Re} \Gamma_{i,j}^{*k} \frac{\partial}{\partial x^k} + \sum_{k=1}^n \operatorname{Im} \Gamma_{i,j}^{*k} \frac{\partial}{\partial y^k}.$$

The terms $O(t^3)$ and $O(t^2)$ can be estimated in the following

Proposition (2·16). *Let ϕ be the mapping given in Proposition (2·15) and let $\delta_0 = \min_{i=1,2,\dots,N} \delta_0^{(i)}$. Then we have*

(1) *Suppose that $g(\partial/\partial x^i, \partial/\partial x^j) \circ \phi = t^m \cdot G(t)$, where $G(t)$ is a real analytic function. Then we obtain*

$$|G(t)| \leq 2C_0 \delta_0^{-m} \text{ for } t \text{ with } \phi(t) \in B_\delta(p),$$

(2) *Suppose that $g(\partial/\partial x^i, \partial/\partial y^j) \circ \phi = t^m \cdot G'(t)$, where $G'(t)$ is a real analytic function. Then we obtain*

$$|G'(t)| \leq 2C_0 \delta_0^{-m} \text{ for } t \text{ with } \phi(t) \in B_\delta(p),$$

(3) *Suppose that $\partial/\partial t(g\partial/\partial x^i, \partial/\partial y^j) \circ \phi = t^m \cdot G''(t)$, where $G''(t)$ is a real analytic function. Then we obtain*

$$|G''(t)| \leq C(m) C_0 \delta_0^{-(m+1)} \text{ for } t \text{ with } \phi(t) \in B_\delta(p),$$

(4) *Suppose that $\Gamma_{i,j}^k \circ \phi = t^m \cdot G'''(t)$, where $G'''(t)$ is a real analytic function. Then we obtain*

$$|G'''(t)| \leq nC(m) C_0^2 \delta_0^{-(m+1)} \text{ for } t \text{ with } \phi(t) \in B_\delta(t),$$

where $C(m) = 4^{m+2} \cdot 3^{-(m-1)}$.

Let

$$2g_{\alpha\bar{\beta}} = \sum a_{i_1 \dots i_n, \bar{j}_1 \dots \bar{j}_n} (z^1)^{i_1} \dots (\bar{z}^n)^{\bar{j}_n}.$$

Then owing to (4) in Lemma (2·10), we find

$$|a_{i_1 \dots i_n, \bar{j}_1 \dots \bar{j}_n} (z^1)^{i_1} \dots (\bar{z}^n)^{\bar{j}_n}| \leq C_0 | (z^1/\delta_0^{(i)})^{i_1} \dots (\bar{z}^n/\delta_0^{(j)})^{\bar{j}_n} | \text{ on } U_{\delta_0, (i)}(z; p).$$

By using (0) in (ii) in Lemma (2·10) and the definition of δ_0 , we obtain the above estimates.

§ 3. Some Propositions Concerning Pseudoconvexity

In this section we summarize some propositions concerning pseudoconvex functions and pseudoconvex domains. Let z^1, z^2, \dots, z^n be a system of local coordinates on some neighborhood of p and let L be a complex line through p which is parametrized as

$$L: z^1 = a_1 \lambda, z^2 = a_2 \lambda, \dots, z^n = a_n \lambda,$$

where

$$|a_1|^2 + |a_2|^2 + \dots + |a_n|^2 = 1.$$

A function φ is called a pseudoconvex function on D , if for any point $p \in D$ and for any complex line L ,

$$\varphi(0) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(re^{i\theta}) d\theta \quad \text{where } \lambda = re^{i\theta}.$$

In what follows we write

$$\mu_{r,p}(\varphi) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(re^{i\theta}) d\theta.$$

When φ is a function of C^2 -class, we find that

$$\Delta\varphi(0) = 4 \lim_{r \rightarrow 0} \frac{\mu_{r,p}(\varphi) - \varphi(0)}{r^2},$$

where

$$\Delta\varphi = 4 \frac{\partial^2 \varphi}{\partial \lambda \partial \bar{\lambda}}.$$

Hence if φ is a pseudoconvex function of C^2 -class, we see that $\Delta\varphi \geq 0$ holds everywhere. More generally a necessary and sufficient condition that φ is a pseudoconvex (resp. s -pseudoconvex) function is that

$$(3.1) \quad W(\varphi(p)) = 4 \inf_L \liminf_{r \rightarrow 0} \frac{\mu_{r,p}(\varphi) - \varphi(0)}{r^2}$$

is non-negative (resp. positive) at every point p in D .

The following theorem is well known (for example, see H. Grauert [4]):

Theorem (3.2). *If D is a relatively compact s -pseudoconvex domain, then D is holomorphically convex.*

Let D be an s -pseudoconvex domain whose boundary is a real submanifold in M . Take a point $p \in D$ near the boundary. Then we may assume that there exist a boundary point q and a geodesic σ between p and q which attains $d(p) = d(p, q)$. So there exist a local coordinate neighborhood U_{δ_0} of p and a system of local coordinates z^1, z^2, \dots, z^n which satisfies (0) \sim (4) in (i) in Lemma (2.10).

Then we can prove the following Proposition, which will be used in § 4. For the proof, see A. Takeuchi [9, Lemma 3, p. 338]:

Proposition (3.3). *There exist a holomorphic function $\beta(z^2, z^3, \dots, z^n)$ on U_{∂_0} and a divisor S through q ,*

$$S: \{z^1 = \beta(z^2, z^3, \dots, z^n)\}$$

such that

$$S \cap D = \phi \text{ and } S \cap \partial D = \{q\}.$$

Now we consider a general pseudoconvex domain D . Let p_0 be a boundary point and let $U(p_0)$ be a Stein neighborhood. Then $D \cap U(p_0)$ is also a Stein manifold. Hence there exists a sequence of domains in $D \cap U(p_0)$, A_j , ($j=1, 2, \dots$) with the following property: (1) $A_j \subset\subset A_{j+1}$, $\bigcup_{j=1}^{\infty} A_j = U(p_0) \cap D$ and (2) A_j is an s -pseudoconvex domain whose boundary is a real submanifold in $D \cap U(p_0)$.

We infer the following Proposition:

Proposition (3.4). *For any point $p_0 \in \partial D$ and for any neighborhood $U(p_0)$ there exists a relatively compact neighborhood $V(p_0)$ in $U(p_0)$ satisfying*

$$d(p) = \inf_{q \in (\partial D \cap V(p_0))} d(p, q) \quad \text{for } p \in V(p_0).$$

For the proof, see H. Grauert [3, Hilfsatz 9, p. 69].

Take an arbitrary point p in $V(p_0)$. Then there exists an integer i_0 such that $p \in A_i$ for $i \geq i_0$. We set

$$(3.5) \quad d_i(p) = \inf_{q \in \partial A_j} d(p, q) \text{ and } \varphi_i(p) = -\log d_i(p) \text{ for } p \in A_j.$$

Then we see that

$$d_i(p) < d_{i+1}(p) \text{ and } d_j(p) \rightarrow d(p) \quad (j \rightarrow \infty),$$

and that

$$\varphi_i(p) > \varphi_{i+1}(p) \text{ and } \varphi_j(p) \rightarrow \varphi(p) \quad (j \rightarrow \infty).$$

We note that if φ_i is a pseudoconvex function (resp. $W(\varphi_i) \geq c > 0$) for $i \geq i_0$, then φ is also a pseudoconvex function (resp. $W(\varphi) \geq c$).

§ 4. Proof of Key Lemma

We shall prove our Key Lemma stated in Introduction. Let D be

an s -pseudoconvex domain whose boundary is a real submanifold in M . Take a positive constant C_0 with $C_0 > 1$. By (ii) in Lemma (2·10), we can find positive constants $\delta, \delta_0^{(1)}, \delta_0^{(2)}, \dots, \delta_0^{(n)}$. As in Proposition (2·16), we define δ_0 . Choosing δ' ($\delta' < \delta$) very small, we may assume that the following condition is satisfied: For any point $p \in D_{\delta'}$, there exist a point $q \in \partial D$ and a geodesic σ between p and q satisfying $d(p) = d(p, q)$.

Now we take a point p in $D_{\delta'}$. Then we can find q and σ as above. By (ii) in Lemma (2·10), we choose a system of local coordinates z^1, z^2, \dots, z^n . Letting $d(p) = l$, we see that $l < \delta$. To get the estimate $W(\varphi)$ on $D_{\delta'}$, we construct another simple variation of σ . Let N be a complex line through p and its holomorphic parametrization is given as follows:

$$(4.1) \quad N: z^1 = a_1\lambda, \quad z^2 = a_2\lambda, \quad \dots, \quad z^n = a_n\lambda,$$

where $|a_1|^2 + |a_2|^2 + \dots + |a_n|^2 = 1$.

We write $\lambda = \xi + \sqrt{-1}\eta$.

Choose a divisor $S: \{z^1 = \beta(z^2, z^3, \dots, z^n)\}$ through q as in Proposition (3·3). By the choice of the coordinates, we find $z^1(q) = l$ and

$$(4.2) \quad l = \operatorname{Re} \beta(0, 0, \dots, 0) \quad \text{and} \quad 0 = \operatorname{Im} \beta(0, 0, \dots, 0).$$

Consider a holomorphic variation ψ of σ between N and S , $\psi: D \times [0, l] \rightarrow M$ as follows, where D is a small disk:

$$(4.3) \quad \begin{cases} z^1 = \left(1 - \frac{t}{l}\right) a_1\lambda + \frac{t}{l} \beta(a_2\lambda, a_3\lambda, \dots, a_n\lambda) \\ z^2 = a_2\lambda \\ \vdots \\ z^n = a_n\lambda, \end{cases}$$

where we say a holomorphic variation ψ of σ if ψ is a holomorphic mapping for every fixed t . Then the real representation of $\psi(\xi, \eta, t)$ is given as follows:

$$(4.4) \quad \begin{cases} x^1 = \left(1 - \frac{t}{l}\right) \operatorname{Re}(a_1\lambda) + \frac{t}{l} \operatorname{Re} \beta(a_2\lambda, a_3\lambda, \dots, a_n\lambda), \\ y^1 = \left(1 - \frac{t}{l}\right) \operatorname{Im}(a_1\lambda) + \frac{t}{l} \operatorname{Im} \beta(a_2\lambda, a_3\lambda, \dots, a_n\lambda), \\ x^i = \operatorname{Re}(a_i\lambda) \\ y^i = \operatorname{Im}(a_i\lambda) \quad (i = 2, 3, \dots, n). \end{cases}$$

First we show

$$(4.5) \quad \|\psi(0, 0, t)\| = 1.$$

In fact, by (4.2) and (4.4) we see that $\dot{\psi}(0, 0, t) = \partial/\partial x^1 \circ \psi(0, 0, t)$. And by (2.13), we prove the assertion.

Next we set

$$X = \psi_*(D_1), \quad Y = \psi_*(D_2) \quad \text{and} \quad Z = \psi_*(D_3),$$

where $D_1 = \partial/\partial t$, $D_2 = \partial/\partial \xi$ and $D_3 = \partial/\partial \eta$. Then X, Y and Z can be expressed as follows:

$$(4.6) \quad X = \left\{ -\frac{1}{l} \operatorname{Re}(a_1 \lambda) + \frac{1}{l} \operatorname{Re} \beta \right\} \frac{\partial}{\partial x^1} \\ + \left\{ -\frac{1}{l} \operatorname{Im}(a_1 \lambda) + \frac{1}{l} \operatorname{Im} \beta \right\} \frac{\partial}{\partial y^1},$$

$$(4.7) \quad Y = \left\{ \left(1 - \frac{t}{l}\right) \operatorname{Re} a_1 + \frac{t}{l} \frac{\partial \operatorname{Re} \beta}{\partial \xi} \right\} \frac{\partial}{\partial x^1} + \left\{ \left(1 - \frac{t}{l}\right) \operatorname{Im} a_1 \right. \\ \left. + \frac{t}{l} \frac{\partial \operatorname{Im} \beta}{\partial \xi} \right\} \frac{\partial}{\partial y^1} + \sum_{i=2}^n \operatorname{Re} a_i \frac{\partial}{\partial x^i} + \sum_{i=2}^n \operatorname{Im} a_i \frac{\partial}{\partial y^i},$$

$$Z = \left\{ \left(1 - \frac{t}{l}\right) (-\operatorname{Im} a_1) + \frac{t}{l} \frac{\partial \operatorname{Re} \beta}{\partial \eta} \right\} \frac{\partial}{\partial x^1} + \left\{ \left(1 - \frac{t}{l}\right) \operatorname{Re} a_1 \right. \\ \left. + \frac{t}{l} \frac{\partial \operatorname{Im} \beta}{\partial \eta} \right\} \frac{\partial}{\partial y^1} + \sum_{i=2}^n (-\operatorname{Im} a_i) \frac{\partial}{\partial x^i} + \sum_{i=2}^n \operatorname{Re} a_i \frac{\partial}{\partial y^i}.$$

Because ψ is a holomorphic variation, we find that

$$(4.8) \quad Z = J(Y).$$

Now we set

$$D^*(\xi, \eta) = \int_0^t \|\dot{\psi}(\xi, \eta, t)\| dt \quad \text{and} \quad \Psi(\xi, \eta) = -\log D^*(\xi, \eta).$$

We shall calculate

$$\Delta \Psi(0, 0) = \left(\frac{\partial^2 \Psi}{\partial \xi^2} + \frac{\partial^2 \Psi}{\partial \eta^2} \right) (0, 0).$$

For this we first calculate $\partial^2 \Psi / \partial \xi^2(0, 0)$ by using

$$\frac{\partial^2 \Psi}{\partial \xi^2}(0, 0) = \frac{1}{(D^*)^2} \left(\frac{\partial D^*}{\partial \xi} \right)^2 (0, 0) - \frac{1}{D^*} \left(\frac{\partial^2 D^*}{\partial \xi^2} \right) (0, 0).$$

Referring to (4.5), we can compute $\partial D^* / \partial \xi(0, 0)$ by Lemma (2.3) and we obtain

$$\frac{\partial D^*}{\partial \xi}(0, 0) = g(Y, X) \circ \psi(0, 0, t)|_0^l.$$

Putting X and Y into $g(X, Y)$ by (4.6) and (4.7) and using (2.14), we get

$$\begin{aligned} g(X, Y) \circ \psi(0, 0, t) &= \left(1 - \frac{t}{l}\right) \operatorname{Re} a_1 + \frac{t}{l} \frac{\partial \operatorname{Re} \beta}{\partial \xi}(0) \\ &+ \sum_{i=2}^n \operatorname{Re} a_i g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^1}\right) \circ \psi(0, 0, t) \\ &+ \sum_{i=2}^n \operatorname{Im} a_i g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial x^1}\right) \circ \psi(0, 0, t). \end{aligned}$$

By (1), (2) in Proposition (2.15) and (2.6), we see that

$$\begin{aligned} g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^1}\right) \circ \psi(0, 0, t) &= 2 \operatorname{Re} K_{i\bar{1}\bar{1}}(0) \cdot t^2 + O(t^3) \quad (i=2, 3, \dots, n), \\ g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial x^1}\right) \circ \psi(0, 0, t) &= -2 \operatorname{Im} K_{i\bar{1}\bar{1}}(0) \cdot t^2 + O(t^3) \quad (i=2, 3, \dots, n). \end{aligned}$$

Moreover, by (4) in the same Proposition, we find $K_{i\bar{1}\bar{1}}(0) = 0$. Therefore we get

$$\begin{aligned} g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial x^1}\right) \circ \psi(0, 0, t) &= t^3 \cdot G_i \quad \text{and} \quad g\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial x^1}\right) \circ \psi(0, 0, t) = t^3 \cdot G_{n+i} \\ &\text{for } i=2, 3, \dots, n. \end{aligned}$$

Hence by (1), (2) in Proposition (2.16), we have

$$|G_i| \leq 2C_0 \delta_0^{-3} \quad \text{and} \quad |G_{n+i}| \leq 2C_0 \delta_0^{-3} \quad (i=2, 3, \dots, n).$$

So we obtain

$$(4.9) \quad \frac{\partial D^*}{\partial \xi}(0, 0) = \left(\frac{\partial \operatorname{Re} \beta}{\partial \xi}(0, 0) - \operatorname{Re} a_1\right) + M^{(1)}(l) \cdot l^3,$$

where

$$|M^{(1)}(l)| \leq 4(n-1)C_0 \delta_0^{-3}.$$

Also $\partial^2 D^* / \partial \xi^2(0, 0)$ can be computed by (ii) in Lemma (2.3) as follows:

$$(4.10) \quad \frac{\partial^2 D^*}{\partial \xi^2}(0, 0) = \int_0^l \{g(\mathcal{V}_D, \tilde{Y}, \mathcal{V}_D, \tilde{Y}) - g(R(Y, X)X, Y)\} \circ \psi(0, 0, t) dt$$

$$+ g(\mathcal{F}_{D_t} Y, X) \circ \psi(0, 0, t)|_0^t.$$

The first term in the left side can be computed as follows: From (2·1) we infer that

$$(\mathcal{F}_{D_t} \tilde{Y}) \circ \psi(0, 0, t) = \mathcal{F}_{D_t} \{ \tilde{Y} \circ \psi(0, 0, t) \}.$$

By using $X \circ \psi(0, 0, t) = \partial / \partial x^1 \circ \psi(0, 0, t)$, we see that

$$\begin{aligned} \tilde{Y} \circ \psi(0, 0, t) &= - \left\{ \sum_{i=2}^n \operatorname{Re} a_i g \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^1} \right) \circ \psi(0, 0, t) \right. \\ &\quad + \sum_{i=2}^n \operatorname{Im} a_i g \left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial x^1} \right) \circ \psi(0, 0, t) \left. \right\} \frac{\partial}{\partial x^1} \circ \psi(0, 0, t) \\ &\quad + \left\{ \left(1 - \frac{t}{l} \right) \operatorname{Im} a_1 + \frac{t}{l} \frac{\partial \operatorname{Im} \beta}{\partial \xi} \right\} \frac{\partial}{\partial y^1} \circ \psi(0, 0, t) \\ &\quad + \sum_{i=2}^n \operatorname{Re} a_i \frac{\partial}{\partial x^i} \circ \psi(0, 0, t) + \sum_{i=2}^n \operatorname{Im} a_i \frac{\partial}{\partial y^i} \circ \psi(0, 0, t). \end{aligned}$$

Also by (4·4) and (2·14), we have

$$\frac{\partial}{\partial t} (x^i \circ \psi(0, 0, t)) = \delta_{i,1} \quad \text{and} \quad \frac{\partial}{\partial t} (y^i \circ \psi(0, 0, t)) = 0$$

$$(i=1, 2, \dots, n),$$

$$\Gamma_{1,1}^j \circ \psi(0, 0, t) = \Gamma_{1,n+1}^j \circ \psi(0, 0, t) = 0 \quad (j=1, 2, \dots, 2n).$$

Hence defining U^i and V^i ($i=1, 2, \dots, n$) by

$$(\mathcal{F}_{D_t} \tilde{Y}) \circ \psi(0, 0, t) = \sum_{i=1}^n U^i \frac{\partial}{\partial x^i} \circ \psi(0, 0, t) + \sum_{i=1}^n V^i \frac{\partial}{\partial y^i} \circ \psi(0, 0, t),$$

we obtain

$$\begin{aligned} U^1 &= - \sum_{i=2}^n \operatorname{Re} a_i \frac{\partial}{\partial t} \left\{ g \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^1} \right) \circ \psi(0, 0, t) \right\} \\ &\quad - \sum_{i=2}^n \operatorname{Im} a_i \frac{\partial}{\partial t} \left\{ g \left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial x^1} \right) \circ \psi(0, 0, t) \right\} \\ &\quad + \sum_{i=2}^n \operatorname{Re} a_i \Gamma_{1,i}^1 \circ \psi(0, 0, t) + \sum_{i=2}^n \operatorname{Im} a_i \Gamma_{1,n+i}^1 \circ \psi(0, 0, t) \\ U^k &= \sum_{i=2}^n \operatorname{Re} a_i \Gamma_{1,i}^k \circ \psi(0, 0, t) + \sum_{i=2}^n \operatorname{Im} a_i \Gamma_{1,n+i}^k \circ \psi(0, 0, t) \\ &\quad (k=2, 3, \dots, n), \end{aligned}$$

$$\begin{aligned}
 V^1 &= \left\{ -\frac{1}{l} \operatorname{Im} a_1 + \frac{1}{l} \frac{\partial \operatorname{Im} \beta}{\partial \xi} \right\} + \sum_{i=2}^n \operatorname{Re} a_i \Gamma_{1,i}^{n+1} \circ \psi(0, 0, t) \\
 &\quad + \sum_{i=2}^n \operatorname{Im} a_i \Gamma_{1,n+i}^{n+1} \circ \psi(0, 0, t), \\
 V^k &= \sum_{i=2}^n \operatorname{Re} a_i \Gamma_{1,i}^{n+k} \circ \psi(0, 0, t) + \sum_{i=2}^n \operatorname{Im} a_i \Gamma_{1,n+i}^{n+k} \circ \psi(0, 0, t) \\
 &\quad (k=2, 3, \dots, n).
 \end{aligned}$$

By (3) in Proposition (2.15) and (2.12), we see that

$$\begin{aligned}
 \Gamma_{1,i}^{n+1} \circ \psi(0, 0, t) &= 2 \operatorname{Im} K_{\mathbb{H}i}(0) \cdot t + O(t^2), \\
 \Gamma_{1,n+i}^{n+1} \circ \psi(0, 0, t) &= 2 \operatorname{Re} K_{\mathbb{H}i}(0) \cdot t + O(t^2).
 \end{aligned}$$

Moreover, by (4) in Proposition (2.16) and (2.6), we see that $K_{\mathbb{H}i}(0) = 0$. Hence we obtain

$$\begin{aligned}
 V^1 &= \left\{ -\frac{1}{l} \operatorname{Im} a_1 + \frac{1}{l} \frac{\partial \operatorname{Im} \beta}{\partial \xi} \right\} + V_1^{(2)} \cdot t^2, \\
 V^i &= V_i^{(1)} \cdot t \quad (i=2, 3, \dots, n), \\
 U^i &= U_i^{(1)} \cdot t \quad (i=1, 2, \dots, n),
 \end{aligned}$$

where $V_1^{(2)}$, $V_i^{(1)}$ and $U_i^{(1)}$ are real analytic functions of t . By (3), (4) in Proposition (2.16), we see that

$$\begin{aligned}
 |V_1^{(2)}| &\leq 2n(n-1)C(2)C_0^2\delta_0^{-3}, \\
 |V_i^{(1)}| &\leq 2n(n-1)C(1)C_0^2\delta_0^{-2} \quad (i=2, 3, \dots, n), \\
 |U_1^{(1)}| &\leq \{2n(n-1)C(1)C_0^2 + 2(n-1)C(1)C_0\}\delta_0^{-2}, \\
 |U_i^{(1)}| &\leq 2n(n-1)C(1)C_0^2\delta_0^{-2} \quad (i=2, 3, \dots, n).
 \end{aligned}$$

In the same manner as in A. Takeuchi [9, p. 343], we have

$$\frac{\partial \beta}{\partial z^i}(0, 0, \dots, 0) = -2g_{i,\bar{i}}(q) \quad \text{for } i=2, 3, \dots, n.$$

Therefore, by (4) in (ii), Lemma (2.10), we obtain

$$\begin{aligned}
 (4.11) \quad &\left| \frac{\partial \operatorname{Im} \beta}{\partial \xi}(0) - \operatorname{Im} a_1 \right| \leq 1 + (n-1)C_0, \\
 &\left| \frac{\partial \operatorname{Re} \beta}{\partial \xi}(0) - \operatorname{Im} a_1 \right| \leq 1 + (n-1)C_0.
 \end{aligned}$$

Hence we obtain

$$\int_0^l g(\mathcal{V}_{D_1} \tilde{Y}, \mathcal{V}_{D_1} \tilde{Y}) \circ \psi(0, 0, t) dt = \frac{1}{l} \left(\frac{\partial \operatorname{Im} \beta}{\partial \xi}(0) - \operatorname{Im} a_1 \right)^2 + H(l) \cdot l^2,$$

where $|H(l)| \leq M^{(2)}$ and $M^{(2)}$ is a constant which depends only on C_0 and δ_0 .

Next we compute the third term in the left side of (4.10). By (2.2) and (4.7) and by using $\partial/\partial \xi(x^i \circ \psi) = Y^i$ and $\partial/\partial \xi(y^i \circ \psi) = Y^{n+i}$, $\mathcal{V}_{D_2} Y$ can be expressed as follows:

$$\mathcal{V}_{D_2} Y = \sum_{i=1}^n V^i \frac{\partial}{\partial x^i} \circ \psi(0, 0, t) + \sum_{i=1}^n W^i \frac{\partial}{\partial y^i} \circ \psi(0, 0, t),$$

where

$$V^1 = \frac{t}{l} \frac{\partial^2 \operatorname{Re} \beta}{\partial \xi^2} + \sum_{i,j=1}^{2n} Y^i Y^j \cdot \Gamma_{i,j}^1 \circ \psi(0, 0, t)$$

$$W^1 = \frac{t}{l} \frac{\partial^2 \operatorname{Im} \beta}{\partial \xi^2} + \sum_{i,j=1}^{2n} Y^i Y^j \cdot \Gamma_{i,j}^{n+1} \circ \psi(0, 0, t)$$

$$V^k = \sum_{i,j=1}^{2n} Y^i Y^j \Gamma_{i,j}^k \circ \psi(0, 0, t)$$

$$(k=2, 3, \dots, n).$$

$$W^k = \sum_{i,j=1}^{2n} Y^i Y^j \Gamma_{i,j}^{n+k} \circ \psi(0, 0, t)$$

By $\Gamma_{j,1}^1 \circ \psi(0, 0, 0) = 0$ and $\Gamma_{j,n+1}^1 \circ \psi(0, 0, 0) = 0$ for $j=1, 2, \dots, 2n$ (see, (3), (4) in Proposition(2.15) and also (2.6) and (2.12)), we have

$$g(\mathcal{V}_{D_2} Y, X) \circ \psi(0, 0, t) \Big|_0^l = \frac{\partial^2 \operatorname{Re} \beta}{\partial \xi^2}(0) + \sum_{i,j=2}^n \operatorname{Re} a_i \operatorname{Re} a_j \Gamma_{i,j}^1 \circ \psi(0, 0, l)$$

$$+ \sum_{i,j=2}^n \operatorname{Re} a_i \operatorname{Im} a_j \Gamma_{i,n+j}^1 \circ \psi(0, 0, l)$$

$$+ \sum_{i,j=2}^n \operatorname{Im} a_i \operatorname{Re} a_j \Gamma_{n+i,j}^1 \circ \psi(0, 0, l)$$

$$+ \sum_{i,j=2}^n \operatorname{Im} a_i \operatorname{Im} a_j \Gamma_{n+i,n+j}^1 \circ \psi(0, 0, l) + H'(l) \cdot l^2,$$

where $|H'(l)| \leq M^{(3)}$ and $M^{(3)}$ is a positive constant which depends only on C_0 and δ_0 . This follows from (1), (2) and (4) in Proposition (2.16) and (4.11).

Hence we obtain

$$\begin{aligned} \frac{\partial^2 D^*}{\partial \xi^2}(0, 0) &= \frac{1}{l} \left(\frac{\partial \operatorname{Im} \beta}{\partial \xi} - \operatorname{Im} a_1 \right)^2 + \frac{\partial^2 \operatorname{Re} \beta}{\partial \xi^2}(0) \\ &\quad - \int_0^l g(R(Y, X) X, Y) \circ \psi dt \\ &\quad + \sum_{i,j=2}^n \operatorname{Re} a_i \operatorname{Re} a_j \Gamma_{i,j}^1 \circ \psi(0, 0, l) + \sum_{i,j=2}^n \operatorname{Re} a_i \operatorname{Im} a_j \Gamma_{i,n+j}^1 \circ \psi(0, 0, l) \\ &\quad + \sum_{i,j=2}^n \operatorname{Im} a_i \operatorname{Re} a_j \Gamma_{n+i,j}^1 \circ \psi(0, 0, l) + \sum_{i,j=2}^n \operatorname{Im} a_i \operatorname{Im} a_j \Gamma_{n+i,n+j}^1 \circ \psi(0, 0, l) \\ &\quad + H''(l) \cdot l^2, \end{aligned}$$

where $|H''(l)| \leq M^{(4)}$ and $M^{(4)}$ is a constant which depends only on C_0 and δ_0 .

In the same manner as above, we compute $\partial^2 \Psi / \partial \eta^2(0, 0)$ as follows

$$\frac{\partial^2 \Psi}{\partial \eta^2}(0, 0) = \frac{1}{D^{*2}} \left(\frac{\partial D^*}{\partial \eta} \right)^2(0, 0) - \frac{1}{D^*} \frac{\partial^2 D^*}{\partial \eta^2}(0, 0),$$

where

$$\frac{\partial D^*}{\partial \eta}(0, 0) = \left(\frac{\partial \operatorname{Re} \beta}{\partial \eta} + \operatorname{Im} a_1 \right) + J(l) \cdot l^3$$

and

$$\begin{aligned} \frac{\partial^2 D^*}{\partial \eta^2}(0, 0) &= \frac{1}{l} \left(\frac{\partial \operatorname{Im} \beta}{\partial \eta} - \operatorname{Re} a_1 \right)^2 + \frac{\partial^2 \operatorname{Re} \beta}{\partial \eta^2}(0) \\ &\quad - \int_0^l g(R(Z, X) X, Z) \circ \psi dt \\ &\quad + \sum_{i,j=2}^n (-\operatorname{Im} a_i) (-\operatorname{Im} a_j) \Gamma_{i,j}^1 \circ \psi(0, 0, l) \\ &\quad + \sum_{i,j=2}^n (-\operatorname{Im} a_i) (\operatorname{Re} a_j) \Gamma_{i,n+j}^1 \circ \psi(0, 0, l) \\ &\quad + \sum_{i,j=2}^n (\operatorname{Re} a_i) (-\operatorname{Im} a_j) \Gamma_{n+i,j}^1 \circ \psi(0, 0, l) \\ &\quad + \sum_{i,j=2}^n (\operatorname{Re} a_i) (\operatorname{Re} a_j) \Gamma_{n+i,n+j}^1 \circ \psi(0, 0, l) \\ &\quad + J'(l) \cdot l^2, \end{aligned}$$

where $|J(l)| \leq M^{(5)}$ and $|J'(l)| \leq M^{(6)}$ and $M^{(5)}, M^{(6)}$ are positive constants which depend only on C_0 and δ_0 .

Therefore by using (4.11) and $|(\partial \operatorname{Re} \beta / \partial \eta) + \operatorname{Im} a_1| \leq (n-1)C_0 + 1$ again, we obtain

$$\begin{aligned}
 \Delta \Psi(0, 0) &= \frac{1}{l^2} \left\{ \left(\frac{\partial D^*}{\partial \xi}(0, 0) \right)^2 + \left(\frac{\partial D^*}{\partial \eta}(0, 0) \right)^2 \right\} - \frac{1}{l} \left\{ \frac{\partial^2 D^*}{\partial \xi^2}(0, 0) \right. \\
 &\quad \left. + \frac{\partial^2 D^*}{\partial \eta^2}(0, 0) \right\} \\
 &= \frac{1}{l^2} \left\{ \left(\frac{\partial \operatorname{Re} \beta}{\partial \xi}(0, 0) - \operatorname{Re} a_1 \right)^2 + \left(\frac{\partial \operatorname{Re} \beta}{\partial \eta}(0, 0) + \operatorname{Im} a_1 \right)^2 \right\} \\
 &\quad - \frac{1}{l^2} \left\{ \left(\frac{\partial \operatorname{Im} \beta}{\partial \xi}(0, 0) - \operatorname{Im} a_1 \right)^2 + \left(\frac{\partial \operatorname{Im} \beta}{\partial \eta}(0, 0) - \operatorname{Re} a_1 \right)^2 \right\} \\
 &\quad - \frac{1}{l} \left\{ \frac{\partial^2 \operatorname{Re} \beta}{\partial \xi^2}(0, 0) + \frac{\partial^2 \operatorname{Re} \beta}{\partial \eta^2}(0, 0) \right\} \\
 &\quad - \frac{1}{l} \left\{ \sum_{i,j=2}^n \operatorname{Re} a_i \operatorname{Re} a_j (\Gamma_{i,j}^1 + \Gamma_{n+i,n+j}^1) \circ \psi(0, 0, l) \right. \\
 &\quad + \sum_{i,j=2}^n \operatorname{Re} a_i \operatorname{Im} a_j (\Gamma_{i,n+j}^1 - \Gamma_{n+i,j}^1) \circ \psi(0, 0, l) \\
 &\quad + \sum_{i,j=2}^n \operatorname{Im} a_i \operatorname{Re} a_j (\Gamma_{n+i,j}^1 - \Gamma_{i,n+j}^1) \circ \psi(0, 0, l) \\
 &\quad \left. + \sum_{i,j=2}^n \operatorname{Im} a_i \operatorname{Im} a_j (\Gamma_{n+i,n+j}^1 + \Gamma_{i,j}^1) \circ \psi(0, 0, l) \right\} \\
 &\quad + \frac{1}{l} \int_0^l \{g(R(Y, X)X, Y) + g(R(Z, X)X, Z)\} \circ \psi(0, 0, t) dt \\
 &\quad + M(l) \cdot l,
 \end{aligned}$$

where $|M(l)| \leq M^n$ and M^n is a positive constant which depends only on C_0 and δ_0 .

Because β is a holomorphic function, we see that

$$\frac{\partial \operatorname{Re} \beta}{\partial \xi} = \frac{\partial \operatorname{Im} \beta}{\partial \eta} \quad \text{and} \quad \frac{\partial \operatorname{Re} \beta}{\partial \eta} = -\frac{\partial \operatorname{Im} \beta}{\partial \xi}.$$

Also by (2.12), we have

$$\Gamma_{i,j}^1 + \Gamma_{n+i,n-j}^1 = 0 \quad \text{and} \quad \Gamma_{n+i,j}^1 - \Gamma_{i,n+j}^1 = 0.$$

Hence all the terms except the last two ones cancel out. Thus we obtain

Lemma (4.12). *There exists a positive constant $M^{(7)}$ which depends only on C_0 and δ_0 such that*

$$\begin{aligned} \Delta\Psi(0, 0) &= \frac{1}{l} \int_0^l \{g(R(Y, X)X, Y) + g(R(Z, X)X, Z)\} \circ \psi(0, 0, t) dt \\ &\quad + F(l) \cdot l, \end{aligned}$$

where $|F(l)| \leq M^{(7)}$.

By (2.8) and (4.8), we have

$$\begin{aligned} (4.13) \quad &\int_0^l \{g(R(Y, X)X, Y) + g(R(Z, X)X, Z)\} \circ \psi(0, 0, t) dt \\ &\geq \rho \left[\int_0^l \{g(X, X)g(Y, Y) - g(X, Y)^2\} \circ \psi(0, 0, t) dt \right. \\ &\quad \left. + \int_0^l \{g(X, X)g(Z, Z) - g(X, Z)^2\} \circ \psi(0, 0, t) dt \right]. \end{aligned}$$

The integrands in the left side can be estimated as follows: By $g(X, X) = 1$, we have

$$\begin{aligned} &\{g(X, X)g(Y, Y) - g(X, Y)^2\} \circ \psi(0, 0, t) \\ &= \left\{ \left(1 - \frac{t}{l}\right) \text{Im } a_1 + \frac{t}{l} \frac{\partial \text{Im } \beta}{\partial \xi} \right\}^2 + \sum_{i=2}^n (\text{Re } a_i)^2 + \sum_{i=2}^n (\text{Im } a_i)^2 + H(t) \cdot t^2, \end{aligned}$$

where $|H(t)| \leq M^{(8)}$ and $M^{(8)}$ is a positive constant which depends only on C_0 and δ_0 . In the same manner, we have

$$\begin{aligned} &\{g(X, X)g(Z, Z) - g(X, Z)^2\} \circ \psi(0, 0, t) \\ &= \left\{ \left(1 - \frac{t}{l}\right) \text{Re } a_1 + \frac{t}{l} \frac{\partial \text{Im } \beta}{\partial \eta} \right\}^2 + \sum_{i=2}^n (\text{Re } a_i)^2 + \sum_{i=2}^n (\text{Im } a_i)^2 + H'(t) \cdot t^2, \end{aligned}$$

where $|H'(t)| \leq M^{(9)}$ and $M^{(9)}$ is a positive constant which depends only on C_0 and δ_0 . By

$$\begin{aligned} &\int_0^l \left\{ \left(1 - \frac{t}{l}\right) \text{Im } a_1 + \frac{t}{l} \frac{\partial \text{Im } \beta}{\partial \xi} \right\}^2 dt \geq \frac{l}{4} (\text{Im } a_1)^2, \\ &\int_0^l \left\{ \left(1 - \frac{t}{l}\right) \text{Re } a_1 + \frac{t}{l} \frac{\partial \text{Im } \beta}{\partial \eta} \right\}^2 dt \geq \frac{l}{4} (\text{Re } a_1)^2, \end{aligned}$$

the right side of (4.13) can be bounded below by the following:

$$\frac{\rho}{4} + H''(l) \cdot l^2,$$

where $|H''(l)| \leq M^{(10)}$ and $M^{(10)}$ is a positive constant which depends only on C_0 and δ_0 .

Finally we obtain

$$(4.14) \quad \Delta\Psi(0, 0) \geq \frac{\rho}{4} + H''(l) \cdot l,$$

where $|H''(l)| \leq M^*$ and M^* is a positive constant which depends only on C_0 and δ_0 .

By (4.14) we prove our Key Lemma. We choose δ^* by the following condition:

$$\delta^* = \min(\rho/8M^*, \delta').$$

Take an arbitrary point $p \in D_{\delta^*}$. Then as we stated in the beginning of this section, we can find a point $q \in \partial D$ such that $d(p) = d(p, q)$ and a geodesic between p and q , which is denoted by σ . We choose a system of local coordinates z^1, z^2, \dots, z^n on $U_{\delta_0(\epsilon)}(z; p)$ as in (ii) in Lemma (2.10) and a divisor S through q as in Proposition (3.3). And for any complex line N , we construct a holomorphic variation ψ of σ as (4.3). Then for any point $r \in N$, we see that

$$d(r) \leq D^*(r) \text{ and } d(p) = D^*(p).$$

By the choice of δ^* ,

$$\Delta\Psi(p) \geq \rho/8.$$

In view of $\varphi(r) = -\log d(r)$ and $\Psi(r) = -\log D^*(r)$, we have

$$\varphi(r) \geq \Psi(r) \text{ and } \varphi(p) = \Psi(p).$$

Therefore by (3.5), we have

$$\Delta\varphi(p) \geq \Delta\Psi(p),$$

which proves by (3.1) that

$$W(\varphi(p)) \geq \rho/8 \text{ for } p \in D_{\delta^*}.$$

This completes the proof.

§ 5. Proof of Main Theorem

Finally we prove our Main Theorem. Let D be a general pseudoconvex domain on M and let C_0 be a positive constant with $C_0 > 1$. Take an arbitrary point $p \in \partial D$ and a geodesic through p . Then by (i) in Lemma (2·10) we can choose neighborhoods, $V(p)$ and $U_{\delta_0}(p)$. Making $V(p)$ smaller, we may assume that Proposition (3·4) holds for $V(p)$ and $U_{\delta_0}(p)$. By using (i) in Lemma (2·10) again, we choose $V(p)$ and $U_{\delta_0}(p)$ for every $p \in \bar{D}$, where we assume that especially when $p \in \partial D$, $V(p)$ satisfies the above condition. Then there exist $\{p_i\}$ $i=1, 2, \dots, N$ such that $\cup_{i=1}^N V(p_i) \supset \bar{D}$. Then setting $\delta = \min_{i=1, 2, \dots, N} \delta^{(i)}$ where $\delta^{(i)}$ is chosen as δ in (i) in Lemma (2·10) at p_i , we see that $\delta, \delta_0^{(1)}, \delta_0^{(2)}, \dots, \delta_0^{(N)}$ satisfy (ii) in Lemma (2·10), where $\delta_0^{(i)}$ is chosen as δ_0 in (i) at p_i . Here we may assume that there exists a subset $\{p_j\}_{j=1, 2, \dots, r}$ such that every p_j is contained in ∂D and $\cup_{j=1}^r V(p_j) \supset \partial D$. By M^* we denote the positive constant which is determined by C_0 and δ_0 in (4·14). We define δ^* by

$$\delta^* = \min(\rho/8M^*, \delta'),$$

where δ' is defined by the same condition as in the beginning of § 4. Making δ^* smaller, we may assume that $D_{\delta^*} \subset \cup_{i=1}^r V(p_i)$. Choosing a point $p \in D_{\delta^*}$. We assume $p \in V(p_i)$. As in § 3, we choose a sequence of domains $\{A_j\}$ of $U_{\delta_0^{(i)}}(Z:p) \cap D$. Then there exists an integer j_0 such that $p \in A_j$ for $j \geq j_0$. As (3·5) we define $d_j(r)$ and $\varphi_j(r)$ for $r \in A_j$. By the choice of δ^* , we see that $d_j(r) < \delta^*$ for $r \in A_j \cap D_{\delta^*}$. So by Key Lemma, we find that

$$W(\varphi_j(p)) \geq \rho/8.$$

As in the end of § 3, we see that $\varphi_i(p) > \varphi_{i+1}(p)$ for $i \geq j_0$ and $\varphi_j(p) \rightarrow \varphi(p)$. Therefore we obtain

$$W(\varphi(p)) \geq \rho/8,$$

which implies

$$W(\varphi(p)) \geq \rho/8 \text{ for } p \in D_{\delta^*}.$$

Then By Theorem (3·2), we prove Main Theorem.

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Note added in proof. After this paper was submitted for publication, the author was informed from Mr. T. Sasaki that the paper of G. Elencwajg "Pseudo-convexité locale dans les variétés kählériennes" appeared in *Ann. Inst. Fourier (Grenoble)*, **25** (1975), 295-314. In the forthcoming paper in this journal, the author will prove an analogous theorem in the case of C^∞ -kähler metrics.