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Cohomological Dimension of Homogeneous Spaces of Complex Lie Groups I[1)]

By

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In [4] we obtained a formula which represents the completeness of complex Lie groups, and in [3] we generalized it to some types of homogeneous spaces of complex Lie groups. The purpose of the present short note is to give a proof of the generalized formula without the assumption in [3].

Throughout we denote (complex) Lie groups by Roman capital letters and their Lie algebras by the corresponding German small letters respectively. Let (G, H) be a pair of a connected complex Lie group G and a connected closed complex subgroup H of G. Let (K, L) be a pair of maximal compact subgroups K and L of G and H respectively. We consider only such K that contains L. We denote the canonical surjection from $\mathfrak{g}$ onto the quotient complex vector space $\mathfrak{g}/\mathfrak{h}$ by π. Denoting the complex dimension of any complex object X by $\delta(X)$, we can give the following indices of a G-homogeneous complex manifold G/H:

$$\alpha(G/H) = \max_{\mathfrak{k}}\{\delta(G/H) - \delta(\pi(\mathfrak{k})^c)\}$$

$$\beta(G/H) = \max_{\mathfrak{k}} \delta(\pi(\mathfrak{k}) \cap \sqrt{-1}\pi(\mathfrak{k}))$$

$$\gamma(G/H) = \dim_R(K/L) \quad (= \text{topological dimension of } K/L)$$

where $\pi(\mathfrak{k})^c = \pi(\mathfrak{k}) + \sqrt{-1}\pi(\mathfrak{k})$.

We define the cohomological dimension of a complex space X (denoted by $cd\ (X)$) as follows:

$cd(X) := \min\{n | H^q(X, F) = 0,\ {}^{\forall}$*coherent analytic sheaf* F *over* X

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and $\forall q>n\}$

We define the completeness of a complex space X (denoted by $c(X)$) as follows:

$c(X):=\min\{q \mid X$ *is* $(q+1)$*-complete in the sense of* [1].$\}$

Theorem. *If* $\alpha(H)=0$ *we have the following equalities*:
$$cd(G/H)=c(G/H)=\beta(G/H).$$

Proof. Without loss of generality we can assume $\alpha(G)=0$. In fact, we have the following fibering (which is not necessarily principal):

$$K^c/H \to G/H \to G/K^c,$$

where K^c is the complex Lie subgroup of G which corresponds to the Lie algebra $\mathfrak{k}^c:=\mathfrak{k}+\sqrt{-1}\mathfrak{k}$. This fibering is determined up to the choice of K since $L^c=H$ is given from the beginning. By [2], G/K^c is (biholomorphic with) a complex affine space $\boldsymbol{C}^{\alpha(G)}$ of dimension $\alpha(G)$. It follows from the "Oka's principle" that this fibering is complex analytically trivial, i.e. G/H is (complex analytically homeomorphic with) the direct product of the base space G/K^c $(\cong \boldsymbol{C}^{\alpha(G)})$ and the fibre K^c/H. Since G/K^c is affine, $cd(G/H)=cd(G/K^c\times K^c/H)=cd(K^c/H)$, and $c(G/H)=c(G/K^c\times K^c/H)=c(K^c/H)$. As for the index β, we have easily $\beta(G/H)=\beta(K^c/H)$. Thus we have only to prove $cd(K^c/H)=c(K^c/H)$ $=\beta(K^c/H)$. Now we assume $\alpha(G)=0$. Let Z be the connected center of G. Then it is easy to verify $\alpha(Z)=0$. The complex subgroup generated by Z and H is not necessarily closed in G. Let $\overline{ZH}$ be the smallest closed complex subgroup of G containing $ZH:=\{ab \mid a\in Z, b\in H\}$. $\overline{ZH}$ does exist, since the intersection of all closed complex Lie subgroups of G which contain ZH is closed in G and the intersection of all the corresponding Lie algebras is a complex Lie subalgebra of $\mathfrak{g}$. Let N be the normalizer of H in G. Then N is a closed complex Lie subgroup of G. Since ZH is contained in N, so is $\overline{ZH}$, which implies H is invariant in $\overline{ZH}$. Thus we obtain the following principal fibering:

$$\overline{ZH}/H \to G/H \to G/\overline{ZH}.$$

Now we shall show $\alpha(\overline{ZH})=0$. Let K_1 be the maximal compact subgroup of Z and K_2 be a maximal compact subgroup of $\overline{ZH}$ which contains the compact subgroup K_1L generated by K_1 and L. Then $\mathfrak{k}_2^c=\mathfrak{k}_2+\sqrt{-1}\mathfrak{k}_2 \supset(\mathfrak{k}_1^c+\mathfrak{l}^c)=\mathfrak{g}+\mathfrak{h}$. Hence K_2^c contains ZH. On the other hand K_2^c is closed in $\overline{ZH}$ by [2]. From the smallestness condition of $\overline{ZH}$ it follows that $\overline{ZH}=K_2^c$, i.e. $\alpha(\overline{ZH})=0$. As

$$G/\overline{ZH}\approx\frac{G/Z}{\overline{ZH}/Z}\quad (\textit{complex analytically homeomorphic})$$

and G/Z is a Stein group, we can conclude that $G/\overline{ZH}$ is also Stein by using the criterion $\alpha(\overline{ZH}/Z)=0$ (which follows directly from $\alpha(\overline{ZH})=0$) in [3]. Now applying Theorem 35 in [4] to the second fibering we obtain $c(G/H)\leqq\beta(\overline{ZH}/H)$. As $\beta(\overline{ZH}/H)\leqq\beta(G/H)$ is evident, we have $c(G/H)\leqq\beta(G/H)$. From the vanishing theorem of cohomology groups in [1] it follows directly that $cd(G/H)\leqq c(G/H)$. On the other hand we have $\beta(G/H)\leqq cd(G/H)$. For letting $\delta=\delta(G/H)$ and $\beta=\beta(G/H)$ etc., we have $\beta+\delta=\alpha+\gamma=\gamma$ since $\alpha(G/H)=0$. G/H is homotopy equivalent to the γ-dimensional (orientable) compact topological manifold K/L. Hence $H^{\gamma}(G/H, \boldsymbol{C})=\boldsymbol{C}\neq\boldsymbol{O}$. This implies $\beta(G/H)\leqq cd(G/H)$ according to [5]. Q.E.D.

Remark. The indices α and β are well defined only for a fixed representation G/H of some homogeneous space X. The author doesn't know whether they are defined for X independent of the choice of the pair (G, H).

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