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# Multiplications in the Complex Bordism Theory with Singularities

*Dedicated to Professor Ryoji Shizuma on his 60-th birthday*

By

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## Introduction

In 1967, D. Sullivan [13] introduced a bordism theory based on manifolds with singularities and successively N. Baas [3], [4] studied and reformulated the theory so that it has been given a more accessible ground.

This theory is considered as a natural generalization of usual bordism theory, and for each given singularity class $\mathscr{S}=\{P_1, P_2, \cdots\}$, a sequence of closed manifolds, one obtains such a theory. Thus there have appeared various interesting generalized homology theories.

For example, as Baas [4] shows, there exists a tower of homology theories and natural transformations connecting complex bordism to ordinary singular homology.

One of main problems with these theories has been to show whether they are multiplicative or not (Baas [6]). And the purpose of the present paper is to study this problem.

For convenience sake, we will restrict ourselves to the case of complex bordism theory $MU(\mathscr{S})_*(\quad)$ with singularity class $\mathscr{S}$. In this theory we introduce natural (external) multiplications (§3 and §6)

$$\mu_{\boldsymbol{E}}: MU(\mathscr{S})_a(X, Y)\otimes MU(\mathscr{S})_b(V, W) \rightarrow MU(\mathscr{S})_{a+b}(X\times V, Y\times V\cup X\times W)$$

which are "admissible" in an analogous sense of Araki-Toda [2], where $\boldsymbol{E}=\{E_1, E_2, \cdots\}$ means a sequence of Morava's manifolds $E_i$ one for each

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$P_i$ (§ 2).

Although these multiplications $\mu_E$ depend on $E$ and primarily on $\mathscr{S}$, all of them are shown to be associative (Theorem 5.25).

As for commutativity we obtain an "obstruction" formula (Theorem 4.12). And for certain favoring cases where the obstructions vanish, $\mu_E$ become commutative (§ 6). It follows, then, that the dual cohomology theories $MU(\mathscr{S})^*(\ )$ are multiplicative and hence the representing spectra $MU(\mathscr{S})$ become commutative ring spectra. Among them, we can show that, the "integral" Brown-Peterson spectrum $BP$ for an odd prime $p$ (Brown-Peterson [7]) is a ring spectrum (Corollary 6.7).

*Added after completion of the manuscript of this paper.*

In a recent issue of Izvestija Akad. Nauk SSSR, Ser. Mat. 39, No. 5 (1975), 1065-1092, the following paper appeared: O. K. Mironov, "Existence of multiplicative structures in the theories of cobordism with singularities" (in Russian).

Although our work has been done independently of his own, there may be some overlaps in the results.

Mironov dealt with the same objects as ours, but in more than one category of manifolds, i.e. with $SO$, $U$, $SU$ and $Sp$ structure respectively. He used an inductive construction (on $n$) for defining multiplications in the bordism theories $\Omega_*^{\Sigma_n}(\ )$ with singularity type $\Sigma_n=\{P_1, \cdots, P_n\}$. ($\Sigma_n$ correspond to our $\mathscr{S}_n$).

On this point his method seems considerably different from ours. As for commutativity and associativity, he treats only the case $n=1$, in the above paper, with detailed analysis for the mod $q$ theories.

## § 1. Baas-Sullivan Bordism Groups

Recall first some notions concerning manifolds with singularities and the bordism theory based on such manifolds ([4]). Throughout this paper, by manifolds we mean compact, weakly complex manifolds with corners (see [8], [9] for manifolds with corners).

**Definition 1.1.** $V$ is a *decomposed manifold* of type $n$, iff there exist submanifolds $\partial_0 V, \partial_1 V, \cdots, \partial_n V$ of codimension zero of the boundary $\partial V$ such that

$$\partial V = \partial_0 V \cup \partial_1 V \cup \cdots \cup \partial_n V$$

where union means identification along common part of boundary.

Each $\partial_i V$ is again a decomposed manifold by defining

$$\partial_j(\partial_i V) = \begin{cases} \partial_j V \cap \partial_i V & \text{for } j \neq i \\ \phi & \text{for } j = i, \end{cases}$$

and we can continue defining $\partial_k(\partial_j(\partial_i V))$ etc.

Let $\mathcal{S}_n = \{P_1, P_2, \cdots, P_n\}$ be a sequence of closed weakly complex manifolds of (real) even dimensions. We call it a *singularity class* (of weakly complex manifolds).

**Definition 1.2.** A manifold $A$ is called an *$\mathcal{S}_n$-manifold,* iff

i) to each ordered set $\omega = (i_1, \cdots i_k)$ of integers in $\{0, 1, \cdots, n\}$, there associates a decomposed manifold (of type $n$) $A(\omega)$ such that

a) $A(\phi) = A$,

b) there exist isomorphisms (of weakly complex manifolds)

$$\alpha(i;\omega):\ \partial_i A(\omega) \approx \begin{cases} P_i \times A(i, \omega) & \text{for } i \notin \omega \\ \phi & \text{for } i \in \omega, \end{cases}$$

where $P_0 = *$ and $(i, \omega)$ denotes the ordered set $(i, i_1, \cdots, i_k) \subset \{0, 1, \cdots, n\}$,

c) $A(\sigma(\omega)) = \varepsilon(\sigma) A(\omega)$ for $\sigma \in S_\omega$,

here $S_\omega$ means the permutation group of the set $\omega$, and $\varepsilon(\sigma)$ the sign of $\sigma$, so that $-A(\omega)$ means the negative of $A(\omega)$, i.e. the manifold $A(\omega)$ with the opposite $U$-structure.

ii) for any $(i, j) \subset \{0, 1, \cdots, n\}$ the following diagram commutes

$$\begin{array}{ccccc} \partial_j\partial_i A(\omega) & \xrightarrow{\alpha(i;\,\omega)|\partial_j} & P_i \times \partial_j A(i, \omega) & \xrightarrow{id \times \alpha(j;\,(i,\omega))} & P_i \times P_j \times A(j, i, \omega) \\ {\scriptstyle id}\downarrow & & & & \downarrow{\scriptstyle T \times id} \\ -\partial_i\partial_j A(\omega) & \xrightarrow{\alpha(j;\,\omega)|\partial_i} & -P_j \times \partial_i A(j, \omega) & \xrightarrow{id \times \alpha(i;\,(j,\omega))} & -P_j \times P_i \times A(i, j, \omega) \end{array}$$

where $T$ is the twisting map and all of the maps are isomorphisms.

An $\mathscr{S}_n$-manifold may be denoted by $\{A(\omega), \alpha(i;\omega)\}$, or simply by $A$, and the dimension of $\mathscr{S}_n$-manifold $A$ is defined to be that of the ambiant manifold $A(\phi)$.

**Definition 1.3.** An *isomorphism* $\varphi: A \to B$ *between* $\mathscr{S}_n$*-manifolds* $\{A(\omega), \alpha(i,\omega)\}$ *and* $\{B(\omega), \beta(i,\omega)\}$ is defined by a system of isomorphisms of weakly complex manifolds $\varphi(\omega)$: $A(\omega) \to B(\omega)$ such that the following diagram commutes

$$\begin{array}{ccc}
A(\omega) & \xrightarrow{\varphi(\omega)} & B(\omega) \\
\uparrow & & \uparrow \\
\partial_i A(\omega) & \xrightarrow{\varphi(\omega)|\partial_i} & \partial_i B(\omega) \\
{\scriptstyle \alpha(i;\omega)}\downarrow & & \downarrow{\scriptstyle \beta(i;\omega)} \\
P_i \times A(i,\omega) & \xrightarrow{id \times \varphi(i,\omega)} & P_i \times B(i,\omega).
\end{array}$$

Let $(X, Y)$ be a pair of topological spaces.

**Definition 1.4.** A *singular* $\mathscr{S}_n$-manifold, $(A, f)$ *in* $(X, Y)$ is given by a system of pairs $\{(A(\omega), f(\omega))\}$ such that

i) $A = \{A(\omega), \alpha(i;\omega)\}$ is an $\mathscr{S}_n$-manifold,

ii) $f(\omega): A(\omega) \to X$ are continuous maps such that the following diagrams commute

a)
$$\begin{array}{ccc}
A(\omega) & \xrightarrow{f(\omega)} & X \supset Y \\
\uparrow & & \nwarrow{\scriptstyle f(i,\omega)} \quad \uparrow{\scriptstyle f(0,\omega)} \\
\partial_i A(\omega) & & \\
{\scriptstyle \alpha(i;\omega)}\downarrow & & \\
P_i \times A(i,\omega) & \xrightarrow{pr.} & A(i,\omega)
\end{array}$$

b)
$$\begin{array}{ccc}
A(\omega) & \xrightarrow{f(\omega)} & X \\
{\scriptstyle id}\downarrow & \nearrow{\scriptstyle f(\sigma(\omega))} & \\
\varepsilon(\sigma) A(\sigma(\omega)) & &
\end{array}$$

.

**Definition 1.5.** An *isomorphism between singular $\mathscr{S}_n$-manifolds $(A, f)$ and $(B, g)$ in $(X, Y)$* is given by an isomorphism $\varphi: A \to B$ of $\mathscr{S}_n$-manifolds such that the following diagrams commute

$$\begin{array}{ccc} A(\omega) & \xrightarrow{f(\omega)} & X \\ \varphi(\omega)\downarrow & \nearrow & g(\omega) \\ B(\omega) & & \end{array} .$$

**Definition 1.6.** A singular $\mathscr{S}_n$-manifold $(A, f)$ in $(X, Y)$ *bords*, iff there exists a singular $\mathscr{S}_n$-manifold $(B, g)$ (we do not require $g(0, \omega)$ to factor through $Y$) such that $A(\omega)$ are submanifold of codimension zero of $B(\omega, 0)$ and

$$g(\omega, 0)|A(\omega) = f(\omega), \qquad g(\omega, 0)(B(\omega, 0) - A(\omega)^\circ) \subset Y$$

where $A(\omega)^\circ$ denotes the interior of $A(\omega)$.

In particular, we define the "boundary" $(\partial_0 A, \partial_0 f) = (A(0), f(0))$ of a singular $\mathscr{S}_n$-manifold $(A, f)$ in $(X, Y)$ by

(1·7) $$(\partial_0 A)(\omega) = A(\omega, 0), \ (\partial_0 f)(\omega) = f(\omega, 0).$$

Then $(\partial_0 A, \partial_0 f)$ becomes a singular $\mathscr{S}_n$-manifold in $Y = (Y, \phi)$.

If $\partial_0 A = \phi$, $A$ is called a closed $\mathscr{S}_n$-manifold.

Now for the formation and basic properties of the Baas-Sullivan bordism groups $MU(\mathscr{S}_n)_*(X, Y)$ we refer to [4]. We cite here the following facts which will be needed below.

First of all, we have

**Proposition 1.8** (Theorem 3.3 in [4]). *The functor $MU(\mathscr{S}_n)_*(\ )$ forms a generalized homology theory.*

The group $MU(\mathscr{S}_n)_*(X, Y)$ is considered as a (two-sided) $MU_*(pt)$-module in a canonical way, that is, in forming the usual product of a closed manifold and an $\mathscr{S}_n$-manifold.

For the next proposition, let $\mathscr{S}_{n-1}^{(i)} = \{P_1, \cdots, \hat{P}_i, \cdots, P_n\}$ be the subsequence of $\mathscr{S}_n$ obtained by deleting the $i$-th singularity manifold $P_i$. For $i \in \{1, 2, \cdots, n\}$, define a homomorphism of degree $p_i$ ($p_i = \dim P_i$):

$$(1\cdot 9) \qquad \beta_i\colon MU(\mathscr{S}_{n-1}^{(i)})_*(X, Y)\to MU(\mathscr{S}_{n-1}^{(i)})_*(X, Y)$$

induced from the above multiplication by the element $[P_i]$ of $MU_*(pt.)$, i.e.

$$\beta_i[A, f]=[P_i\times A, f\circ pr]$$

where $[A, f]$ denotes the bordism class of $(A, f)$. Further, let

$$(1\cdot 10) \qquad \gamma_i\colon MU(\mathscr{S}_{n-1}^{(i)})_*(X, Y)\to MU(\mathscr{S}_n)_*(X, Y)$$

be defined by regarding $\mathscr{S}_{n-1}^{(i)}$-manifold $A$ naturally as an $\mathscr{S}_n$-manifold $A$ with $\partial_i A=\phi$.

Finally the Bockstein operator of degree $-p_i-1$

$$(1\cdot 11) \qquad \delta_i\colon MU(\mathscr{S}_n)_*(X, Y)\to MU(\mathscr{S}_{n-1}^{(i)})_*(X, Y)$$

is defined by

$$\delta_i[A, f]=[A(i), f(i)].$$

Then we have Bockstein exact sequence.

**Proposition 1.12** (Theorem 3.2 in [4]). *The sequence*

$$\cdots\longrightarrow MU(\mathscr{S}_{n-1}^{(i)})_*(X, Y)\xrightarrow{\beta_i}MU(\mathscr{S}_{n-1}^{(i)})_*(X, Y)\xrightarrow{\gamma_i}MU(\mathscr{S}_n)_*(X, Y)$$
$$\xrightarrow{\delta_i}MU(\mathscr{S}_{n-1}^{(i)})_*(X, Y)\xrightarrow{\beta_i}\cdots$$

*is exact.*

For the coefficient group, we have

**Proposition 1.13** (Theorem 4.1 in [4]). *If the sequence* $\{[P_1], \cdots, [P_n]\}$ *of the bordism classes* $[P_i]$, $P_i\in\mathscr{S}_n$, *constitutes a regular sequence in* $MU_*(pt)$, *we have an isomorphism*

$$MU(\mathscr{S}_n)_*(pt)\approx MU_*(pt)/([P_1], \cdots, [P_n])$$

*as an* $MU_*(pt)$*-module, where the right hand side means the quotient algebra of* $MU_*(pt)$ *by the ideal generated by* $[P_1], \cdots, [P_n]$.

*Notations.* The above homomorphisms $\beta_i$, $\gamma_i$ and $\delta_i$ will be sometimes denoted by $\beta_{P_i}$, $\gamma_{P_i}$ and $\delta_{P_i}$ respectively, and the notations $\delta_{i_1\cdots i_k} = \delta_{i_1}\delta_{i_2}\cdots\delta_{i_k}$, $\delta_{i_1\cdots i_k}A = A(i_1\cdots i_k) = A(P_{i_1}\cdots P_{i_k})$ will be also used.

*Remark.* If singularity manifolds $Q_1, \cdots, Q_n$ are bordant to $P_1, \cdots, P_n$ respectively, then the corresponding groups $MU(Q_1, \cdots, Q_n)_*(\quad)$ and $MU(P_1, \cdots, P_n)_*(\quad)$ are isomorphic.

## § 2. Some Geometric Constructions and Basic Lemmas

Recall first a geometric technique due to Morava (cf. Johnson-Wilson [10], Appendix).

Let $P$ be a closed, weakly complex manifold of dimension $p$. Put

$$E(0) = I \times P \times P$$

where $I$ denotes the unit interval $[0, 1]$. This manifold $E(0)$ is to be given the canonical weakly complex structure induced from that of $P \times P$.

Define

$$(2\cdot1) \qquad \partial_1 E(0) = \partial E(0) = -(0 \times P \times \underline{P}) \cup (1 \times \underline{P} \times P),$$
$$E(10) = -(0 \times P) \cup (1 \times P), \ \varepsilon(1;0)\colon \partial_1 E(0) \approx \underline{P} \times E(10),$$

where $\varepsilon(1;0)$ denotes the twisting isomorphism defined by

$$(2\cdot1)' \qquad \varepsilon(1;0)(i, p, q) = \begin{cases} (q, 0, p) & \text{for} \quad i=0 \\ (p, 1, q) & \text{for} \quad i=1\,. \end{cases}$$

Here we have used the convention $\partial I = (-0) \cup (1)$.

Then $E(0)$ is a closed $\{P\}$-manifold of dimension $2p+1$. Since $MU(P)_*(pt) \approx MU_*(pt)/([P])$ (Proposition 1.11) and the odd dimension part of this group is zero, we can choose a $\{P\}$-manifold $E$ with $\partial_0 E = E(0)$.

We will call such a manifold $E$ *Morava's manifold* for $P$ and sometimes denote it by $E_P$.

**Lemma 2.2.** *Let $E$ and $E'$ be Morava's manifolds for a fixed $P$. Then $E$ and $E'$ determine a closed $\{P\}$-manifold $b(E, E') = E \cup_{id} (-E')$ representing an element $\beta(E, E')$ of $MU(P)_{2p+2}(pt)$. Conversely, for any element $\beta$ of $MU(P)_{2p+2}(pt)$ and any choice of Morava's manifold $E$ for $P$, there exists another $E'$ such that $\beta(E, E') = \beta$.*

*Proof.* By $E\cup_{id}(-E')$ we mean the manifold obtained from $E\cup(-E')$ by identifying the part $E(0)$ with the corresponding part $E'(0)$. The first half of the lemma is obvious. For the second half, we note that $\beta(E,E)=0$. For a given $\beta$, choose a closed manifold $B$, of which image in $MU_*(pt)/([P])$ gives $\beta$. Take $E\cup(-B)$ as $E'$, then we have $\beta(E,E')=\beta$.

Now we will introduce several lemmas which are needful for our purpose.

These lemmas aim to analyse the groups $MU(\mathcal{S}_n)_*(X,Y)$, especially for the case of singularity class $\mathcal{S}_n$ having duplicated members.

Let $\mathcal{S}_n=\{P_1,\cdots,P_n\}$ be a sequence of closed manifolds as before, and suppose $P_i=P_j=P$ for some pair $(i,j)$, $1\leqq i<j\leqq n$. Let $\mathcal{S}_{n-1}^{(i)}=\{P_1,\cdots,\hat{P}_i,\cdots P_n\}$ be the subsequence $\mathcal{S}_n-\{P_i\}$.

Utilizing Morava's manifold $E$ for $P$, we shall define a homomorphism of degree $p+1$

$$s_{ij}=s_{ij}^E\colon MU(\mathcal{S}_{n-1}^{(i)})_*(X,Y)\to MU(\mathcal{S}_n)_*(X,Y) \tag{2·3}$$

as follows. Given a singular $\mathcal{S}_{n-1}^{(i)}$-manifold $(B,g)$ in $(X,Y)$, and put

$$s_{ij}(B)=I\times P\times B\underset{j0}{\cup}E\times B(j) \tag{2·3$'$}$$

with identification of isomorphic parts of boundaries:

$$\begin{array}{ccc} I\times P\times\partial_j B & & \partial_0 E\times B(j)\\ \approx\Big\downarrow 1\times 1\times\beta(j) & & \Big\|\\ I\times P\times P_j\times B(j) & \xrightarrow{id} & I\times P\times P\times B(j), \end{array}$$

here it should be noticed that we will make a convenience of labelling parts of $\partial B$ as if $B$ were an $\mathcal{S}_n$-manifold with $\partial_i B=\phi$. Thus $\partial_j$, $B(j)$ mean $\partial_{P_j}$, $B(P_j)$ respectively (See § 1). Then $s_{ij}(B)$ is regarded as an $\mathcal{S}_n$-manifold by defining

$$\partial_k s_{ij}(B)=\begin{cases} -0\times P\times B & \text{for } k=i\\ 1\times P\times B\underset{j0}{\cup}\partial_1 E\times B(j) & \text{for } k=j\\ -I\times P\times\partial_k B\underset{j0}{\cup}E\times\partial_k B(j) & \text{for } k\neq i,j, \end{cases} \tag{2·4}$$

$$s_{ij}(B)(\omega)=\begin{cases} -I\times P\times B(\omega)\cup E\times B(\omega,j) & \text{for } \omega\not\ni i,j \\ -\varepsilon(\sigma)(0\times B(\omega')) & \text{for } \sigma(\omega)=(\omega',i) \\ \varepsilon(\tau)(1\times B(\omega'')\cup(-1)^{|\omega''|}E(1)\times B(\omega'',j) & \\ & \text{for } \tau(\omega)=(\omega'',j), \\ & |\omega''|=\text{length of } \omega'', \end{cases}$$

$$s_{ij}(g)|I\times P\times B=g\circ\text{proj}.$$

$$s_{ij}(g)|E\times B(j)=g(j)\circ\text{proj}.$$

Inspection shows this is well-defined. .

The following lemma follows directly from $(2\cdot4)$.

**Proposition 2.5.** *Let $\mathcal{S}_n$ and $\mathcal{S}_{n-1}^{(i)}$ be as above, and suppose $P_i=P_j=P$ for some $j$, $j>i$. Put $p=\dim P$. Then the Bockstein long exact sequence* (Proposition 1.12) *breaks in split exact sequences*

$$0\longrightarrow MU(\mathcal{S}_{n-1}^{(i)})_l(X,Y)\xrightarrow{\gamma_i}MU(\mathcal{S}_n)_l(X,Y)$$

$$\underset{s_{ij}^E}{\overset{\delta_i}{\rightleftarrows}}MU(\mathcal{S}_{n-1}^{(i)})_{l-p-1}(X,Y)\longrightarrow 0$$

*such that $\delta_i s_{ij}=-id$.*

This means, in particular, that the homomorphism $\beta_i$ in Proposition 1.12 vanishes for the present case. But, this implies, in turn,

**Corollary 2.6** ($2\cdot4$ in [10]). *For any singularity class $\mathcal{S}_n=\{P_1, \cdots, P_n\}$, the group $MU(\mathcal{S}_n)_*(X,Y)$ is a module over the quotient ring $MU_*(pt)/([P_1],\cdots,[P_n])$. The module structure is induced from the usual action of $MU_*(pt)$* (*cf.* Proposition 1.13).

**Proposition 2.7.** *Under the same assumption as in Proposition* 2.5, *we have*

i) $\delta_j\gamma_i=\gamma_i\delta_j$ *for* $i\neq j$, $\quad s_{ij}\gamma_k=\gamma_k s_{ij}$ *for* $k\neq i,j$,

ii) $$\partial_k s_{ij}^E = \begin{cases} -id & \text{for} \quad k=i \\ (ij)^* & \text{for} \quad k=j \\ -s_{ij}^E \partial_k & \text{for} \quad k \neq i, j, \end{cases}$$

*where*

(2·8) $$(ij)^* : MU(\mathscr{S}_{n-1}^{(i)})_*(X) \to MU(\mathscr{S}_{n-1}^{(j)})_*(X)$$

*means the isomorphism induced by the transposition* $(ij)$ *on the labels, and hence, by the exchange of* $P_j$ *for* $P_i$, $\partial_j$ *for* $\partial_i$ *and etc.* (*cf.* (2·12))

*Proof.* The equalities are directly seen from (2·4), except for the case $k=j$ in ii). For this case, we observe from (2·4)

$$\partial_j s_{ij}(B) = s_{ij}(B)(j) = 1 \times B \underset{j0}{\cup} E(1) \times B(j).$$

It will be sufficient to prove that $(ij)^* B \sim s_{ij}(B)(j)$ as $\mathscr{S}_{n-1}^{(j)}$-manifolds.

First note that $\partial E(1) = (0 \times P) \cup -(1 \times P)$, hence $E(1)$ is a $\{P\}$-manifold of odd dimension in virtue of the assumption dim $P$=even (§ 1).

Let $I_1$ be a copy of the unit interval $[0,1]$. Consider $E(1) \cup I_1 \times P$ with identification $\partial E(1) \equiv -\partial(I_1 \times P)$ (*cf.* (2·1)). Then this is a closed manifold representing an element of $MU_{odd}(pt) = 0$. Therefore there exists a manifold $L$ such that $\partial L = E(1) \cup I_1 \times P$.

Construct

$$V = [0, \varepsilon]_2 \times B \cup -L \times B(j)$$

with identification of isomorphic parts of boundaries:

$$\begin{array}{ccc} [0,\varepsilon]_2 \times B & & -L \times B(j) \\ \cup & & \cup \\ [0,\varepsilon]_2 \times P \times B(j) & \approx & -[1-\varepsilon, 1]_1 \times P \times B(j) \\ \Cup & & \Cup \\ (t, p, x) & \longleftrightarrow & (1-t, p, x) \end{array}$$

where we assume $0 < \varepsilon < 1 - \varepsilon$.

Put

$$\partial_0 V = -(0_2 \times B \cup E(1) \times B(j)) \perp\!\!\!\perp (\varepsilon_2 \times B \cup -[\varepsilon, 1-\varepsilon]_1 \times P \times B(j)),$$

$$\partial_i V = -[0, \varepsilon]_1 \times P \times B(j), \qquad \partial_j V = \phi,$$

$$\partial_k V = -([0, \varepsilon]_2 \times \partial_k B \cup L \times \partial_k B(j)) \quad \text{for} \quad k \neq 0, i, j,$$

where $\amalg$ means disjoint union.

We would like to omit to give the explicit forms of $V(\omega)$.

Thus $V$ is an $\mathscr{S}_{n-1}^{(j)}$-manifold and

$$\partial_0 V = -s_{ij}(B)(j) + (ij)^* B,$$

which was to be shown.

For the next proposition, we insert a small construction and a lemma.

Chose a Morava's manifold $E = E_P$ for a closed manifold $P$.

Define

$$\varphi: I = [0,1] \to I, \qquad \text{by } \varphi(t) = 1-t,$$

$$(2\cdot 9) \qquad T: P \times P \to P \times P, \quad \text{by } T(p,q) = (q,p).$$

Then we have an isomorphism

$$(2\cdot 9)' \qquad \varphi \times T: E(0) = I \times P \times P \approx -E(0)$$

and we construct a closed $\{P\}$-manifold

$$(2\cdot 10) \qquad a(E) = E \bigcup_{\varphi \times T} E$$

from two copies of $E$, identifying the parts $E(0)$'s by the isomorphism $(2\cdot 9)'$.

The following lemma follows easily.

**Lemma 2.11.** *Let $E$ and $E'$ be Morava's manifolds for the same $P$. Then, we have*

$$\alpha(E) - \alpha(E') = 2\beta(E, E'), \ \alpha(E) = [a(E)],$$

*in $MU(P)_*(pt)$. (See Lemma 2.2 for the definition of $\beta(E, E')$.)*

Let now $\mathscr{S}_n$ be as before and assume $P_i = P_j = P$, $i < j$, for some $i$, $j$ in $\{1, 2, \cdots, n\}$. Let $\sigma = (ij)$ be the transposition. This induces a canonical isomorphism

$$(2\cdot 12) \qquad (\underline{ij})^*: MU(\mathscr{S}_n)_*(\ \ ) \to MU(\sigma(\mathscr{S}_n))_*(\ \ ),$$

where $\sigma(\mathscr{S}_n) = \{P_1, \cdots \overset{i}{P_j} \cdots \overset{j}{P_i} \cdots P_n\}$. Explicitly, for a singular $\mathscr{S}_n$-manifold $(A, f)$ it gives

$$(A,f)\mapsto(A^*,f^*)=(\underline{\sigma}^*A,\ \underline{\sigma}^*f),$$

$\partial_i A^*=\partial_{\sigma(i)}A$, $A^*(\omega)=A(\sigma(\omega))$ and $f^*(\omega)=f(\sigma(\omega))$ etc.

Composing this isomorphism $(\underline{ij})^*$ with the trivial identification.

$$(2\cdot12)' \qquad \iota\colon MU(\sigma(\mathscr{S}_n))_*(\ \ )\equiv MU(\mathscr{S}_n)_*(\ \ )$$

induced from the obvious identification $\sigma(\mathscr{S}_n)\equiv\mathscr{S}_n$, we have an involution

$$(2\cdot12)'' \qquad \bar{\sigma}^*=(\overline{ij})^*=\iota\circ(\underline{ij})^*\colon MU(\mathscr{S}_n)_*(\ \ )\to MU(\mathscr{S}_n)_*(\ \ ).$$

Then the following proposition is one of key lemmas in our method.

**Proposition 2.13.** *Under the above assumption on $\mathscr{S}_n$, we have*

i) $$\delta_k(\overline{ij})^*=\begin{cases}(ji)^*\delta_j & \text{for } k=i\\ (ij)^*\delta_i & \text{for } k=j\\ (\overline{ij})^*\delta_k & \text{for } k\neq i,j,\end{cases}$$

ii) $$(\overline{ij})^*-1=s^E_{ij}\delta_i(1-(\overline{ij})^*)+\alpha(E_j)\times\delta_i\delta_j$$

*on $MU(\mathscr{S}_n)_*(\ \ )$. More explicitly,* ii) *means*

$$(A^*,f^*)-(A,f)\sim$$
$$s^E_{ij}(A(i),f(i))-s^E_{ij}(A^*(i),f^*(i))+(a(E_j)\times A(ij),f(ij)\circ pr)$$

*for a singular $\mathscr{S}_n$-manifold $(A,f)$, where the last term means the singular $\mathscr{S}_n$-manifold defined by*

$$\begin{array}{lll} & \overset{\partial_i}{\nearrow} & \phi \\ a(E_j)\times A(ij) & \overset{\partial_j}{\longrightarrow} & \partial_1 a(E)\times A(ij) \quad (\textit{cartesian product}) \\ & \underset{\partial_k}{\searrow} & a(E)\times\partial_k A(ij) \quad \textit{for} \quad k\neq i,j. \end{array}$$

In this proposition the assertion i) is easily verified from the definitions of the terms involved there. (See (1·11), (2·8) and (2·12)″.) For the proof of the assertion ii), we have to prepare some lemmas.

**Lemma 2.14.** *Under the same situation as in Proposition* 2.13, *we have*

$$(\overline{ij})^*A-A\sim I\times(\partial_i A\cup\partial_j A),$$

*where the right hand side means the $\mathscr{S}_n$-manifold defined by*

$$
I\times(\partial_i A\cup\partial_j A)\;
\begin{cases}
\xrightarrow{\partial_i} -0\times\partial_i A\cup 1\times\partial_j A\\[4pt]
\xrightarrow{\partial_j} -0\times\partial_j A\cup 1\times\partial_i A\\[4pt]
\xrightarrow{\partial_k} -I\times\partial_k(\partial_i A\cup\partial_j A) \quad \textit{for} \quad k\neq i,j
\end{cases}
$$

*Proof.* First replace $A$ with $A'=A\cup[0,\varepsilon]\times\partial A$, so that $A'$ has a collared neighborhood of boundary and itself is isomorphic to $A$. Construct $I\times A'$ as an $\mathscr{S}_n$-manifold:

$$
I\times A'\;
\begin{cases}
\xrightarrow{\partial_0} \{-0\times(A\cup[0,\varepsilon]\times\partial_0 A)\cup -I\times\varepsilon\times\partial_0 A\cup 1\\
\qquad\quad \times(A\cup[0,\varepsilon]\times\partial_0 A)\}\cup -I\times\varepsilon\times(\partial_i A\cup\partial_j A)\\[4pt]
\xrightarrow{\partial_i} -0\times[0,\varepsilon]\times\partial_i A\cup 1\times[0,\varepsilon]\times\partial_j A\\[4pt]
\xrightarrow{\partial_j} -0\times[0,\varepsilon]\times\partial_j A\cup 1\times[0,\varepsilon]\times\partial_i A\\[4pt]
\xrightarrow{\partial_k} -0\times[0,\varepsilon]\times\partial_k A\cup -I\times\varepsilon\times\partial_k A\cup 1\times[0,\varepsilon]\times\partial_k A,\\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for} \quad k\neq 0,i,j
\end{cases}
$$

Then, by inspection we see that $(-A)\cup(\overline{ij})^*A\cup(-I\times(\partial_i A\cup \partial_j A))$ bords $I\times A'$ (See Definition 1.6).

**Lemma 2.15.** *Under the same assumption as above, we have*

$$I\times(\partial_i A\cup\partial_j A)\sim s_{ij}^E(A(i))+s'(A(j))+a(E_j)\times A(ij)$$

*where*

$$(2\cdot 16)\qquad s'(A(j))=I\times P\times A(j)\underset{\partial_i(\varphi\times 1)\partial_0}{\cup} E\times A(ji)$$

*is defined by identification*

$$-I\times P\times P\times A(ij)\underset{\approx}{\overset{\varphi\times 1\times 1}{\longrightarrow}} E(0)\times A(ji)$$

*and it is considered as an $\mathscr{S}_n$-manifold:*

$$(2\cdot 17)\qquad s'(A(j)) \begin{cases} \xrightarrow{\partial_i} 1\times P\times A(j) \\ \xrightarrow{\partial_j} -0\times P\times A(j)\cup\partial_1 E\times A(ji) \\ \xrightarrow{\partial_k} -I\times P\times\partial_k A(j)\cup E\times\partial_k A(ji) \quad \textit{for} \quad k\neq i,j. \end{cases}$$

*Proof.* Since the detailed proof is considerably tedious, we will only outline the proof. Let $I_1$ and $I_2$ be copies of the unit interval $[0,1]$, and let $E'$ and $E''$ be copies of Morava's manifold $E_P$.

Construct a "manifold"

$$W=I_1\times I_2\times(\partial_i A\cup\partial_j A)\cup I_3\times(E'\underset{\varphi\times T}{\cup}E'')\times A(ij)$$

with identification:

$$(2\cdot 18)\qquad \begin{array}{ccc} 1_1\times I_2\times\partial_j\partial_i A & & -0_3\times E'(0)\times A(ij) \\ \| & & \| \\ 1_1\times I_2\times P\times P\times A(ji) & \equiv & 0_3\times I\times P\times P\times A(ji) \end{array}$$

Exactly speaking, this could not be called a manifold. In order to obtain a genuine manifold, we need a certain "thickening" process in the above attachment $(2\cdot 18)$.

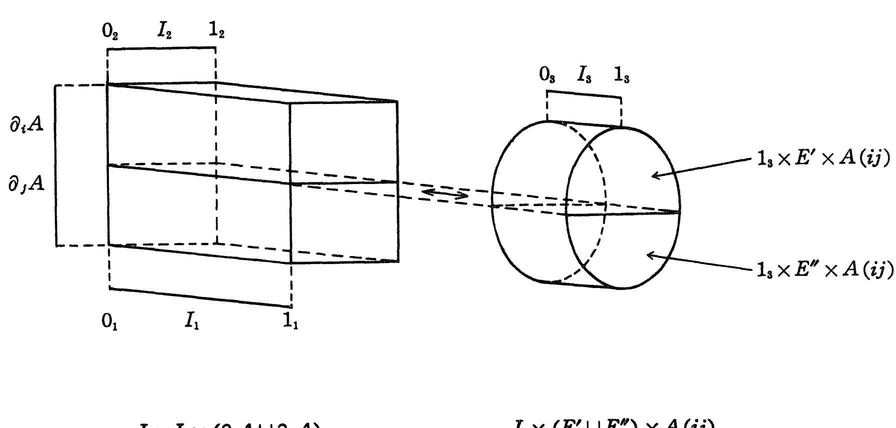


Figure 1

But, for simplicity's sake, we shall content with the pseudo-manifold $W$. Now $W$ is considered as an $\mathscr{S}_n$-"manifold" as follows:

$$\partial_0 W = I_1 \times I_2 \times \partial_0 (\partial_i A \cup \partial_j A) \cup -0_1 \times I_2 \times (\partial_i A \cup \partial_j A)$$
$$\cup (1_1 \times I_2 \times \partial_i A \cup -0_3 \times E' \times A(ij)) \cup (1_1 \times I_2 \times \partial_j A \cup$$
$$-0_3 \times E'' \times A(ij)) \cup 1_3 \times (E' \cup E'') \times A(ij)$$

$$\partial_i W = I_1 \times 0_2 \times \partial_i A \cup -I_1 \times 1_2 \times \partial_j A$$

$$\partial_j W = I_1 \times 0_2 \times \partial_j A \cup -I_1 \times 1_2 \times \partial_j A \cup -I_3 \times \partial_1 (E' \cup E'') \times A(ij)$$

$$\partial_k W = I_1 \times I_2 \times \partial_k (\partial_i A \cup \partial_j A) \quad \text{for} \quad k \neq 0, \text{i, j}.$$

And there are isomorphisms:

a) $s^E_{ij}(A(i)) \approx 1_1 \times I_2 \times \partial_i A \cup 0_3 \times E' \times A(ji) \subset \partial_0 W$

b) $s'(A(j)) \approx 1_1 \times I_2 \times \partial_j A \cup 0_3 \times E'' \times A(ji) \subset \partial_0 W$

c) $a(E_j) \times A(ij) \approx 1_3 \times (E' \underset{\varphi \times T}{\cup} E'') \times A(ij) \subset \partial_0 W$.

In fact, the involved identifications will be seen in the following commutative diagram

$$\begin{array}{lcccc}
s^E_{ij}(A(i)): & I \times P \times A(i) & \cup & E' \times A(ji) \\
 & \uparrow \partial_i & id & \uparrow \partial_0 \\
 & -I \times P \times P \times A(ji) & \longleftarrow & E'(0) \times A(ji) \\
(2\cdot 19) & \downarrow 1 \times T \times 1 & \varphi \times 1 \times 1 & \downarrow \varphi \times T \times 1 \\
 & -I \times P \times P \times A(ij) & \longleftarrow & E''(0) \times A(ji) \\
 & \downarrow \partial_i & & \downarrow \partial_0 \\
s'(A(j)): & I \times P \times A(j) & \cup & E'' \times A(ji).
\end{array}$$

Then Lemma 2.15 follows easily.

**Lemma 2.20.** *In the above lemma, we have an isomorphism*

$$s'(A(j)) \approx -s^E_{ij}(A^*(i))$$

*where* $A^* = (\overline{ij})^* A$.

*Proof.* Define the following map between $s'(A(j))$ and $-s^E_{ij}(A^*(i))$ (cf. (2·4), (2·16), (2·17)):

$$\begin{array}{ccccccc} s'(A(j)): I\times P\times A(j) & \xleftarrow{\partial_i} & -I\times P\times P\times A(ij) & \xrightarrow{\varphi\times 1\times 1} & E(0)\times A(ji) & \xrightarrow{\partial_0} & E\times A(ji) \\ \downarrow{\scriptstyle \varphi\times 1} & & \downarrow{\scriptstyle \varphi\times 1\times 1} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle id} \\ -s^E_{ij}(A^*(i)): -I\times P\times A^*(i) & \xleftarrow{\partial_i} & I\times P\times P\times A^*(ji) & \rightarrow & -E(0)\times A^*(ji) & \rightarrow & -E\times A^*(ji) \end{array}$$

This map gives the isomorphism in the lemma.

*Proof of Proposition* 2.13, ii). Lemma 2.14, 2.15 and 2.20 give the desired proof.

## § 3. Multiplication in $MU(\mathscr{S}_n)_*(\ )$

Let $\mathscr{S}_n=\{P_1, \cdots, P_n\}$ and $\mathscr{T}_m=\{Q_1, \cdots, Q_m\}$ be singularity classes as defined in § 1. Recall that all $P_i$ and $Q_j$ are closed, weakly complex manifolds of even dimensions.

Let $(A, f)$ be a singular $\mathscr{S}_n$-manifold in $(X, Y)$, and $(B, g)$ a singular $\mathscr{T}_m$-manifold in $(V, W)$. Put $a=\dim A$ and $b=\dim B$. Denote by $\mathscr{S}_n+\mathscr{T}_m=\{P_1, \cdots, P_n, Q_1, \cdots, Q_m\}$ the sequence obtained by juxtaposing $\mathscr{S}_n$ and $\mathscr{T}_m$ in their proper orders (cf. [3]).

**Definition 3.1.** The *cross product* $(A, f)\times(B, g)$ is defined as the singular $(\mathscr{S}_n+\mathscr{T}_m)$-manifold $(A\times B, f\times g)$ in $(X\times V, Y\times V\cup X\times W)$, which is given by the following data:

i) $$\partial_i(A\times B)=\begin{cases}\partial_0 A\times B\cup(-1)^a A\times\partial_0 B & \text{for } i=0\\ \partial_i A\times B & \text{for } 1\leqq i\leqq n\\ (-1)^a A\times\partial_{i-n}B & \text{for } n+1\leqq i\leqq n+m,\end{cases}$$

ii) $(A\times B)(\omega', \bar{\omega}'')=(-1)^{a|\omega''|}A(\omega')\times B(\omega'')$

and

$$(A\times B)(\omega', \bar{\omega}'', 0)=(-1)^{a|\omega''|}(A(\omega', 0)\times B(\omega'')\cup(-1)^a A(\omega')\times B(\omega'', 0))$$

for $\omega'\subset\{1, \cdots, n\}$, $\omega''=(j_1, \cdots, j_l)\subset\{1, \cdots, m\}$, $|\omega''|=l$, $\bar{\omega}''=(n+j_1, \cdots, n+j_l)$,

iii) $(f\times g)(\omega', \bar{\omega}'')=f(\omega')\times g(\omega'')$

and

$$(f\times g)\,(\omega', \overline{\omega}'', 0)=\begin{cases} f(\omega', 0)\times g(\omega'') & \text{on} \quad A(\omega', 0)\times B(\omega'') \\ f(\omega')\times g(\omega'', 0) & \text{on} \quad A(\omega')\times B(\omega'', 0) \end{cases}$$

for $\omega'$, $\omega''$, $\overline{\omega}''$ as above.

*Remark.* We may also take $\{P_1, Q_1, P_2, Q_2, \cdots, P_k, Q_k, \cdots\}$ as $\mathscr{S}_n+\mathscr{T}_m$. In this case, the above definition 3.1 should receive a change of labelling in $\partial_i$, $\delta_i$ etc.

The cross product induces a bilinear map

$$(3\cdot 2)\qquad \times: MU(\mathscr{S}_n)_a(X, Y)\times MU(\mathscr{T}_m)_b(V, W)$$
$$\to MU(\mathscr{S}_n+\mathscr{T}_m)_{a+b}(X\times V, Y\times V\cup X\times W)$$
$$[A,f]\times[B,g]=[(A,f)\times(B,g)]$$

or a linear map

$$(3\cdot 2)'\qquad \times: MU(\mathscr{S}_n)_a(X, Y)\otimes MU(\mathscr{T}_m)_b(V, W)$$
$$\to MU(\mathscr{S}_n+\mathscr{T}_m)_{a+b}(X\times V, Y\times V\cup X\times W)$$

which are natural and compatible with the multiplication by elements of $MU_*(pt)$.

Moreover the cross product is apparently associative:

$$(3\cdot 3)\qquad \{(A,f)\times(B,g)\}\times(C,h)=(A,f)\times\{(B,g)\times(C,h)\}.$$

Hereafter, we consider the case of $\mathscr{T}_m=\mathscr{S}_n$.

For convenience, denote by $\mathscr{S}_n'=\{P_1', \cdots, P_n'\}$ a copy of $\mathscr{S}_n$ so that $P_i'=P_i$ for $i=1, 2, \cdots, n$.

Then we have endomorphisms

$$(3\cdot 4)\qquad \pi_{12}^{P_i}: MU(\mathscr{S}_n+\mathscr{S}_n')_*(\quad)\to MU(\mathscr{S}_n+\mathscr{S}_n')_*(\quad)$$

defined by

$$(3\cdot 4)'\qquad \pi_{12}^{P_i}=1+s_{P_iP_i'}^{E_i}\delta_{P_i}=1+s_{ii'}\delta_i$$

where 1 means the identity map and $E_i=E_{P_i}$ means a Morava's manifold for $P_i$. (See § 2).

**Lemma 3.5.** *Let* $\mathscr{S}_{n-2}^{(ij)}=\{P_1, \cdots, \hat{P}_i, \cdots, \hat{P}_j, \cdots, P_n\}$ *be the* *subsequence of* $\mathscr{S}_n$ *which is obtained from* $\mathscr{S}_n$ *by deleting its two members* $P_i$ *and* $P_j$, $i\neq j$. *Then we have*

$$s_{P_iP_i'}\circ s_{P_jP_j'}=-s_{P_jP_j'}\circ s_{P_iP_i'}$$

*as homomorphisms*: $MU(\mathscr{S}_{n-2}^{(ij)}+\mathscr{S}_n')_*(\ )\to MU(\mathscr{S}_n+\mathscr{S}_n')_*(\ )$.

*Proof.* Take an $(\mathscr{S}_{n-2}^{(ij)}+\mathscr{S}_n')$-manifold $C$. It will be sufficient to show that $s_{P_iP_i'}s_{P_jP_j'}C$ is isomorphic to $-s_{P_jP_j'}s_{P_iP_i'}C$ as an $(\mathscr{S}_n+\mathscr{S}_n')$-manifold.

By $(2\cdot3)'$, we see

$$(3\cdot6)\qquad s_{P_iP_i'}s_{P_jP_j'}C=I_1\times P_i\times(I_2\times P_j\times C\cup E_j\times C(P_j'))$$
$$\cup E_i\times(-I_2\times P_j\times C(P_i')\cup E_j\times C(P_i'P_j'))$$

and, on the other hand, exchanging the order of pastings we have

$$(3\cdot7)\qquad s_{P_jP_j'}s_{P_iP_i'}C=I_2\times P_j\times(I_1\times P_i\times C\cup E_i\times C(P_i'))$$
$$\cup E_j\times(-I_1\times P_i\times C(P_j')\cup E_i\times C(P_j'P_i'))$$
$$=\{I_2\times P_j\times I_1\times P_i\times C\cup -E_j\times I_1\times P_i\times C(P_j')\}$$
$$\cup\{I_2\times P_j\times E_i\times C(P_i')\cup E_j\times E_i\times C(P_j'P_i')\}.$$

Comparing this expression with $(3\cdot6)$, we have an isomorphism

$$(3\cdot8)\qquad \psi: s_{P_iP_i'}s_{P_jP_j'}C\approx -s_{P_jP_j'}s_{P_iP_i'}C$$

which is induced by the twisting maps:

$$(I_1\times P_i)\times(I_2\times P_j)\to(I_{\_}\times P_j)\times I_1\times P_i,$$
$$(I_1\times P_i)\times E_j\to E_j\times(I_1\times P_i),\quad E_i\times(I_2\times P_j)\to(I_2\times P_j)\times E_i,$$

and

$$E_i\times E_j\to E_j\times E_i$$

on the four pairs of corresponding parts in $(3\cdot6)$ and $(3\cdot7)$.

**Proposition 3.9.** *The endomorphisms* $\pi_{12}^{P_i}$ *in* $(3\cdot4)$ *satisfy*

$$\pi_{12}^{P_i}\circ\pi_{12}^{P_j}=\begin{cases}\pi_{12}^{P_j}\circ\pi_{12}^{P_i} & \text{for}\quad i\neq j\\ \pi_{12}^{P_j} & \text{for}\quad i=j,\end{cases}$$

*that is,* $\pi_{12}^{P_i}$, $i=1, 2, \cdots, n$, *are mutually commutable idempotent endomorphisms of the group* $MU(\mathscr{S}_n+\mathscr{S}_n')_*(\ \ )$.

**Proposition 3.10.** *We have*

i) $$\delta_{P_i}\circ\pi_{12}^{P_j}=\begin{cases}\pi_{12}^{P_j}\circ\delta_{P_i} & \text{for } i\neq j\\ 0 & \text{for } i=j,\end{cases}$$

ii) $$\pi_{12}^{P_i}\circ s_{P_jP_j'}=\begin{cases}s_{P_jP_j'}\circ\pi_{12}^{P_i} & \text{for } i\neq j\\ 0 & \text{for } i=j,\end{cases}$$

*and*

iii) $$\operatorname{Im}\pi_{12}^{P_i}=\operatorname{Ker}\delta_{P_i}=\operatorname{Im}\gamma_{P_i}.$$

**Proposition 3.11.** *Let* $\gamma: MU(\mathscr{S}_n')_*(\ \ )\to MU(\mathscr{S}_n+\mathscr{S}_n')_*(\ \ )$ *be the canonical monomorphism defined by the composition* $\gamma=\gamma_{P_1}\cdots\gamma_{P_n}$ *of the monomorphisms* $\gamma_{P_i}$ *(See Proposition* 2.5). *Then we have a direct sum decomposition of the group* $MU(\mathscr{S}_n+\mathscr{S}_n')_*(\ \ )$:

i) $$MU(\mathscr{S}_n+\mathscr{S}_n')_*(\ \ )=\operatorname{Im}\gamma\oplus(\sum_{i=1}^{n}\operatorname{Im}s_{P_iP_i'}^{E_i})$$

*where* $s_{P_iP_i'}: MU(\mathscr{S}_{n-1}^{(i)}+\mathscr{S}_n')_*(\ \ )\to MU(\mathscr{S}_n+\mathscr{S}_n')_*(\ \ )$.

*Moreover, we have*

ii) $\operatorname{Im}\gamma=\operatorname{Im}\pi_{\boldsymbol{E}}$, $\operatorname{Ker}\pi_{\boldsymbol{E}}=\sum_i\operatorname{Im}s_{P_iP_i'}$, $\pi_{\boldsymbol{E}}=\pi_{12}^{P_1}\cdots\pi_{12}^{P_n}$,

*where* $\boldsymbol{E}=\{E_1, \cdots, E_n\}$ *is the system of Morava's manifolds* $E_i$.

*Proof of the above three propositions.* Propositions 3.9 and 3.10 follow from Lemma 3.5, Propositions 2.5 and 2.7 by easy calculations.

As for Proposition 3.11, the first half i) is obtained by iterated uses of Propositions 2.5 and 2.7, i). To prove the second half, we first note that

$$\pi_{12}^{P_i}\gamma_{P_j}=\begin{cases}\gamma_{P_j}\pi_{12}^{P_i} & \text{for } i\neq j\\ \gamma_{P_j} & \text{for } i=j.\end{cases}$$

Using this and Proposition 3.10, ii), iii), we conclude $\pi_{\boldsymbol{E}}|\operatorname{Im}\gamma=id$, and

$$\operatorname{Ker}\pi_{\boldsymbol{E}}=\sum_{i=1}^{n}\operatorname{Im}s_{P_iP_i'}^{E_i}.$$

Thus we have proved the assertion ii).

Now, we have come to introduce multiplications in the bordism theory $MU(\mathscr{S}_n)_*(\ )$.

Let $(A,f)$ and $(B,g)$ be representatives of elements of $MU(\mathscr{S}_n)_a(X, Y)$ and $MU(\mathscr{S}_n')_b(V, W)$ respectively. Let $(A\times B, f\times g)$ be the cross product of $(A,f)$ and $(B,g)$, which represents an element of $MU(\mathscr{S}_n+\mathscr{S}_n')_{a+b}(X\times V, Y\times V\cup X\times W)$.

Let

$$\gamma: MU(\mathscr{S}_n')_{a+b}(\ )\to MU(\mathscr{S}_n+\mathscr{S}_n')_{a+b}(\ )$$

and

$$\pi_{\boldsymbol{E}}: MU(\mathscr{S}_n+\mathscr{S}_n')_{a+b}(\ )\to MU(\mathscr{S}_n+\mathscr{S}_n')_{a+b}(\ )$$

be as in Proposition 3.11.

Then, by Proposition 3.11, ii), there is a singular $\mathscr{S}_n'$-manifold $(A\cdot_{\boldsymbol{E}}B, f\cdot_{\boldsymbol{E}}g)$ in $(X\times V, Y\times V\cup X\times W)$ such that

$$\gamma(A\cdot_{\boldsymbol{E}}B, f\cdot_{\boldsymbol{E}}g)=\pi_{\boldsymbol{E}}(A\times B, f\times g), \tag{3·12}$$

or

$$[A\cdot_{\boldsymbol{E}}B, f\cdot_{\boldsymbol{E}}g]=[\gamma^{-1}\circ\pi_{\boldsymbol{E}}(A\times B, f\times g)],$$

where $\boldsymbol{E}=\{E_1, \cdots, E_n\}$ is the system of Morava's manifolds $E_i$ for $P_i$ which were used in defining $\pi_{12}^{P_i}$ (See (3·4)′).

The bordism class $[A\cdot_{\boldsymbol{E}}B, f\cdot_{\boldsymbol{E}}g]$ depends on the choices of the system $\boldsymbol{E}$ of Morava's manifolds, but, for a fixed $\boldsymbol{E}$, it is uniquely determined from the given bordism classes $[A,f]$ and $[B,g]$.

Thus we have defined a bilinear map

$$\mu_{\boldsymbol{E}}: MU(\mathscr{S}_n)_a(X, Y)\times MU(\mathscr{S}_n')_b(V, W) \to MU(\mathscr{S}_n')_{a+b}(X\times V, Y\times V\cup X\times W) \tag{3·13}$$

or a linear map

$$\mu_{\boldsymbol{E}}: MU(\mathscr{S}_n)_a(X, Y)\otimes MU(\mathscr{S}_n)_b(V, W) \to MU(\mathscr{S}_n)_{a+b}(X\times V, Y\times V\cup X\times W) \tag{3·13)′$$

by putting

$$\mu_{\boldsymbol{E}}([A,f], [B,g])=[A\cdot_{\boldsymbol{E}}B, f\cdot_{\boldsymbol{E}}g].$$

**Theorem 3.14.** *Let $\mathscr{S}_n=\{P_1, \cdots, P_n\}$ be a singularity class. Then, for any choice of system $\boldsymbol{E}=\{E_1, \cdots, E_n\}$ of Morava's manifolds $E_i$ one for each $P_i$, there exists a natural multiplication $\mu_{\boldsymbol{E}}$ in the bordism theory $MU(\mathscr{S}_n)_*(\ )$:*

$$\mu_{\boldsymbol{E}}: MU(\mathscr{S}_n)_a(X, Y)\otimes MU(\mathscr{S}_n)_b(V, W) \to MU(\mathscr{S}_n)_{a+b}(X\times V, Y\times V\cup X\times W)$$

*which has the bilateral unit $1\in MU(\mathscr{S}_n)_0(pt)$, the canonical image of the unit element of the ring $MU_*(pt)$. Further the multiplication $\mu_{\boldsymbol{E}}$ is admissible in the following sense (cf. Araki-Toda [2]);*

i) *$\mu_{\boldsymbol{E}}$ is compatible with the usual multiplication by elements of $MU_*(pt)$.*

ii) *For the boundary homomorphism $\partial_0$, we have*

$$j_*\partial_0\mu_{\boldsymbol{E}}(x\otimes y)=j_{1*}\mu_{\boldsymbol{E}}(\partial_0 x\otimes y)+(-1)^{\dim x}j_{2*}\mu_{\boldsymbol{E}}(x\otimes\partial_0 y)$$

*where*

$$\begin{array}{ccc} & & (Y\times V, Y\times W) \\ & & \downarrow j_1 \\ (Y\times V\cup X\times W, \phi) & \xrightarrow{j} & (Y\times V\cup X\times W, Y\times W) \\ & & \uparrow j_2 \\ & & (X\times W, Y\times W) \end{array}$$

*are the natural inclusion maps.*

iii) *For the Bockstein operator $\delta_i$, we have*

$$\gamma_i\delta_i\mu_{\boldsymbol{E}}(x\otimes y)=\mu_{\boldsymbol{E}}(\gamma_i\delta_i x\otimes y)+(-1)^{\dim x}\mu_{\boldsymbol{E}}(x\otimes\gamma_i\delta_i y),\ for\ i=1, \cdots n,$$

*that is, the operations $\gamma_i\delta_i$ are derivations.*

iv) *For $x, y\in MU(\mathscr{S}_{n-1}^{(i)})_*(\ )$, we have*

$$\gamma_i\mu_{\boldsymbol{E}^{(i)}}(x\otimes y)=\mu_{\boldsymbol{E}}(\gamma_i x\otimes\gamma_i y)$$

*where $\boldsymbol{E}^{(i)}=\{E_1, \cdots\widehat{E}_i\cdots E_n\}=\boldsymbol{E}-\{E_i\}$,*

v) *If one of $x$, $y$, $z$ is the image of an element of $MU_*(pt)$, then*

$$\mu_{\boldsymbol{E}}((x\otimes y)\otimes z))=\mu_{\boldsymbol{E}}(x\otimes(y\otimes z)).$$

*Proof.* $\mu_{\boldsymbol{E}}(1\otimes x)=x$ follows directly from the definition of $\mu_{\boldsymbol{E}}$. To calculate $\mu_{\boldsymbol{E}}(x\otimes 1)$, take a representative $(A, f)$ of an element $x$ of

$MU(\mathscr{S}_n)_a(X, Y)$. Note that

$$\pi_{12}^i((A,f)\times 1)=(A,f)+s_{ii'}\delta_i(A,f)=(A,f)+(I\times P\times A(i),$$
$$f(i)\circ pr)\sim\sigma_{ii'}^*(A,f) \text{ in } MU(\mathscr{S}_n+\mathscr{S}_n')_*(X, Y).$$

Since $\pi_{12}^i\sigma_{jj'}^*=\sigma_{jj'}^*\pi_{12}^i$ for $i\neq j$, we have

$$\pi_E(x\times 1)=\pi_{12}^n\circ\cdots\circ\pi_{12}^1(x\times 1)=\sigma_{nn'}^*,\cdots\sigma_{11'}^*(x\times 1)=1\times x'=\gamma(x')$$

where $x'$ denotes a copy of $x$ in $MU(\mathscr{S}_n')_a(X, Y)$ identified with $MU(\mathscr{S}_n)_a(X, Y)$. It follows that

$$\mu_E(x\otimes 1)=\gamma^{-1}\cdot\pi_E(x\times 1)=x'\,.$$

Similar argument applies to the case when the unit 1 is replaced by any element $[M]$ of $MU_*(pt)$. The assertions i), ii), iv) and v) follows then easily and we will omit their proofs. For the proof of iii), we deduce the following equations:

$$\begin{aligned}\gamma\gamma_{i'}\delta_{i'}\mu_E&=\gamma_{i'}\delta_{i'}\pi_E=\pi_E{}^{(i)}\gamma_{i'}\delta_{i'}\pi_{12}^i=\pi_E{}^{(i)}\gamma_{i'}(\delta_{i'}+\sigma_{ii'}^*\delta_i)\\&=\pi_E{}^{(i)}\gamma_{i'}\delta_{i'}+\pi_E{}^{(i)}\bar{\sigma}_{ii'}^*\gamma_i\delta_i\,,\\ \gamma\mu_E(\gamma_i\delta_i\otimes 1)&=\pi_E{}^{(i)}\gamma_i\delta_i\,,\end{aligned}$$

and

$$\begin{aligned}\gamma\mu_E(1\otimes\gamma_i\delta_i)&=\pi_E\gamma_{i'}\delta_{i'}=\pi_E{}^{(i)}\pi_{12}^i\gamma_{i'}\delta_{i'}\\&=\pi_E{}^{(i)}\gamma_{i'}\delta_{i'}+\pi_E{}^{(i)}s_{ii'}\gamma_{i'}\delta_i\delta_{i'}\,,\end{aligned}$$

where $\pi_E{}^{(i)}=\pi_{12}^n\cdots\widehat{\pi_{12}^i}\cdots\pi_{12}^1$ is the composition of $\pi_{12}^j$, $j\neq i$. Using Proposition 2.13 and the fact that $\sigma_{ii'}^*=id$ on $MU(\mathscr{S}_{n-2}^{(ii')})_*(\ \ )$, and comparing the right hand sides of the above equations, we have

$$\begin{aligned}&\gamma\gamma_{i'}\delta_{i'}\mu_E-\gamma\mu_E(\gamma_i\delta_i\otimes 1)-\gamma\mu_E(1\otimes\gamma_i\delta_i)\\&=\pi_E{}^{(i)}(\bar{\sigma}_{ii'}^*\gamma_i\delta_i-\gamma_i\delta_i-s_{ii'}\gamma_{i'}\delta_i\delta_{i'})\\&=-\pi_E{}^{(i)}s_{ii'}(\sigma_{ii'}^*\delta_{i'}\gamma_i\delta_i+\gamma_{i'}\delta_i\delta_{i'})=0\,.\end{aligned}$$

This proves the assertion iii) and the proof of the theorem is completed.

## § 4. Commutativity

Let $\mathscr{S}_n'$ be a copy of a singularity class $\mathscr{S}_n$. Recall that in

$MU(\mathscr{S}_n+\mathscr{S}_n')_*(\ \ )$ we have the involutions

$$(4\cdot 1)\qquad \bar{\sigma}_i^* = (\overline{i, i'})^* : MU(\mathscr{S}_n+\mathscr{S}_n')_*(\ \ ) \to MU(\mathscr{S}_n+\mathscr{S}_n')_*(\ \ )$$

and also we have the isomorphisms

$$(4\cdot 2)\qquad \underline{\sigma}_i^* = (\underline{i, i'})^* : MU(\mathscr{S}_n+\mathscr{S}_n')_*(\ \ ) \to MU(\sigma_i(\mathscr{S}_n+\mathscr{S}_n'))_*(\ \ ).$$

These are related in the following forms

$$(4\cdot 3)\qquad \bar{\sigma}_i^* = \iota\circ\underline{\sigma}_i^*$$

where $\iota$ is the obvious identification $MU(\mathscr{S}_n+\mathscr{S}_n')_*(\ \ ) \equiv MU(\sigma_i(\mathscr{S}_n+\mathscr{S}_n'))_*(\ \ )$. (See (2·12)-(2·12)″.)

In order to check commutativity for the product $x\cdot_{E} y$, $x\in MU(\mathscr{S}_n)_a(X)$, $y\in MU(\mathscr{S}_n')_b(Y)$, we consider the following diagram

(4·4)

$$\begin{array}{ccccc}
MU(\mathscr{S}_n)_a(X)\otimes MU(\mathscr{S}_n')_b(Y) & \xrightarrow{\times} & MU(\mathscr{S}_n+\mathscr{S}_n')_{a+b}(X\times Y) & \xrightarrow{\gamma^{-1}\circ\pi_E} & MU(\mathscr{S}_n')_{a+b}(X\times Y) \\
 & & \underline{\sigma}_n^*\circ\cdots\circ\underline{\sigma}_1^* \downarrow \ \uparrow \iota & \text{iii)} & \Vert\ \iota \\
\tau \downarrow & \text{i)} & MU(\mathscr{S}_n'+\mathscr{S}_n)_{a+b}(X\times Y) & \xrightarrow{(\gamma')^{-1}\circ\pi_E'} & MU(\mathscr{S}_n)_{a+b}(X\times Y) \\
 & & T_* \uparrow & \text{ii)} & \uparrow T_* \\
MU(\mathscr{S}_n')_b(Y)\otimes MU(\mathscr{S}_n)_a(X) & \xrightarrow{\times} & MU(\mathscr{S}_n'+\mathscr{S}_n)_{a+b}(Y\times X) & \xrightarrow[(\gamma')^{-1}\circ\pi_E']{} & MU(\mathscr{S}_n)_{a+b}(Y\times X)
\end{array}$$

where $\tau(x\otimes y) = (-1)^{ab} y\otimes x$ and $T: Y\times X\to X\times Y$ is the twisting map. (See 3.11 for $\gamma$, $\pi_E$.)

**Lemma 4.5.** *The squares* i), ii) *and* iii) *in* (4·4) *are all commutative.*

*Proof.* The commutativities of ii) and iii) are trivial.

In order to prove the commutativity of i), take representatives $(A, f)$ and $(B, g)$ of $x\in MU(\mathscr{S}_n)_a(X)$ and $y\in MU(\mathscr{S}_n')_b(Y)$ respectively.

Then we can define an isomorphism between $(\mathscr{S}_n'+\mathscr{S}_n)$-manifolds in $X\times Y$:

$$\tau' : \underline{\sigma}^*(A\times B, f\times g) \to ((-1)^{ab} B\times A, T\circ(g\times f))$$

$$(4\cdot 6)\qquad \begin{array}{ccc} \underline{\sigma}^*(A\times B) & \xrightarrow[\approx]{\tau'} & (-1)^{ab}B\times A \\ & \searrow^{f\times g} & \downarrow{\scriptstyle g\times f} \\ & & Y\times X \\ & & \downarrow{\scriptstyle T} \\ & X\times Y & \end{array}$$

where $\underline{\sigma}^*=\underline{\sigma}_n{}^*\circ\cdots\circ\underline{\sigma}_1{}^*$ and $\tau'$ is induced from the exchange of factors of the original cartesian product $A\times B$.

This means the commutativity of the diagram i) in (4·4).

From Lemma 4.5, it follows

$$(4\cdot 7)\qquad (-1)^{ab}T_*(y\cdot_E x)=(-1)^{ab}T_*(\gamma')^{-1}\circ\pi_E{}'(y\times x)$$
$$=(-1)^{ab}(\gamma')^{-1}\circ\pi_E{}'\circ T_*(y\times x)=(\gamma')^{-1}\circ\pi_E{}'\underline{\sigma}^*(x\times y)$$
$$=(\gamma^{-1})\circ\pi_E\circ\bar{\sigma}^*(x\times y)$$

On the other hand, we can compute in the following form. By Prop. 2.13 ii), we have

$$\bar{\sigma}_i{}^*(x\times y)=x\times y+s^{E_i}_{P_iP_i'}\delta_{P_i}(1-\bar{\sigma}_i{}^*)(x\times y)+\alpha(E_{P_i'})\times\delta_{P_i}\delta_{P_i'}(x\times y)$$

or simply writing,

$$(4\cdot 8)\qquad \bar{\sigma}_i{}^*(x\times y)=x\times y+\alpha(E_i')\times\delta_{ii'}(x\times y)\quad(\mathrm{mod}\ \mathrm{Im}\, s_{P_iP_i'})$$
$$\text{for}\quad i=1,\cdots,n.$$

In virtue of the commutativities

$$(4\cdot 9)\qquad \bar{\sigma}_i{}^*s_{P_jP_j'}=s_{P_jP_j'}\bar{\sigma}_i{}^*\quad\text{for}\quad i\neq j,$$

we deduce, from (4·8), inductively

$$(4\cdot 10)\qquad \bar{\sigma}^*(x\times y)=\bar{\sigma}_n{}^*\cdots\bar{\sigma}_1{}^*(x\times y)\equiv x\times y$$
$$+\sum_{k=1}^{n}\sum_{1\leqq i_1<\cdots<i_k\leqq n}\alpha(E_{i_1'})\times\cdots\times\alpha(E_{i_k'})\times\delta_{i_1i_1'i_2i_2'\cdots i_ki_k'}(x\times y)$$
$$\left(\mathrm{mod}\ \sum_{i=1}^{n}\mathrm{Im}\, s_{P_iP_i'}\right)$$

or by using 3.11, ii),

$$(4\cdot 11)\qquad \bar{\sigma}^*(x\times y)\equiv x\times y$$

$$+\sum_{k=1}^{n}\varepsilon_k\sum_{1\leq i_1<\cdots<i_k\leq n}\delta_{i_1\cdots i_k}x\times(\alpha(E_{i_1})\times\cdots\times\alpha(E_{i_k})\times\delta_{i_1\cdots i_k}y)'$$

$$(\bmod \operatorname{Ker}\pi_E)$$

where $\varepsilon_k=(-1)^{a+(a-1)+\cdots+(a-k+1)}$.

Applying $\gamma^{-1}\circ\pi_E$ to the both sides of $(4\cdot 11)$ and comparing $(4\cdot 7)$, we have

**Theorem 4.12.** *For any singularity class $\mathscr{S}_n$ of weakly complex manifolds, the exterior multiplication $\mu_E(x\otimes y)=x\cdot_E y$ in $MU(\mathscr{S}_n)_*(\ \ )$ satisfies*

$$(-1)^{ab}T_*(y\cdot_E x)-x\cdot_E y$$

$$=\sum_{k=1}^{n}\varepsilon_k\sum_{1\leq i_1<\cdots<i_k\leq n}\delta_{i_1\cdots i_k}x\cdot_E(\alpha(E_{i_1})\times\cdots\times\alpha(E_{i_k})\times\delta_{i_1\cdots i_k}y)$$

*where* $\varepsilon_k=(-1)^{a+(a-1)+\cdots+(a-k+1)}$, $a=\dim x$.

## § 5. Associativity

We shall first deal with the case $n=1$, $\mathscr{S}_1=\{P\}$. For convenience, put $P=P'=P''$ and consider the group $MU(P,P',P'')_*(\ \ )$.

The following idempotent endomorphisms

$$\pi_{12}=\pi_{12}^P=1+s_{PP'}^E\delta_P,\ \ \pi_{23}=1+s_{P'P''}^E\delta_{P'},\ \ \pi_{13}=1+s_{PP''}^E\delta_P$$

of $MU(P,P',P'')_*(\ \ )$ are similarly defined as in $(3\cdot 4)$.

**Lemma 5.1.** *Let $\gamma: MU(P'')_*(\ \ )\to MU(P,P',P'')_*(\ \ )$ be the canonical monomorphism defined by the composition $\gamma=\gamma_P\circ\gamma_{P'}$. Then we have a direct sum decomposition*

i) $\quad MU(P,P',P'')_*(\ \ )=\operatorname{Im}\gamma\oplus\operatorname{Im}s_{PP''}\oplus\operatorname{Im}s_{P'P''}\circ\gamma_P$,

*and*

ii) $\quad \operatorname{Im}\gamma=\operatorname{Ker}\delta_1\cap\operatorname{Ker}\delta_2(=\operatorname{Ker}\delta_P\cap\operatorname{Ker}\delta_{P'})$.

*Proof.* This is an easy consequence of Proposition 2.5.

**Lemma 5.2.**

$$\operatorname{Im} \gamma \supset \operatorname{Im} \pi_{23}\pi_{12}, \qquad \operatorname{Im} \gamma \supset \operatorname{Im} \pi_{13}\pi_{23}.$$

*Proof.* A direct calculations plus Lemma 5.1, ii) prove the lemma.

**Proposition 5.3.**

$$\pi_{23}\pi_{12} = \pi_{13}\pi_{23}.$$

We will postpone the proof of this proposition.

Now let $[A,f]$, $[B,g]$ and $[C,h]$ be elements of $MU(P)_*(X)$, $MU(P')_*(Y)$ and $MU(P'')_*(Z)$ respectively. We have products

$[A \cdot_E B, f \cdot_E g]$ in $MU(P')_*(X \times Y)$,

$[B \cdot_E C, g \cdot_E h]$ in $MU(P'')_*(Y \times Z)$, $[(A \cdot_E B) \cdot_E C, (f \cdot_E g) \cdot_E h]$

and

$$[A \cdot_E (B \cdot_E C), f \cdot_E (g \cdot_E h)] \text{ in } MU(P'')_*(X \times Y \times Z).$$

For simplicity, we will omit the mappings in the bordism classes and only deal with manifolds.

In $MU(P, P', P'')_*(X \times Y \times Z)$, we have

$$(5\cdot4) \quad \gamma[[A \cdot_E B] \cdot_E C] = \gamma_P \gamma_{P'}[(A \cdot_E B) \cdot_E C] = \gamma_P \pi_{23}[(A \cdot_E B) \times C]$$
$$= \pi_{23}\gamma_P[(A \cdot_E B) \times C] = \pi_{23}[\pi_{12}(A \times B) \times C] = \pi_{23}\pi_{12}[A \times B \times C],$$
$$\gamma[A \cdot_E (B \cdot_E C)] = \gamma_{P'}\gamma_P[A \cdot_E (B \cdot_E C)] = \gamma_{P'}\pi_{13}[A \times (B \cdot_E C)]$$
$$= \pi_{13}[A \times \pi_{23}(B \times C)] = \pi_{13}\pi_{23}[A \times B \times C].$$

Then, by Proposition 5.3, we obtain

**Theorem 5.5.** *The exterior multiplication $\mu_E$ in $MU(P)_*(\ )$ is associative*:

$$(x \cdot_E y) \cdot_E z = x \cdot_E (y \cdot_E z).$$

The next lemma plays a key role in the proof of Prop. 5.3.

**Lemma 5.6.** *For $[C, h] \in MU(P'')_*(\ )$, we have*

$$(s_{13}^E s_{23}^E + s_{23}^E s_{13}^E)[C, h] = \pi_{13}(\alpha(E_3) \times (13)^*[C, h])$$

$$(=\pi_{13}(13)^*(\alpha(E_1)\times[C,h]))$$

*in* $MU(P,P',P'')_*(\ )$. *Here we are considering in the situation*: $\mathscr{S}_3=\{P,P',P''\}$ *with* $P=P'=P''$, $[C,h]\in MU(\mathscr{S}_1^{(12)})_*(\ )$. (*See* § 2.)

*Proof.* Construct a $\{P''\}$-manifold

$$W=s_{13}s_{23}C\bigcup_{\psi}s_{23}s_{13}C \tag{5·7}$$

with identifications shown in the following diagram:

(5·8)

$$s_{13}s_{23}C=I_1\times P\times\overbrace{(I_2\times P\times C\bigcup_{\partial_3\partial_0}E_2\times C(3))}^{s_{23}C}\bigcup_{\partial_3\partial_0}E_1\times\overbrace{(1_2\times C\cup E_2(1)\times C(3))}^{(s_{23}C)\,(3)}$$

$$\begin{array}{ll}
\partial_1\searrow & \partial_2\searrow \\
-0_1\times\underline{P}\times I_2\times P\times C\cup-0_1\times\underline{P}\times E_2\times C(3) & \\
\approx\ \downarrow\psi & I_1\times P\times 0_2\times\underline{P}\times C\cup E_1\times 0_2\times\underline{P}\times C(3) \\
I_2\times P\times 0_1\times\underline{P}\times C\cup E_2\times 0_1\times\underline{P}\times C(3) & \approx\ \downarrow\psi \\
 & -0_2\times\underline{P}\times I_1\times P\times C\cup-0_2\times\underline{P}\times E_1\times C(3) \\
\partial_1\nwarrow & \partial_2\nearrow
\end{array}$$

$$s_{23}s_{13}C=I_2\times P\times\underbrace{(I_1\times P\times C\bigcup_{\partial_3\partial_0}E_1\times C(3))}_{s_{13}C}\bigcup_{\partial_3\partial_0}E_2\times\underbrace{(1_1\times C\cup E_1(1)\times C(3))}_{(s_{13}C)\,(3)}$$

where $E_1$, $E_2$ are copies of the Morava manifold $E$ for $P$. The following lemma follows easily.

**Lemma 5.9.**

$$s_{13}s_{23}C+s_{23}s_{13}C\sim\gamma(W)$$

*where* $\gamma(W)$ *denotes the* $\{P,P',P''\}$*-manifold* $W$ *with* $\partial_1W=\partial_2W=\phi$.

Now $\gamma(W)$ is considered as the union of two $\{P,P''\}$-manifolds $U$ and $V$ as follows:

$$\gamma_1(W)=U\bigcup_{\partial_1\partial_1}V,\quad U=U(E)_{(3)}\times C_{(1)},\quad V=V(E)_{(13)}\times C(3) \tag{5·10}$$

where

$$(5\cdot 11)\quad U(E)=(I_1\times P\times I_2\times P\cup E_1\times 1_2)\underset{\psi}{\cup}(I_2\times P\times I_1\times P\cup E_2\times 1_1),$$

with identification indicated in $(5\cdot 8)$. Here $U(E)$ is considered as a closed $\{P''\}$-manifold such that

$$\partial_3 U(E)=(1_1\times \underline{P}\times I_2\times P\cup \underline{P}\times E_1(1)\times 1_2)$$
$$\underset{\psi}{\cup}(1_2\times \underline{P}\times I_1\times P\cup \underline{P}\times E_2(1)\times 1_1),$$

$U=U(E)\times C$ can be considered as a $\{P, P''\}$-manifold:

$$\partial_1 U=U(E)\times \partial_3 C$$
$$\partial_3 U=\partial_3 U(E)\times C.$$

Similarly we put

$$(5\cdot 12)\quad V(E)=(I_1\times P\times E_2\cup E_1\times E_2(1))\underset{\psi}{\cup}(I_2\times P\times E_1\cup E_2\times E_1(1)).$$

We consider $V(E)$ as a $\{P, P''\}$-manifold:

$$\partial_1 V(E)=-\{(I_1\times P\times I_2\times P\times \underline{P}\cup E_1\times 1_2\times \underline{P})$$
$$\underset{\psi}{\cup}(I_2\times P\times I_1\times P\times \underline{P}\cup E_2\times 1_1\times \underline{P})\}$$
$$\partial_3 V(E)=(1_1\times \underline{P}\times E_2\cup \underline{P}\times E_1(1)\times E_2(1))$$
$$\underset{\psi}{\cup}(1_2\times \underline{P}\times E_1\cup \underline{P}\times E_2(1)\times E_1(1)),$$

and $V=V(E)\times C(3)$ is the cross product of $V(E)$ and $C(3)$.

Note that

$$(5\cdot 13)\qquad \partial_1 V(E)=-U(E).$$

By $(5\cdot 10)$, we have, as $\{P, P''\}$-manifolds,

$$(5\cdot 14)\qquad \gamma_1(W)\sim U+V.$$

By simple calculations, we have

$$[U]=\pi_{13}[U(E)_{(3)}\times C_{(1)}]-s_{13}[U(E)_{(3)}\times C(3)]$$
$$=\pi_{13}(13)^*[U(E)_{(1)}\times C]-s_{13}[U(E)_{(3)}\times C(3)],$$

and by using $(5\cdot 13)$, we have

$$[V]=\pi_{13}[V(E)_{(13)}\times C(3)]+s_{13}[U(E)_{(3)}\times C(3)].$$

But there is a unique element $[v(E)]$ in $MU(P'')_*(pt)$ such that

$$\pi_{13}[V(E)\times C(3)]=\pi_{13}[V(E)]\times[C(3)]=\gamma_1[v(E)]\times[C(3)],$$

and $\dim v(E)=3\dim P+3$ is odd, therefore $[v(E)]=0$ in $MU(P'')_*(pt)$.

From the above, we have

$$(5\cdot15)\qquad \gamma_1[W]=[U]+[V]=\pi_{13}(13)^*[U(E)_{(1)}\times C].$$

**Lemma 5.16.** *There is an isomorphism*

$$U(E)\approx a(E)=E\bigcup_{\varphi\times T}E$$

*(See* (2·10) *for* $a(E)$.)

*Proof.* From (5·11) and (5·8), we see

$$U(E)=E_1\times 1_2\cup(I_1\times P\times I_2\times P\bigcup_{\psi}I_2\times P\times I_1\times P)\cup E_2\times 1_1\,.$$

The isomorphism in Lemma 5.16 will be visualized in the following figure:

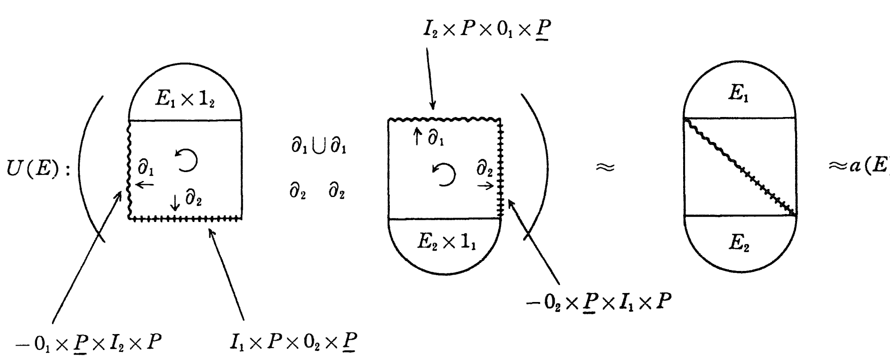


Figure 2

Now the proof of Lemma 5.6 follows from Lemma 5.9, (5·15) and Lemma 5.16.

*Proof of Proposition* 5.3. We shall prove, in $MU(P,P',P'')_*(\ )$,

$$\pi_{23}\pi_{12}-\pi_{13}\pi_{23}=(1+s_{23}\delta_2)(1+s_{12}\delta_1)-(1+s_{13}\delta_1)(1+s_{23}\delta_2)=0$$

or equivalently

$$(5\cdot17)\qquad (*)\equiv s_{12}\delta_1-s_{13}\delta_1+s_{23}\delta_2s_{12}\delta_1-s_{13}\delta_1s_{23}\delta_2=0\,.$$

Using Proposition 2.7, we have

$$(5\cdot 18)\qquad (*)=s_{12}\delta_1-s_{13}\delta_1+s_{23}(12)^*\delta_1+s_{13}s_{23}\delta_1\delta_2\,.$$

As for the first term of $(5\cdot 18)$, we first note that

$$s_{12}=(\overline{23})^*s_{13}(\overline{23})^*\,,$$

and using Proposition 2.13 we have

$$\begin{aligned}(5\cdot 19)\qquad s_{12}\delta_1&=(\overline{23})^*s_{13}(\overline{23})^*\delta_1=s_{13}(\overline{23})^*\delta_1+s_{23}\delta_2(s_{13}(\overline{23})^*\delta_1-s_{12}\delta_1)\\&+\alpha(E)_{(3)}\times\delta_2\delta_3s_{13}(\overline{23})^*\delta_1=s_{13}(\overline{23})^*\delta_1-s_{23}s_{13}\delta_2(\overline{23})^*\delta_1\\&-s_{23}(12)^*\delta_1+\alpha(E)_{(3)}\times\delta_2\delta_3s_{13}(\overline{23})^*\delta_1\,.\end{aligned}$$

Putting $(5\cdot 19)$ into $(5\cdot 18)$, we have

$$\begin{aligned}(5\cdot 20)\qquad (*)&=s_{13}((\overline{23})^*-1)\delta_1+s_{13}s_{23}\delta_1\delta_2-s_{23}s_{13}\delta_2(\overline{23})^*\delta_1\\&+\alpha(E)_{(3)}\times\delta_2(13)^*(\overline{23})^*\delta_1\,.\end{aligned}$$

Using again Proposition 2.13, from $(5\cdot 20)$ we have

$$\begin{aligned}(5\cdot 21)\qquad (*)&=-(s_{23}s_{13}+s_{13}s_{23})(23)^*\delta_3\delta_1+s_{13}(\alpha(E)_{(3)}\times\delta_2\delta_3\delta_1)\\&+\alpha(E)_{(3)}\times(13)^*(23)^*\delta_3\delta_1\,.\end{aligned}$$

Then, by Lemma 5.6,

$$\begin{aligned}(*)&=-\pi_{13}(13)^*(\alpha(E)_{(1)}\times(23)^*\delta_3\delta_1)+s_{13}(\alpha(E)_{(3)}\times\delta_2\delta_3\delta_1)\\&\quad+\alpha(E)_{(3)}\times(13)^*(23)^*\delta_3\delta_1\\&=-(13)^*(\alpha(E)_{(1)}\times(23)^*\delta_3\delta_1)-s_{13}\delta_1(13)^*(\alpha(E)_{(1)}\times(23)^*\delta_3\delta_1)\\&\quad+s_{13}(\alpha(E)_{(3)}\times\delta_2\delta_3\delta_1)+\alpha(E)_{(3)}\times(13)^*(23)^*\delta_3\delta_1\\&=-s_{13}(13)^*(\alpha(E)_{(1)}\times\delta_3(23)^*\delta_3\delta_1)+s_{13}(\alpha(E)_{(3)}\times\delta_2\delta_3\delta_1)\\&=-s_{13}(\alpha(E)_{(3)})\times((13)^*(23)^*-1)\delta_2\delta_3\delta_1\,.\end{aligned}$$

But, $(13)^*(23)^*=id$ on $\operatorname{Im}\delta_2\delta_3\delta_1$, thus we have

$$(*)\equiv\pi_{23}\pi_{12}-\pi_{13}\pi_{23}=0\,,$$

and hence we have completed the proof of Proposition 5.3.

We have come to prove the associativity of the external multiplications

$$\mu_E\colon MU(\mathscr{S}_n)_a(X)\otimes MU(\mathscr{S}_n)_b(Y)\to MU(\mathscr{S}_n)_{a+b}(X\times Y)$$

for general $n \geqq 1$.

Let $\mathscr{S}_n = \{P_1, \cdots, P_n\}$ be a singularity class as before and let $\mathscr{S}_n' = \{P_1', \cdots, P_n'\}$, $\mathscr{S}_n'' = \{P_1'', \cdots, P_n''\}$ be copies of $\mathscr{S}_n$ such that $P_i = P_i' = P_i''$, $1 \leqq i \leqq n$.

Take up bordism classes

$$x \in MU(\mathscr{S}_n)_a(X), \quad y \in MU(\mathscr{S}_n')_b(Y), \quad z \in MU(\mathscr{S}_n'')_c(Z).$$

Then we have the cross product (§ 3)

$$x \times y \times z \text{ in } MU(\mathscr{S}_n + \mathscr{S}_n' + \mathscr{S}_n'')_{a+b+c}(X \times Y \times Z).$$

Choose a system $\boldsymbol{E} = \{E_1, \cdots, E_n\}$ of Marava's manifolds $E_i$ for $P_i$. Then, we have the endomorphisms

$$\pi_{12}^i = 1 + s_{P_i P_i'}^{E_i} \delta_{P_i}, \ \pi_{23}^i = 1 + s_{P_i' P_i''}^{E_i} \delta_{P'_i}, \ \pi_{13}^i = 1 + s_{P_i P_i''}^{E_i} \delta_{P_i}, \quad i = 1, \cdots, n, \tag{5·22}$$

in the group $MU(P_1, \cdots, P_n, P_1', \cdots, P_n', P_1'', \cdots, P_n'')_*(X \times Y \times Z)$.

**Lemma 5.23.** *For the above endomorphisms* $\pi_{12}^i$, $\pi_{23}^i$, $\pi_{13}^i$, *we have*

i) $\pi_{kl}^i \circ \pi_{st}^j = \pi_{st}^j \circ \pi_{kl}^i$ *for* $i \neq j$,

ii) $\pi_{23}^i \circ \pi_{12}^i = \pi_{13}^i \circ \pi_{23}^i$ *for* $i = 1, \cdots, n$.

*Proof.* The first assertion is trivial, and the second assertion and its proof are essentially the same as those of Proposition 5.3.

Let $\gamma: MU(\mathscr{S}_n'')_*(\ ) \to MU(\mathscr{S}_n + \mathscr{S}_n' + \mathscr{S}_n'')_*(\ )$ be the canonical monomorphism as before. Then, by Lemma 5.23, we have

$$\begin{aligned} \gamma((x \cdot_{\boldsymbol{E}} y) \cdot_{\boldsymbol{E}} z) &= (\pi_{23}^n \circ \cdots \circ \pi_{23}^1) \circ (\pi_{12}^n \circ \cdots \circ \pi_{12}^1)(x \times y \times z) \\ &= (\pi_{23}^n \circ \pi_{12}^n) \circ (\pi_{23}^{n-1} \circ \pi_{12}^{n-1}) \circ \cdots \circ (\pi_{23}^1 \circ \pi_{12}^1)(x \times y \times z) \\ &= (\pi_{13}^n \circ \pi_{23}^n) \circ (\pi_{13}^{n-1} \circ \pi_{23}^{n-1}) \circ \cdots \circ (\pi_{13}^1 \circ \pi_{23}^1)(x \times y \times z) \\ &= (\pi_{13}^n \circ \cdots \circ \pi_{13}^1) \circ (\pi_{23}^n \circ \cdots \circ \pi_{23}^1)(x \times y \times z) = \gamma(x \cdot_{\boldsymbol{E}} (y \cdot_{\boldsymbol{E}} z)). \end{aligned} \tag{5·24}$$

Thus we have proved

**Theorem 5.25.** *For any singularity class* $\mathscr{S}_n$ *of weakly complex* *manifolds and for any choice of the system* $\boldsymbol{E}$, *the exterior multiplication* $\mu_{\boldsymbol{E}}$ *in* $MU(\mathscr{S}_n)_*(\ )$ *is associative.*

**Corollary 5.26.** *The dual cohomology theory* $MU(\mathscr{S}_n)^*(\ )$ *has also associative multiplications compatible with the module structure over* $MU^*(pt)$.

## § 6. Conclusions

Corresponding to any singularity class $\mathscr{S}_n=\{P_1, \cdots, P_n\}$ of closed, weakly complex manifolds and any choice of system $\boldsymbol{E}=\{E_1, \cdots, E_n\}$ of Morava's manifolds $E_i$ one for each $P_i$, we have obtained a natural multiplication

$$\mu_{\boldsymbol{E}}: MU(\mathscr{S}_n)_*(X, Y)\otimes MU(\mathscr{S}_n)_*(V, W)\to MU(\mathscr{S}_n)_*(X\times V, Y\times V\cup X\times W)$$

which is admissible (Theorem 3.14) and associative (Theorem 5.25).

And, if the obstruction classes $\alpha(E_i)$ or their canonical images in the coefficient group $MU(\mathscr{S}_n)_*(pt)$ happen to vanish, then $\mu_{\boldsymbol{E}}$ is commutative (Theorem 4.12). We can consider such favoring situations corresponding to suitable choices of $\mathscr{S}_n$ and $\boldsymbol{E}$.

In the below, for convenience, we consider only the case when the singularity class $\mathscr{S}_n$ represents a regular sequence in $MU_*(pt)$.

**Corollary 6.1.** *Let* $\mathscr{S}_n=\{P_1, \cdots, P_n\}$ *represent a regular sequence in* $MU_*(pt)$. *Then any multiplication* $\mu_{\boldsymbol{E}}$ *as above induces a unique ring structure in the coefficient group* $MU(\mathscr{S}_n)_*(pt)$ *which is canonically isomorphic to the quotient ring* $MU_*(pt)/([P_1], \cdots, [P_n])$. (*Cf. Proposition* 1.13).

For the next theorem, suppose $\alpha(E_i)\equiv 0 \pmod 2$ in $MU(\mathscr{S}_n)_*(pt)$ for $i=1, 2, \cdots, n$. Then, by Lemmas 2.2 and 2.11, we can choose another Morava's manifold $E_i'$ in place of $E_i$ for each $i$ such that $\alpha(E_i')=0$ in $MU(\mathscr{S}_n)_*(pt)$. Thus we have.

**Theorem 6.2.** *Let* $\mathscr{S}_n$ *represent a regular sequence in* $MU_*(pt)$. *Assume that every element of* $MU(\mathscr{S}_n)_{2p_i+2}(pt)$, $p_i=\dim P_i$, *is divisible* *by* 2 *for each* $i=1, 2, \cdots, n$.

*Then there is an external multiplication* $\mu_E$ *in the bordism theory* $MU(\mathscr{S}_n)_*(\ )$, *which is admissible, associative and commutative. Therefore the dual cohomology theory* $MU(\mathscr{S}_n)^*(\ )$ *is multiplicative and the representing spectrum* $MU(\mathscr{S}_n)$ *is a (commutative) ring spectrum.*

**Corollary 6.3.** *Assume that* $\mathscr{S}_n$ *represent a regular sequence in* $MU_*(pt)$ *and contain a member, say,* $P_1=\boldsymbol{Z}/2k+1$, *the point set of odd number of elements. (In this case* $MU(\mathscr{S}_n)_*(\ )$ *is a* $(\boldsymbol{Z}/2k+1)$-*module.)*

*Then there is an admissible, associative and commutative multiplication in* $MU(\mathscr{S}_n)_*(\ )$ *(In such a case we shall call the homology theory* $MU(\mathscr{S}_n)_*(\ )$ *multiplicative), and so is the cohomology theory* $MU(\mathscr{S}_n)^*(\ )$. (*cf.* [2], [11]).

So far we assumed $\mathscr{S}_n$ is a finite sequence, but this restriction will be unnecessary and we can generalize, without difficulty, all the above argument to the case $n=\infty$. In fact, if we take an infinite sequence $\mathscr{S}=\{P_1, P_2, \cdots\}$ of closed, weakly complex manifolds as a singularity class, the corresponding homology theory $MU(\mathscr{S})_*(\ )$ can be defined as the direct limit $\varinjlim MU(\mathscr{S}_n)_*(\ )$ of the sequence

$$MU_*(\ )\xrightarrow{\gamma_1}MU(\mathscr{S}_1)_*(\ )\xrightarrow{\gamma_2}MU(\mathscr{S}_2)_*(\ )\longrightarrow\cdots$$

for the sections $\mathscr{S}_n=\{P_1, \cdots, P_n\}$ of $\mathscr{S}$. Or, we may start, from the outset, with singular $\mathscr{S}$-manifolds, just as in the case of singular $\mathscr{S}_n$-manifolds but for $n$ unrestricted, and proceed as usual to defining bordism theory (See Baas [4]).

**Corollary 6.4** (Corollary 5.1 in [4]). *If* $\mathscr{S}=\{P_1, P_2, \cdots\}$ *represents a generating system* $\{x_1, x_2, \cdots\}$ *of the polynomial algebra* $MU_*(pt)=\boldsymbol{Z}[x_1, x_2, \cdots]$, $x_n=[P_n]$, $\dim x_n=2n$, *then we have*

$$MU(\mathscr{S})_*(\ )=H_*(\ ,\boldsymbol{Z})$$

*where the right hand side is the ordinary singular homology theory.*

If we take $\mathscr{S}=\{P_1, P_3, \cdots, P_{2k+1}, \cdots\}$, $[P_{2k+1}]=x_{2k+1}$, $k\geqq 0$, as a sequence representing $\{x_1, x_3, \cdots, x_{2k+1}, \cdots\}$ in $MU_*(pt)=\boldsymbol{Z}[x_1, x_2, \cdots]$, then the coefficient group $MU(\mathscr{S})_*(pt)$ vanishes in dimensions $4l+2(l\geqq 0)$ and is isomorphic to $\boldsymbol{Z}[x_2, x_4, \cdots, x_{2k}, \cdots]$. In such a case, we write also $MU(\mathscr{S})_*(\ )=MU\langle x_2, x_4, \cdots, x_{2k}, \cdots\rangle_*(\ )$. Since $\dim \alpha(E_{2k+1})=8k+4+2$ in this case, the images of $\alpha(E_{2k+1})$ in $MU(\mathscr{S})_*(pt)$ are all zero. So we have

**Theorem 6.5.** *If $\mathscr{S}=\{P_1, P_3, \cdots, P_{2k-1}, \cdots\}$ represents the subsequence $\{x_1, x_3, \cdots, x_{2k+1}, \cdots\}$ of the polynomial basis of $MU_*(pt)=\boldsymbol{Z}[x_1, x_2\cdots]$, then the homology theory $MU(\mathscr{S})_*(\ )=MU\langle x_2, x_4, \cdots, x_{2k}, \cdots\rangle_*(\ )$ is multiplicative, and so the cohomology theory $MU(\mathscr{S})^*(\ )$.*

Similarly we have

**Theorem 6.6.** *The cohomology theory $MU\langle x_{2k_i}| 1\leqq k_1<k_2<\cdots\rangle^*(\ )$ is multiplicative.*

Fix an odd prime $p$. Taking $x_i=u_{i+1}$, $u_{p^i}=v_i$, $i\geqq 1$, the Hazewinkel generators of $MU_*(pt)$ ([16], cf. [15] and [17]), we can deduce, from [5],

**Corollary 6.7.** *The (integral) Brown-Peterson cohomology theory $BP^*(\ )=MU\langle v_i| i\geqq 1\rangle^*(\ )$ for an odd prime $p$ is multiplicative and the spectrum $BP$ is a ring spectrum. (The adjective "integral" means "not localized at $p$".) And similarly for $BP\langle v_{i_1}, v_{i_2}, \cdots| i_1<i_2<\cdots\rangle^*(\ )$ and, in particular, $k\langle n\rangle^*(\ )=BP\langle v_n\rangle^*(\ )$, the connective extraordinary K-theory, as well $K\langle n\rangle^*(\ )=(v_n)^{-1}k\langle n\rangle^*(\ )$, the periodic cohomology after Morava* ([10], [12]).

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