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Higher-Order Approximate Solutions of Neumann Problems by Isoparametric Finite Element Method with Relevant Lumping Operator

By

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§ 1. Introduction

In 1966, Friedrichs and Keller [1] proposed a generalized finite difference scheme for the Neumann problem. Today, their method can be regarded as a finite element method using the piecewise linear "chapeau" bases. From the viewpoint of computations, however, their method is not convenient enough, except the case of a polygonal domain, since it requires to perform all the integrations which appear in obtaining Ritz coefficients and right-hand sides of determinate linear equations, in an exact domain and on an exact boundary.

The purpose of this paper is to consider a higher-order approximate procedure for the Neumann problem in an approximate domain, using isoparametric finite elements. This forces us to change slightly the data functions to guarantee the solvability of the determinate linear system. We shall study some kinds of errors, which are caused by the selection of the basis functions (approximation errors), by the change in domain, by the approximation of data functions and by the change of data functions for the solvability. Errors caused by the selection of the basis functions are, of course, inevitable. It is desirable that errors due to other reasons mentioned above do not destroy the order of accuracy which the basis functions could achieve. In this sense, it appears to be reasonable to expect that, when the boundary Γ is curved, the optimal order of accuracy can be achieved by the use of isoparametric technique. Also, we may apply the lumping technique to the body force term f, which may simplify

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the computation of the inner product including it. Our main conclusion is that the scheme using the isoparametric elements guarantees the optimal order of accuracy in H^1-norm, and that it may not be destroyed by the introduction of a class of lumping operators. L^2-estimates are also obtained using Nitsche's trick [2]. Our results can be extended also to the Dirichlet problem straightforwards.

Strang and Fix [8] have discussed the error due to the change in domain for the equation $-\Delta u+u=f$ with the Neumann condition, in which there is no problem on the solvability of approximate equations. With regards to the Dirichlet problem, Ciarlet and Raviart [4], Nitsche [3], and Strang and Verger [7] have investigated the effects of approximate boundary conditions. Numerical integrations, which may produce errors of different type, are also taken into account in [4]. Aubin [10] has discussed external approximations from the theoretical standpoint. The lumping operator in this paper may also be considered as one of them.

In § 2, we introduce the problem. In § 3, we obtain general error estimates for approximate solutions with a lumping operator in an approximate domain. In § 4, after the isoparametric finite element procedure and the relevant lumping operator are described in detail, the convergence orders are obtained in terms of the maximum side-length of the triangles. Although the isoparametric finite elements we treat here are of Lagrangean type only, it may be clear that we can extend the results to the case of elements of Hermitian type. In the last section, we show some examples. For the results of the numerical experiments of these examples, readers are referred to [12].

§ 2. Preliminaries

The problem we consider is

$$(2\cdot 1)\qquad \begin{cases} -\Delta u=f & \text{in } \Omega, \\ \dfrac{du}{dn}=g & \text{on } \Gamma, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^2$ and Γ is its boundary. We assume that Γ is sufficiently smooth and that it is expressed as

$$(2\cdot 2)\qquad \Gamma=\{(X(t),\, Y(t));\, \dot{X}(t)^2+\dot{Y}(t)^2\equiv 1\}.$$

In (2·1), $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2$ and n refers to the outer normal to Γ. Let $H^m(\Omega)$ denote the usual Sobolev space with the norm given by

$$\|u\|_{m,\Omega}=\left\{\sum_{|\alpha|\leq m}\int_\Omega |D^\alpha u|^2 dxdy\right\}^{1/2}.$$

Here $\alpha=(\alpha_1, \alpha_2)$, and α_1 and α_2 are non-negative integers such that

$$|\alpha|=\alpha_1+\alpha_2 \quad \text{and} \quad D^\alpha u=\frac{\partial^{|\alpha|}u}{\partial x^{\alpha_1}\partial y^{\alpha_2}}.$$

$H^0(\Omega)$ is also denoted by $L^2(\Omega)$. Similarly, $H^m(\Gamma)$ and $L^2(\Gamma)$ are defined to be the Sobolev spaces defined on Γ. Here and later, c indicates a generic numerical positive constant, independent of h, u, f and g, which may be different at different places. (h is a parameter depending on a subdivision of Ω.)

An approximate domain Ω_h is obtained as a union of finite elements. (Ω_h is not always a polygonal domain. For details, see § 4.) Let $\{P_i\}_{i=1}^{N_1}$ be all the nodal points in $\bar{\Omega}_h$. Let $S(\Omega_h)$ be a subspace of $H^1(\Omega_h)$ spanned by $\{\hat{\phi}_i(x,y)\}_{i=1}^{N_1}$ such that

$$\hat{\phi}_i\in C(\bar{\Omega}_h)\cap H^1(\Omega_h), \tag{2·3}$$

$$\hat{\phi}_i(P_j)=\delta_{ij} \quad \text{for} \quad i,j=1,\cdots,N_1, \tag{2·4}$$

and

$$\sum_{i=1}^{N_1}\hat{\phi}_i\equiv 1 \qquad \text{in } \Omega_h. \tag{2·5}$$

Let $T(\Omega_h)$ be a subspace of $L^2(\Omega_h)$ spanned by $\{\bar{\phi}_i(x,y)\}_{i=1}^{N_2}$ $(N_2\leqq N_1)$ such that

$$\bar{\phi}_i\in L^2(\Omega_h) \tag{2·6}$$

and

$$\sum_{i=1}^{N_2}\bar{\phi}_i\equiv 1 \qquad \text{almost everywhere in } \Omega_h. \tag{2·7}$$

We define two linear mappings P and Q from $C(\bar{\Omega}_h)$ into $S(\Omega_h)$ and $T(\Omega_h)$, respectively,

$$P: C(\bar{\Omega}_h)\to S(\Omega_h) \qquad \text{such that}$$

$$Pu=\sum_{i=1}^{N_1}u(P_i)\hat{\phi}_i, \tag{2·8}$$

$$Q: C(\bar{\Omega}_h)\to T(\Omega_h) \qquad \text{such that}$$

$$(2\cdot 9)\qquad Qu=\sum_{i=1}^{N_2} u(\mathrm{Q}_i)\bar{\phi}_i\,,$$

where $\{\mathrm{Q}_i\}_{i=1}^{N_2}$ is a subset of $\{\mathrm{P}_i\}_{i=1}^{N_1}$. Q is called a lumping operator associated with P.

Remark 2.1. For $\bar{\phi}_i$, the condition as $(2\cdot 4)$ is not required.

Remark 2.2. If we take $\bar{\phi}_i=\hat{\phi}_i$ and $\mathrm{Q}_i=\mathrm{P}_i$ $(i=1, \cdots, N_2=N_1)$, then $Q=P$.

Remark 2.3. $P^2=P$ and $QP=Q$.

Remark 2.4. We assume that Γ_h (the boundary of Ω_h) approximates Γ so naturally that the following conditions are satisfied:

i) There exists a bounded domain $\tilde{\Omega}$ such that $\bar{\Omega}\subset\tilde{\Omega}$ and $\bar{\Omega}_h\subset\tilde{\Omega}$ for all h,

ii) for any $f\in H^1(\Omega_h)$, there exists a $\tilde{f}\in H^1(\tilde{\Omega})$ such that

$$(2\cdot 10)\qquad \|\tilde{f}\|_{i,\tilde{\Omega}}\leqq c\|f\|_{i,\Omega_h}\quad \text{for}\quad i=0, 1,$$

and

iii) for any $f\in H^1(\Omega_h)$, it holds that

$$(2\cdot 11)\qquad \|f\|_{0,\Gamma_h}\leqq c\|f\|_{1,\Omega_h}\,.$$

These conditions may be satisfied when Ω is approximated by a polygon Ω_h whose vertices are all on Γ, or when isoparametric finite element procedures are used.

We use the following notations throughout this paper:

$$(f, v)=\int_{\Omega} f\cdot v\, dxdy\quad \text{for}\quad f, v\in L^2(\Omega),$$

$$a(u, v)=(\partial u/\partial x, \partial v/\partial x)+(\partial u/\partial y, \partial v/\partial y)\quad \text{for}\quad u, v\in H^1(\Omega),$$

$$[g, v]=\int_{\Gamma} g\cdot v\, dt\quad \text{for}\quad g, v\in L^2(\Gamma)$$

and

$$|\nabla u|=\{|\partial u/\partial x|^2+|\partial u/\partial y|^2\}^{1/2}\ \text{for}\ u\in H^1,$$

and we define $(f, v)_h$, $a_h(u, v)$ and $[g, v]_h$ by replacing Ω and Γ with Ω_h and Γ_h in (f, v), $a(u, v)$ and $[g, v]$, respectively.

§ 3. General Error Estimates for Approximate Solutions

In order that a solution of $(2\cdot 1)$ exists, the condition on f and g,

$$(f, 1) + [g, 1] = 0, \tag{3·1}$$

is required. Conversely, if f and g satisfy (3·1) and have appropriate smoothness, a solution exists in $H^m(\Omega)$ uniquely under the condition

$$(u, 1) = 0, \tag{3·2}$$

for some m. Let $\tilde{u} \in H^m(\tilde{\Omega})$ be a smooth extension of u such that

$$\|\tilde{u}\|_{m,\tilde{\Omega}} \leqq c\|u\|_{m,\Omega}, \tag{3·3}$$

and u_1 be $\tilde{u} - k_1$, where

$$k_1 = (\tilde{u}, 1)_h / \mathrm{mes}(\Omega_h). \tag{3·4}$$

We give two functions $f_A \in L^2(\Omega_h)$ and $g_A \in L^2(\Gamma_h)$, which may approximate f and g in some sense. (An example of f_A and g_A will be given later in Theorem 4.1.) Let f_1 be $f_A - k_2$, where

$$k_2 = \{(f_A, 1)_h + [g_A, 1]_h\} / \mathrm{mes}(\Omega_h). \tag{3·5}$$

The finite element solution of (2·1) and (3·2) is defined to be the function $\hat{u} \in S(\Omega_h)$ such that

$$a_h(\hat{u}, \hat{\phi}) = (f_1, Q\hat{\phi})_h + [g_A, \hat{\phi}]_h \quad \text{for} \quad {}^\forall\hat{\phi} \in S(\Omega_h) \tag{3·6}$$

and

$$(\hat{u}, 1)_h = 0. \tag{3·7}$$

Theorem 3. 1. *We assume that Γ_h is a natural approximation of Γ in the sense of Remark 2. 4. Further, we assume (2·3), (2·4) and (2·5) for $S(\Omega_h)$, and (2·6) and (2·7) for $T(\Omega_h)$. Then, the system (3·6) and (3·7) has a unique solution $\hat{u}$ in $S(\Omega_h)$, and*

$$\|\hat{u} - \tilde{u}\|_{1,\Omega_h} \leqq c\Big\{|k_1| + \|P\tilde{u} - \tilde{u}\|_{1,\Omega_h} + (|k_2| + \|f_A + \Delta\tilde{u}\|_{0,\Omega_h}) \times \frac{\|Q(\hat{u} - P\tilde{u})\|_{0,\Omega_h}}{\|\hat{u} - P\tilde{u}\|_{1,\Omega_h}} + \frac{|(\Delta\tilde{u}, (I-Q)(\hat{u} - P\tilde{u}))_h|}{\|\hat{u} - P\tilde{u}\|_{1,\Omega_h}} + \left\|g_A - \frac{d\tilde{u}}{dn_h}\right\|_{0,\Gamma_h}\Big\}, \tag{3·8}$$

where n_h refers to the outer normal to Γ_h.

For the proof of Theorem 3.1, the following lemma is used.

Lemma 3. 1. *Let $E(\Omega)$ be a subspace of $H^1(\Omega)$ such that*

$$E(\Omega) = \{u \in H^1(\Omega)\,;\,(u, 1) = 0\}.$$

Then, $\{a(u, u)\}^{1/2}$ is equivalent to $\|u\|_{1,\Omega}$ in $E(\Omega)$.

Proof. It is sufficient to show

$$\|u\|_{0,\Omega} \leqq c|u|_A \quad \text{for} \quad \forall u \in E(\Omega), \tag{3·9}$$

where $|u|_A = \{a(u, u)\}^{1/2}$. Let k be any constant. It holds that

$$(u, u) = (u, u-k) \leqq \|u\|_{0,\Omega}\|u-k\|_{0,\Omega} \quad \text{for} \quad \forall u \in E(\Omega).$$

Then we have

$$\|u\|_{0,\Omega} \leqq \inf_k \|u-k\|_{0,\Omega} \leqq c|u|_A .$$

Here, the last inequality follows from the fact that $|u|_A$ is equivalent to the usual norm of the quotient space $H^1(\Omega)/P_0$, where P_0 is the set of all the constant functions. (See [11].)

Proof of Theorem 3.1. (3·6) forms N_1 linear equations,

$$a_h(\hat{u}, \hat{\phi}_i) = (f_1, Q\hat{\phi}_i)_h + [g_A, \hat{\phi}_i]_h \quad \text{for} \quad i = 1, \cdots, N_1 . \tag{3·10}$$

Summing up these N_1 equations, we obtain a trivial relation

$$a_h(\hat{u}, 1) = (f_1, 1)_h + [g_A, 1]_h = 0 \tag{3·11}$$

from (2·5), (2·7) and (3·5). Therefore, the system (3·6) and (3·7) is not overdetermined. Suppose that there exists a nontrivial solution $\hat{u}$ for $f_A = g_A = 0$. Then,

$$a_h(\hat{u}, \hat{u}) = 0 \quad \text{and} \quad (\hat{u}, 1)_h = 0 . \tag{3·12}$$

These conclude $u \equiv 0$ by Lemma 3.1, which is a contradiction. This shows that the system (3·6) and (3·7) is solvable for any data f_A and g_A and that it has a unique solution in $S(\Omega_h)$.

Let us prove (3·8). First,

$$\|\hat{u} - \tilde{u}\|_{1,\Omega_h} \leqq \|\hat{u} - u_1\|_{1,\Omega_h} + c|k_1| . \tag{3·13}$$

By Lemma 3.1, we have

$$\|\hat{u} - u_1\|^2_{1,\Omega_h}/c \leqq a_h(\hat{u} - u_1, \hat{u} - u_1) \tag{3·14}$$

$$=a_h(\hat{u}-\tilde{u},\ \hat{u}-\tilde{u})$$

$$=a_h(u-\tilde{u},\ P\tilde{u}-\tilde{u})+a_h(\hat{u}-\tilde{u},\ \hat{u}-P\tilde{u}).$$

Each term of the right-hand side is estimated as follows:

$$(3\cdot 15)\qquad |\text{1st term}|\leqq\|\hat{u}-\tilde{u}\|_{1,\Omega_h}\|P\tilde{u}-\tilde{u}\|_{1,\Omega_h},$$

$$(3\cdot 16)\qquad |\text{2nd term}|=\Big|(f_1,\ Q(\hat{u}-P\tilde{u}))_h+[g_A,\ \hat{u}-P\tilde{u}]_h-(-\Delta\tilde{u},\ \hat{u}-P\tilde{u})_h-\left[\frac{d\tilde{u}}{dn_h},\ \hat{u}-P\tilde{u}\right]_h\Big|$$

$$\leqq|(f_1+\Delta\tilde{u},\ Q(\hat{u}-P\tilde{u}))_h|+|(\Delta\tilde{u},\ (I-Q)(\hat{u}-P\tilde{u}))_h|+\left|\left[g_A-\frac{d\tilde{u}}{dn_h},\ u-P\tilde{u}\right]_h\right|$$

$$\leqq\|f_1+\Delta\tilde{u}\|_{0,\Omega_h}\|Q(\hat{u}-P\tilde{u})\|_{0,\Omega_h}+|(\Delta\tilde{u},\ (I-Q)(\hat{u}-P\tilde{u}))_h|+c\left\|g_A-\frac{d\tilde{u}}{dn_h}\right\|_{0,\Gamma_h}\|\hat{u}-P\tilde{u}\|_{1,\Omega_h}.$$

Combining (3·13)~(3·16) and the trivial inequality

$$(3\cdot 17)\qquad \|\hat{u}-P\tilde{u}\|_{1,\Omega_h}\leqq\|u-\tilde{u}\|_{1,\Omega_h}+\|\tilde{u}-P\tilde{u}\|_{1,\Omega_h},$$

we obtain (3·8). This completes the proof.

Let us estimate $\|\hat{u}-\tilde{u}\|_{0,\Omega_h}$, using Nitsche's trick. Let e be $\hat{u}-\tilde{u}\in H^1(\Omega_h)$, $\tilde{e}\in H^1(\tilde{\Omega})$ be a smooth extension of e such that

$$(3\cdot 18)\qquad \|\tilde{e}\|_{i,\tilde{\Omega}}\leqq c\|e\|_{i,\Omega_h}\quad\text{for}\quad i=0,\ 1,$$

and e_1 be $\tilde{e}-k_3$, where

$$(3\cdot 19)\qquad k_3=(1,\ \tilde{e})/\text{mes}(\Omega).$$

Let $w\in H^3(\Omega)$ be the unique solution of

$$(3\cdot 20)\qquad \begin{cases} -\Delta w=e_1 & \text{in}\quad \Omega,\\ \dfrac{dw}{dn}=0 & \text{on}\quad \Gamma,\\ (w,\ 1)=0, & \end{cases}$$

and $\tilde{w}\in H^2(\tilde{\Omega})$ be a smooth extension of w such that

$$(3\cdot 21)\qquad \|\tilde{w}\|_{2,\tilde{\Omega}}\leqq c\|w\|_{2,\Omega}.$$

Theorem 3.2. *In the same assumptions of Theorem* 3.1, *the following estimate is obtained,*

$$(3\cdot 22)\qquad \|\hat{u}-\tilde{u}\|_{0,\Omega_h}\leqq c\Big\{\|e\|_{1,\Omega_h}\frac{\|\tilde{w}-P\tilde{w}\|_{1,\Omega_h}}{\|\tilde{w}\|_{2,\Omega_h}}+(\|f_A+\Delta\tilde{u}\|_{0,\Omega_h}+|k_2|)$$

$$\times\frac{\|Q\tilde{w}\|_{0,\Omega_h}}{\|\tilde{w}\|_{2,\Omega_h}}+\frac{|(\Delta\tilde{u},(P-Q)\tilde{w})_h|}{\|\tilde{w}\|_{2,\Omega_h}}+\Big\|g_A-\frac{d\tilde{u}}{dn_h}\Big\|_{0,\Gamma_h}$$

$$\times\frac{\|P\tilde{w}-\tilde{w}\|_{1,\Gamma_h}}{\|\tilde{w}\|_{2,\Omega_h}}+\frac{\Big|\Big[g_A-\frac{d\tilde{u}}{dn_h},\tilde{w}\Big]_h\Big|}{\|\tilde{w}\|_{2,\Omega_h}}+\frac{J_1}{\|\tilde{w}\|_{2,\Omega_h}}+J_2+|k_3|\Big\},$$

where $J_1=\int_{\Omega-\Omega_h}|\nabla\tilde{e}||\nabla w|dxdy+\int_{\Omega_h-\Omega}|\nabla e||\nabla\tilde{w}|dxdy$ *and* $J_2=\|e\|_{0,\Omega_h-\Omega}+\|\tilde{e}\|_{0,\Omega-\Omega_h}$.

Proof. From (3·20) and (3·21), we obtain

$$(3\cdot 23)\qquad \|\tilde{w}\|_{2,\Omega_h}\leqq c\|e_1\|_{0,\Omega}$$

and

$$(3\cdot 24)\qquad \begin{aligned}\|e_1\|^2_{0,\Omega}&=(e_1,-\Delta w)\\&=a(e_1,w)\\&=a_h(e_1,\tilde{w}-P\tilde{w})+a_h(e_1,P\tilde{w})+(\nabla\tilde{e},\nabla w)_{\Omega-\Omega_h}\\&\quad-(\nabla e,\nabla\tilde{w})_{\Omega_h-\Omega}.\end{aligned}$$

Each term of the right-hand side is estimated as follows:

$$(3\cdot 25)\qquad |\text{1st term}|\leqq\|e\|_{1,\Omega_h}\|\tilde{w}-P\tilde{w}\|_{1,\Omega_h},$$

$$(3\cdot 26)\qquad \begin{aligned}|\text{2nd term}|&=\Big|(f_1,QP\tilde{w})_h+[g_A,P\tilde{w}]_h\\&\qquad-(-\Delta\tilde{u},P\tilde{w})_h-\Big[\frac{d\tilde{u}}{dn_h},P\tilde{w}\Big]_h\Big|\\&=\Big|(f_1+\Delta\tilde{u},Q\tilde{w})_h+(\Delta\tilde{u},(P-Q)\tilde{w})_h\\&\qquad+\Big[g_A-\frac{d\tilde{u}}{dn_h},P\tilde{w}-\tilde{w}\Big]_h+\Big[g_A-\frac{d\tilde{u}}{dn_h},\tilde{w}\Big]_h\Big|\\&\leqq c\|f_1+\Delta\tilde{u}\|_{0,\Omega_h}\|Q\tilde{w}\|_{0,\Omega_h}+|(\Delta\tilde{u},(P-Q)\tilde{w})_h|\\&\qquad+c\Big\|g_A-\frac{d\tilde{u}}{dn_h}\Big\|_{0,\Gamma_h}\|P\tilde{w}-\tilde{w}\|_{1,\Omega_h}+\Big|\Big[g_A-\frac{d\tilde{u}}{dn_h},\tilde{w}\Big]_h\Big|,\end{aligned}$$

and

(3·27) $\qquad |3\text{rd term}+4\text{th term}| \leqq J_1$.

Combining (3·23)~(3·27) and the inequality

(3·28) $$\|e\|_{0,\Omega_h} \leqq \|\tilde{e}\|_{0,\Omega} + cJ_2$$
$$\leqq \|e_1\|_{0,\Omega} + c\{J_2 + |k_3|\},$$

we obtain (3·22). This completes the proof.

§ 4. Isoparametric Finite Elements and Relevant Lumping Operators

We shall illustrate the isoparametric finite element procedure. Let T_0 be a closed fundamental triangle with vertices $A_1(0, 1)$, $A_2(0, 0)$ and $A_3(1, 0)$ in (ξ, η)-plane. In T_0, there exist k fundamental nodal points, A_i, $i=1, \cdots, k$, including the three vertices. Let $\{\hat{\varphi}_i(\xi, \eta)\}_{i=1}^{k}$ be a set of real functions defined in T_0 such that

i) $\hat{\varphi}_i \in C(T_0) \cap H^1(T_0^\circ)$, where T_0° is an interior of T_0,

ii) $\hat{\varphi}_i(A_j) = \delta_{ij}$ for $i, j=1, \cdots, k$,

iii) $\sum_{i=1}^{k} \hat{\varphi}_i \equiv 1$,

iv) there exist two sets of real numbers $\{\alpha_i\}_{i=1}^{k}$ and $\{\beta_i\}_{i=1}^{k}$ such that

$$\sum_{i=1}^{k} \alpha_i \hat{\varphi}_i \equiv \xi \text{ and } \sum_{i=1}^{k} \beta_i \hat{\varphi}_i \equiv \eta ,$$

v) $\hat{\varphi}_i = 0$ on any side which does not contain A_i, and

vi) the set $\{(A_i, \hat{\varphi}_i);\ A_i$ is on the side of $T_0\}$ is symmetric with respect to the barycenter of T_0, i.e., if $A_i = (\lambda_1^i, \lambda_2^i, \lambda_3^i)^T$, $\hat{\varphi}_i(\lambda) = \hat{\varphi}_i((\lambda_1, \lambda_2, \lambda_3)^T)$ by the barycentric coordinate expression and $\lambda_1^i \lambda_2^i \cdot \lambda_3^i = 0$, then $(S\lambda^i, \hat{\varphi}_i(S^{-1}\lambda))$ coincides with some nodal point and its corresponding function, where S is any 3×3 permutation matrix.

We triangulate Ω to obtain the set of closed triangles $\{\Delta_i\}_{i=1}^{N_T}$ and the set of vertices $\{P_i\}_{i=1}^{N_0}$ such that

i) $P_i \in \bar{\Omega}$,

ii) $\Delta_i \cap \Delta_j$ is empty or equal to a common completely overlapping side, for any $i \neq j$.

iii) $\inf_i$ {minimum angle of Δ_i} $\geqq c$.

and

iv) all vertices of the polygon $\bigcup_{i=1}^{N_T}\Delta_i$ are on Γ.

With each triangulation, h is associated as follows:

$$h=\max_i \{\text{maximum side-length of } \Delta_i\}.$$

$\{\Delta_i\}_{i=1}^{N_T}$ consists of two types of triangles, i.e., interior triangles and boundary triangles. An interior triangle has at most one vertex on Γ and a boundary triangle has just two vertices on Γ. We add required nodal points $\{P_i\}_{i=N_0+1}^{N_1}$ in $\bar{\Omega}$ so that there exist k nodal points $\{B_i^j\}_{i=1}^{k}$ in a neighbourhood of each triangle Δ_j satisfying the following four conditions.

i) $\{B_i^j\}_{i=1}^{k}$ is a subset of $\{P_i\}_{i=1}^{N_1}$.

ii) B_1^j, B_2^j and B_3^j are vertices of Δ_j.

iii) For the interior triangle, $B_i^j=\tilde{B}_i^j$ for $i=4, \cdots, k$, where $\tilde{B}_i^j$ is a point which has the same barycentric coordinates as A_i.

iv) For the boundary triangle, B_1^j and B_3^j are on Γ. (B_2^j is in Ω.) $B_i^j=\tilde{B}_i^j$ for all A_i lying on A_1A_2 and A_2A_3. For each A_i on A_3A_1, B_i^j is taken on Γ near the point C_i^j, which is on Γ and satisfies

$$\widehat{B_3^jC_i^j} : \widehat{C_i^jB_1^j} = \overline{A_3A_i} : \overline{A_iA_1},$$

where $\widehat{B_l^jC_i^j}$ is the length from B_l^j to C_i^j along Γ. Each B_i^j of the other nodal points is taken at the position which $\tilde{B}_i^j$ occupies after adding an appropriate small shift to $\tilde{B}_i^j$. (This shift is chosen so that $(4\cdot1)$ may hold.)

With each Δ_j, the mapping F_j from T_0 into $\mathbb{R}^2$ are associated as follows:

$$F_j=\sum_{i=1}^{k} B_i^j\hat{\varphi}_i .$$

Let $\{K_j\}_{j=1}^{N_T}$ be the set of finite elements, where

$$K_j=F_j(T_0),$$

and Ω_h be an approximate domain of Ω such that

$$\Omega_h=\text{an interior of } \bigcup_{j=1}^{N_T}K_j .$$

Remark 4.1. For interior elements, it is clear that

$$F_j=B_1^j\eta+B_2^j(1-\xi-\eta)+B_3^j\xi \text{ and } K_j=\Delta_j .$$

Therefore, the Jacobians of F_j for interior elements are constant.

Let $S(\Omega_h)$ be a N_1 dimensional subspace of $H^1(\Omega_h)$ spanned by

$\{\hat{\phi}_i(x, y)\}_{i=1}^{N_1}$. Here

$$\hat{\phi}_i = \sum_{l=1}^{k} \delta(\mathrm{P}_i, \mathrm{B}_l^j) \hat{\phi}_l(F_j^{-1}) \quad \text{in} \quad K_j,$$

where $\delta(\mathrm{P}_i, \mathrm{B}_l^j) = 1$ if $\mathrm{P}_i = \mathrm{B}_l^j$ and $= 0$ otherwise. It should be noted that $\hat{\phi}_i$ satisfies the conditions (2·3), (2·4) and (2·5). Let P be an interpolating operator from $C(\bar{\Omega}_h)$ into $S(\Omega_h)$ defined by (2·8). As the boundary is smooth, there exists a one to one correspondence ν from Γ onto Γ_h such that $\mathrm{P}(t)\cdot\nu(\mathrm{P}(t))$ is normal to Γ for $\mathrm{P}(t) = (X(t), Y(t)) \in \Gamma$. Let $(\delta(t), \varepsilon(t))$ be the components of the vector $\nu(\mathrm{P}(t)) - \mathrm{P}(t)$ and $\lambda(t)$ be its length.

Then, we say that an isoparametric finite element procedure is of order α, if the following four conditions are satisfied:

i) For any $f \in H^{\alpha+1}(\tilde{\Omega}) \cap C(\tilde{\Omega})$ and $i = 0, 1$,

$$\|(I-P)f\|_{i,\Omega_h} \leqq ch^{\alpha+1-i} \|f\|_{\alpha+1,\Omega_h}. \tag{4·1}$$

ii) For any $f \in H^{\alpha}(\tilde{\Omega}) \cap C(\tilde{\Omega})$,

$$\|(I-P)f\|_{0,\Omega_h} \leqq ch^{\alpha} \|f\|_{\alpha,\Omega_h}. \tag{4·2}$$

iii) For $i = 0, 1$,

$$\left| \frac{d^i \lambda(t)}{dt^i} \right| \leqq ch^{\alpha+1-i}. \tag{4·3}$$

iv) For any $v \in H^{\alpha+2}(\tilde{\Omega})$,

$$\|(I-P)v\|_{0,\Gamma_h} \leqq ch^{\alpha+1} \sum_{|\beta| \leqq \alpha+1} \|D^{\beta} v\|_{0,\Gamma_h}. \tag{4·4}$$

Next, we define the lumping operator Q. Let $\{\bar{\psi}_i(\xi, \eta)\}_{i=1}^{k}$ be a set of real measurable functions defined in T_0 such that

i) $\bar{\psi}_i \in L^2(T_0^{\circ})$,

ii) $\sum_{i=1}^{k} \bar{\psi}_i \equiv 1$ almost everywhere in T_0

and

iii) the set $\{\mathrm{A}_i; \mathrm{A}_i$ *is on the side of* T_0 *and* $\bar{\psi}_i \not\equiv 0\}$ is symmetric with respect to the barycenter.

Renumbering the nodal points which belong to

$$\{\mathrm{P}_i; \exists \mathrm{B}_l^j \text{ such that } \mathrm{B}_l^j = \mathrm{P}_i \text{ and } \bar{\psi}_l \not\equiv 0, \ i = 1, \cdots, N_1\},$$

we obtain $\{\mathrm{Q}_i\}_{i=1}^{N_2}$. Let $T(\Omega_h)$ be a subspace of $L^2(\Omega_h)$ spanned by $\{\bar{\phi}_i(x, y)\}_{i=1}^{N_2}$, where

$$(4\cdot 5)\qquad \bar{\phi}_i=\textstyle\sum_{l=1}^{k}\delta(\mathrm{Q}_i,\mathrm{B}_l^j)\bar{\psi}_l(F_j^{-1})\ \text{ in }\ K_j\,.$$

$T(\Omega_h)$ satisfies the condition $(2\cdot 6)$ and $(2\cdot 7)$. Let Q be a linear operator from $C(\bar{\Omega}_h)$ into $T(\Omega_h)$ defined by $(2\cdot 9)$. We call Q a lumping operator of type (β,γ) associated with P, if the following two conditions are satisfied:

i) For any $f\in H^i(\tilde{\Omega})\cap C(\tilde{\Omega})$,

$$(4\cdot 6)\qquad \|(I-Q)f\|_{0,\Omega_h}\leqq ch^i\|f\|_{i,\Omega_h}$$

are satisfied for $i=1$ if $\beta=0$ and $i=1,2$ if $\beta\geqq 1$.

ii) There exists a continuous linear operator R_o from $C(T_0)$ into $C(T_0)$ such that,

a) for $u,\ v\in C(T_0)$,

$$(4\cdot 7)\qquad (R_0v,\ (P_0-Q_0)u)_{T_0}=0\,,$$

where $P_0u=\sum_{i=1}^{k}u(\mathrm{A}_i)\hat{\psi}_i$ and $Q_0u=\sum_{i=1}^{k}u(\mathrm{A}_i)\bar{\psi}_i$,

b) for any $f\in H^{\beta+1}(\tilde{\Omega})\cap C(\tilde{\Omega})$,

$$(4\cdot 8)\qquad \|(I-R)f\|_{0,\Omega_h}\leqq ch^{\beta+1}\|f\|_{\beta+1,\Omega_h}\,,$$

where R is a linear mapping from $C(\bar{\Omega}_h)$ into $L^2(\Omega_h)$ such that

$$(4\cdot 9)\qquad Ru=(R_0u(F_j))(F_j^{-1})\ \text{ in }\ K_j\,,$$

and

c) there exists a continuous function $H_j(\xi,\eta)$ for each boundary element K_j such that

$$(4\cdot 10)\qquad |H_j(\xi,\eta)|\leqq ch^{\gamma+3}$$

and

$$(4\cdot 11)\qquad ((J_j(\xi,\eta)-H_j(\xi,\eta))R_0v,\ (P_0-Q_0)u)_{T_0}=0\quad\text{for}\quad u,\ v\in C(T_0),$$

where $J_j(\xi,\eta)$ is the Jacobian of F_j.

Theorem 4.1. *In* $(3\cdot 6)$ *and* $(3\cdot 7)$, *we use the isoparametric finite element procedure of order* $\alpha(\geqq 1)$ *and the lumping operator of type* (β,γ) $(\beta\geqq 0)$. *Assuming* $f\in C(\bar{\Omega})$, $g\in C(\Gamma)$ *and* $(3\cdot 1)$, *we take* $f_A=Pf$ *and* $g_A=Pg$. *Then* $(3\cdot 6)$ *and* $(3\cdot 7)$ *has a unique solution and the following estimates hold*:

$$(4\cdot 12)\qquad \|\hat{u}-\tilde{u}\|_{1,\Omega_h}\leqq c\{h^{\alpha}(\|u\|_{\alpha+2,\Omega}+\|u\|_{4,\Omega})+h^{\beta+2}\|u\|_{\beta+3,\Omega}+h^{\gamma+5/2}\|u\|_{4,\Omega}\}$$

and

$$(4\cdot13)\qquad \|\hat{u}-\tilde{u}\|_{0,\Omega_h}\leqq\begin{cases} c\{h^{\alpha+1}\|u\|_{\alpha+3,\Omega}+h^{\beta+2}\|u\|_{\beta+3,\Omega}+h^{\gamma+5/2}\|u\|_{4,\Omega}\} \\ \qquad\text{if}\quad \beta=0, \\ c\{h^{\alpha+1}\|u\|_{\alpha+3,\Omega}+h^{\beta+3}\|u\|_{\beta+3,\Omega}+h^{\gamma+7/2}\|u\|_{4,\Omega}\} \\ \qquad\text{if}\quad \beta\geqq1, \end{cases}$$

where $\tilde{u}$ is a smooth extension over $\tilde{\Omega}$ of the smooth solution u of (2·1).

Remark 4.2. From the condition v) of $\hat{\varphi}_i$, Pg is well-defined on Γ_h.

Remark 4.3. If $Q=P$, Q is of type (α,∞). This is shown by taking $R_0=P_0$ and $H_j\equiv0$. Then, (4·12) and (4·13) are reduced to

$$(4\cdot12')\qquad \|\hat{u}-\tilde{u}\|_{1,\Omega_h}\leqq ch^{\alpha}(\|u\|_{\alpha+2,\Omega}+\|u\|_{4,\Omega})$$

and

$$(4\cdot13')\qquad \|\hat{u}-\tilde{u}\|_{0,\Omega_h}\leqq ch^{\alpha+1}\|u\|_{\alpha+3,\Omega}\,.$$

Proof of Theorem 4.1. We can estimate each term of the right-hand side of (3·8) as follows:

$$(4\cdot14)\qquad |k_1|\leqq ch^{\alpha+1}\|u\|_1\,,$$

$$(4\cdot15)\qquad \|P\tilde{u}-\tilde{u}\|_{1,\Omega_h}\leqq ch^{\alpha}\|u\|_{\alpha+1}\,,$$

$$(4\cdot16)\qquad |k_2|\leqq c\{h^{\alpha}\|u\|_{\alpha+2}+h^{\alpha+1}\|u\|_4\}\,,$$

$$(4\cdot17)\qquad \|f_A+\Delta\tilde{u}\|_{0,\Omega_h}\leqq ch^{\alpha}\|u\|_{\alpha+2}\,,$$

$$(4\cdot18)\qquad \|Q(\hat{u}-P\tilde{u})\|_{0,\Omega_h}/\|\hat{u}-P\tilde{u}\|_{1,\Omega_h}\leqq c\,,$$

$$(4\cdot19)\qquad |(\Delta\tilde{u},\,(I-Q)(\hat{u}-P\tilde{u}))_h|/\|\hat{u}-P\tilde{u}\|_{1,\Omega_h}\leqq c\{h^{\beta+2}\|u\|_{\beta+3} + h^{\gamma+5/2}\|u\|_4\}\,,$$

and

$$(4\cdot20)\qquad \left\|g_A-\frac{d\tilde{u}}{dn_h}\right\|_{0,\Gamma_h}\leqq ch^{\alpha}\|u\|_{\alpha+2}\,.$$

(4·12) follows from these inequalities. (4·15) and (4·17) are direct consequences of (4·1) and (4·2). (4·14) and (4·18) are proved easily.

For $(4\cdot16)$, it holds that

$$(4\cdot21)\qquad k_2=\left\{(f_A+\Delta\tilde{u},1)_h-\int_{\Omega-\Omega_h}f\,dxdy-\int_{\Omega_h-\Omega}\Delta\tilde{u}\,dxdy\right.$$
$$\left.+[g_A,1]_h-[g,1]\right\}\Big/\mathrm{mes}\,(\Omega_h).$$

The first three terms of the right-hand side of $(4\cdot21)$ are bounded by $ch^\alpha\|u\|_{\alpha+2}$, $ch^{\alpha+1}\|u\|_3$ and $ch^{\alpha+1}\|u\|_3$. To estimate the last two terms we use the functions $\delta(t)$ and $\varepsilon(t)$. From $(2\cdot2)$, it holds that

$$\delta(t)=\lambda(t)\dot{Y}(t)\quad\text{and}\quad\varepsilon(t)=-\lambda(t)\dot{X}(t).$$

We introduce $v(x,y)$ which is defined in $\tilde{\Omega}$ as follows: In a neighbourhood of Γ,

$$(4\cdot22)\qquad v(x,y)=\frac{\partial\tilde{u}}{\partial x}(x,y)\dot{Y}(t)-\frac{\partial\tilde{u}}{\partial y}(x,y)\dot{X}(t),$$

where $(x-X(t),y-Y(t))$ is the normal to Γ, and $v(x,y)$ is extended smoothly over $\tilde{\Omega}$. It is easy to see that $v(x,y)$ is equal to g on Γ. Now we have

$$(4\cdot23)\qquad [Pg,1]_h-[g,1]=[(P-I)v,1]_h$$
$$+\int_\Gamma v(\nu(\mathrm{P}(t)))\{\sqrt{(\dot{X}+\dot{\delta})^2+(\dot{Y}+\dot{\varepsilon})^2}-1\}dt$$
$$+\int_\Gamma\{v(\nu(\mathrm{P}(t)))-v(\mathrm{P}(t))\}dt.$$

Using the following inequalities,

$$|[(P-I)v,1]_h|\leqq c\|(P-I)v\|_{1,\Omega_h}\leqq ch^\alpha\|u\|_{\alpha+2},$$
$$|\sqrt{(\dot{X}+\dot{\delta})^2+(\dot{Y}+\dot{\varepsilon})^2}-1|\leqq ch^{\alpha+1},$$

and

$$|v(\nu(\mathrm{P}(t)))-v(\mathrm{P}(t))|\leqq\left|\left\{\frac{\partial\tilde{u}}{\partial x}(X+\delta,Y+\varepsilon)-\frac{\partial\tilde{u}}{\partial x}(X,Y)\right\}\dot{Y}\right|$$
$$+\left|\left\{\frac{\partial\tilde{u}}{\partial y}(X+\delta,Y+\varepsilon)-\frac{\partial\tilde{u}}{\partial y}(X,Y)\right\}\dot{X}\right|,$$

we estimate each term of the right-hand side of $(4\cdot23)$ by $ch^\alpha\|u\|_{\alpha+2}$, $ch^{\alpha+1}\|u\|_3$ and $ch^{\alpha+1}\|u\|_4$. For $(4\cdot19)$,

$$(4\cdot24)\qquad (\Delta\tilde{u}, (I-Q)(\hat{u}-P\tilde{u}))_h=((I-R)\Delta\tilde{u}, (I-Q)(\hat{u}-P\tilde{u}))_h + (R\Delta\tilde{u}, (I-Q)(\hat{u}-P\tilde{u}))_h .$$

Each term of (4·24) is estimated as follows:

$$(4\cdot25)\qquad |\text{1st term}|\leqq\|(I-R)\Delta\tilde{u}\|_{0,\Omega_h}\|(I-Q)(\hat{u}-P\tilde{u})\|_{0,\Omega_h} \leqq ch^{\beta-2}\|u\|_{\beta+3}\|\hat{u}-P\tilde{u}\|_{1,\Omega_h} ,$$

and

$$(4\cdot26)\qquad |\text{2nd term}|=\left|\sum\int_{K_j}R\Delta\tilde{u}\cdot(P-Q)(\hat{u}-P\tilde{u})dxdy\right|$$
$$=\left|\sum\int_{T_0}R_0\Delta\tilde{u}\cdot(P_0-Q_0)(\hat{u}-P\tilde{u})J_j(\xi,\eta)d\xi d\eta\right|$$
$$=\left|\sum{}_B\int_{T_0}R_0\Delta\tilde{u}\cdot(P_0-Q_0)(\hat{u}-P\tilde{u})H_j(\xi,\eta)d\xi d\eta\right|$$
$$\leqq\frac{1}{\sqrt{2}}\max\left(\left|R_0\Delta\tilde{u}\right|\frac{|H_j(\xi,\eta)|}{\sqrt{J_j(\xi,\eta)}}\right)$$
$$\times\sum{}_B\left\{\int_{T_0}|(P_0-Q_0)(\hat{u}-P\tilde{u})|^2J_j(\xi,\eta)d\xi d\eta\right\}^{1/2}$$
$$\leqq c\|u\|_4h^{\gamma-2}\{\textstyle\sum_B 1\}^{1/2}\|(P-Q)(\hat{u}-P\tilde{u})\|_{0,\Omega_h}$$
$$\leqq c\|u\|_4h^{\gamma-5/2}\|\hat{u}-P\tilde{u}\|_{1,\Omega_h} ,$$

where $\sum_B$ means the summation over all the boundary elements. Here we have used

$$|J_j(\xi,\eta)|\geqq ch^2 \quad\text{and}\quad \{\textstyle\sum_B 1\}\leqq ch^{-1} .$$

(4·19) follows from (4·24), (4·25) and (4·26). For (4·20), with (4·1) and $v(x,y)$ of (4·22), we obtain

$$(4\cdot27)\qquad \left\|g_A-\frac{d\tilde{u}}{dn_h}\right\|_{0,\Gamma_h}\leqq\|Pv-v\|_{0,\Gamma_h}+\left\|v-\frac{d\tilde{u}}{dn_h}\right\|_{0,\Gamma_h} \leqq c\{h^{\alpha}\|u\|_{\alpha+2}+h^{\alpha}\|u\|_3\} .$$

Let us prove (4·13). This is concluded from (3·22) with (4·12), (4·20) and the following estimates:

$$(4\cdot28)\qquad \|P\tilde{w}-\tilde{w}\|_{1,\Omega_h}/\|\tilde{w}\|_{2,\Omega_h}\leqq ch ,$$

$$(4\cdot29)\qquad \|f_A+\Delta\tilde{u}\|_{0,\Omega_h}\leqq ch^{\alpha+1}\|u\|_{\alpha-3} ,$$

$$(4\cdot 30)\qquad |k_2|\leqq ch^{\alpha+1}\{\|u\|_{\alpha+3}+\|u\|_4\},$$

$$(4\cdot 31)\qquad \|Q\tilde{w}\|_{0,\Omega_h}/\|\tilde{w}\|_{2,\Omega_h}\leqq c,$$

$$(4\cdot 32)\qquad |(\Delta\tilde{u},(P-Q)\tilde{w})_h|/\|\tilde{w}\|_{2,\Omega_h} \leqq \begin{cases} c\{h^{\beta+2}\|u\|_{\beta+3}+h^{\gamma+5/2}\|u\|_4\} & \text{if}\quad \beta=0,\\ c\{h^{\beta+3}\|u\|_{\beta+3}+h^{\gamma+7/2}\|u\|_4\} & \text{if}\quad \beta\geqq 1,\end{cases}$$

$$(4\cdot 33)\qquad \left|\left[g_A-\frac{d\tilde{u}}{dn_h},\ \tilde{w}\right]_h\right|\Big/\|\tilde{w}\|_{2,\Omega_h}\leqq ch^{\alpha+1}\|u\|_{\alpha+3},$$

$$(4\cdot 34)\qquad J_1/\|\tilde{w}\|_{2,\Omega_h}\leqq ch^{(\alpha+1)/2}\|e\|_{1,\Omega_h},$$

$$(4\cdot 35)\qquad J_2\leqq ch^{(\alpha+1)/2}\|e\|_{1,\Omega_h},$$

and

$$(4\cdot 36)\qquad |k_3|\leqq ch^{\alpha+1}\{\|e\|_{1,\Omega_h}+\|u\|_1\}.$$

(4·28), (4·29), (4·31) and (4·34)∼(4·36) are shown easily. (4·32) is proved by the same way as (4·19) in the case $\beta=0$. In $\beta\geqq 1$, using the inequality

$$\|(P-Q)\tilde{w}\|_{0,\Omega_h}\leqq\|(I-P)\tilde{w}\|_{0,\Omega_h}+\|(I-Q)\tilde{w}\|_{0,\Omega_h} \leqq ch^2\|\tilde{w}\|_{2,\Omega_h},$$

we obtain (4·32). For (4·33), it holds that

$$(4\cdot 37)\qquad \left[g_A-\frac{d\tilde{u}}{dn_h},\ \tilde{w}\right]_h=\{[g_A,\tilde{w}]_h-[g,w]\} +\{(-\Delta\tilde{u},\tilde{w})_h-(f,w)\}+\{-a_h(\tilde{u},\tilde{w})+a(u,w)\}.$$

The second and the third terms are bounded by $ch^{\alpha+1}\|u\|_3\|\tilde{w}\|_{1,\Omega_h}$ and $ch^{\alpha+1}\|u\|_2\|\tilde{w}\|_{2,\Omega_h}$, respectively. The first term is divided into

$$(4\cdot 38)\qquad [(P-I)v,\tilde{w}]_h+\{[v,\tilde{w}]_h-[v,w]\}.$$

The second term of (4·38) is bounded by $ch^{\alpha+1}\|u\|_4\|\tilde{w}\|_{2,\Omega_h}$ similarly in (4·23). For the first term of (4·38), using (4·4), we obtain

$$(4\cdot 39)\qquad |[(P-I)v,\tilde{w}]_h|\leqq\|(P-I)v\|_{0,\Gamma_h}\|\tilde{w}\|_{0,\Gamma_h} \leqq ch^{\alpha+1}\textstyle\sum_{|\beta|\leqq\alpha+1}\|D^\beta v\|_{0,\Gamma_h}\|\tilde{w}\|_{0,\Gamma_h} \leqq ch^{\alpha+1}\|u\|_{\alpha+3,\Omega}\|\tilde{w}\|_{1,\Omega_h}.$$

From these estimates follows (4·33). Using the same argument as (4·39) to estimate the first term of the right-hand side of (4·23), we obtain (4·30). This completes the proof.

§ 5. Examples

Now let us consider some examples of P and Q.

Example 5. 1. Let $k=3$, and $\{\hat{\varphi}_i\}_{i=1}^3$ be taken as follows:

$$\hat{\varphi}_1=\eta\ ,$$

$$\hat{\varphi}_2=1-\xi-\eta\ ,$$

and

$$\hat{\varphi}_3=\xi\ .$$

These are the usual piecewise linear bases. (4·1)∼(4·4) are satisfied for $\alpha=1$. Then, three kinds of Q may be considered. The first $\{\bar{\varphi}_i\}_{i=1}^3$ is as follows:

$$\bar{\varphi}_1=1 \text{ in } S_1 \text{ and } =0 \text{ otherwise}\ ,$$

$$\bar{\varphi}_2=1 \text{ in } S_2 \text{ and } =0 \text{ otherwise}\ ,$$

and

$$\bar{\varphi}_3=1 \text{ in } S_3 \text{ and } =0 \text{ otherwise}\ ,$$

where S_1 is the quadrilateral with vertices A_1, A_4, G and A_6, and S_2 and S_3 are taken similarly. (See Fig. 5. 1.) The second case is:

$$\bar{\varphi}_1=\bar{\varphi}_2=\bar{\varphi}_3=\frac{1}{3}\ .$$

In these cases, Q is of type $(0, \infty)$ by taking

$$R_0 u=2\int_{T_0} u(\xi, \eta)\, d\xi d\eta\ . \tag{5·1}$$

In fact, as $J_j(\xi, \eta)$ is constant for each j and

$$(1, \hat{\varphi}_i-\bar{\varphi}_i)_{T_0}=0 \quad \text{for} \quad i=1, 2, 3, \tag{5·2}$$

we may take $H_j(\xi, \eta)\equiv 0$. Therefore, we obtain

$$\|\hat{u}-\tilde{u}\|_{1, \Omega_h}=O(h) \quad \text{and} \quad \|\hat{u}-\tilde{u}\|_{0, \Omega_h}=O(h^2)\ . \tag{5·3}$$

The third case is:

$$\bar{\varphi}_i = \hat{\varphi}_i \quad \text{for} \quad i=1, 2, 3.$$

Then (5·3) is valid from (4·12′) and (4·13′).

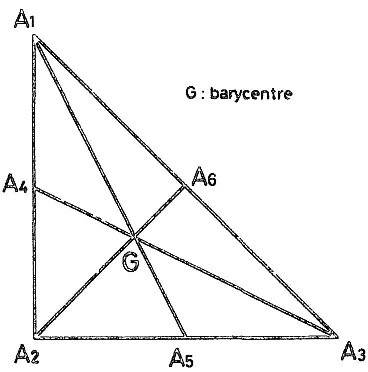

Fig. 5. 1. Fundamental Triangle.

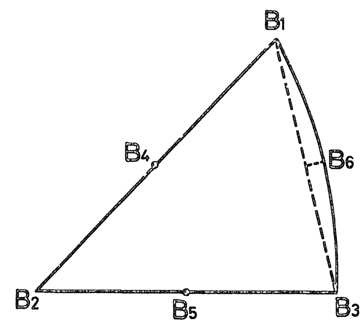

Fig. 5. 2. Curved Element.

Example 5. 2. Let $k=6$. Let A_4, A_5 and A_6 be taken at the midpoints of A_1A_2, A_2A_3 and A_3A_1, respectively. $\{\hat{\varphi}_i\}_{i=1}^{6}$ is as follows:

$$\hat{\varphi}_1 = \eta(2\eta-1),$$

$$\hat{\varphi}_2 = (1-\xi-\eta)\{1-2(\xi+\eta)\},$$

$$\hat{\varphi}_3 = \xi(2\xi-1),$$

$$\hat{\varphi}_4 = 4\eta(1-\xi-\eta),$$

$$\hat{\varphi}_5 = 4\xi(1-\xi-\eta),$$

and

$$\hat{\varphi}_6 = 4\xi\eta .$$

These are the usual piecewise quadratic bases. B_6^j is taken on Γ in such a way that $\overline{B_6^j\tilde{B}_6^j}$ is perpendicular to $\overline{B_3^jB_1^j}$. Then, (4·1)$\sim$(4·4) are satisfied for $\alpha=2$. (See [6] for (4·1) and (4·2).) The Jacobian of F_j for the boundary element K_j is

$$J_j(\xi, \eta) = c_0 + c_1\xi + c_2\eta , \tag{5·4}$$

where

$$c_0 = (x_3-x_2)(y_1-y_2) - (x_1-x_2)(y_3-y_2),$$

$$c_1 = 4\left\{(x_3-x_2)\left(y_6 - \frac{y_1+y_3}{2}\right) - \left(x_6 - \frac{x_1+x_3}{2}\right)(y_3-y_2)\right\},$$

$$c_2=4\left\{\left(x_6-\frac{x_1+x_3}{2}\right)(y_1-y_2)-(x_1-x_2)\left(y_6-\frac{y_1+y_3}{2}\right)\right\},$$

and (x_i, y_i) is the coordinates of B_i^j for $i=1, \cdots, 6$. Four kinds of Q are considered. In the first case $\{\bar{\psi}_i\}_{i=1}^6$ is as follows:

$$\bar{\psi}_1=\bar{\psi}_2=\bar{\psi}_3=0\,,$$

$$\bar{\psi}_4=1 \text{ in } S_4 \text{ and } =0 \text{ otherwise}\,,$$

$$\bar{\psi}_5=1 \text{ in } S_5 \text{ and } =0 \text{ otherwise}\,,$$

and

$$\bar{\psi}_6=1 \text{ in } S_6 \text{ and } =0 \text{ otherwise}\,,$$

where S_4 is the triangle with vertices A_1, A_2 and G, and S_5 and S_6 are taken similarly. In the second case,

$$\bar{\psi}_1=\bar{\psi}_2=\bar{\psi}_3=0 \quad \text{and} \quad \bar{\psi}_4=\bar{\psi}_5=\bar{\psi}_6=\frac{1}{3}\,.$$

In these cases, taking R_0 of (5·1) and $H_j=c_1\xi+c_2\eta$ from (5·4) for the boundary element, we find Q to be of type (0, 0). Therefore, we obtain

$$(5\cdot5) \qquad \|\hat{u}-\tilde{u}\|_{1,\Omega_h}=O(h^2) \quad \text{and} \quad \|\hat{u}-\tilde{u}\|_{0,\Omega_h}=O(h^2).$$

In the third case,

$$\bar{\psi}_1=-\frac{1}{5}+\frac{3}{5}\eta,$$

$$\bar{\psi}_2=\frac{2}{5}-\frac{3}{5}\xi-\frac{3}{5}\eta,$$

$$\bar{\psi}_3=-\frac{1}{5}+\frac{3}{5}\xi,$$

$$\bar{\psi}_4=\frac{3}{5}-\frac{4}{5}\xi,$$

$$\bar{\psi}_5=\frac{3}{5}-\frac{4}{5}\eta,$$

and

$$\bar{\psi}_6=-\frac{1}{5}+\frac{4}{5}\xi+\frac{4}{5}\eta\,.$$

Taking R_0 equal to P_0 of Example 5.1, and H_j being the same as the first and the second cases, we find Q to be of type $(1, 0)$. This is shown by $(5\cdot2)$ and

$$(5\cdot6) \qquad (\xi, \hat{\varphi}_i - \bar{\varphi}_i)_{T_0} = 0 \quad \text{and} \quad (\eta, \hat{\varphi}_i - \bar{\varphi}_i)_{T_0} = 0 \quad \text{for} \quad i = 1, \cdots, 6.$$

Therefore, we obtain

$$(5\cdot7) \qquad \|\hat{u} - \tilde{u}\|_{1, \Omega_h} = O(h^2) \quad \text{and} \quad \|\hat{u} - \tilde{u}\|_{0, \Omega_h} = O(h^3).$$

In the fourth case,

$$\bar{\varphi}_i = \hat{\varphi}_i \quad \text{for} \quad i = 1, \cdots, 6.$$

From $(4\cdot12')$ and $(4\cdot13')$, we obtain $(5\cdot7)$.

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