Publ. RIMS, Kyoto Univ.
12 (1976), 379-382

Mappings of Nonpositively Curved Manifolds[1)]

By

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§ 1. Introduction

In recent papers with S. S. Chern [3] and T. Ishihara [4], the author studied both the volume- and distance-decreasing properties of harmonic mappings thereby obtaining real analogues and generalizations of the classical Schwarz-Ahlfors lemma, as well as Liouville's theorem and the little Picard theorem. The domain M in the first case was the open ball with the hyperbolic metric of constant negative curvature, and the target was a negatively curved Riemannian manifold with sectional curvature bounded away from zero. In this paper, it is shown that M may be taken to be any complete simply connected Riemannian manifold of nonpositive curvature. Details will appear elsewhere.

Theorem. *Let $f: M \rightarrow N$ be a harmonic K-quasiconformal mapping of Riemannian manifolds of dimensions m and n, respectively. If M is complete and simply connected, and (a) the sectional curvatures of M are nonpositive and bounded below by a negative constant $-A$, and (b) the sectional curvatures of N are bounded above by the constant $-((m-1)/(k-1))kAK^4$, $k=\min(m, n)$, then f is distance-decreasing. If $m=n$ and (b) is replaced by the condition (b′) the sectional curvatures of N are bounded away from zero by $-AK^4$, then f is volume-decreasing.*

§ 2. Harmonic and K-quasiconformal mappings

Let M and N be C^∞ Riemannian manifolds of dimensions m and n,

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Communicated by S. Nakano, October 22, 1975.
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1) This research was supported in part by the National Science Foundation.

respectively. Let $f:M\to N$ be a C^∞ mapping. The Riemannian metrics of M and N can be written locally as $ds_M^2=\omega_1^2+\cdots+\omega_m^2$ and $ds_N^2=\omega_1^{*2}+\cdots+\omega_n^{*2}$, where $\omega_i(1\leqq i\leqq m)$ and $\omega_a^*(1\leqq a\leqq n)$ are linear differential forms in M and N, respectively. The structure equations in M are

$$d\omega_i=\sum_j \omega_j\wedge\omega_{ji}\,,$$

$$d\omega_{ij}=\sum_j \omega_{ik}\wedge\omega_{kj}-\frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k\wedge\omega_l\,.$$

Similar equations are valid in N and we will denote the corresponding quantities in the same notation with asterisks. Let $f^*\omega_a^*=\sum_i A_i^a\omega_i$. Then the covariant differential of A_i^a is defined by

$$DA_i^a\equiv dA_i^a+\sum_j A_j^a\omega_{ji}+\sum_b A_i^b\omega_{ba}^*\equiv\sum_j A_{ij}^a\omega_j$$

with $A_{ij}^a=A_{ji}^a$. The mapping f is called harmonic if $\sum_i A_{ii}^a=0$.

The differential f_* of f is extended to the mapping $\wedge^p f_*:\wedge^p T(M)\to\wedge^p T(N)$. $\wedge^p f_*$ is also regarded as an element of $\wedge^p T^*(M)\otimes\wedge^p T(N)$ on which a norm is defined in terms of the metrics of M and N. The norm $\|\wedge^p f_*\|$ is regarded as the ratio function of intermediate volume elements of M and N. As in [4] we consider the Laplacian Δ of $\|f_*\|^2$ and obtain the following formula when f is harmonic.

$$(1)\qquad (1/2)\Delta\|f_*\|^2=\sum_{a,i,j}(A_{ij}^a)^2+\sum_{a,i,j}R_{ij}A_i^aA_j^a-\sum_{a,b,c,d,i,j}R_{abcd}^*A_i^aA_j^bA_i^cA_j^d\,,$$

where R_{ij} is the Ricci tensor of M.

At each point $x\in M$, let A be the matrix representation of $(f_*)_x$ relative to orthonormal bases of $T_x(M)$ and $T_{f(x)}(N)$ and let ${}^tA$ be the transpose of A. In the sequel, we assume $\operatorname{rank} f_*=\operatorname{rank} A=k$ at every point. Then, $k\leqq\min(m,n)$ and rank $G=k$, where G is the positive semi-definite symmetric matrix ${}^tAA$. Let $\lambda_1\geqq\cdots\geqq\lambda_k>\lambda_{k+1}=\cdots=\lambda_m=0$ be the eigenvalues of G. The norm $\|\wedge^p f_*\|$ is represented as

$$(2)\qquad \|\wedge^p f_*\|^2=\sum_{i_1<\cdots<i_p}\lambda_{i_1}\cdots\lambda_{i_p}\,.$$

Lemma 1. *If $k\leqq\min(m,n)$ and rank f_* is k everywhere on M, then*

$$\left(\|\wedge^p f_*\|^2\Big/\binom{k}{p}\right)^{1/p}\geqq\left(\|\wedge^q f_*\|^2\Big/\binom{k}{q}\right)^{1/q},\quad 1\leqq p\leqq q\leqq k\,.$$

At each point $x \in M$, let S^{k-1} be a unit $(k-1)$-sphere in $T_x(M)$. If $(f_*)_x$ has maximal rank k, the image of S^{k-1} under $(f_*)_x$ is an ellipsoid of dimension $k-1$. Let f be a C^∞ mapping of maximal rank k and $K \geqq 1$. f is *K-quasiconformal* if at each point x of M, the ratio of the largest to the smallest axis of the ellipsoid $\leqq K$. One may verify that f is *K-quasiconformal* if and only if $\lambda_1/\lambda_k \leqq K^2$ at each point. Hence, from (2) we obtain

Lemma 2. *If f is K-quasiconformal, then*

$$\left(\|\wedge^p f_*\|^2 \Big/ \binom{k}{p}\right)^{1/p} \leqq K^2 \left(\|\wedge^q f_*\|^2 \Big/ \binom{k}{q}\right)^{1/q}, \quad 1 \leqq p < q \leqq k.$$

§ 3. Proof of Theorem

Let $d\tilde{s}_M^2$ be a Riemannian metric on M conformally related to ds_M^2. Then, there is a function $p>0$ on M such that $d\tilde{s}_M^2 = p^2 ds_M^2$. Let $\tilde{u} = \sum (\tilde{A}_i^a)^2 = p^{-2} \sum (A_i^a)^2$. Then

Lemma 3. *Let $f: M \to N$ be harmonic with respect to (ds_M^2, ds_N^2), and let $\tilde{u}$ attain its maximum at $x \in M$. If the symmetric matrix function $X_{ij} = p_{ij} + \delta_{ij} \sum (p_k)^2 - 2p_i p_j$, where p_i is given by $d \log p = \sum p_i \omega_i$ and p_{ij} is its covariant derivative, is positive semi-definite everywhere on M, then $-\sum R^*_{abcd} \tilde{A}_i^a \tilde{A}_j^b \tilde{A}_i^c \tilde{A}_j^d \leqq -\sum \tilde{R}_{ij} \tilde{A}_i^a \tilde{A}_j^a$ at x.*

Let y be a point of M and denote by $d(x, y)$ the distance-from-y function. Then, $t(x) = (d(x, y))^2$, $x \in M$, is C^∞ and convex on M (see [2]). The function $\tau(x) = d(x, y)$ is also convex, but it is only continuous on M. The convex open submanifolds $M_\rho = \{x \in M \mid t(x) < \rho\}$ of M exhaust M, that is $M = \bigcup_{\rho<\infty} M_\rho$ (see [5]). The nonnegative function $v_\rho = \log \dfrac{\rho}{\rho - t}$ is a C^∞ convex function.

Consider the metric $d\tilde{s}^2 = e^{2v_\rho} ds^2$ on M_ρ. Then $\tilde{u} = e^{-2v_\rho} u = \left(\dfrac{\rho - t}{\rho}\right)^2 u$ is nonnegative and continuous on the closure $\overline{M}_\rho$ of M_ρ and vanishes on ∂M_ρ. Since $\overline{M}_\rho$ is compact, $\tilde{u}$ has a maximum in M_ρ. Since the function $t(x)$ is convex the matrix X_{ij} is positive semi-definite, so we obtain the conclusion of Lemma 3.

Relating the Ricci tensors of $d\tilde{s}_M^2$ and ds_M^2, we obtain

$$\sum \tilde{R}_{ij}\tilde{A}_i^a\tilde{A}_j^a=\left(\frac{\rho-t}{\rho}\right)^2\sum R_{ij}\tilde{A}_i^a\tilde{A}_j^a-\frac{\rho-t}{\rho^2}(m-2)\sum t_{ij}\tilde{A}_i^a\tilde{A}_j^a$$

$$-\frac{\rho-t}{\rho^2}\Delta t\|f_*\|_\rho^2-\frac{m-1}{\rho^2}\langle dt, dt\rangle\|f_*\|_\rho^2 .$$

Lemma 4. *For each* ρ, *there exists a positive constant* $\varepsilon(\rho)$ *such that the inequality*

$$-\sum\tilde{R}_{ij}\tilde{A}_i^a\tilde{A}_j^a\leqq[(m-1)A+\varepsilon(\rho)]\tilde{u}$$

holds on M_ρ. *Moreover* $\varepsilon(\rho)\to 0$ *as* $\rho\to\infty$.

To see that $\Delta\tau$ is bounded as $\tau\to\infty$, observe that the level hypersurfaces of τ are spheres S with y as center. The hessian $D^2\tau$ of τ can be identified with the second fundamental form h of those spheres,e xtended to be 0 in the normal direction. If follows that $\Delta\tau=$ trace $D^2\tau=$ trace h $=(m-1)\cdot$mean relative curvature of S. If the curvature $K\geqq a^2$, then from [1; pp. 247-255], $\Delta\tau\leqq(m-1)a\dfrac{\cos a\tau}{\sin a\tau}$. If we put $a^2=-\alpha^2$, then $\Delta\tau\leqq(m-1)\alpha\coth\alpha\tau$.

The rest of the proof of the theorem is now a consequence of Lemmas 1-4.

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