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Supplement to "Pseudoconvex Domains on a Kähler Manifold with Positive Holomorphic Bisectional Curvature"

By

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Introduction

In this note a kähler manifold is always assumed to have a kähler metric of C^∞-class. The purpose of this note is to prove the following

Theorem. *Let M be a kähler manifold with positive holomorphic bisectional curvature. Then every relatively compact pseudoconvex domain in M is Stein.*

By definition, a kähler manifold with positive sectional curvature has positive holomorphic bisectional curvature. So the following Corollary is a direct consequence of Theorem:

Corollary. *Let M be a kähler manifold with positive sectional curvature. Then every relatively compact pseudoconvex domain in M is Stein.*

In O. Suzuki [3], the author proved that if M has a real analytic kähler metric with positive holomorphic bisectional curvature, then every relatively compact pseudoconvex domain in M is holomorphically convex. After the completion of O. Suzuki [3], the paper of G. Elencwajg [1] appeared. There he proved the same result as in O. Suzuki [3] in the case of kähler metrics of C^∞-class. Therefore we see that both results

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obtained in O.Suzuki [3] and G. Elencwajg [1] are included in our Theorem.

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§ 1. A Lemma on Kähler Metrics

Let U' be a domain on $\mathbb{C}^n$. The Euclidian coordinates are denoted by $z^1, z^2, \cdots, z^n$. Suppose that a kähler metric is given by using a real potential function $\Psi^{(0)}$ of C^∞-class on U' as follows:

$$(1\cdot1)\qquad \begin{cases} ds^2 = 2\sum g^{(0)}_{i,\bar{j}}\, dz^i \cdot d\bar{z}^j, \\ 2g^{(0)}_{i,\bar{j}} = \dfrac{\partial^2 \Psi^{(0)}}{\partial z^i \partial \bar{z}^j}. \end{cases}$$

Take a relatively compact domain U in U'. Following Whitney, for any $\varepsilon_1 (\varepsilon_1 > 0)$ and for any non-negative integer α_0 there exists a real analytic function $\Psi^{(1)}$ on the closure $\bar{U}$ of U satisfying $\|D^\alpha(\Psi^{(0)} - \Psi^{(1)})\|_{\bar{U}} < \varepsilon$ for any multiorder α with $|\alpha| \leqq \alpha_0$, where $D^\alpha = \dfrac{\partial^{\alpha_1}}{\partial z^{\alpha_1}} \cdot \dfrac{\partial^{\alpha_2}}{\partial z^{\alpha_2}} \cdots \dfrac{\partial^{\bar{\alpha}_n}}{\partial \bar{z}^{\alpha_n}}$, $|\alpha| = \alpha_1 + \cdots + \bar{\alpha}_n$ and the norm $\| \;\; \|$ means the supremum norm.

Choose a sequence of positive numbers $\{\varepsilon_\nu\}$ with $\varepsilon_\nu \to 0$ $(\nu \to \infty)$. Similarly, we take a $\Psi^{(\nu)}$ for ε_ν and α_0 and set

$$(1\cdot2)_{(\nu)}\qquad \begin{cases} ds^2 = 2\sum g^{(\nu)}_{i,\bar{j}}\, dz^i \cdot d\bar{z}^j, \\ 2g^{(\nu)}_{i,\bar{j}} = \dfrac{\partial^2 \Psi^{(\nu)}}{\partial z^i \partial \bar{z}^j}. \end{cases}$$

We may assume that $(1\cdot2)_{(\nu)}$ gives a kähler metric for every ν.

For two points p and q in M, we denote its distance by $d(p, q)$ (resp. $d_\nu(p, q)$) with respect to the metric $(1\cdot1)$ (resp. $(1\cdot2)_{(\nu)}$). We write $B_\delta(p) = \{q \in M : d(p, q) < \delta\}$.

First we prove the following Lemma (compare with Lemma 2 in A. Takeuchi [4]):

Lemma $(1\cdot3)_{(\nu)}$. (i) *For any point $p_0 \in U$, there exist positive constants δ, M_0 and a neighborhood V of p_0 such that the following* *holds for every* ν: *For any point* p *in* V *and for any geodesic* σ_ν *through* p *with respect to* $(1\cdot 2)_{(\nu)}$, *there exist a neighborhood* U_ν *of* p *and a system of local coordinates* $z_\nu^1, z_\nu^2, \cdots, z_\nu^n$ *at* p *on* U_ν *satisfying the following* $(1)\sim(4)$:

(1) σ_ν *is expressed as* $\operatorname{Im} z_\nu^1=0,\ z_\nu^2=0,\ \cdots,\ z_\nu^n=0$,

(2) $\{|\operatorname{Re} z_\nu^1|<\delta, \operatorname{Im} z_\nu^1=0, z_\nu^2=0, \cdots, z_\nu^n=0\}\subset U_\nu$,

(3) *The metric tensor with respect to* $z_\nu^1, z_\nu^2, \cdots, z_\nu^n$ *is expressed as*

$$2g^{(\nu)}_{i,\bar{j}}=\delta_{i,j}+2\sum K^{(\nu)}_{i\bar{j}k\bar{l}}(0)z_\nu^k\bar{z}_\nu^l+\cdots,$$

(4) *Let* $\phi_\nu: I_\delta\to U_\nu$ *be* $z_\nu^1=t, z_\nu^2=0, \cdots, z^n=0$ *where* $I_\delta=\{t\in\mathbf{R}:|t|<\delta\}$. *Then* $\|D_t^\alpha(g^{(\nu)}_{i,\bar{j}}\circ\phi_\nu)\|\leqq M_0$ *and* $\|D_t^\alpha(g^{i,\bar{j}}_{(\nu)}\circ\phi_\nu)\|\leqq M_0$ *for* $\alpha\leqq\alpha_0$, *where* $(g^{i,\bar{j}}_{(\nu)})$ *denotes the inverse matrix of* $(g^{(\nu)}_{i,\bar{j}})$.

(ii) *For any relatively comact domain* V *in* U, *there exist positive constants* δ *and* M_0 *satisfying* $(1)\sim(4)$ *for any point* $p\in V$.

Proof of (i). We choose a neighborhood V of p_0 so that there exists a system of real analytic sections $e_1, e_2, \cdots, e_n$ on V of the holomorphic tangent bundle which are linearly independent at every point in V. Making V smaller, we fix an orthonormal system $e_1^{(\nu)}, e_2^{(\nu)}, \cdots, e_n^{(\nu)}$ with respect to $(1\cdot 2)_{(\nu)}$ in the following manner:

$$e_i^{(\nu)}=\sum_{k=1}^{i}\alpha_k^{(\nu)}e_k\quad(i=1,2,\cdots,n),$$

where $\alpha_k^{(\nu)}$ $(k=1,2,\cdots,i)$ are determined by the condition $(e_k^{(\nu)}, e_j^{(\nu)})_\nu=\delta_{k,j}$ $(k,j=1,2,\cdots,i)$. Here $(\ ,\ )_\nu$ denotes the inner product defined by $(1\cdot 2)_{(\nu)}$. Also we define an orthonormal system $e_1^{(0)}, e_2^{(0)}, \cdots, e_n^{(0)}$ on V with respect to $(1\cdot 1)$ in the same manner. Let

$$e_i^{(0)}=\sum\lambda_i^{(0)k}\frac{\partial}{\partial z^k}\quad\text{and}\quad e_i^{(\nu)}=\sum\lambda_i^{(\nu)k}\frac{\partial}{\partial z^k}\quad(i=1,2,\cdots,n).$$

Then for any positive ε we can find ν_1 such that

$$\|\lambda_i^{(0)k}-\lambda_i^{(\nu)k}\|_V<\varepsilon\quad(\nu\geqq\nu_1).\tag{1·3}$$

Take a point $p\in V$ and write $z^i(p)=z_0^i$. After A. Takeuchi [4, p. 327], we can choose local coordinates $z_\nu'^1, z_\nu'^2, \cdots, z_\nu'^n$ at p by

$$z^i-z_0^i=\sum\lambda^{(\nu)}{}_k{}^i(p)z_\nu'^k\quad(i=1,2,\cdots,n).$$

Making V smaller, we may assume that these give local coordinates on $B_\delta(p)$ with δ independent of ν. Choose a unitary matrix (α_j^i) and set

$$w_\nu^i = \sum \alpha_j^i z_\nu'^j + \sum \beta^i_{(\nu)jk} z_\nu'^j z_\nu'^k + \sum \gamma^i_{(\nu)jkl} z_\nu'^j z_\nu'^k z_\nu'^l .$$

$$(i=1, 2, \cdots, n)$$

We shall choose positive constants δ, M_0 independent of ν and determine $\{\beta^i_{(\nu)jk}\}$, $\{\gamma^i_{(\nu)jkl}\}$ so that $w_\nu^1, w_\nu^2, \cdots, w_\nu^n$ are local coodinates on $B_\delta(p)$ with the following properties:

$$(1\cdot 4) \qquad 2\tilde{g}^{(\nu)}_{i,\bar{j}} = \delta_{\cdot,\bar{j}} + \sum 2\tilde{K}^{(\nu)}_{i\bar{j}s\bar{t}}(0)\, w_\nu^s \bar{w}_\nu^t + \cdots ,$$

where $\tilde{g}^{(\nu)}_{i,\bar{j}}$ denotes the metric tensor with respect to $w_\nu^1, w_\nu^2, \cdots, w_\nu^n$,

$$(1\cdot 5) \qquad |\beta^i_{(\nu)jk}| \leqq M_0 \quad \text{and} \quad |\gamma^i_{(\nu)jkl}| \leqq M_0 .$$

By A. Takeuchi [4, (8), (9) in p. 328], we can choose $\{\beta^i_{(\nu)jk}\}$, $\{\gamma^i_{(\nu)jkl}\}$ satisfying $(1\cdot 4)$. By the choices of $\Psi^{(\nu)}$ and $\{\beta^i_{(\nu)jk}\}$, $\{\gamma^i_{(\nu)jkl}\}$, we can find M_0 satisfying $(1\cdot 5)$. By this we can choose a required δ. Making M_0 larger, from $(1\cdot 3)$ and $(1\cdot 5)$ we see that

$$(1\cdot 6) \qquad \|D^\alpha \tilde{g}^{(\nu)}_{i,\bar{j}}(w_\nu)\| \leqq M_0 \quad \text{and} \quad \|D^\alpha g^{i,\bar{j}}_{(\nu)}(w_\nu)\| \leqq M_0 \quad \text{on } B_\delta(p)$$

$$\text{for} \quad |\alpha| \leqq \alpha_0 \quad \text{and} \quad \nu \geqq \nu_1 .$$

Now we take a geodesic σ_ν through p. Choosing a suitable (α_j^i), we may assume that σ_ν satisfies

$$\begin{cases} \dfrac{d^2 w_\nu^i}{ds_\nu^2} + \sum \Gamma^{i(\nu)}_{k,h} \dfrac{dw_\nu^k}{ds_\nu} \dfrac{dw_\nu^h}{ds_\nu} = 0 , \\ w_\nu^i(0) = 0 \qquad\qquad (i=1, 2, \cdots, n), \\ \dfrac{dw_\nu^i}{ds_\nu}(0) = \delta_{i,1} \end{cases}$$

where s_ν denotes the length of σ_ν and $\Gamma^{i(\nu)}_{j,h}$ denote the connection coefficients. The solution is denoted by $w_\nu^i = \varphi_\nu^i(s_\nu)$ $(i=1, 2, \cdots, n)$. Setting $\varphi_\nu^i(-s_\nu)$, we get the expression of σ_ν in the opposite direction. In what follows, we assume that the parameter s_ν is extended to some interval containing the origin. Then we can find a constant δ independent of ν such that $\dfrac{d\varphi_\nu^1}{ds_\nu} \neq 0$ for $|s_\nu| < \delta$. Let $z_\nu^1 = s_\nu + \sqrt{-1}\; t_\nu$ and make a holomorphic extension $\varphi_\nu^1(z_\nu^1)$ of $\varphi_\nu^1(s_\nu)$ on $U_\nu^1 = \{z_\nu^1 : |\operatorname{Re} z_\nu^1| < \delta,$

$|\mathrm{Im}\, z_\nu^1| < \varepsilon_\nu\}$, where ε_ν is a positive constant. We may assume that $\varphi_\nu^1(z_\nu^1)$ is a univalent function on U_ν^1. $z_\nu^1 = \phi_\nu^1(w_\nu^1)$ denotes the inverse of $\varphi_\nu^1(z_\nu^1)$. Define a new system of local coordinates $z_\nu^1, z_\nu^2, \cdots, z_\nu^n$ on $U_\nu = U_\nu^1 \times \{|z_\nu^2| < \delta, \cdots, |z_\nu^n| < \delta\}$ by

$$z_\nu^1 = \phi^1(w_\nu^1),\ z_\nu^2 = w_\nu^2,\ \cdots,\ z_\nu^n = w_\nu^n .$$

By A. Takeuchi [4, p.332-333], the conditions (1), (2) and (3) are satisfied. Taking acount that

$$\begin{cases} g_{i,\bar{j}}^{(\nu)}(z_\nu) = \sum \tilde{g}_{k,\bar{l}}^{(\nu)}(w_\nu) \dfrac{\partial w_\nu^k}{\partial z_\nu^i} \overline{\dfrac{\partial w_\nu^l}{\partial z_\nu^j}} \\ \sum \tilde{g}_{i,\bar{j}}^{(\nu)}(w_\nu) \dfrac{dw_\nu^i}{ds_\nu} \overline{\dfrac{dw_\nu^j}{ds_\nu}} = 1 , \end{cases}$$

and by using $(1\cdot 6)$, we can easily see (4).

The proof of (ii) is easily done by using (i) and the compactness of $\overline{V}$.

§ 2. Proof of Theorem

In this section, M is assumed to be a kähler manifold with positive holomorphic bisectional curvature. Let D be a relatively compact domain in M. We set

$$d(p) = \inf_{q \in \partial D} d(p, q) \quad \text{and} \quad \varphi(p) = -\log d(p) \quad \text{for } p \in D .$$

Also we set $D_\delta = \{q \in D : d(q) < \delta\}$. Then we have the following

Theorem (2·1). *Let M be a kähler manifold with positive holomorphic bisectional curvature. For a compact set K there exists a positive constant δ such that the following inequality holds for any pseudoconvex domain D in K:*

$$W(\varphi)(p) \geqq \rho/16 \ \textit{for}\ p \in D_\delta ,$$

where $W(\varphi)(p)$ means the minimum of the eigenvalues of the hessian of φ at p and ρ is the minimum of the holomorphic bisectional curvature on K.

For the proof of Theorem (2·1), it is sufficient to show the following

Lemma (2·2). In fact, replacing Lemma 5 in A. Takeuchi [4] by this Lemma and using Lemma 6 in A. Takeuchi [4], we prove the assertion.

Let U' be a domain in $\mathbf{C}^n$ and consider a kähler metric (1·1) on U'. We fix a real analytic approximation of (1·1) as $(1\cdot 2)_{(\nu)}$ on U with $U \Subset U'$. Let D be an s-pseudoconvex domain V in U whose boundary is of C^∞-class. Take a relatively compact domain in U. Making δ so small that (1) $d(p, U^c) > 2\delta$ for $p \in V$, where U^c means the complement of U and (2) (ii) in $(1\cdot 3)_{(\nu)}$ holds for V. Then we have

Lemma (2·2). *There exists a positive* δ_* *such that* $W(\varphi)(p) \geqq \rho/16$ *for* $p \in V \cap D_{\delta_*}$, *where* ρ *is the infimum of the holomorphic bisectional curvature on* U.

Proof. Choosing α_0 sufficiently large, we may assume that the infimum ρ_ν of the holomorphic bisectional curvature on U with respect to $(1\cdot 2)_{(\nu)}$ satisfying $\rho_\nu > \rho/2$ for large ν. Take a point $p \in V \cap D_\delta$. Then for every ν, we can find a point $q_\nu \in \partial D$ and a geodesic σ_ν between p and q_ν which attains $d(p) = d(p, q_\nu)$. For σ_ν choose a system of local coordinates $z_\nu^1, z_\nu^2, \cdots, z_\nu^n$ as in Lemma $(1\cdot 3)_{(\nu)}$. Then by O. Suzuki [3],

$$W(\varphi_\nu)(p) \geqq \rho/8 - F_\nu(p) \cdot d_\nu(p),$$

where $\varphi_\nu(p) = -\log d_\nu(p)$. The estimate of $F_\nu(p)$ can be done by using the estimates of $\|G_{i,j}^{(\nu)}\|, \|G_{i,j}^{(\nu)\prime}\|, \|G_{i,j}^{(\nu)k}\|$ on $0 \leqq t < \delta$ which are defined by

$$g_{i,\bar{j}}^{(\nu)} \circ \phi_\nu(t) = t^k G_{i,\bar{j}}^{(\nu)}(t),$$

$$\frac{d}{dt} g_{i,\bar{j}}^{(\nu)} \circ \phi_\nu(t) = t^l G_{i,\bar{j}}^{(\nu)\prime}(t),$$

$$\Gamma_{i,j}^{k(\nu)} \circ \phi_\nu(t) = t^s G_{i,j}^{(\nu)k}(t),$$

where $G_{i,\bar{j}}^{(\nu)}, G_{i,\bar{j}}^{(\nu)\prime}, G_{i,j}^{(\nu)k}$ are real analytic functions of t and k, l, m are non-negative integers less than 4. Making α_0 large again, we can estimate these functions by using M_0 which is defined in (3).(i) in Lemma $(1\cdot 3)_{(\nu)}$. Then we can find a positive constant M_* which is determined only by M_0 satisfying $|F_\nu^{(\nu)}(p)| \leqq M_*$ for $p \in V \cap D_\delta$. Define $\delta_* = \min(\delta, \rho/16M_*)$. Then we conclude that $W(\varphi)(p) \geqq \rho/16$ for $p \in V \cap D_{\delta_*}$. By Theorem (2·1), we obtain

Theorem (2·3). *Let M be a kähler manifold with positive holomorphic bisectional curvature. Then M admits no exceptional analytic sets in the sense of H. Grauert* [2].

Proof. Suppose that M admits an exceptional analytic set E. We consider a connected component of E which is also denoted by the same letter E. By H. Grauert [2], E has an s-pseudoconvex neighborhood system $\{V_\varepsilon(E)\}$. Then there exist a compact set K and ε_0 satisfying $V_\varepsilon(E)\subset K$ for $\varepsilon<\varepsilon_0$. By Theorem (2·1), we choose δ for K. Now making ε smaller, we may assume that $V_\varepsilon=(V_\varepsilon)_\delta$. Then by Theorem (2·1), $W(\varphi)(p)\geqq\rho/16$ on V_ε. Then φ is s-pseudoconvex on E. This contradicts the existence of E.

Our Theorem stated in Introduction is nothing but the combination of Theorem (2·1) with Theorem (2·3). Hereby we also complete the proof of Theorem.

References

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