Publ. RIMS, Kyoto Univ.
12 (1976), 447-492

On Asymptotic Behaviour of the Spectra of a One-Dimensional Hamiltonian with a Certain Random Coefficient

By

Shin'ichi KOTANI*

§ 0. Introduction

Let $\{Q(x)\}$ be a random process with stationary independent increments. We consider a second order differential operator L defined by

$$L\varphi = -\frac{d\frac{d\varphi}{dx} - \varphi dQ(x)}{dx}.$$

Let $N(\lambda, I)$ be the number of eigenvalues not exceeding λ for a certain boundary value problem of the operator L in the interval I. We define the spectral distribution function of L by

$$N(\lambda) = \lim_{|I|\to\infty} \frac{1}{|I|} N(\lambda, I),$$

if it exists, where $|I|$ is the length of I.

The operator L has been used as a Schrödinger operator describing a motion of an electron in a one-dimensional random array of atoms (cf. M. Lax-J. C. Phillips [1], I. M. Lifšic [2]). We are concerned with the study of asymptotic properties of $N(\lambda)$ at the edges of the support. One interest is in making clear the influences caused by the randomness of potentials. One of them is the exponential decay of $N(\lambda)$ at the left edge, which was shown by many authors for various potentials (cf. H. L. Frisch-S. P. Lloyd [3], M. M. Benderskii-L. A. Pastur [4], [5], T. P. Eggarter [6]). L. A. Pastur [7] is a survey written mainly from mathematical points of view and gives us good informations about the problems arising

Communicated by K. Itô, April 23, 1976.
* Department of Mathematics, Kyoto University, Kyoto 606, Japan.

from random differential equations.

The author, suggested by H. L. Frisch-S. P. Lloyd [3], has succeeded in developing their results to obtain a sharper estimate of $N(\lambda)$ at the left end point. The purpose of the present paper is to give a complete proof of a formula obtained by H. L. Frisch-S. P. Lloyd [3] for potentials belonging to a slightly wider class and to obtain the estimate of $N(\lambda)$ by analyzing the formula on the pure imaginary axis. In the proof of the formula, the author was given valuable suggestions from M. Fukushima-S. Nakao [8]. Further the author would like to remark that S. Nakao [9] has obtained satisfactory results in several dimensional case by making use of the result of Donsker-Varadhan on "Wiener sausage".

Now, we explain the content of this paper. In § 1, as a preparation for the latter sections, we shall prove some properties relating to the zeros of eigenfunctions of a generalized differential operator

$$\frac{d\frac{d\varphi}{dx}-\varphi dQ(x)}{dM(x)}.$$

In § 2, we shall prove the ergodic property of the solution of a Ricatti equation with a random coefficient

$$dz(x)=(z(x)^2+\lambda)dx-dQ(x),$$

where $\lambda>0$ and Q is a process with stationary independent increments [Theorem 2.5]. With the help of this theorem, the Rice formula and the Frisch-Lloyd formula will be proved [Corollary 2.6 and 2.8]. In § 3, we restrict ourselves to the case when the process Q is increasing and express the spectra $N(\lambda)$ in a simpler form [Theorem 3.2]. In § 4, applying this form, we shall obtain the main result for the asymptotic behaviour of $N(\lambda)$ at the origin [Theorem 4.7]. In § 5, we shall give an expression of the spectral distribution $N(\lambda)$ of an equation

$$\frac{d}{dM}\frac{d\varphi}{dx}=-\lambda\varphi,$$

where M is an increasing process with stationary independent increments. In § 6, we shall give a comment on the spectral distribution of an equation defined on the whole line $\boldsymbol{R}^1$.

The author wishes to express thanks to Prof. M. Fukushima for teaching him the problem and to Prof. S.Watanabe for his constant attention to the present work.

§ 1. On the Behaviour of Zeros of Eigenfunctions of Generalized Second Order Differential Equations

First of all, let us introduce necessary notations and terminologies. Let $[a, b]$ be a finite closed interval. We denote $f\in V[a, b]$, if f consists of a function $f(x)$ of bounded variation in $[a, b]$ and two additional numbers $\{f(a-0), f(b+0)\}$. To any $f\in V[a, b]$, a complex measure df corresponds in such a way that

$$\begin{cases} df = \textit{the usual one in } (a, b), \\ df(a) = f(a+0) - f(a-0), \quad df(b) = f(b+0) - f(b-0). \end{cases}$$

We remark here that it is possible to define $g(x) = \int_{[a,x]} f(y)\, dQ(y) \in V[a, b]$ for any $Q\in V[a, b]$ and $f\in L^1(|dQ|, [a, b])$ if we put $g(a-0) = 0$ and $g(b+0) = \int_{[a,b]} f(y)\, dQ(y)$.

Let M and $Q\in V[a, b]$. Throughout this section we assume that dM defines a nonnegative measure and dQ defines a real one. We denote the right (left) derivative of a function f at x by $f^+(x)$ (resp. $f^-(x)$). Put

$D[a, b] = \{f\in C[a, b]$; *there exists an* $f^+\in V[a, b]$ *coinciding with* $f^+(x)$ *for every* $x\in[a, b)$ *and the measure* $df^+ - fdQ$ *is absolutely continuous with respect to* dM *and its density belongs to* $L^2(dM, [a, b])$.$\}$

$D_{\alpha,\beta}[a, b] = \{f\in D[a, b];\ f(a)\cos\alpha + f^+(a-0)\sin\alpha = 0,\ \ f(b)\cos\beta + f^+(b+0)\sin\beta = 0.\}$.

It is not difficult to see that any element of $D[a, b]$ is absolutely continuous with respect to the Lebesgue measure in $[a, b]$ and has the left derivative at each point in $(a, b]$. Let L denote an operator defined by

$$Lf = -\frac{df^+ - fdQ}{dM}$$

for $f\in D[a, b]$. Let $\varphi_\alpha(x, \lambda)$ and $\psi_\beta(x, \lambda)$ be solutions of the following integral equations

$$(1\cdot 1)\quad \varphi_\alpha(x,\lambda)=-\sin\alpha+(x-a)\cos\alpha+\int_{[a,x]}(x-y)\varphi_\alpha(y,\lambda)dQ(y)$$
$$-\lambda\int_{[a,x]}(x-y)\varphi_\alpha(y,\lambda)dM(y),$$
$$\psi_\beta(x,\lambda)=-\sin\beta+(x-b)\cos\beta+\int_{[x,b]}(y-x)\psi_\beta(y,\lambda)dQ(y)$$
$$-\lambda\int_{[x,b]}(y-x)\psi_\beta(y,\lambda)dM(y).$$

It is easy to see that these functions may be determined as the unique solutions of the following equations respectively

$$\begin{cases} L\varphi=\lambda\varphi, & \varphi(a)=-\sin\alpha, \quad \varphi^+(a-0)=\cos\alpha, \\ L\psi=\lambda\psi, & \psi(b)=-\sin\beta, \quad \psi^+(b+0)=\cos\beta. \end{cases}$$

Here we note the well-known identity between those functions of $D[a,b]$.

Lemma 1.1. *Let $f,\ g\in D[a,b]$ and let $c,\ d\in[a,b]$. Then we have*

$$\int_{[c\pm 0,\,d\pm 0]}\{Lf(x)g(x)-f(x)Lg(x)\}dM(x)$$
$$=[f(x)g^+(x)-f^+(x)g(x)]_{c\pm 0}^{d\pm 0},$$

where $[c+0,d+0]=(c,d]$, $[c-0,d-0]=[c,d)$ and so on.

The following comparison relation between zeros of solutions will be very usefull.

Lemma 1.2. *Let φ, ψ be nontrivial solutions of $L\varphi=\lambda\varphi$, $L\psi=\mu\psi$ in (x_1,x_2). Suppose $\psi(x_1)=\psi(x_2)=0$ and ψ has no zeros in (x_1,x_2). If $\lambda\geqq\mu$, then either φ has at least one zero in (x_1,x_2) or φ is a constant multiple of ψ in (x_1,x_2). Here the latter case occurs only if $\lambda=\mu$ or $dM=0$ in (x_1,x_2).*

Proof. In Lemma 1.1, substituting $c=x_1$, $d=x_2$, $f=\varphi$ and $g=\psi$, we have

$$(1\cdot 2)\quad (\lambda-\mu)\int_{(x_1,x_2)}\varphi(x)\psi(x)dM(x)=\varphi(x_2)\psi^+(x_2-0)-\varphi(x_1)\psi^+(x_1+0),$$

where we have used the assumption $\psi(x_1)=\psi(x_2)=0$. Since ψ has no zeros in (x_1, x_2), we may assume $\psi(x)>0$ for any $x\in(x_1, x_2)$. Then we have

$$\psi^+(x_2-0)<0 \text{ and } \psi^+(x_1+0)>0 . \tag{1·3}$$

For at any fixed point x, it is impossible that $\psi(x)$ and $\psi^{\pm}(x)$ vanish simultaneously.

Now we consider the two cases separately.

1°) Either $\lambda=\mu$ or $dM=0$ in (x_1, x_2). Then from (1·2) it follows that $\varphi(x_2)\psi^+(x_2-0)=\varphi(x_1)\psi^+(x_1+0)$. Noting (1·3), we see that either $\varphi(x_1)=\varphi(x_2)=0$ or $\varphi(x_1)\varphi(x_2)<0$ holds. Since under the condition 1°) φ and ψ satisfy the same equation, the first case implies that φ is a constant multiple of ψ. In the second case, from the continuity of φ we see that there exists at least one zero of φ.

2°) $\lambda>\mu$ and $dM=0$ in (x_1, x_2). Suppose φ has no zeros in (x_1, x_2), hence assume $\varphi(x)>0$ for any $x\in(x_1, x_2)$. Then from (1·2) we have

$$\varphi(x_2)\psi^+(x_2-0)>\varphi(x_1)\psi^+(x_1+0)\geqq 0 ,$$

which implies $\varphi(x_2)<0$. This contradicts the assumption $\varphi(x)>0$ in (x_1, x_2). This proves the lemma.

Since $\varphi_\alpha(x, \lambda)$ is a solution of $L\varphi=\lambda\varphi$, the set of zeros of $\varphi_\alpha(x, \lambda)$ in $[a, b]$ has no accumulating points. Let $\tau_n(\alpha, \lambda)$ be the n-th zero from the left end point a of $[a, b]$, where $n=1, 2, \cdots$. We denote the support of dM by F_M and put

$$a_0=\inf F_M , \ b_0=\sup F_M . \tag{1·4}$$

Let $\varphi_\alpha(x)$ be $\varphi_\alpha(x, 0)$ and let $\{x_1, \cdots x_{n_0}\}$ be the set of zeros of $\varphi_\alpha(x)$ in $[a, a_0]$. Then it is obvious that every solution $\varphi_\alpha(x, \lambda)$ has the common zeros $\{x_1, x_2, \cdots x_{n_0}\}$.

Proposition 1. 3.

(1) *If $\lambda>\mu$, then $\tau_n(\alpha, \lambda)\leqq\tau_n(\alpha, \mu)$ for $n=1, 2, \cdots$. Moreover $\tau_{n+n_0}(\alpha, \lambda)=\tau_{n+n_0}(\alpha, \mu)$ holds for some $n\geqq 1$ if and only if there exists a sequence $\{a_k\}_{k=1}^{n}$ such that*

$$\begin{cases} a_k = \tau_{k+n_0}(\alpha, \lambda) & for \quad \boldsymbol{R}^1, \ k=1, 2, \cdots n, \\ F_M \cap [a, a_n] \subset \{a_0, a_1, a_2, \cdots a_n\}. \end{cases}$$

and theer exists a nontrivial solution φ_α *of the equation*

$$(1\cdot 5) \qquad \begin{cases} d\varphi^+ = \varphi dQ \quad in \quad [a, a_n] \\ \varphi(a)\cos\alpha + \varphi^+(a-0)\sin\alpha = 0 \\ \varphi(a_k) = 0 \quad for \quad k=0, 1, 2, \cdots n. \end{cases}$$

(2) $\tau_n(\alpha, \lambda)$ *is continuous in* λ *for every fixed* α.

Proof. First we consider the case $n=1$. Let $\xi = \tau_{n_0+1}(\alpha, \lambda)$ and $\eta = \tau_{n_0+1}(\alpha, \mu)$. Suppose $\xi > \eta$. In Lemma 1.1, substituting $c=a$, $d=\eta$, $f(x) = \varphi_\alpha(x, \lambda)$ and $g(x) = \varphi_\alpha(x, \mu)$, we have

$$(1\cdot 6) \quad (\lambda-\mu) \int_{[a_0, \eta)} \varphi_\alpha(x, \lambda)\varphi_\alpha(x, \mu)\, dM(x) = \varphi_\alpha(\eta, \lambda)\varphi_\alpha^+(\eta-0, \mu),$$

here we have used the facts that $\varphi_\alpha(a, \lambda) = \varphi_\alpha(a, \mu) = -\sin\alpha$. $\varphi_\alpha^+(a-0, \lambda) = \varphi_\alpha^+(a-0, \mu) = \cos\alpha$, $\varphi_\alpha(\eta, \mu) = 0$ and $dM=0$ in $[a, a_0)$. Once we have obtained the identity $(1\cdot 6)$, the situation becomes quite similar to the one of Lemma 1. 2, so we stop going into details.

Next we consider the case $n>1$. It follows from Lemma 1.1 that $\tau_{n_0+n}(\alpha, \lambda) \leqq \tau_{n_0+n}(\alpha, \mu)$. Suppose $\tau_{n_0+n}(\alpha, \lambda) = \tau_{n_0+n}(\alpha, \mu)$. Put $a_k = \tau_{n_0+k}(\alpha, \mu)$ for $k=1, 2, \cdots n$. Unless $\varphi_\alpha(x, \lambda)$ is a constant multiple of $\varphi_\alpha(x, \mu)$ in (a_{n-1}, a_n), $\tau_{n_0+n}(\alpha, \lambda) < \tau_{n_0+n}(\alpha, \mu)$. Hence it is necessary that $\varphi_\alpha(x, \lambda)$ and $\varphi_\alpha(x, \mu)$ are lineary dependent in (a_{n-1}, a_n) and $dM=0$ in (a_{n-1}, a_n). Consequently we have $\tau_{n_0+n-1}(\alpha, \lambda) = \tau_{n_0+n-1}(\alpha, \mu)$. Continuing this argument until $n=1$, we may prove (1).

Now let us prove (2). If $\alpha \equiv 0 \pmod{\pi}$, then $\tau_1(\alpha, \lambda) = a$ for every λ, hence the continuity is trivial. So we may suppose $\alpha \not\equiv 0 \pmod{\pi}$. First we consider the case when

$$a < \tau_1(\alpha, \lambda) < \tau_2(\alpha, \lambda) < \cdots < \tau_n(\alpha, \lambda) < b.$$

For brevity we put $\varphi_\lambda(x) = \varphi_\alpha(x, \lambda)$ and $\tau_k(\lambda) = \tau_k(\alpha, \lambda)$. $\varphi_\lambda^+(x)$ is continuous at each point $\tau_k(\lambda)$. This is because we have

$$\varphi_\lambda^+(\tau_k(\lambda)) - \varphi_\lambda^-(\tau_k(\lambda)) = \varphi_\lambda(\tau_k(\lambda))\{dQ(\tau_k(\lambda)) - \lambda dM(\tau_k(\lambda))\} = 0.$$

Therefore for any fixed sufficiently small $\varepsilon > 0$, φ_λ in $[\tau_k(\lambda) - \varepsilon, \tau_k(\lambda))$

and φ_λ in $(\tau_k(\lambda), \tau_k(\lambda)+\varepsilon]$ have different signs. Since $\varphi_\mu(x)$ converges to $\varphi_\lambda(x)$ uniformly in $[a, b]$ as $\mu\to\lambda$, there exists $\delta>0$ such that for any μ satisfying $|\mu-\lambda|<\delta$, $\varphi_\mu(x_k)\varphi_\mu(y_k)<0$ holds for some $x_k\in[\tau_k(\lambda)-\varepsilon, \tau_k(\lambda))$ and $y_k\in(\tau_k(\lambda), \tau_k(\lambda)+\varepsilon]$. Hence φ_μ has at least one zero in $[\tau_k(\lambda)-\varepsilon, \tau_k(\lambda)+\varepsilon]$ for every μ, $|\mu-\lambda|<\delta$. Again from the continuity of $\varphi_\lambda^+(x)$ at $\tau_k(\lambda)$, we may assume that $\varphi_\lambda^+(x)$ does not vanish in $[\tau_k(\lambda)-\varepsilon, \tau_k(\lambda)+\varepsilon]$. On the other hand, it is easy to see that $\varphi_\mu^+(\varphi_\mu^-)$ also converges uniformly to φ_λ^+(resp. φ_λ^-). Hence $\varphi_\mu^\pm(x)$ have the same sign as $\varphi_\lambda^+(x)$ for every $x\in[\tau_k(\lambda)-\varepsilon, \tau_k(\lambda)+\varepsilon]$ for $k=1, 2, \cdots n$. Hence $|\tau_k(\lambda)-\tau_k(\mu)|<\varepsilon$ for every μ such that $|\lambda-\mu|<\delta$.

In case $\tau_n(\lambda)=b$, we have only to extend the measures dQ and dM to a closed interval including $[a, b]$ in which $\varphi_\alpha(x, \lambda)$ has more than $n+1$ zeros. Then the problem may be reduced to the above case. This completes the proof of (2).

Since $\varphi_\alpha(x, \lambda)$ and $\varphi_\alpha^+(x, \lambda)$ do not vanish simultaneously, we may define

$$z_\alpha(x) = -\frac{\varphi_\alpha^+(x, \lambda)}{\varphi_\alpha(x, \lambda)}$$

as a function taking the values in $\boldsymbol{R}^1\cup\infty$.

Lemma 1.4. *As far as $\varphi_\alpha(x, \lambda)$ dose not vanish, we have*

$$(1)\qquad \begin{cases} dz_\alpha(x) = z_\alpha(x)^2dx - dQ(x) + \lambda dM(x) \\ z_\alpha(a-0) = \cot\alpha\,, \end{cases}$$

(2) *$z_\alpha(x)$ is continuous at $\tau_k(\alpha, \lambda)$ and*

$$z_\alpha(\tau_k(\alpha, \lambda)-0) = +\infty\,, \quad z_\alpha(\tau_k(\alpha, \lambda)+0) = -\infty\,.$$

Proof. (1) may be proved by easy calculations. As for (2), we note that $\varphi_\alpha^+(x, \lambda)$ is continuous at the zeros of $\varphi_\alpha(x, \lambda)$, which was proved in the argument in Proposition 1.3. Hence $z_\alpha(x)$ is continuous at $\tau_k(\alpha, \lambda)$. The last equalities are obvious, so we omit the proof.

Proposition 1. 5.

(1) *If* $0 \leqq \alpha < \beta < \pi$, *then* $\tau_n(\alpha, \lambda) < \tau_n(\beta, \lambda) < \tau_{n+1}(\alpha, \lambda)$.

(2) $\tau_n(\alpha, \lambda)$ *is continuous in* $\alpha \in [0, \pi)$.

(3) $\tau_n(\pi - 0, \lambda) = \tau_{n+1}(0, \lambda)$ *for* $n = 1, 2, \cdots$.

Proof. We restrict α and β to $[0, \pi)$ in the following argument and for brevity denote $\tau_n(\alpha) = \tau_n(\alpha, \lambda)$. First it should be noted that in any subinterval of $[a, b]$, $\varphi_\alpha(x, \lambda)$ and $\varphi_\beta(x, \lambda)$ are linearly independent if $\alpha \neq \beta$. This is because every solution of $L\varphi = \lambda\varphi$ is uniquely determined by the values $\{\varphi(c), \varphi^{\pm}(c)\}$ at arbitrary fixed point $c \in [a, b]$. Hence from Lemma 1.2, there exists at least one zero of $\varphi_\alpha(x, \lambda)$ between two successive zeros of $\varphi_\beta(x, \lambda)$ and vice versa. Hence in order to prove (1) it is sufficient to verify $\tau_1(\alpha) < \tau_1(\beta)$. If $\alpha = 0$ and $0 < \beta < \pi$, then $0 = \tau_1(\alpha) < \tau_1(\beta)$, which proves (1). Suppose $0 < \alpha < \beta < \pi$ and put $\delta(x) = z_\alpha(x) - z_\beta(x)$, $\delta_0 = \cot \alpha - \cot \beta$. Then from (1) of Lemma 1. 4 we have

$$\begin{cases} d\delta(x) = \{z_\alpha(x) + z_\beta(x)\}\delta(x)dx \\ \delta(a-0) = \delta_0 . \end{cases}$$

Hence we see

$$\delta(x) = \delta_0 \exp\left\{\int_a^x (z_\alpha(y) + z_\beta(y))dy\right\}, \tag{1·7}$$

as long as both $\varphi_\alpha(x, \lambda)$ and $\varphi_\beta(x, \lambda)$ do not vanish. Since $\delta_0 > 0$, we have by (1·7)

$$z_\alpha(x) > z_\beta(x). \tag{1·8}$$

Suppose $\tau_1(\alpha) > \tau_1(\beta)$. Then combining (2) of Lemma 1.4 and (1·8), we have

$$+\infty > z_\alpha(\tau_1(\beta) - 0) \geqq z_\beta(\tau_1(\beta) - 0) = +\infty ,$$

which is a contradiction. Consequently, noting φ_α and φ_β have no common zeros, we see $\tau_1(\alpha) < \tau_1(\beta)$. This proves (1).

Let us fix $\alpha \in [0, \pi)$. Suppose $\tau_n(\alpha) < \tau_n(\beta)$ for some $\beta \in (0, \pi)$. Because $\varphi_\alpha(x, \lambda)$ may have as many zeros as we need by extending the measures dQ and dM appropriately, this assumption becomes no restriction. From (1), we have $\tau_n(\alpha + 0) \geqq \tau_n(\alpha)$. Since $\varphi_\alpha(x, \lambda)$ is continuous in (x, α), we have $\varphi_\alpha(\tau_n(\alpha+0), \lambda)=0$. Suppose $\tau_n(\alpha+0)>\tau_n(\alpha)$. Then there exists only one zero of $\varphi_\beta(x, \lambda)$ in $(\tau_n(\alpha), \tau_n(\alpha+0))$, which is equal to $\tau_n(\beta)$. This contradicts the fact $\tau_n(\alpha+0)<\tau_n(\beta)$. Hence $\tau_n(\alpha+0)=\tau_n(\alpha)$. We may prove also $\tau_n(\alpha-0)=\tau_n(\alpha)$ for $\alpha>0$. This proves the continuity of $\tau_n(\alpha)$.

In the inequality

$$\tau_n(\alpha)<\tau_n(\beta)<\tau_{n+1}(\alpha),$$

for $0\leqq\alpha<\beta<\pi$, letting $\beta\to\pi$ and $\alpha=0$, we have

$$\tau_n(\alpha)<\tau_n(\pi-0)\leqq\tau_{n+1}(0).$$

Since $\tau_n(\pi-0)$ is a zero of $\varphi_0(x, \lambda)$, we have necessarily $\tau_n(\pi-0)=\tau_{n+1}(0)$. This completes the proof.

Definition 1. 6. *The pair $(L, D_{\alpha,\beta}[a, b])$ is said to have the E_0-property if F_M consists of finite elements and the equation*

$$(1\cdot 9)\qquad \begin{cases} d\varphi^+=\varphi dQ \quad in \quad [a, b] \\ \varphi(a)\cos\alpha+\varphi^+(a-0)\sin\alpha=0 \\ \varphi(b)\cos\beta+\varphi^+(b+0)\sin\beta=0 \\ \varphi(x)=0 \quad for\ every \quad x\in F_M \end{cases}$$

has a nontrivial solution.

Let $\Delta_{\alpha,\beta}(\lambda)$ be the Wronskian of $\varphi_\alpha(x, \lambda)$ and $\psi_\beta(x, \lambda)$, namely

$$\begin{aligned} \Delta_{\alpha,\beta}(\lambda) &= \psi_\beta^+(a-0, \lambda)\sin\alpha+\psi_\beta(a, \lambda)\cos\alpha \\ &= -\varphi_\alpha(b, \lambda)\cos\beta-\varphi_\alpha^+(b+0, \lambda)\sin\beta\,. \end{aligned}$$

Proposition 1. 7. *$(L, D_{\alpha,\beta}[a, b])$ has the E_0-property if and only if $\Delta_{\alpha,\beta}=0$. Moreover this is equivalent to that $\varphi_\alpha(x, \lambda)=0$, a.e. dM in $[a, b]$ and $\Delta_{\alpha,\beta}(\lambda)=0$ for some (or every) λ.*

Proof. In Lemma 1. 1, substituting $f(x)=\varphi_\alpha(x, \lambda)$, $g(x)=\varphi_\alpha(x, \mu)$, $c=a$ and $d=b$, we have

$$(1\cdot 10)\qquad (\lambda-\mu)\int_{[a,b]}\varphi_\alpha(x, \lambda)\varphi_\alpha(x, \mu)\,dM(x)=\varphi_\alpha(b, \lambda)\varphi_\alpha^+(b+0, \mu)$$

$$-\varphi_\alpha(b,\mu)\varphi_\alpha^+(b+0,\lambda).$$

Suppose $\Delta_{\alpha,\beta}(\lambda)=0$ identically. Then by the definition of $\Delta_{\alpha,\beta}$, we have

$$\varphi_\alpha(b,\lambda)\cos\beta+\varphi_\alpha^+(b+0,\lambda)\sin\beta=0. \tag{1·11}$$

Combining (1·10) and (1·11), we have

$$\int_{[a,b]}\varphi_\alpha(x,\lambda)\varphi_\alpha(x,\mu)\,dM(x)=0$$

for every λ, μ. Hence we see $\varphi_\alpha(x,\lambda)=0$, a.e. dM in $[a,b]$ for every λ. In particular, putting $\varphi(x)=\varphi_\alpha(x,0)$, we see that φ satisfies (1·9). Since φ is nontrivial, this is possible only when F_M has no accumulating points in $[a,b]$. Hence F_M should be a finite set.

Conversely assume (1·9) has a nontrivial solution φ. Then for any λ, we have

$$d\varphi^+=\varphi dQ-\lambda\varphi dM.$$

Noting $\varphi(x)$ and $\varphi_\alpha(x,\lambda)$ satisfy the same boundary condition at a, we see that there exists a constant C depending on λ such that $\varphi_\alpha(x,\lambda)=C\varphi(x)$ holds for any $x\in[a,b]$. Then we have

$$\begin{aligned}\Delta_{\alpha,\beta}(\lambda)&=C(\varphi(b)\cos\beta+\varphi^+(b+0)\sin\beta)\\&=0,\end{aligned}$$

which completes the proof.

For simplicity, we denote

$$\varphi_0(x)=\varphi_\alpha(x,0),\quad \psi_0(x)=\psi_\beta(x,0).$$

Define

$$V(x,y)=\frac{1}{\Delta_{\alpha,\beta}(0)}\{\varphi_0(x)\psi_0(y)-\varphi_0(y)\psi_0(x)\},$$

except for the case $\Delta_{\alpha,\beta}(0)\neq0$.

Lemma 1.8. *Suppose $\Delta_{\alpha,\beta}(0)\neq0$. Then $\varphi_\alpha(x,\lambda)$ and $\psi_\beta(x,\lambda)$ satisfy the integral equations*

$$(1\cdot 12)\qquad \begin{cases} \varphi_\alpha(x,\lambda)=\varphi_0(x)-\lambda\displaystyle\int_{[a,x)}V(x,y)\varphi_\alpha(y,\lambda)dM(y) \\ \psi_\beta(x,\lambda)=\psi_0(x)+\lambda\displaystyle\int_{(x,b]}V(x,y)\psi_\beta(x,\lambda)dM(y). \end{cases}$$

Moreover $\varphi_\alpha(x,\lambda)$ *and* $\psi_\beta(x,\lambda)$ *have the estimates*

$$(1\cdot 13)\qquad |\varphi_\alpha(x,\lambda)|\leqq C\cosh(c(x-a)M_+(x)|\lambda|)^{1/2}$$

$$|\psi_\beta(x,\lambda)|\leqq C\cosh(c(b-x)M_-(x)|\lambda|)^{1/2},$$

where $M_+(x)=\int_{[a,x)}dM(y)$, $M_-(x)=\int_{(x,b]}dM(y)$ *and* C *is a constant.*

Proof. (1·12) is easy to verify, so we omit the proof. Noting

$$\Delta_{\alpha,\beta}(0)\,V(x,y)=\int_x^y\{\varphi_0(x)\psi_0^-(u)-\psi_0(x)\varphi_0^+(u)\}\,du,$$

we have

$$|V(x,y)|\leqq C|x-y|$$

for every $x,y\in[a,b]$. Then referring to I. S. Kac-M. G. Krein [10], we may obtain the inequality (1·13).

Definition 1.9. *The pair* $(L,\ D_{\alpha,\beta}[a,b])$ *is said to have the* E_1*-property if* F_M *consists of finite elements and the equation*

$$(1\cdot 14)\qquad d\varphi^+=\varphi dQ \quad in\ [a,b]$$

$$\begin{cases} \{\varphi(a)\cos\alpha+\varphi^+(a-0)\sin\alpha\}\,\{\varphi(b)\cos\beta+\varphi^+(b+0)\sin\beta\}=0 \\ \{\varphi(a)\cos\alpha+\varphi^+(a-0)\sin\alpha\}+\{\varphi(b)\cos\beta+\varphi^+(b+0)\sin\beta\}\neq 0 \\ \varphi(x)=0,\quad for\quad x\in F_M \end{cases}$$

has a solution.

Proposition 1.10. *The entire function* $\Delta_{\alpha,\beta}$ *has no zeros if and only if* $(L,D_{\alpha,\beta}[a,b])$ *has the* E_1*-property. Moreover this is equivalent to that either* $\varphi_\alpha(x,\lambda)$ *or* $\psi_\beta(x,\lambda)$ *is equal to zero, a.e.* dM *in* $[a,b]$ *for some (or every)* λ *and* $\Delta_{\alpha,\beta}$ *is nontrivial.*

Proof. From (1·13), it follows that $\Delta_{\alpha,\beta}(\lambda)=-\varphi_\alpha(b,\lambda)\cos\beta-$

$\varphi_\alpha^+(b+0,\lambda)\sin\beta$ is an entire function of order at most 1/2. Hence according to the Hadamard factorization theorem, $\Delta_{\alpha,\beta}$ has no zeros if and only if $\Delta_{\alpha,\beta}$ is a nonzero constant δ. Without loss of generality, we may assume $\sin\beta\neq 0$. Then we have

$$\varphi_\alpha^+(b+0,\lambda)=-\delta\operatorname{cosec}\beta-\varphi_\alpha(b,\lambda)\cot\beta\,.$$

Hence from the identity $(1\cdot 10)$, we have

$$(\lambda-\mu)\int_{[a,b]}\varphi_\alpha(x,\lambda)\varphi_\alpha(x,\mu)\,dM(x)=\delta\operatorname{cosec}\beta[\varphi_\alpha(b,\mu)-\varphi_\alpha(b,\lambda)]$$

for every λ, μ. Putting $\mu=\bar{\lambda}$, we have

$$(\operatorname{Im}\lambda)\int_{[a,b]}|\varphi_\alpha(x,\lambda)|^2dM(x)=-\delta\operatorname{cosec}\beta\operatorname{Im}\varphi_\alpha(b,\lambda).$$

Hence we see that $\operatorname{Im}\varphi_\alpha(b,\lambda)\geqq 0$ or $\leqq 0$ in $\boldsymbol{C}_+$ according as $\delta\operatorname{cosec}\beta<0$ or >0. Since $\varphi_\alpha(b,\lambda)$ is an entire function, $\varphi_\alpha(b,\lambda)=p\lambda+q$ for some real numbers p, q (cf. B.Ja. Levin [11] p. 230). Consequently we have

$$\int_{[a,b]}\varphi_\alpha(x,\lambda)\varphi_\alpha(x,\mu)\,dM(x)=-p\delta\operatorname{cosec}\beta\,,$$

for every λ, μ. From this identity, it follows that for every λ

$$\varphi_\alpha(x,\lambda)=\varphi_\alpha(x,0)=\varphi_0(x)$$

for every $x\in F_M$ holds. It is obvious that by the same argument as above we have for ψ_β

$$\psi_\beta(x,\lambda)=\psi_\beta(x,0)=\psi_0(x)$$

for every $x\in F_M$. Hence from $(1\cdot 12)$ it follows that

$$(1\cdot 15)\quad \varphi_0(x)\int_{[a,x]}\psi_0(y)\varphi_0(y)\,dM(y)=\psi_0(x)\int_{[a,x]}\varphi_0(y)^2dM(y)$$

$$(1\cdot 16)\quad \varphi_0(x)\int_{[x,b]}\psi_0(y)^2dM(y)=\psi_0(x)\int_{[x,b]}\varphi_0(y)\psi_0(y)\,dM(y)$$

hold for every $x\in F_M$.

First we show that F_M is a finite set. Assume F_M is infinite. Then F_M has an accumulating point x_0. Without loss of generality we may assume that x_0 is a right accumulating point. Then taking the right derivatives in $(1\cdot 15)$ and $(1\cdot 16)$ we have

$$(1\cdot 17)\quad \varphi_0^+(x_0)\int_{[a,x_0]}\psi_0(y)\varphi_0(y)\,dM(y)=\psi_0^+(x_0)\int_{[a,x_0]}\varphi_0(y)^2dM(y)$$

$$(1\cdot 18)\quad \varphi_0^+(x_0)\int_{(x_0,b]}\psi_0(y)^2dM(y)=\psi_0^+(x_0)\int_{(x_0,b]}\varphi_0(y)\psi_0(y)\,dM(y).$$

Noting $\Delta_{\alpha,\beta}(0)=\delta\neq 0$, we see that $\varphi_0(x_0)$ and $\psi_0(x_0)$ do not vanish simultaneously. Hence

$$\int_{[a,x_0]}\varphi_0(y)^2dM(y)+\int_{(x_0,b]}\psi_0(y)^2dM(y)\neq 0\,.$$

We may assume for instance $\int_{[a,x_0]}\varphi_0(y)^2dM(y)\neq 0$. In this case, from (1·15) and (1·17), it follows that φ_0 and ψ_0 are linearly dependent, which contradicts $\Delta_{\alpha,\beta}(0)\neq 0$. In this way, we may prove that F_M is a finite set.

Let $F_M=\{x_1, x_2, \cdots x_n\}$, where $a\leqq x_1<x_2\cdots<x_n\leqq b$. Put $\varphi_0(x_k)=\alpha_k$, $\psi_0(x_k)=\beta_k$ and $dM(x_k)=m_k$. Then (1·15) and (1·16) turn to the equations

$$(1\cdot 19)\qquad \alpha_k\sum_{j=1}^{k}\alpha_j\beta_j m_j=\beta_k\sum_{j=1}^{k}\alpha_j^2 m_j$$

$$(1\cdot 20)\qquad \alpha_k\sum_{j=k}^{n}\beta_j^2 m_j=\beta_k\sum_{j=k}^{n}\alpha_j\beta_j m_j\,.$$

Supposing $\alpha_k=0$ for some $k\geqq 1$, we have from (1·19)

$$\beta_k\sum_{j=1}^{k}\alpha_j^2 m_j=0\,.$$

Noting $\varphi_0(x)$ and $\psi_0(x)$ do not vanish simultaneously, we have

$$\sum_{j=1}^{k}\alpha_j^2 m_j=0\,,$$

which implies $\alpha_j=0$ for every $1\leqq j\leqq k$. The parallel argument is possible also for β_j, hence we may assume that there exists a finite subset (possibly empty) S of $\{1, 2, \cdots n\}$ such that

$$\begin{cases}\alpha_j\neq 0,\ \beta_j\neq 0 & \text{for every}\quad j\in S\\ \alpha_j=0 & \text{for every}\quad j<n_1\\ \beta_j=0 & \text{for every}\quad j>n_2\,,\end{cases}$$

where $n_1=\min S$ and $n_2=\max S$. In this case, the equations (1·19) and (1·20) are valid even if we change n to n_2 and 1 to n_1. Then it is easy to prove inductively that

$$\beta_j/\alpha_j=\gamma \text{ for every } j\in S.$$

However putting $k=n_1-1$ in (1·20), we obtain

$$\alpha_{n_1-1}\sum_{j=n_1-1}^{n_2}\beta_j^2 m_j=\beta_{n_1-1}\sum_{j=n_1-1}^{n_2}\alpha_j\beta_j m_j .$$

Noting $\alpha_{n_1-1}=0$ and $\beta_{n_1-1}\neq 0$, we have

$$\sum_{j=n_1}^{n_2}\alpha_j\beta_j m_j=0 .$$

Substituting $\beta_j=\gamma\alpha_j$ into the above identity, we see

$$\gamma\sum_{j=n_1}^{n_2}\alpha_j^2 m_j=0 .$$

However this is possible only when S is empty, which implies $\alpha_j=0$, $\beta_j\neq 0$ for every $j=1, 2, \cdots n$. Hence in this case $\varphi(x)=\varphi_\alpha(x, 0)$ satisfies the equation (1·14).

Next suppose $\alpha_j\neq 0$ and $\beta_j\neq 0$ for every $j=1, 2, \cdots n$. Then as we have seen in the above discussions, from (1·19) and (1·20) there exists some constant γ such that

$$\beta_j=\gamma\alpha_j \text{ for every } j=1, 2, \cdots n .$$

Put $\varphi(x)=\psi_\beta(x, 0)-\gamma\varphi_\alpha(x, 0)$. Since ψ_β and φ_α are linearly independent, φ does not vanish identically. By the definition of φ, φ satisfies the equation

$$d\varphi^{+}=\varphi dQ-\lambda\varphi dM. \tag{1·21}$$

However we have assumed $\Delta_{\alpha,\beta}(\lambda)\neq 0$, hence $\varphi_\alpha(x, \lambda)$ and $\psi_\beta(x, \lambda)$ are linearly independent solutions of (1·21). Therefore there exist some constants $p(\lambda)$ and $q(\lambda)$ such that

$$\varphi(x)=p(\lambda)\psi_\beta(x, \lambda)-q(\lambda)\varphi_\alpha(x, \lambda)$$

holds for every $x\in[a, b]$. As has been seen,

$$0=\varphi(x_k)=p(\lambda)\beta_k-q(\lambda)\alpha_k=\{\gamma p(\lambda)-q(\lambda)\}\alpha_k$$

holds for every $x_k\in F_M$. Since $\alpha_k\neq 0$, we have

$$q(\lambda)=\gamma p(\lambda).$$

Hence $\varphi(x) = p(\lambda)\{\psi_\beta(x, \lambda) - \gamma\varphi_\alpha(x, \lambda)\}$. Taking $x_0 \in [a, b]$ such that $\varphi(x_0) \neq 0$, we have

$$p(\lambda)^{-1} = \varphi(x_0)^{-1}\{\psi_\beta(x_0, \lambda) - \gamma\varphi_\alpha(x_0, \lambda)\}.$$

From (1·13) we see that the right hand side is an entire function of order at most 1/2. Since $p(\lambda)$ has no zeros, $p(\lambda)$ should be a nonzero constant. Since $p(0) = 1$, we see $p(\lambda) = 1$ identically. Consequently we have

$$\begin{cases} \varphi(x) = \psi_\beta(x, \lambda) - \gamma\varphi_\alpha(x, \lambda) \\ \varphi^+(x) = \psi_\beta{}^+(x, \lambda) - \gamma\varphi_\alpha{}^+(x, \lambda) \end{cases}$$

for every $x \in [a, b]$ and λ. Putting $x = b + 0$, we have

$$\varphi_\alpha(b, \lambda) = C_1, \quad \varphi_\alpha{}^+(b+0, \lambda) = C_2,$$

where C_1 and C_2 are constants independent of λ. Choosing β' such that $C_1 \cos\beta' + C_2 \sin\beta' = 0$, we consider a boundary value problem

$$\begin{cases} f(a)\cos\alpha + f^+(a-0)\sin\alpha = 0 \\ f(b)\cos\beta' + f^+(b+0)\sin\beta' = 0\,. \end{cases}$$

Then by the definition of β', we have $\Delta_{\alpha,\beta'}(\lambda) = 0$ for every λ. Hence applying Proposition 1.7, we see $\varphi_\alpha(x, 0) = 0$, a.e. dM in $[a, b]$, which contradicts the assumption that $\varphi_\alpha(x, 0)$ has no zeros in F_M.

Conversely suppose $(L, D_{\alpha,\beta}[a, b])$ has the E_1-property. We may assume a solution of (1·14) satisfies the equation

$$\begin{cases} \varphi(a) = -\sin\alpha, \quad \varphi^+(a-0) = \cos\alpha \\ \varphi(b)\cos\beta + \varphi^+(b-0)\sin\beta \neq 0\,. \end{cases}$$

Since $d\varphi^+ = \varphi dQ - \lambda\varphi dM$ for every λ, we have

$$\varphi_\alpha(x, \lambda) = \varphi(x)$$

for every $x \in [a, b]$ and λ. Hence we have

$$\begin{aligned} \Delta_{\alpha,\beta}(\lambda) &= -\cos\beta\varphi_\alpha(b, \lambda) - \sin\beta\varphi_\alpha{}^+(b+0, \lambda) \\ &= -\cos\beta\varphi(b) - \sin\beta\varphi^+(b+0), \end{aligned}$$

which proves the proposition.

Theorem 1.11. *Suppose that $\Delta_{\alpha,\beta}(\lambda)$ is a nontrivial function pos-*

sessing at least one zero. Let $\{\lambda_n\}$ *be the set of zeros of* $\Delta_{\alpha,\beta}$ *and arrange them according to magnitude. Let* N_n *be the number of zeros of* $\varphi_\alpha(x, \lambda_n)$. *Then we have*

(1) *There exists the minimum* λ_0 *of* $\{\lambda_n\}$.

(2) $N_n = n + \#\{x \in [a, b];\ \varphi_\alpha(x, \lambda_0) = 0\}$.

Proof. From (1) of Proposition 1.3 it follows that N_n is a non-decreasing sequence. First we show that N_n is strictly increasing. Suppose $N_n = N_{n+1}$ for some n. Then from (1) of Proposition 1.3 it follows that

$$\tau_k(\alpha, \lambda_{n+1}) \leqq \tau_k(\alpha, \lambda_n), \quad k = 1, 2, \cdots N_n.$$

However the similar argument as in Proposition 1. 3 is possible also for ψ_β, hence we obtain

$$\tau_k(\alpha, \lambda_{n+1}) \geqq \tau_k(\alpha, \lambda_n), \quad k = 1, 2, \cdots N_n.$$

Consequently we have

$$\tau_k(\alpha, \lambda_{n+1}) = \tau_k(\alpha, \lambda_n), \quad k = 1, 2, \cdots N_n.$$

Applying (1) of Proposition 1. 3, we see that $(L, D_{\alpha,\beta}[a, b])$ has the E_0-property, which contradicts the assumption that $\Delta_{\alpha,\beta}$ is nontrivial. Hence N_n should be strictly increasing.

Let us prove the equality

$$N_{n+1} = N_n + 1. \tag{1·22}$$

First we consider the case $\beta = 0$. Then the boundary condition at b becomes $\varphi(b) = 0$. Let λ_n be any fixed zero of $\Delta_{\alpha,\beta}(\lambda)$. We consider the two cases below:

1°) For any $\lambda > \lambda_n$, the number of zeros of $\varphi_\alpha(x, \lambda)$ in $[a, b]$ remains unchanged. Then since we have shown that N_n is strictly increasing, we see that there exists no greater zeros than λ_n.

2°) For some $\lambda > \lambda_n$, the number of zeros of $\varphi_\alpha(x, \lambda)$ in $[a, b]$ increases. Let Q and M extend to the right hand side of b so that $\varphi_\alpha(x, \lambda)$ may have as many zeros in $[a, c]$ for some $c > b$ as we need. Let $\tau_k(\lambda)$ be the k-th zero of $\varphi_\alpha(x, \lambda)$ in $[a, c]$. Suppose $\varphi_\alpha(x, \lambda_n)$ has p zeros in $[a, b]$. Then we have obviously

$$\tau_{p+1}(\lambda_n) > \tau_p(\lambda_n) = b\,.$$

On the other hand, from the assumption

$$\tau_{p+1}(\lambda) \leqq \tau_p(\lambda_n) = b$$

for some $\lambda > \lambda_n$ follows. Since we have proved that $\tau_{p+1}(\lambda)$ is continuous (see Proposition 1.3), there exists $\mu > \lambda_n$ such that

$$\tau_{p+1}(\mu) = b\,.$$

Noting $\varphi_\alpha(b, \mu) = 0$, we see $\mu \geqq \lambda_{n+1}$. Remembering that N_n is increasing, we have $\mu = \lambda_{n+1}$, which proves (1·22) in case $\beta = 0$.

In case $\beta \not\equiv 0 \pmod \pi$, we may reduce the problem to the case $\beta = 0$, by extending dM and dQ to $[a, x_1]$ as follows:

$$dM = 0 \text{ in } (b, x_1]$$

$$dQ(x) = q_0 \delta_{\{x_0\}}(dx),$$

where $x_1 > x_0 > b$, $q_0 \in \boldsymbol{R}^1$ and $\delta_{\{x_0\}}(dx)$ is the Dirac measure at x_0. Here three constants should be chosen to satisfy the identity

$$\{1 + (x_1 - x_0)q_0\}\sin\beta = \{(x_1 - b) + (x_0 - x_1)q_0\}\cos\beta\,.$$

This is because for $x > x_0$, $\varphi_\alpha(x, \lambda)$ satisfies the equation

$$\varphi_\alpha(x, \lambda) = \varphi_\alpha(b, \lambda) + (x - b)\varphi_\alpha{}^+(b + 0, \lambda) + \int_{(b, x]} (x - y)\varphi_\alpha(y, \lambda)\, dQ(y)\,.$$

Consequently the equality (1·22) has been established. This completes the proof.

Corollary 1. 12. *Under the same condition as Theorem* 1.11, *we have*

$$\#\{n; \lambda_n \leqq \lambda\} = \varepsilon(\lambda) + \#\{x \in [a, b]; \varphi_\alpha(x, \lambda) = 0\} - \#\{x \in [a, b]; \varphi_\alpha(x, \lambda_0) = 0\},$$

where $|\varepsilon(\lambda)| \leqq 2$.

It is complicated to count the number of zeros of $\varphi_\alpha(x, \lambda_0)$ explicitly. However, for instance in case $dM(x) = dx$ or $dQ(x) \geqq 0$, that number becomes zero in (a_0, b_0).

§ 2. Ergodic Property of the Solution of a Ricatti Equation with a Random Coefficient

Let $\{Q(x); x \in [0, \infty)\}$ be a random process with stationary independent increments whose characteristic function $\psi(\xi)$ defined by

$$E(e^{i\xi Q(x)}) = e^{x\psi(\xi)}$$

may be expressed as follows:

$$\psi(\xi) = \int_{-\infty}^{\infty} (e^{i\xi u} - 1)\, n(du),$$

where $n(du)$ is a measure on $\mathbf{R}^1$ such that

$$\int_{-\infty}^{\infty} \min(1, |u|)\, n(dn) < \infty .$$

Under this condition, $Q(x)$ becomes a function of bounded variation in each finite interval with probability one. Hence we may define the operator L by

$$L\varphi = -\frac{d\frac{d\varphi}{dx} - \varphi dQ(x)}{dx} .$$

Let φ be the solution of the equation

$$\begin{cases} L\varphi = \lambda\varphi \\ \varphi(0) = -\sin\alpha, \quad \varphi^-(0) = \cos\alpha . \end{cases}$$

Put $\xi(x) = \varphi(x)$, $\eta(x) = \varphi^+(x)$ and $\zeta(x) = (\xi(x), \eta(x))$. Then $\zeta(x)$ satisfies the following stochastic differential equation

$$(2\cdot1) \qquad \begin{cases} d\xi(x) = \eta(x)\, dx \\ d\eta(x) = -\lambda\xi(x)\, dx + \xi(x)\, dQ(x). \end{cases}$$

Without loss of generality, we may assume that $Q(x)$ is continuous at the origin almost surely. Hence the initial value of $\zeta(x)$ is $\zeta(0) = (-\sin\alpha, \cos\alpha)$.

Now let G be a continuous map from $\mathbf{R}^2 \setminus \{0\}$ to $\mathbf{R}^1 \cup \infty$ defined by

$$G(\zeta) = -\frac{\eta}{\xi}, \quad for \quad \zeta = (\xi, \eta).$$

Since the equation $(2\cdot1)$ is linear, we have immediately the following

Lemma 2. 1. *Let* $\zeta(x, \zeta_i)$ *be the solution of* $(2\cdot1)$ *with an initial value* ζ_i $(i=1,2)$. *Suppose* $G(\zeta_1)=G(\zeta_2)$. *Then* $G(\zeta(x,\zeta_1))=G(\zeta(x,\zeta_2))$ *for any* $x\geqq0$.

Let us define a process $\{z(x)\}$ by

$$z(x)=G(\zeta(x)).$$

Since $\{\zeta(x)\}$ is a strong Markov process on $\boldsymbol{R}^2\setminus\{0\}$, from the above lemma it follows that $\{z(x)\}$ also becomes a strong Markov process on $\boldsymbol{R}^1\cup\infty$. The generator A of the process has the form

$$\begin{cases} Af(z)=(z^2+\lambda)\dfrac{df}{dz}+\displaystyle\int_{-\infty}^{\infty}\{f(z-u)-f(z)\}\,n(du) & for \quad z\neq\infty, \\ Af(\infty)=-\displaystyle\lim_{z\to\infty} z\{f(z)-f(\infty)\}. \end{cases}$$

Let us define a sequence of random times by

$$\begin{cases} \tau_1=\inf\{x>0;\, z(x)=\infty\}, \\ \tau_n=\inf\{x>\tau_{n-1};\, z(x)=\infty\}. \end{cases}$$

Since $z(x)$ is continuous at each τ_n by (2) of Lemma 1.4, it is easy to see that every τ_n becomes a Markov time.

We prepare some lemmas for our theorem.

Lemma 2. 2. *Let* n *be a measure on* $\boldsymbol{R}^1$ *such that* $\int_{-\infty}^{\infty}\min(1,|u|)\,n(du)<\infty$. *Then for every fixed* $\lambda>0$, *we have the estimate*

$$\int_{-\infty}^{\infty}n(du)\left|\int_z^{z-u}\frac{dy}{y^2+\lambda}\right|=O\left\{|z|^{-3/2}+\int_{|u|>|z|^{1/2}}n(du)\right\}$$

as $|z|\to\infty$.

Proof. First we consider the case $z\to+\infty$. We divide the above integral into two parts.

$$\int_{-\infty}^{0}n(du)\int_z^{z-u}\frac{dy}{y^2+\lambda}=\int_{-\infty}^{-z^{1/2}}n(du)\int_z^{z-u}\frac{dy}{y^2+\lambda}+\int_{-z^{1/2}}^{-1}n(du)\int_z^{z-u}\frac{dy}{y^2+\lambda}$$

$$+\int_{-1}^{0} n(du)\int_{z}^{z-u}\frac{dy}{y^2+\lambda}$$

$$\leqq \int_{-\infty}^{-z^{1/2}} n(du)\int_{z}^{\infty}\frac{dy}{y^2+\lambda}+\int_{-\infty}^{-1} n(du)\int_{z}^{z+z^{1/2}}\frac{dy}{y^2+\lambda}+\frac{1}{z^2+\lambda}\int_{-1}^{0}|u|n(du)$$

$$=O\left\{z^{-3/2}+\int_{-\infty}^{-z^{1/2}} n(du)\right\}.$$

$$\int_{0}^{\infty} n(du)\int_{z-u}^{z}\frac{dy}{y^2+\lambda}=\int_{0}^{1} n(du)\int_{z-u}^{z}\frac{dy}{y^2+\lambda}+\int_{1}^{z^{1/2}} n(du)\int_{z-u}^{z}\frac{dy}{y^2+\lambda}$$

$$+\int_{z^{1/2}}^{\infty} n(du)\int_{z-u}^{z}\frac{dy}{y^2+\lambda}$$

$$=O\left\{z^{-3/2}+\int_{z^{1/2}}^{\infty} n(du)\right\}.$$

As for the case $z\to-\infty$, we have only to put $\check{n}(du)=n(-du)$, then we have

$$\int_{-\infty}^{\infty} n(du)\int_{-z}^{-z-u}\frac{dy}{y^2+\lambda}=\int_{-\infty}^{\infty}\check{n}(du)\int_{z}^{z-u}\frac{dy}{y^2+\lambda}.$$

Consequently we may obtain the expected estimate.

Lemma 2. 3. *Let n be the same measure as in Lemma 2. 2. Let $C_b(\boldsymbol{R}^1)$ be the space of bounded continuous functions on $\boldsymbol{R}^1$ with the supremum norm. For $g\in C_b(\boldsymbol{R}^1)$, we define*

$$Ng(z)=\int_{-\infty}^{\infty} n(du)\int_{z}^{z-u}\frac{g(y)\,dy}{y^2+\lambda}.$$

Then for given $h\in C_b(\boldsymbol{R}^1)$, the following equation is uniquely solvable in $C_b(\boldsymbol{R}^1)$.

$$(2\cdot2)\qquad g(z)+Ng(z)=h(z).$$

Proof. First we prove that N is completely continuous in $C_b(\boldsymbol{R}^1)$. Let B be the unit ball in $C_b(\boldsymbol{R}^1)$. Applying Lemma 2.2, we have the estimate

$$(2\cdot3)\qquad |Ng(z)|\leqq\int_{-\infty}^{\infty} n(du)\left|\int_{z}^{z-u}\frac{dy}{y^2+\lambda}\right|=O\left\{|z|^{-3/2}+\int_{|u|\geqq|z|^{1/2}} n(du)\right\}$$

uniformly with respect to $g\in B$. Putting

$$\delta(z, z_0; u) = \sup_{g\in B}\left|\int_z^{z-u}\frac{g(y)}{y^2+\lambda}dy - \int_{z_0}^{z_0-u}\frac{g(y)}{y^2+\lambda}dy\right|$$

for $z, z_0 \in \boldsymbol{R}^1$, we have the estimates

$$\delta(z, z_0; u) \leqq 2\min\left\{|u|/\lambda, \int_{-\infty}^{\infty}\frac{dy}{y^2+\lambda}\right\},$$

and

$$\delta(z, z_0; u) \leqq (2/\lambda)|z-z_0|.$$

Therefore, owing to the dominated convergence theorem, we have

$$(2\cdot4) \qquad \sup_{g\in B}|Ng(z) - Ng(z_0)| \leqq \int_{-\infty}^{\infty}\delta(z, z_0; u)\, n(du) \to 0$$

as $z\to z_0$. From $(2\cdot3)$ and $(2\cdot4)$ we may conclude that the image $N(B)$ is relatively compact in $C_b(\boldsymbol{R}^1)$, hence N is completely continuous. In order to prove that the equation $(2\cdot2)$ is uniquely solvable, it is sufficient to show that $\mathrm{Ker}(I+N)=0$. For $g\in\mathrm{Ker}(I+N)$, put

$$f(z) = -\int_z^{\infty}\frac{g(y)}{y^2+\lambda}dy\,.$$

Then we see

$$(z^2+\lambda)\frac{df}{dz} + \int_{-\infty}^{\infty}\{f(z-u)-f(z)\}\, n(du) = g(z) + \int_{-\infty}^{\infty}n(du)\int_z^{z-u}\frac{g(y)}{y^2+\lambda}dy$$
$$= (I+N)\,g(z) = 0\,.$$

Here we make use of the Markov process $\{z(x)\}$. The Dynkin formula gives us the identity

$$E_z(f(z(T_k))) - f(z) = E_z\left(\int_0^{T_k}(Af)(z(x))\,dx\right)$$

for any $z\neq\infty$, where $T_k=\min(k, \tau_1)$. As has been verified in the above argument, $Af(z)=0$, for any $z\neq\infty$. Hence we have

$$f(z) = E_z(f(z(T_k)))\,.$$

Remembering (2) of Lemma 1.4, we see by letting k to $+\infty$

$$f(z) = E_z(f(z(\tau_1-0))) = E_z(f(+\infty)) = 0\,.$$

This implies $\mathrm{Ker}(I+N)=0$, which completes the proof.

Lemma 2.4. *For any* $z \in \boldsymbol{R}^1 \cup \infty$ and $\lambda > 0$, $E_z(\tau_1)$ *is finite.* $f(z) = E_z(\tau_1)$ *is a unique solution of the equation*

$$(2\cdot 5)\qquad \begin{cases} (z^2+\lambda)\dfrac{df}{dz} + \displaystyle\int_{-\infty}^{\infty} \{f(z-u) - f(z)\}\, n(du) = -1 \\ f(+\infty) = \lim\limits_{z\to+\infty} f(z) = 0\,, \quad |f(-\infty)| < \infty\,. \end{cases}$$

Proof. Let us consider the integral equation

$$g(z) + \int_{-\infty}^{\infty} n(du) \int_z^{z-u} \frac{g(y)}{y^2+\lambda} dy = -1\,.$$

Applying Lemma 2. 3, we have a unique solution $g(z)$ in $C_b(\boldsymbol{R}^1)$. Put

$$f(z) = -\int_z^{\infty} \frac{g(y)}{y^2+\lambda} dy\,.$$

Then f satisfies the equation

$$\begin{cases} (z^2+\lambda)\dfrac{df}{dz} + \displaystyle\int_{-\infty}^{\infty} \{f(z-u) - f(z)\}\, n(du) = -1 \\ f(+\infty) = 0\,, \quad |f(-\infty)| < \infty\,. \end{cases}$$

For $z \neq \infty$, the Dynkin formula leads us to

$$E_z(f(z(T_k))) - f(z) = E_z\left(\int_0^{T_k} (Af)(z(x))\, dx \right).$$

Since, by the definition of A, $Af(z) = -1$ for any $z \neq \infty$, we have

$$E_z(f(z(T_k))) - f(z) = -E_z(T_k)\,.$$

Letting k to $+\infty$ and observing $z(\tau_1 - 0) = +\infty$, we have

$$E_z(\tau_1) = f(z) - E_z(f(+\infty)) = f(z)$$

for any $z \neq \infty$. Applying Proposition 1.5, we have

$$E_\infty(\tau_1) = \lim_{z\to-\infty} E_z(\tau_1) = f(-\infty)\,.$$

This completes the proof.

Now we may prove the ergodic property of $\{z(x)\}$.

Theorem 2.5. *For any fixed* $\lambda > 0$, $\{z(x)\}$ *is ergodic, namely* *for any continuous function* φ *in* $\boldsymbol{R}^1\cup\infty$, *the equality*

$$\lim_{l\to\infty}\frac{1}{l}\int_0^l\varphi(z(x))\,dx=\frac{1}{E_\infty(\tau_1)}E_\infty\left(\int_0^{\tau_1}\varphi(z(x))\,dx\right) \tag{2·6}$$

holds almost surely for any P_z. *Moreover the process has an invariant measure* $T(z)\,dz$ *which may be determined as a unique solution of the equation*

$$\begin{cases}\dfrac{d}{dz}\{(z^2+\lambda)T(z)\}=\displaystyle\int_{-\infty}^{\infty}\{T(z+u)-T(z)\}n(du)\\ \displaystyle\int_{-\infty}^{\infty}T(z)\,dz=1\,.\end{cases} \tag{2·7}$$

Proof. Since we have observed in (2) of Lemma 1.4 that $z(\tau_n)=\infty$, from the strong Markov property of $\{z(x)\}$ it follows that the sequence of random variables

$$\int_{\tau_n}^{\tau_{n-1}}\varphi(z(x))\,dx\,,\quad n=1,2,\cdots$$

is independent and has the same distribution as

$$P_\infty\left\{\int_0^{\tau_1}\varphi(z(x))\,dx<a\right\}$$

with respect to any probability measure P_z. Since we have verified the finiteness of $E_\infty(\tau_1)$ in Lemma 2.4, the strong law of large numbers gives us the identity

$$\begin{aligned}\lim_{l\to\infty}\frac{1}{l}\int_0^l\varphi(z(x))\,dx&=\lim_{n\to\infty}\frac{1}{\tau_n}\int_0^{\tau_n}\varphi(z(x))\,dx\\&=\lim_{n\to\infty}\frac{n}{\tau_n}\frac{1}{n}\sum_{k=1}^{n}\int_{\tau_{k-1}}^{\tau_k}\varphi(z(x))\,dx\\&=\frac{1}{E_\infty(\tau_1)}E_\infty\left(\int_0^{\tau_1}\varphi(z(x))\,dx\right)\end{aligned}$$

almost surely for P_z, which proves the identity (2·6).

Let g be a unique solution of the equation

$$g(z)+\int_{-\infty}^{\infty}n(du)\int_z^{z-u}\frac{g(y)}{y^2+\lambda}dy=-\varphi(z),$$

whose existence was assured in Lemma 2. 3. Putting $h(z)=-\int_z^\infty \frac{g(y)}{y^2+\lambda}dy$, we have

$$(z^2+\lambda)\frac{dh}{dz}+\int_{-\infty}^{\infty}\{h(z-u)-h(z)\}n(du)=-\varphi(z).$$

As in the proof of Lemma 2. 4, the Dynkin formula gives the identity

$$h(z)=E_z\Big(\int_0^{\tau_1}\varphi(z(x))dx\Big).$$

Here we define a function $S(z)$ by

$$S(-z)=-\frac{d}{dz}E_z(\tau_1),$$

then by $(2\cdot 5)$ S satisfies the equation

$$(2\cdot 8)\qquad (z^2+\lambda)S(z)+\int_{-\infty}^{\infty}n(du)\int_z^{z+u}S(y)dy=1.$$

The following formal calculation may be easily justified through the approximation of the measure n by those with compact supports.

$$\begin{aligned}\int_{-\infty}^{\infty}\varphi(z)S(z)dz &= -\int_{-\infty}^{\infty}(z^2+\lambda)\frac{dh}{dz}S(z)dz \\ &\quad -\int_{-\infty}^{\infty}S(z)dz\int_{-\infty}^{\infty}\{h(z-u)-h(z)\}n(du) \\ &= h(-\infty)+\int_{-\infty}^{\infty}h(z)dz\Big\{\frac{d}{dz}(z^2+\lambda)S(z) \\ &\quad -\int_{-\infty}^{\infty}(S(z+u)-S(z))n(du)\Big\},\end{aligned}$$

where we have used the fact $h(+\infty)=0$ and $(z^2+\lambda)S(z)\to 1$ as $|z|\to\infty$. The identity $(2\cdot 8)$ implies that the second term vanishes identically, hence we have

$$\int_{-\infty}^{\infty}\varphi(z)S(z)dz=h(-\infty)=E_\infty\Big(\int_0^{\tau_1}\varphi(z(x))dx\Big).$$

Then it is easy to see that $T(z)=\frac{S(z)}{E_\infty(\tau_1)}$ satisfies $(2\cdot 7)$. Thus we obtain the theorem.

Here we define a spectral distribution function. Let $N(\lambda, I)$ be the number of eigenvalues not exceeding λ for a certain boundary value problem of L in the interval I.

Corollary 2. 6. *(Rice formula) For any $\lambda>0$, we have the identity*

$$\lim_{l\to\infty}\frac{1}{l}N(\lambda,[0,l])=\lim_{|z|\to\infty}z^2T(z), \tag{2·9}$$

almost surely for any P_z.

Proof. From Corollary 1.12, we have

$$N(\lambda,[0,l])=\#\{n;\tau_n\leqq l\}+\varepsilon(\lambda),$$

where $|\varepsilon(\lambda)|\leqq 2$. Hence we obtain by the law of large numbers

$$\lim_{l\to\infty}\frac{1}{l}N(\lambda,[0,l])=\lim_{n\to\infty}n/\tau_n$$

$$=1/E_\infty(\tau_1)\quad a.s.\ P_z\,.$$

Noting $T(z)=S(z)/E_\infty(\tau_1)$ and $z^2S(z)\to 1$ as $|z|\to\infty$, we have easily the identity (2·9).

Definition 2. 7. *We call the function $\lim_{l\to\infty}\frac{1}{l}N(\lambda,[0,l])$ the spectral distribution function of L and denote it by $N(\lambda)$.*

Corollary 2. 8. *(Frisch-Lloyd [3]) Suppose*

$$\int_{|u|>1}\log|u|n(du)<\infty\,.$$

Then the function

$$\varphi(s)=\int_{-\infty}^{\infty}e^{-isz}T(z)\,dz$$

satisfies the equations

$$\begin{cases}\dfrac{d^2\varphi}{ds^2}(s)=\left\{\lambda-\dfrac{\psi(s)}{is}\right\}\varphi(s) & in\quad \boldsymbol{R}^1\setminus\{0\}\\ \varphi(\pm\infty)=0\,,\quad \varphi(0)=1\,,\end{cases} \tag{2·10}$$

and

(2·11) $$N(\lambda) = -\frac{1}{\pi} \operatorname{Re} \frac{d\varphi}{ds}(0+),$$

where

$$\psi(s) = \int_{-\infty}^{\infty} (e^{isu}-1)\, n(du).$$

Proof. Since $T(z)$ is integrable, we may take the Fourier transform of the both sides of (2·7) in the Schwartz distribution sense to obtain

$$is\Big(-\frac{d^2}{ds^2}\varphi(s) + \lambda\varphi(s)\Big) = \psi(s)\varphi(s).$$

The identities $\varphi(\pm\infty)=0$ and $\varphi(0)=1$ follow from the Riemann-Lebesgue theorem and the definition of $T(z)$ respectively, which proves (2·10). On the other hand, the assumption for $n(du)$ implies

$$\int_{|z|\geqq 1} \frac{dz}{|z|} \int_{|u|\geqq |z|^{1/2}} n(du) < \infty .$$

Noting $|T(z)| \leqq \dfrac{C}{z^2+\lambda}$ for some constant C, we see by Lemma 2. 2

(2·12) $$\int_{-\infty}^{\infty} \frac{|z|\,dz}{z^2+\lambda} \int_{-\infty}^{\infty} n(du) \left| \int_z^{z+u} T(x)\,dx \right| < \infty .$$

By the definition of $N(\lambda)$ and $T(z)$, the equality

$$T(z) = \frac{N(\lambda)}{z^2+\lambda} + \frac{1}{z^2+\lambda} \int_{-\infty}^{\infty} n(du) \int_z^{z+u} T(x)\,dx$$

holds. Applying the Fourier transform to the both sides, we have

$$\varphi(s) = \frac{\pi}{\lambda^{1/2}} N(\lambda)\, e^{-\lambda^{1/2}|s|} + \int_{-\infty}^{\infty} \frac{e^{-isz}}{z^2+\lambda}\,dz \int_{-\infty}^{\infty} n(du) \int_z^{z+u} T(x)\,dx .$$

Observing (2·12), we may differentiate the both sides to obtain

$$\frac{d\varphi}{ds} = -\pi N(\lambda)\, e^{-\lambda^{1/2}s} - i \int_{-\infty}^{\infty} \frac{z e^{-isz}}{z^2+\lambda}\,dz \int_{-\infty}^{\infty} n(du) \int_z^{z+u} T(x)\,dx .$$

Since $T(x)$ is real, we have immediately

$$\operatorname{Re} \frac{d\varphi}{ds}(0+) = -\pi N(\lambda),$$

which completes the proof.

§ 3. Analytic Continuation of $\varphi(s)$ to the Upper Half Plane When the Support of n Is Contained in $(0, \infty)$

We assume that the support of n is contained in $(0, \infty)$ thereafter, whence the process $\{Q(x)\}$ is nondecreasing. Put

$$V(s) = \frac{1}{is} \int_0^\infty (e^{ius} - 1)\, n(du).$$

Then since $V(s)$ is holomorphic in the upper half plane $\boldsymbol{C}_+$, the Frisch-Lloyd formula $(2\cdot 10)$ may be studied on the pure imaginary axis. On this axis $V(s)$ is real valued and behaves like the function $s^{-\alpha}$, which makes it possible to apply the methods used in the scattering theory.

For this we need the following

Lemma 3. 1. *Let $V(z)$ be a holomorphic function in $\boldsymbol{C}_+$ satisfying*

(1) $\int_1^\infty \left|\frac{dV}{dz}(re^{i\theta})\right| dr < \infty$, *for each* $0 < \theta < \pi$, *and*

(2) $V(iy)$ *is real valued for every* $y > 0$,

(3) $\sup\{|V(z)|;\ |z| \geqq M,\ \theta \leqq \arg z \leqq \pi - \theta\} \to 0$ *as* $M \to \infty$, *for each* $0 < \theta < \pi/2$,

Then for each fixed $\lambda > 0$, *there exist unique linearly independent solutions* $g_\pm(z)$ *of the equation*

$$\frac{d^2}{dz^2} g(z) = \{\lambda - V(z)\}\, g(z)$$

such that

$$(3\cdot 1) \qquad \begin{cases} g_\pm(z) \sim \exp\left\{\pm \int_{z_0}^z (\lambda - V(s))^{1/2} ds\right\} \\ \dfrac{d}{dz} g_\pm(z) \sim \pm \lambda^{1/2} \exp\left\{\pm \int_{z_0}^z (\lambda - V(s))^{1/2} ds\right\} \end{cases}$$

as $z \to \infty$ *in each angle* $\{z; \theta \leqq \arg z \leqq \pi - \theta\}$, $0 < \theta < \pi/2$, *where we choose as a branch of* $(\lambda - V(s))^{1/2}$ *the one tending to* $\lambda^{1/2}$ *as* $s \to \infty$, *and* z_0 *is an arbitrary point in* $\boldsymbol{C}_+$ *such that* $\lambda - V(z_0) \neq 0$.

Proof. Since it is not difficult to extend the way of R. Bellman

[12] (ch. 2, Th. 8, p.50) to the above complex variable case, we omit the proof. Q.E.D.

Since in C_+ we have

$$V(z)=\int_0^\infty e^{iuz}du\int_u^\infty n(dt),$$

the inequality

$$\int_1^\infty \left|\frac{dV}{dz}(re^{i\theta})\right|dr \leqq \frac{1}{\sin\theta}\int_0^\infty \exp(-u\sin\theta)\,du\int_u^\infty n(dt)<\infty$$

follows from $\int_0^1 du\int_u^\infty n(dt)<\infty$. Hence $V(z)$ satisfies the all conditions of Lemma 3. 1. Put

$$U(x)=V(ix).$$

Theorem 3. 2. *Suppose*

$$\int_1^\infty \log u\, n(du)<\infty .$$

Then for each $\lambda>0$, *we have*

$$N(\lambda)=\frac{\lambda^{1/2}}{\pi|f_-(0,\lambda)|^2}, \tag{3·2}$$

where f_- *is a unique solution of the equation*

$$\begin{cases} \dfrac{d^2}{dx^2}f_-(x)=\{-\lambda+U(x)\}f_-(x) \\ f_-(x)\sim\exp\left\{-i\displaystyle\int_{x_0}^x(\lambda-U(y))^{1/2}dy\right\} \end{cases} \tag{3·3}$$

as $x\to+\infty$. x_0 *is any positive number such that* $\lambda-U(x_0)\neq 0$.

Proof. First we prove that the analytic continuation of φ, which will be denoted by the same notation, is linearly dependent on g_- of Lemma 3.1. Since $g_\pm$ are linearly independent solutions, there exist two constants a, b such that

$$\varphi(z)=ag_+(z)+bg_-(z).$$

Since $g_\pm$ are bounded on the upper pure imaginary axis, so is φ. Moreover, by the definition of φ, φ must be bounded also on $\boldsymbol{R}^1$. The estimate (3·1) implies that φ is a holomorphic function in $\boldsymbol{C}_+$ with exponential type at most $\lambda^{1/2}$. Hence according to the Phragmén-Lindelöf theorem, φ is bounded in $\boldsymbol{C}_+$. On the other hand, noting that for each fixed $0<\theta<\pi/2$, $g_+(re^{i\theta})$ is of exponential growth and $g_-(re^{i\theta})$ is of exponential decay, we may conclude that a is zero, whence we have

$$\varphi(z)=bg_-(z). \tag{3·4}$$

$\varphi(z)$ may be irregular at the origin. However the assumption $\int_1^\infty \log u\, n(du)<\infty$ implies

$$\int_{0+}|U(re^{i\theta})|dr<\infty$$

for each $0\leqq\theta\leqq\pi$, whence we may avoid that possibility. In particular, we have $\frac{d\varphi}{ds}(0+)=\frac{d\varphi}{ds}(i0+)$, which together with (2·11) gives us the identity

$$N(\lambda)=-\frac{1}{\pi}\operatorname{Re}\frac{d\varphi}{ds}(i0+).$$

Now put $f_-(x)=g_-(ix)$. Then from (3·4) and $\varphi(0)=1$, we have

$$\varphi(z)=\frac{f_-(-iz)}{f_-(0)},\quad \frac{d\varphi}{dz}=-i\frac{\frac{d}{dz}f_-(-iz)}{f_-(0)}.$$

By the definition, f_- is a solution of the equation

$$\begin{cases} \frac{d^2}{dx^2}f_-(x)=\{-\lambda+U(x)\}f_-(x) \\ f_-(x)\sim\exp\left\{-i\int_{x_0}^x(\lambda-U(y))^{1/2}dy\right\} \\ \frac{d}{dx}f_-(x)\sim -i\lambda^{1/2}\exp\left\{-i\int_{x_0}^x(\lambda-U(y))^{1/2}dy\right\}, \end{cases} \tag{3·5}$$

as $x\to+\infty$. Since $U(x)$ is real valued for each positive number x, the conjugate function $\bar{f}_-$ of f_- is also a solution of (3·5) with the conjugate asymptotic behaviour of f_-. Hence according to the invariance of the Wronskian, we have

$$\begin{aligned}\frac{d}{dx}f_-(0+)\overline{f_-(0)}-f_-(0)\overline{\frac{d}{dx}f_-(0+)}&=\frac{d}{dx}f_-(x)\overline{f_-(x)}-f_-(x)\overline{\frac{d}{dx}f_-(x)}\\&=-2i\lambda^{1/2}.\end{aligned}$$

Consequently we have

$$N(\lambda) = -\frac{1}{\pi}\,\mathrm{Im}\,\frac{\frac{d}{dx}f_-(0+)}{f_-(0)} = \frac{\lambda^{1/2}}{\pi|f_-(0)|^2},$$

which completes the proof.

§ 4. Asymptotic Behaviour of $N(\lambda)$ at the Origin

In the previous section, our problem was reduced to the analysis of a familiar equation

$$(4\cdot 1) \qquad \frac{d^2 f}{dx^2} = \{-\lambda + U(x)\} f(x).$$

Observing $U(x) = \int_0^\infty e^{-xu} \int_u^\infty n(dt)$, we see that U is strictly monotone decreasing and $U(+\infty) = 0$. Let $x(\lambda)$ denote a unique solution of $U(x) = \lambda$. As long as we are concerned with small $\lambda > 0$, there are no problems whether the solution exists or not for large λ. Define ζ by

$$\zeta(x) = \begin{cases} \int_{x(\lambda)}^{x} (\lambda - U(y))^{1/2} dy, & \text{for} \quad x \geqq x(\lambda), \\ e^{-3\pi i/2} \int_x^{x(\lambda)} (U(y) - \lambda)^{1/2} dy, & \text{for} \quad 0 < x < x(\lambda). \end{cases}$$

Put

$$\begin{cases} Q(x) = -\frac{1}{4}\frac{d^2U/dx^2}{(\lambda - U(x))^2} - \frac{5}{16}\frac{(dU/dx)^2}{(\lambda - U(x))^3} + \frac{5}{36}\frac{1}{\zeta(x)^2}, \\ \eta_0(\zeta) = (\pi\zeta)^{1/2} H_\nu^{(2)}(\zeta), \quad \eta_1(\zeta) = (\pi\zeta)^{1/2} H_\nu^{(1)}(\zeta), \end{cases}$$

where $\nu = 1/3$ and $H_\nu^{(i)}$ is the Hankel function. Define a Green function

$$G(\zeta, \theta) = \frac{1}{4i}\{\eta_1(\zeta)\eta_0(\theta) - \eta_1(\theta)\eta_0(\zeta)\}.$$

Then the Liuoville transformation of $(4\cdot 1)$ by the variable $\zeta(x)$ leads us to the integral equation

$$(4\cdot 2) \quad g(x) = \eta_0(\zeta(x)) - \int_x^\infty G(\zeta(x), \zeta(y)) g(y) Q(y) (\lambda - U(y))^{1/2} dy,$$

where $g(x) = (\lambda - U(x))^{1/4} f(x)$.

This procedure, which was found out by R. E. Langer, is stated in

E. C. Titchmarsh [13] (p. 356∼) and very effective in studing the behaviour of the solution of (4·1) with respect to (x, λ) simultaneously.

For the latter purpose, we give here the asymptotic behaviours of η_k

$$(4\cdot 3)\qquad \begin{cases} \eta_0(\zeta) \sim 2^{1/2} \exp\left(i\zeta - \dfrac{2\nu+1}{4}\pi i\right), \\ \eta_1(\zeta) \sim 2^{1/2} \exp\left(-i\zeta + \dfrac{2\nu+1}{4}\pi i\right) \end{cases}$$

as $\zeta \to \infty$. G and $\dfrac{\partial G}{\partial \zeta}$ also have the estimates

$$(4\cdot 4)\qquad \begin{cases} |G(\zeta, \theta)| \leqq C \\ \left|\dfrac{\partial G(\zeta, \theta)}{\partial \zeta}\right| \leqq C((1+\zeta)/\zeta)^{5/6}, \quad \text{for} \quad \zeta, \theta \geqq 0, \end{cases}$$

$$(4\cdot 5)\qquad |G(\zeta(x), \zeta(y))| \leqq C\exp(|\zeta(x)| - |\zeta(y)|), \text{ for } 0 < x \leqq y \leqq x(\lambda),$$

where C is a constant independent of both λ and the variable x.

Here we give some lemmas for later use.

Lemma 4. 1.

(1) $$\lim_{x\to\infty} xU(x) = -\lim_{x\to\infty} x^2 \frac{dU}{dx} = \int_0^\infty n(du).$$

(2) $\dfrac{dU}{dx}\Big/U(x) \geqq \dfrac{d^2U}{dx^2}\Big/\dfrac{dU}{dx} \geqq \cdots$, *and each one is negative monotone increasing function.*

(3) $$-\frac{d^{k+1}U}{dx^{k+1}}\Big/\frac{d^kU}{dx^k} \leqq \frac{k+1}{x}, \ \textit{for}\ k=0, 1, 2, \cdots.$$

Proof. From the identities

$$xU(x) = \int_0^\infty (1-e^{-xu})\, n(du) \quad \textit{and} \quad -x^2\frac{dU}{dx} = \int_0^\infty (1-e^{-xu} - xue^{-xu})\, n(du),$$

(1) follows immediately. (2) results from the Schwarz inequality. (3) may be proved by simple computations, whence we omit the proof.

For further developments we must impose a restriction on $U(x)$, namely

$$-x\frac{dU}{dx}/U(x)\geqq c>0 \tag{4·6}$$

holds for every sufficiently large x.

Lemma 4. 2. *Under the condition* (4·6), *we have the estimates for every* $a>1$ *and* $b<1$,

$$a^{-1/c}x(\lambda)\leqq x(a\lambda)\leqq a^{-1}x(\lambda), \tag{4·7}$$

$$a^{-1}U(x)\leqq U(ax)\leqq a^{-c}U(x), \tag{4·8}$$ *for every sufficiently small* λ *and large* x, *and*

$$|\zeta(x(a\lambda))|\to\infty \text{ and } |\zeta(x(b\lambda))|\to\infty \text{ as } \lambda\to 0. \tag{4·9}$$

Proof. From Lemma 4. 1, we have $-x\frac{dU}{dx}\leqq U(x)$. Putting $x=x(\lambda)$, we see

$$-x(\lambda)\frac{dU}{dx}(x(\lambda))\leqq U(x(\lambda))=\lambda\,.$$

Further noting $\frac{dx}{d\lambda}=\frac{dU}{dx}(x(\lambda))^{-1}$, we have $-\frac{dx}{d\lambda}/x(\lambda)\geqq 1/\lambda$. This implies the inequality

$$\log(x(\lambda)/x(a\lambda))=-\int_{\lambda}^{a\lambda}\frac{dx}{du}/x(u)\,du\geqq\int_{\lambda}^{a\lambda}\frac{du}{u}=\log a\,,$$

which proves the inequality (4·7). We may show (4·8) similarly.

Since we have for $y<x(\lambda)$

$$U(y)-\lambda=-\int_{y}^{x(\lambda)}\frac{dU}{du}du\geqq-\frac{dU}{dx}(x(\lambda))\,(x(\lambda)-y),$$

whence noting (4·7) we obtain

$$|\zeta(x(a\lambda))|=\int_{x(a\lambda)}^{x(\lambda)}(U(y)-\lambda)^{1/2}dy\geqq\frac{2}{3}\Big(-\frac{dU}{dx}(x(\lambda))\Big)^{1/2}(x(\lambda)-x(a\lambda))^{3/2}$$

$$\geqq\frac{2}{3}(1-a^{-1})^{3/2}\Big(-x(\lambda)^2\frac{dU}{dx}(x(\lambda))\Big)^{1/2}x(\lambda)^{1/2}\,.$$

Since $x(\lambda)\to\infty$ as $\lambda\to 0$ and $-x(\lambda)^2\frac{dU}{dx}(x(\lambda))$ is bounded from below by some positive number as $\lambda\to 0$, we have $|\zeta(x(a\lambda))|\to\infty$ as $\lambda\to 0$. The second one may be proved similarly.

Lemma 4. 3. *Under the condition* $(4\cdot 6)$ *we obtain the estimates*

$$\int_{x(a\lambda)}^{\infty}|Q(y)\,(\lambda-U(y))^{1/2}|dy=o(1), \tag{4·10}$$

$$\int_{x}^{x(a\lambda)}|Q(y)\,(\lambda-U(y))^{1/2}|dy=o(1) \tag{4·11}$$

as $\lambda\to 0$ *for every sufficiently small* $a>1$.

Proof. Put

$$\begin{cases} I_1=\displaystyle\int_{x(b\lambda)}^{\infty}|Q(y)\,(\lambda-U(y))^{1/2}|dy\,, & I_2=\displaystyle\int_{x(\lambda)}^{x(b\lambda)}|Q(y)\,(\lambda-U(y))^{1/2}|dy\,, \\ I_3=\displaystyle\int_{x(a\lambda)}^{x(\lambda)}|Q(y)\,(\lambda-U(y))^{1/2}|dy\,, & I_4=\displaystyle\int_{x}^{x(a\lambda)}|Q(y)\,(\lambda-U(y))^{1/2}|dy\,. \end{cases}$$

Then we have

$$\int_{x(a\lambda)}^{\infty}|Q(y)\,(\lambda-U(y))^{1/2}|dy=I_1+I_2+I_3\,.$$

Since we may treat I_1 and I_4 (resp. I_2 and I_3) analogously, we prove only the estimates for I_1 and I_2.

By the definition of Q, we have

$$I_1\leqq\frac{1}{4}\int_{x(b\lambda)}^{\infty}\frac{d^2U/dx^2}{(\lambda-U(x))^{3/2}}dx+\frac{5}{16}\int_{x(b\lambda)}^{\infty}\frac{(dU/dx)^2}{(\lambda-U(x))^{5/2}}dx$$
$$+\frac{5}{36}\int_{x(b\lambda)}^{\infty}\frac{(\lambda-U(x))^{1/2}}{\zeta(x)}dx\,,$$

which we denote by J_1, J_2 and J_3 respectively. Noting the inequality

$$\lambda-U(x)=(1/b)\,U(x(b\lambda))-U(x)\geqq(1/b-1)\,U(x),$$

in order to verify that J_1 and J_2 tend to zero as $\lambda\to 0$, we have only to show

$$\int_{1}^{\infty}\left(\frac{d^2U/dx^2}{U(x)^{3/2}}+\frac{(dU/dx)^2}{U(x)^{5/2}}\right)dx<\infty\,. \tag{4·12}$$

From Lemma 4. 1, it follows that

$$\int_1^\infty \frac{d^2U/dx^2}{U(x)^{3/2}}dx = \int_1^\infty \frac{d^2U/dx^2}{-dU/dx}\frac{1}{U(x)}\frac{-dU/dx}{U(x)^{1/2}}dx$$

$$\leqq \int_1^\infty \frac{2}{xU(x)}\frac{-dU/dx}{U(x)^{1/2}}dx \leqq CU(1)^{1/2} < \infty ,$$

and

$$\int_1^\infty \frac{(dU/dx)^2}{U(x)^{5/2}}dx = \int_1^\infty \frac{-dU/dx}{U(x)}\frac{1}{U(x)}\frac{-dU/dx}{U(x)^{1/2}}dx$$

$$\leqq C\int_1^\infty \frac{-dU/dx}{U(x)^{1/2}}dx \leqq 2CU(1)^{1/2} < \infty .$$

On the other hand, the identity

$$J_3 = \frac{5}{36}\int_{x(b\lambda)}^\infty \frac{d\zeta/dx}{\zeta(x)}dx = \frac{5}{36}\frac{1}{\zeta(x(b\lambda))}$$

and the estimate (4·9) imply $J_3 \to 0$ as $\lambda \to 0$. This proves that $I_1 \to 0$ as $\lambda \to 0$.

Put

$$\delta(x) = \frac{2}{5}\frac{dU/dx}{(\lambda - U(x))^{3/2}}\int_{x(\lambda)}^x \frac{(\lambda - U(y))^{3/2}d^2U/dy^2}{(dU/dy)^2}dy .$$

Then $Q(x)$ may be expressed by $\delta(x)$ as follows:

$$Q(x) = \frac{5}{16}\frac{(dU/dx)^2}{(\lambda - U(x))^3}\frac{\delta(x)^2(3+2\delta(x))}{1+\delta(x)} .$$

First note the inequality

$$|\delta(x)| \leqq \frac{2}{5}\frac{-dU/dx}{(\lambda - U(x))^{3/2}}(\lambda - U(x))^{3/2}\int_{x(\lambda)}^x \frac{d^2U/dy^2}{(dU/dy)^2}dy$$

$$= \frac{2}{5}\frac{1}{-\frac{dU}{dx}(x(\lambda))}\left\{\frac{dU}{dx}(x) - \frac{dU}{dx}(x(\lambda))\right\}$$

$$= \frac{2}{5}\frac{1}{-\frac{dU}{dx}(x(\lambda))}\int_{x(\lambda)}^x \frac{d^2U}{dy^2}dy \leqq \frac{2}{5}\frac{d^2U/dx^2}{-dU/dx}(x(\lambda))(x - x(\lambda))$$

$$\leqq \frac{4}{5}\frac{x - x(\lambda)}{x(\lambda)} . \tag{4·13}$$

This together with (4·7) gives us the inequality for $x(\lambda) \leqq x \leqq x(b\lambda)$

$$|\delta(x)|\leqq\frac{4}{5}(b^{-1/c}-1).$$

Choose $b<1$ so that $|\delta(x)|\leqq\frac{1}{2}$ may hold for every $x(\lambda)\leqq x\leqq x(b\lambda)$. Then the main term of Q becomes

$$Q_0(x)=\frac{25}{16}\frac{(du/dx)^2}{(\lambda-U(x))^3}\delta(x)^2.$$

Noting for $x\geqq x(\lambda)$

$$\lambda-U(x)=-\int_{x(\lambda)}^{x}\frac{dU}{dy}dy\geqq-\frac{dU}{dx}(x-x(\lambda))$$

and applying (4·13), we have

$$\begin{aligned}
&\int_{x(\lambda)}^{x(b\lambda)}Q_0(x)\,(\lambda-U(x))^{1/2}dx\\
&\leqq\frac{1}{x(\lambda)^2}\int_{x(\lambda)}^{x(b\lambda)}\frac{(dU/dx)^2}{(\lambda-U(x))^3}(\lambda-U(x))^{1/2}(x-x(\lambda))^2dx\\
&\leqq\frac{1}{x(\lambda)^2}\int_{x(\lambda)}^{x(b\lambda)}\left(-\frac{dU}{dx}(x-x(\lambda))\right)^{-1/2}dx\\
&\leqq x(\lambda)^{-2}\left(-\frac{dU}{dx}(x(b\lambda))\right)^{-1/2}\int_{x(\lambda)}^{x(b\lambda)}(x-x(\lambda))^{-1/2}dx\\
&=2x(\lambda)^{-2}\left(-\frac{dU}{dx}(x(b\lambda))\right)^{-1/2}(x(b\lambda))-x(\lambda))^{1/2}\\
&\leqq C\left(-x(b\lambda)^2\frac{dU}{dx}(x(b\lambda))\right)^{-1/2}x(b\lambda)^{-1/2}.
\end{aligned}$$

Applying Lemma 4.1, we see that the last term tends to zero as $\lambda\to0$, which implies $I_2=o(1)$ as $\lambda\to0$. This completes the proof.

Here we give the estimates of the solution of the equation (4·2).

Lemma 4.4. *(The estimate for $x\geqq x(\lambda)$.) Let $g(x)$ be a solution of* (4·2). *Then we have for $x\geqq x(\lambda)$*

(1) $$|g(x)|\leqq Ce^{CA(x)}$$

(2) $$|g(x)-\eta_0(\zeta(x))|\leqq C(e^{CA(x)}-1),$$

where $A(x)=\int_x^\infty|Q(y)\,(\lambda-U(y))^{1/2}|\,dy$ and C is a constant independent *of* λ, x.

Proof. Since we have the estimates (4·3) and (4·4), the expected estimates may be obtained by ordinary calculations, so we omit the proof.

Put

$$B(x)=\int_x^{x(\lambda)}|Q(y)\,(\lambda-U(y))^{1/2}|dy$$

and let $h(x)$ be a unique solution of the integral equation

$$h(x)=h_0(x)-\int_x^{\infty}G_0(x,y)Q_0(y)h(y)\,(-U(y))^{1/2}dy\,, \tag{4·14}$$

where

$$\begin{cases} Q_0(x)=-\dfrac{1}{4}\,\dfrac{d^2U/dx^2}{U(x)^2}+\dfrac{5}{16}\,\dfrac{(dU/dx)^2}{U(x)^3}\,,\\ h_0(x)=2^{1/2}\exp\Big(\dfrac{2\nu+1}{4}\pi i-\displaystyle\int_0^x(U(y))^{1/2}dy\Big),\\ G_0(x,y)=\dfrac{1}{2i}\Big\{\exp\Big(-\displaystyle\int_x^y(U(u))^{1/2}du\Big)-\exp\Big(\displaystyle\int_x^y(U(u))^{1/2}du\Big)\Big\},\ \textit{for}\ y\geqq x\,. \end{cases}$$

Lemma 4.5. *(The estimate for* $0<x\leqq x(\lambda)$*.) Under the condition* (4·6), $g(x)$ *has the estimate*

$$|g(x)|\leqq C\exp(|\zeta(x)|+CB(x)) \tag{4·15}$$

for every $0<x\leqq x(\lambda)$ *and* $0<\lambda\leqq 1$, *moreover* $h(x,\lambda)=e^{-|\zeta(0)|}g(x)$ *tends to* $h(x)$ *as* $\lambda\to 0$.

Proof. Let

$$\begin{cases} \alpha=\dfrac{1}{4i}\displaystyle\int_{x(\lambda)}^{\infty}\eta_1(\zeta(y))\,g(y)\,Q(y)\,(\lambda-U(y))^{1/2}dy\,,\\ \beta=-\dfrac{1}{4i}\displaystyle\int_{x(\lambda)}^{\infty}\eta_0(\zeta(y))\,g(y)\,Q(y)\,(\lambda-U(y))^{1/2}dy\,. \end{cases}$$

Then by the definition of g we have for $0<x\leqq x(\lambda)$

$$g(x)=(1+\alpha)\eta_0(\zeta(x))+\beta\eta_1(\zeta(x))$$
$$-\int_x^{x(\lambda)}G(\zeta(x),\zeta(y))Q(y)g(y)(\lambda-U(y))^{1/2}dy.$$

Put

$$\xi_0(\zeta)=(1+\alpha)\eta_0(\zeta)+\beta\eta_1(\zeta).$$

Then g satisfies the equation

$$g(x)=\xi_0(\zeta(x))-\int_x^{x(\lambda)}G(\zeta(x),\zeta(y))Q(y)g(y)(\lambda-U(y))^{1/2}dy. \tag{4·16}$$

By Lemma 4. 2, we have the estimate

$$|\alpha|+|\beta|\leqq C(e^{CA(x(\lambda))}-1). \tag{4·17}$$

Lemma 4. 3 implies $A(x(\lambda))=o(1)$ as $\lambda\to 0$, whence by (4·17) we obtain

$$\alpha=o(1),\ \beta=o(1)\ \text{ as } \lambda\to 0. \tag{4·18}$$

This together with (4·3) gives the estimate

$$|\xi_0(\zeta(x))|\leqq Ce^{|\zeta(x)|} \tag{4·19}$$

for $0<\lambda\leqq 1$ and $0<x\leqq x(\lambda)$. Put

$$\begin{cases} g_0(x)=\xi_0(\zeta(x)), \\ g_n(x)=-\int_x^{x(\lambda)}G(\zeta(x),\ \zeta(y))Q(y)g_{n-1}(y)(\lambda-U(y))^{1/2}dy. \end{cases}$$

Then using (4·5) and (4·19), we obtain

$$|g_n(x)|\leqq C\frac{(CB(x))^n}{n!}e^{|\zeta(x)|}, \tag{4·20}$$

which gives the estimate (4·15).

Now we proceed to the proof of the latter half of the lemma. Put

$$\begin{cases} k_n(x)=e^{-|\zeta(0)|}g_n(x), \\ h_n(x)=-\int_x^{\infty}G_0(x,y)h_{n-1}(y)Q_0(y)(-U(y))^{1/2}dy. \end{cases}$$

We must show that each k_n tends to h_n as $\lambda\to 0$. First we see by (4·18) and (4·3) $k_0(x)\to h_0(x)$ as $\lambda\to 0$. Assume $k_n\to h_n$ as $\lambda\to 0$. By the definition of k_n, we have

$$k_{n+1}(x)=I_n+J_n\,,$$

where

$$\begin{cases} I_n=-\displaystyle\int_{x(a\lambda)}^{x(\lambda)} G(\zeta(x),\ \zeta(y))k_n(y)Q(y)\,(\lambda-U(y))^{1/2}dy\,, \\ J_n=-\displaystyle\int_{x}^{x(a\lambda)} G(\zeta(x),\ \zeta(y))k_n(y)Q(y)\,(\lambda-U(y))^{1/2}dy\,. \end{cases}$$

From (4·20) we have the inequality

$$|I_n|\leqq\frac{(CB(x(a\lambda)))^{n+1}}{(n+1)!}\exp\Big(\int_0^x(U(y)-\lambda)^{1/2}dy\Big).$$

which together with Lemma 4. 3 gives the estimate

$$(4\cdot21)\qquad I_n=o(1)\ \text{ as }\lambda\to0\,.$$

Here remember the definition of Q:

$$Q(x)=-\frac{1}{4}\frac{d^2U/dx^2}{(\lambda-U(x))^2}-\frac{5}{16}\frac{(dU/dx)^2}{(\lambda-U(x))^3}+\frac{5}{36}\frac{1}{\zeta(x)^2}\,.$$

In the estimate for J_n, the last term is negligible. For we have

$$\int_x^{x(a\lambda)}|G(\zeta(x),\ \zeta(y))k_n(y)\,(\lambda-U(y))^{1/2}\zeta(y)^{-2}|dy$$

$$\leqq\frac{C}{n!}\exp\Big(\int_0^x(U(y)-\lambda)^{1/2}dy\Big)\int_x^{x(a\lambda)}(CB(y))^n|\zeta(y)|^{-2}(-d|\zeta(y)|)$$

$$\leqq C\frac{(CB(x))^n}{n!}|\zeta(x(a\lambda))|^{-1}\exp\Big(\int_0^x(U(y)-\lambda)^{1/2}dy\Big)$$

as $\lambda\to0$ from Lemmas 4. 2 and 4. 3. Noting the inequality for $0<y\leqq x(a\lambda)$

$$U(y)-\lambda=U(y)-a^{-1}U(x(a\lambda))\geqq U(y)\,(1-a^{-1}),$$

we have

$$(U(y))^{1/2}|Q(y)-\frac{5}{36}\zeta(y)^{-2}|\leqq C\Big(\frac{d^2U/dy^2}{U(y)^2}+\frac{(dU/dy)^2}{U(y)^3}\Big)(U(y))^{1/2}\,.$$

(4·12) implies that the right hand side is integrable in each (c,∞), whence applying the dominated convergence theorem and noting (4·21) we see that $k_{n+1}(x)$ converges to $h_{n+1}(x)$ as $\lambda\to0$ for each $x>0$. Consequently we see $k_n\to h_n$ as $\lambda\to0$ for each n. On the other hand from Lemma 4.3 and (4·20) we have the estimate

$$|k_n(x)| \leqq \frac{C^{n+1}}{n!} \exp\Big(-\int_0^x (U(y)-\lambda)^{1/2} dy\Big)$$

for every $x>0$ and $0<\lambda\leqq 1$. Then it is easy to see that $h(x,\lambda)=\sum_{n=0}^{\infty} k_n(x)$ tends to $h(x)=\sum_{n=0}^{\infty} h_n(x)$, which proves the lemma.

Define $f(x,\lambda)$ by

$$f(x,\lambda)=\frac{g(x,\lambda)}{2^{1/2}(\lambda-U(x))^{1/4}}\exp\Big(-\frac{2\nu+1}{4}\pi i-|\zeta(0)|\Big),$$

where $g(x,\lambda)=g(x)$. Then as we stated before, $f(x,\lambda)$ satisfies the equation

$$\frac{d^2}{dx^2}f(x,\lambda)=(-\lambda+U(x))f(x,\lambda).$$

Lemma 4. 6. $f(0,0+)\neq 0$ *and* $f(x,0+)$ *satisfies the equation*

$$\frac{d^2}{dx^2}f(x,0+)=U(x)f(x,0+). \tag{4·22}$$

Proof. Fix $a>0$. Let $\varphi(x,\lambda)$, $\psi(x,\lambda)$ be the solutions of (4·1) satisfying the initial conditions

$$\begin{cases}\varphi(a,\lambda)=1, & \psi(a,\lambda)=0,\\ \dfrac{d}{dx}\varphi(a,\lambda)=0, & \dfrac{d}{dx}\psi(a,\lambda)=1.\end{cases}$$

Since from the assumption $\int^{+\infty}\log u\, n(du)<\infty$, $U(x)$ is integrable in the neighbourhood of the origin, φ and ψ are continuous with respect to the variable $(x,\lambda)\in[0,\infty)\times\boldsymbol{R}^1$. Note the identity

$$f(x,\lambda)=f(a,\lambda)\varphi(x,\lambda)+\frac{d}{dx}f(a,\lambda)\psi(x,\lambda) \tag{4·23}$$

for any $x\geqq 0$, $\lambda\in\boldsymbol{R}^1$. Lemma 4.5 implies the existence of $f(x,0+)$ for any $x>0$. Observing $\psi(x,0)\neq 0$, we see by (4·23) that $\frac{d}{dx}f(a,0+)$ also exists. Therefore it is evident from (4·23) that $f(x,0+)$ exists for any $x\geqq 0$ and has the form:

$$f(x,0+)=f(a,0+)\varphi(x,0)+\frac{d}{dx}f(a,0+)\psi(x,0).$$

This implies that $f(x, 0+)$ is a solution of the equation $\frac{d^2}{dx^2}f(x)=U(x)\cdot f(x)$. By (4·14) $f(x, 0+)$ has the estimate

$$(4\cdot 24)\qquad |f(x,0+)|\leqq CU(x)^{-1/4}\exp\left(-\int_0^x (U(y))^{1/2}dy\right).$$

Noting the inequality

$$U(x)^{1/4}\exp\left(\int_0^x (U(y))^{1/2}dy\right)\geqq U(x)^{1/4}\int_0^x (U(y))^{1/2}dy\geqq U(x)^{3/4}x\,,$$

we see by Lemma 4. 1 that the right hand side of (4·24) tends to zero as $x\to\infty$, whence we have

$$(4\cdot 25)\qquad f(x,0+)\to 0 \text{ as } x\to+\infty\,.$$

Suppose $f(0, 0+)=0$. Then $f(x, 0+)$ satisfies the equation

$$f(x,0+)=\frac{d}{dx}f(0,0+)x+\int_0^x (x-y)f(y,0+)U(y)dy\,.$$

Noting $U(y)>0$ and $\frac{d}{dx}f(0,0+)\neq 0$, we have $|f(x,0+)|\geqq\left|\frac{d}{dx}f(0,0+)\right|x$ for $x>0$, which contradicts (4·25). Thus we complete the proof.

With the help of these lemmas we obtain the main

Theorem 4. 7. *Suppose* $U(x)=\frac{1}{x}\int_0^\infty (1-e^{-xu})n(du)$ *satisfies*

$$\int_{0+} U(x)dx<\infty\,,\quad \frac{-xdU/dx}{U(x)}\geqq c>0$$

for every sufficiently large x. *Then the spectral distribution* $N(\lambda)$ *has the asymptotic behaviour*:

$$(4\cdot 26)\qquad N(\lambda)\sim\frac{1}{\pi|f(0)|^2}\exp\left(-2\int_0^{x(\lambda)}(U(y)-\lambda)^{1/2}dy\right)\ \ as\ \lambda\to 0\,,$$

where f *is a unique solution of the equation*

$$\frac{d^2}{dx^2}f(x)=U(x)f(x),\quad f(x)\sim U(x)^{-1/4}\exp\left(-\int_0^x (U(y))^{1/2}dy\right)$$

$$as\quad x\to+\infty\,,$$

and $x(\lambda)$ *is the inverse function of* $U(x)$.

Proof. All we have to do is to find the relation between $f(x,\lambda)$ and $f_-(x,\lambda)$ of Theorem 3.2. Noting $A(x)=o(1)$ as $x\to\infty$, we have by Lemma 4. 4,

$$f(x,\lambda)\sim\frac{\eta_0(\zeta(x))}{2^{1/2}(\lambda-U(x))^{1/4}}\exp\left(-\frac{2\nu+1}{4}\pi i-|\zeta(0)|\right)$$

$$\sim\lambda^{-1/4}\exp(-i\zeta(x)-|\zeta(0)|)$$

as $x\to\infty$. Therefore by the definition of $f_-(x,\lambda)$ we see

$$f_-(x,\lambda)=\lambda^{1/4}e^{|\zeta(0)|}f(x,\lambda).$$

Hence Theorem 3.2 together with Lemma 4.6 gives us

$$N(\lambda)=\frac{\lambda^{1/2}}{\pi|f_-(0,\lambda)|^2}=\frac{e^{-2|\zeta(0)|}}{\pi|f(0,\lambda)|^2}$$

$$\sim\frac{1}{\pi|f(0,0+)|^2}\exp\left(-2\int_0^{x(\lambda)}(U(y)-\lambda)^{1/2}dy\right)$$

as $\lambda\to0$. This complete the proof.

Corollary 4. 8. *Suppose* Q *is a stable process with index* α, *namely* $U(x)=nx^{-(1-\alpha)}(0<\alpha<1)$. *Then we have the asymptotic form*:

$$N(\lambda)\sim B_\alpha n^{\frac{1}{1+\alpha}}\exp(-C_\alpha n^{\frac{1}{1-\alpha}}\lambda^{-\frac{1}{1-\alpha}+\frac{1}{2}}),$$

where

$$B_\alpha=(1+\alpha)^{-\frac{1-\alpha}{1+\alpha}}\Gamma\left(\frac{1}{1+\alpha}\right)^{-2},\quad C_\alpha=\pi^{1/2}\Gamma\left(\frac{1}{1-\alpha}-\frac{1}{2}\right)\Gamma\left(\frac{1}{1-\alpha}\right)^{-1}.$$

Proof. Note $2\int_0^{x(\lambda)}(U(y)-\lambda)^{1/2}dy=C_\alpha n^{\frac{1}{1-\alpha}}\lambda^{-\frac{1+\alpha}{2(1-\alpha)}}$ and

$$\begin{cases} f(x)=2(\pi(1+\alpha))^{-1/2}x^{1/2}K_{\frac{1}{1+\alpha}}\left(\dfrac{2n^{1/2}}{1+\alpha}x^{(1+\alpha)/2}\right) \\ f(0)=\pi^{1/2}(1+\alpha)^{(1-\alpha)/2(1+\alpha)}n^{-1/2(1+\alpha)}\Gamma\left(\dfrac{1}{1+\alpha}\right), \end{cases}$$

where K_ν is the modified Bessel function, which proves the corollary.

The above system is a continuous analogue of the one considered by M. Fukushima [14] and the result gives us some suggestions to a discrete system.

Corollary 4.9. *Suppose*

$$\int_0^\infty n(du) + \int^{+\infty} \log u \, n(du) < \infty .$$

Then

$$N(\lambda) \sim \frac{1}{|f(0)|^2} \exp\left(-2 \int_0^{x(\lambda)} (U(y) - \lambda)^{1/2} dy\right)$$

as $\lambda \to 0$, *and further we have*

$$(4\cdot 27) \qquad 2 \int_0^{x(\lambda)} (U(y) - \lambda)^{1/2} dy = n\pi\lambda^{-1/2} + o(\lambda^{-1/2}),$$

where $n = \int_0^\infty n(du)$.

Proof. In this case the second condition of Theorem 4.7 is satisfied automatically in view of Lemma 4.1. The estimate (4·27) may be verified without difficulty.

Remark. When Q is a Poisson process, we may obtain a more explicit asymptotic form:

$$N(\lambda) \sim \frac{1}{|f(0)|^2} e^{-n\pi\lambda^{-1/2}} .$$

This is a completion of the result of T. P. Eggarter [6]. We remark that S. Nakao [9] obtained the estimate $\lambda^{1/2} \log N(\lambda) \to -n\pi$ by the different method from ours.

Although it is more desirable to consider this problem for general additive process, we have not succeeded yet. However we remark that M.Fukushima and S.Nakao [8] treated the white Gaussian noise potential and gave a precise formulation of the eigenvalue problem to obtain the explicit form of $N(\lambda)$.

§ 5. A Remark to the Spectral Distribution Function of a One Dimensional Hamiltonian with a Random Weight

In this section we treat an operator

$$L = -\frac{d}{dM}\frac{d}{dx},$$

where $\{M(x)\}$ is a increasing process of stationary independent increments. We study the spectral distribution function of L. However the method is similar to the one of § 4, we avoid going into details.

Let ψ be the exponent of the characteristic function of M, namely

$$E(e^{-\xi M(x)}) = e^{-x\psi(\xi)},$$

and suppose that ψ has the form:

$$\psi(\xi) = a\xi + \int_0^\infty (1-e^{-\xi u})\, n(du),$$

where $a \geqq 0$ and $\int_0^\infty \min(1, u)\, n(du) < \infty$. First we consider the case $a > 0$. Define an additive process M_0 with the exponent $\int_0^\infty (1-e^{-\xi u})\, n(du)$ by the equation

$$M(x) = ax + M_0(x).$$

Let $(\xi(x), \eta(x))$ be the solutions of the equation

$$\begin{cases} d\xi(x) = \eta(x)\, dx \\ d\eta(x) = -\lambda a \xi(x)\, dx - \lambda \xi(x)\, dM_0(x). \end{cases}$$

Then Corollary 1. 12 gives us the identity

$$N(\lambda, [0, l]) = \#\{x \in [0, l];\ \xi(x) = 0\} + \varepsilon(\lambda),$$

where $|\varepsilon(\lambda)| \leqq 2$. Regarding λ as λa and Q as $-\lambda M_0$ in (2·1) of § 2, we may study the spectral distribution of L in the same way as in the previous section if we perform the analytic continuation of the solution of the Frisch-Lloyd formula to the lower half plane. Namely we have

$$N(\lambda) = \frac{\lambda^{1/2}}{\pi |f(0, 1/\lambda)|^2} \tag{5·1}$$

where $f(x, \lambda)$ is a unique solution of the equation

$$(5\cdot2)\qquad \begin{cases} \dfrac{d^2}{dx^2}f(x,\lambda) = -\lambda U(x)f(x,\lambda), \\ f(x,\lambda)\sim U(x)^{-1/4}\exp\left(i\lambda^{1/2}\displaystyle\int_0^x (U(y))^{1/2}dy\right), \end{cases}$$

as $x\to\infty$, where $U(x)=a+\dfrac{1}{x}\displaystyle\int_0^\infty(1-e^{-xu})n(du)$.

Letting $a\to0$ in the equation (5·2), we see that the solutions $f(x,\lambda)$ converges to the solution of the equation (5·2) regarding a as 0. Hence it may be concluded that the process $\{z(x)\}$ is ergodic even in the case $a=0$. Summing up these arguments we obtain

Theorem 5. 1. *Suppose $U(x)$ satisfies*

$$(5\cdot3)\qquad \int_{0+} U(x)\,dx<\infty .$$

Then the spectral distribution $N(\lambda)$ may be expressed as (5·1) *by the solution of* (5·2).

Example 1. $a=0$, $U(x)=nx^{-(1-\alpha)}$ $(0<\alpha<1)$.

$$N(\lambda)=n^{1/1+\alpha}\Gamma\left(\frac{1}{1+\alpha}\right)^{-2}(1+\alpha)^{-(1-\alpha)/(1+\alpha)}\lambda^{\alpha/(1+\alpha)} .$$

Example 2. $a=0$, $U(x)=\dfrac{n}{x+c}(c>0)$, namely $dn(x)=cne^{-cx}dx$.

$$N(\lambda)=\frac{\lambda}{c\pi^2}|H_1^{(1)}(2(nc/\lambda)^{1/2})|^{-2},$$

where $H_1^{(1)}$ is the Hankel function of the first kind.

§ 6. A Remark to the Spectral Distribution of Equations Defined on the Whole Line R^1

In the previous section we have considered the case when the equations are defined on the half line. In this section we remark a relation between the spectral function on the half line and the one on the whole line.

Let L be one of the operators

$$-\frac{d\frac{d}{dx}-dQ}{dx}, \quad -\frac{d}{dM}\frac{d}{dx},$$

where Q is a function of bounded variation in each finite interval of $\boldsymbol{R}^1$ and M is a nondecreasing function on $\boldsymbol{R}^1$. Put

$$\begin{cases} N_+(\lambda) = \lim_{l\to\infty} \frac{1}{l} N(\lambda, [0, l]), \\ N_-(\lambda) = \lim_{l\to\infty} \frac{1}{l} N(\lambda, [-l, 0]), \\ N(\lambda) = \lim_{l\to\infty} \frac{1}{2l} N(\lambda, [-l, l]), \end{cases}$$

if they exist. Then we have the following

Theorem 6. 1. *Suppose $N_\pm(\lambda)$ exists for each λ. Then $N(\lambda)$ also exists and has the form:*

$$N(\lambda) = \frac{1}{2}(N_+(\lambda) + N_-(\lambda)).$$

Proof. Let $N_0(\lambda, [a, b])$ and $N_\infty(\lambda, [a, b])$ be the number of eigenvalues not exceeding λ for the boundary value problem

$$f(a) = f(b) = 0 \quad \text{and} \quad \frac{df}{dx}(a) = \frac{df}{dx}(b) = 0,$$

respectively. Then these functions satisfy the inequalities

$$\begin{cases} 0 \leqq N_\infty(\lambda, [a, b]) - N_0(\lambda, [a, b]) \leqq 2, \\ N_0(\lambda, [a, b]) \leqq N(\lambda, [a, b]) \leqq N_\infty(\lambda, [a, b]), \end{cases}$$

by the results of § 1. Further the mini-max principle gives the inequalities for $a < c < b$

$$\begin{cases} N_\infty(\lambda, [a, b]) \leqq N_\infty(\lambda, [a, c]) + N_\infty(\lambda, [c, b]), \\ N_0(\lambda, [a, b]) \geqq N_0(\lambda, [a, c]) + N_0(\lambda, [c, b]). \end{cases}$$

Then the theorem is easily obtained.

Owing to this theorem, we may apply the results of § 4∼5 to the spectral distribution of equations on the whole line.

References

[1] Lax, M. and Phillips, J. C., One-dimensional impurity bands, *Phys. Rev.*, **110** (1958), 41–49.

[2] Lifšic, I. M., Energy spectrum structure and quantum states of disordered condensed systems, *Soviet Phys. Uspekhi*, **7** (1965), 549–573.

[3] Frisch, H. L. and Lloyd, S. P., Electron levels in one-dimensional lattice, *Phys. Rev.*, **120** (1960), 1175–1189.

[4] Benderskii, M. M. and Pastur, L. A., On the spectrum of the one-dimensional Schrödinger equation with a random potential, *Mat. Sb.*, **82** (1970), 273–284.

[5] ———, Calculation of the average number of states in a model problem, *Soviet Phys. JETP*, **30** (1970), 158–162.

[6] Eggarter, T. P., Some exact results on electron energy levels in certain one-dimensional random potentials, *Phys. Rev.*, *B*, **5** (1972), 3863–3865.

[7] Pastur, L. A., Spectra of random self-adjoint operator, *Russian Math. Surveys*, **28** (1973), 1–69.

[8] Fukushima, M. and Nakao, S., On spectra of the Schrödinger operator with a white Gaussian noise potential, to appear in *Z. Wahrscheinlichkeitstheorie verw. Gebiete*.

[9] Nakao, S., On the spectral distribution of the Schrödinger operator with random potential, to appear.

[10] Kac, I. S. and Krein, M. G., On spectral functions of a string, *Amer. Math. Soc. Transl.* (2), **103** (1974), 19–102.

[11] Levin, B. Ja., Distribution of zeros of entire functions, *Transl. Math. Monographs* **5**, Amer. Math. Soc., Providence, R. I., 1964.

[12] Bellman, R., *Stability theory of differential equations*, Dover Publications, Inc. New York, 1953.

[13] Titchmarsh, E. C., *Eigenfunction expansions, part* 2, Oxford, 1962.

[14] Fukushima, M., On the spectral distribution of a disordered system and the range of a random walk, *Osaka J. Math.*, **11** (1974), 73–85.