

On the Cauchy Problem for Weakly Hyperbolic Systems

By

Hideo YAMAHARA*

§ 1. Introduction

In this paper we consider the \mathcal{E} -well-posedness for the Cauchy problem of the first order system:

$$(1.1) \quad M[u] = \frac{\partial}{\partial t} u - \sum_{j=1}^l A_j(x, t) \frac{\partial}{\partial x_j} u - B(x, t) u = f(x, t),$$

$$(x, t) \in \mathcal{Q} = \mathbf{R}_x^l \times [0, T],$$

$$u(x, t_0) = u_0(x), \quad 0 \leq t_0 < T,$$

where the coefficients $A_j(x, t)$ and $B(x, t)$ are (m, m) matrices whose elements belong to the class $\mathcal{B}(\mathcal{Q})$ (in the sense of L. Schwartz [8]).

As is well-known, for the \mathcal{E} -well-posedness of (1.1) the characteristic roots $\lambda_i(x, t; \xi)$, ($i=1, 2, \dots, m$) (the roots of $\det(\tau I - \sum_{j=1}^l A_j(x, t) \xi_j) = 0$) must be real at any point $(x, t; \xi) \in \mathcal{Q} \times \mathbf{R}_\xi^l$ (S. Mizohata [4], P. D. Lax [2]). Moreover, under the assumption that the roots $\lambda_i(x, t; \xi)$ have constant multiplicities, the matrix $A(x, t; \xi) = \sum_{j=1}^l A_j(x, t) \xi_j$ must be diagonalizable at any point $(x, t; \xi)$ for the problem (1.1) to be \mathcal{E} -well-posed for every lower order term $B(x, t)$ (K. Kajitani [1]).

Now we consider the case that $A(x, t; \xi)$ is not diagonalizable. To discuss such a case V. M. Petkov has used the method of asymptotic expansions ([9], [10]). Quite different from his, our approach to this problem is much due to the so-called energy estimates (see S. Mizohata [4], S. Mizohata and Y. Ohya [6], [7]).

The results of our present article can briefly be stated as follows.

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* Osaka Electro-Communication University, Neyagawa, Osaka, 572, Japan

Under the assumptions of constant multiplicity and of constant rank on the operator (see (2.1)~(2.3)), we have obtained a necessary and sufficient condition for the Cauchy problem (1.1) to be \mathcal{C} -well-posed. It takes the form (C.B) in terms of eigenvectors and corresponds to the so-called Levi condition for single operator case.

In the preceding paper ([11]), only the outline of our arguments are given. The purpose of this paper is to give more detailed proofs.

§ 2. Levi's Condition and an Energy Estimate

As indicated in § 1, throughout this paper we assume the followings:

(2.1) The multiplicities of the characteristic roots are constant and at most double, more precisely

$$\det(\tau I - A(x, t; \xi)) = \prod_{i=1}^s (\tau - \lambda_i(x, t; \xi))^2 \prod_{j=s+1}^{m-s} (\tau - \lambda_j(x, t; \xi)).$$

(2.2) The characteristic roots $\lambda_i(x, t; \xi)$ are real and distinct for $(x, t; \xi) \in \mathcal{Q} \times (R_\xi^l \setminus \{0\})$ and $i=1, 2, \dots, m-s$, moreover $\inf_{\substack{(x, t) \in \mathcal{Q}, |\xi|=1 \\ i \neq j}} |\lambda_i(x, t; \xi) - \lambda_j(x, t; \xi)| > 0$.

(2.3) For $i=1, 2, \dots, s$, $\text{rank}(\lambda_i(x, t; \xi) I - A(x, t; \xi)) = m-1$, independently of $(x, t; \xi) \in \mathcal{Q} \times (R_\xi^l \setminus \{0\})$.

From these assumptions, we have

Proposition 2.1. *Suppose (2.1)~(2.3), then there exists an (m, m) matrix $N(x, t; \xi)$ which satisfies the following (i)~(iv):*

(i) $N(x, t; \xi) A(x, t; \xi) = D(x, t; \xi) N(x, t; \xi)$,

where $D(x, t; \xi)$ is a Jordan canonical form, namely

$$D(x, t; \xi) = \left(\begin{array}{cc|ccc} \lambda_1(x, t; \xi) & a_1(x, t; \xi) & & & \\ 0 & \lambda_1(x, t; \xi) & & & \\ \hline & & \dots & & \\ & & & \begin{array}{cc} \lambda_s & a_s \\ 0 & \lambda_s \end{array} & \\ & & & & \lambda_{s+1} \\ & & & & \dots \\ & & & & \lambda_{m-s} \end{array} \right)$$

and $a_i(x, t; \xi)$ are not zero and are homogeneous of degree 1 in ξ for $(x, t; \xi) \in \Omega \times (\mathbb{R}_\xi^l \setminus \{0\})$, $(i=1, 2, \dots, s)$.

- (ii) $N(x, t; \xi)$ is homogeneous of degree 0 in ξ .
- (iii) $|\det N(x, t; \xi)| \geq \delta (>0)$ for $(x, t; \xi) \in \Omega \times (\mathbb{R}_\xi^l \setminus \{0\})$.
- (iv) $N(x, t; \xi)$ is smooth for $(x, t; \xi) \in \Omega \times (\mathbb{R}_\xi^n \setminus \{0\})$.

This follows from the fact that the generalized eigenspaces corresponding to the eigenvalues $\lambda_i(x, t; \xi)$, $(i=1, 2, \dots, s)$, are smooth, namely that we can choose the smooth bases of the generalized eigenspace. And the eigenspaces corresponding to the double eigenvalues are of dimension one (see, for example, the Proposition 6.4 in S. Mizohata [5]).

Now we consider the equation:

$$(2.4) \quad M[u] = \left\{ \frac{\partial}{\partial t} - i\mathcal{A}(x, t; D) - B(x, t) \right\} u(x, t) = f(x, t),$$

where $\sigma(\mathcal{A}(x, t; D)) = A(x, t; \xi)$.

Operate to this the pseudo-differential operator $\mathcal{N}(x, t; D)$ defined by $\sigma(\mathcal{N}(x, t; D)) = N(x, t; \xi)$, then we have

$$(2.4)' \quad \left\{ \frac{\partial}{\partial t} - i\mathcal{D}(x, t; D) - \mathcal{B}_1(x, t; D) \right\} (\mathcal{N}u) = \mathcal{C}(x, t; D)u + \mathcal{N}f,$$

where $\mathcal{D}(x, t; D)$ is a p.d.op., homogeneous of order 1 such that $\sigma(\mathcal{D}(x, t; D)) = D(x, t; \xi)$, $\mathcal{B}_1(x, t; D)$ is a p.d.op., homogeneous of order 0 and $\mathcal{C}(x, t; D)$ is of order -1 . Here we have written in short p.d.op. instead of the pseudo-differential operator.

$$\text{Put } \sigma(\mathcal{B}_1(x, t; D)) = B_1(x, t; \xi) = \begin{bmatrix} b_{1,1}^{(1)}(x, t; \xi) & \dots & b_{1,m}^{(1)}(x, t; \xi) \\ & \dots & \\ b_{m,1}^{(1)}(x, t; \xi) & \dots & b_{m,m}^{(1)}(x, t; \xi) \end{bmatrix}$$

and let us introduce the following condition:

- (C·A) All the symbols $b_{2i, 2i-1}^{(1)}(x, t; \xi)$ are identically zero for $(x, t; \xi) \in \Omega \times (\mathbb{R}_\xi^l \setminus \{0\})$, $(i=1, 2, \dots, s)$.

Next define a p.d.op. $\mathcal{J}(D)$ such that

$$\mathcal{J}(D) = \left(\begin{array}{c|c|c} \boxed{\begin{matrix} (1+A)^{-1} & 0 \\ 0 & 1 \end{matrix}} & & \\ & \dots & \\ & & \boxed{\begin{matrix} (1+A)^{-1} & 0 \\ 0 & 1 \end{matrix}} & \\ & & & \begin{matrix} 1 \\ \dots \\ 1 \end{matrix} \end{array} \right)$$

where A is a p.d.op. with the symbol $|\xi|$, and apply this to (2.4)', then we have

$$(2.5) \quad \left\{ \frac{\partial}{\partial t} - i\mathcal{D}_1(x, t; D) - \mathcal{B}_2(x, t; D) \right\} (\mathcal{J}\mathcal{N}u) = \mathcal{J}\mathcal{E}u + \mathcal{J}\mathcal{N}f,$$

where $\mathcal{D}_1(x, t; D)$ is a p.d.op., homogeneous of order 1 and $\mathcal{B}_2(x, t; D)$ is of order 0. And under the condition (C·A) the symbol of \mathcal{D}_1 has a following structure:

$$\sigma(\mathcal{D}_1) = D_1(x, t; \xi) = \left(\begin{array}{cc|cccc|c} \lambda_1 & 0 & 0 & 0 & \cdots & 0 & 0 & \\ 0 & \lambda_1 & * & 0 & \cdots & * & 0 & \\ \hline 0 & 0 & \lambda_2 & 0 & \cdots & 0 & 0 & \\ * & 0 & 0 & \lambda_2 & \cdots & * & 0 & 0 \\ \cdots & & & \ddots & & & & \\ 0 & 0 & 0 & 0 & \cdots & \lambda_s & 0 & \\ * & 0 & * & 0 & \cdots & 0 & \lambda_s & \\ \hline * & 0 & * & 0 & \cdots & * & 0 & \lambda_{s+1} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & 0 \\ * & 0 & * & 0 & \cdots & * & 0 & 0 & \lambda_{m-s} \end{array} \right)$$

From the representation of $D_1(x, t; \xi)$ we can easily see that $D_1(x, t; \xi)$ is diagonalizable. By $N_1(x, t; \xi)$ denote the diagonalizator of $D_1(x, t; \xi)$ and define a p.d.op. $\mathcal{N}_1(x, t; D)$ whose symbol is $N_1(x, t; \xi)$ ($|\det N_1(x, t; \xi)| \geq \delta_1 > 0$).

Here we introduce the norm (see [3]):

$$(2.6) \quad (L_k u, u) \equiv (\mathcal{N}_1 A^k \mathcal{J}\mathcal{N}u, \mathcal{N}_1 A^k \mathcal{J}\mathcal{N}u) + \beta_1 (\mathcal{J}\mathcal{N}u, \mathcal{J}\mathcal{N}u) + \beta_2 (u, u),$$

where k is a non-negative integer, β_1 and β_2 are sufficiently large constants and $(,)$ is the usual inner product on $L_2(\mathbf{R}_x^l)$.

First we have

Lemma 2.1. For $u \in \mathcal{D}_{L^2}^k$ ($k \geq 1$) there exist positive constants c_1 and c_2 such that

$$(2.7) \quad c_1 \|u\|_{k-1} \leq (L_k u, u)^{1/2} \leq c_2 \|u\|_k,$$

where $\|\cdot\|_k$ is the usual norm on $\mathcal{D}_{L^2}^k(\mathbf{R}_x^l)$.

This lemma will be proved if we apply the following lemma to the p.d.op.s $\mathcal{N}(x, t; D)$ and $\mathcal{N}_1(x, t; D)$.

Lemma 2.2. *(The Corollary of the Lemma 2.2 in S. Mizohata [3]). Let \mathcal{P} be an (m, m) matrix, whose elements ρ_{ij} are singular integral operators belonging to $C_{\beta}^{\infty}(\beta > 0)$: $\sigma(\rho_{ij}) \in C_{\beta}^{\infty}$. By $\sigma(\mathcal{P})$ we denote a matrix $(\sigma(\rho_{ij})(x; \xi))$. Suppose that*

$$(2.8) \quad |\sigma(\mathcal{P})\alpha| \geq \delta|\alpha|, \text{ for any } x \in \mathbf{R}_x^l, \text{ any } \xi \in \mathbf{R}_{\xi}^l, \text{ and for any } \alpha = {}^t(\alpha_1, \alpha_2, \dots, \alpha_m) \in \mathbf{C}^n$$

where $|\alpha| = \sqrt{|\alpha_1|^2 + \dots + |\alpha_m|^2}$. Then for any $u = {}^t(u_1, u_2, \dots, u_m) \in \mathcal{D}_{L^2}^k$, k being any non-negative integer, we have the following inequality:

$$(2.9) \quad \|\mathcal{P}A^k u\|^2 \geq \sigma \|A^k u\|^2 - \gamma_k \|u\|^2, \quad \sigma > 0.$$

Next, apply A^k to (2.5), then we have

$$(2.10) \quad \left\{ \frac{\partial}{\partial t} - i\mathcal{D}_1(x, t; D) \right\} (A^k \mathcal{J} \mathcal{N}u) \\ = i(A^k \mathcal{D}_1 - \mathcal{D}_1 A^k) \mathcal{J} \mathcal{N}u + A^k \mathcal{B}_2 \mathcal{J} \mathcal{N}u + A^k \mathcal{J} \mathcal{E}u + A^k \mathcal{J} \mathcal{N}f.$$

Moreover applying $\mathcal{N}_1(x, t; D)$ to (2.10) we have

$$(2.11) \quad \frac{\partial}{\partial t} (\mathcal{N}_1 A^k \mathcal{J} \mathcal{N}u) = i\mathcal{D}_2 \mathcal{N}_1 A^k \mathcal{J} \mathcal{N}u + \mathcal{B}_3 A^k \mathcal{J} \mathcal{N}u \\ + i\mathcal{N}_1 (A^k \mathcal{D}_1 - \mathcal{D}_1 A^k) \mathcal{J} \mathcal{N}u + \mathcal{N}_1 A^k \mathcal{B}_2 \mathcal{J} \mathcal{N}u \\ + \mathcal{N}_1 A^k \mathcal{J} \mathcal{E}u + \mathcal{N}_1 A^k \mathcal{J} \mathcal{N}f,$$

where $\mathcal{B}_3 = \mathcal{B}_3(x, t; D)$ is a p.d.op. of order 0 and $\mathcal{D}_2 = \mathcal{D}_2(x, t; D)$ is a p.d.op., homogeneous of order 1 with the following symbol:

$$\sigma(\mathcal{D}_2) = D_2(x, t; \xi) = \left[\begin{array}{ccccccc} \lambda_1(x, t; \xi) & & & & & & \\ & \lambda_1(x, t; \xi) & & & & & \\ & & \ddots & & & & \\ & & & \lambda_s & & & \\ & & & & \lambda_s & & \\ & & & & & \lambda_{s+1} & \\ & & & & & & \ddots \\ & & & & & & & \lambda_{m-s} \end{array} \right].$$

From (2.5) and (2.11) we have

$$\begin{aligned} \frac{d}{dt}(L_k u, u) &\leq \text{const.} \{ \|\mathcal{N}_1 A^k \mathcal{G} \mathcal{N} u\|^2 + \|u\|_{k-1} \|\mathcal{N}_1 A^k \mathcal{G} \mathcal{N} u\| \\ &\quad + \beta_1 \|u\|_1 \|\mathcal{G} \mathcal{N} u\| + \beta_2 \|u\|_1 \|u\| \} + 2 \{ \|\mathcal{N}_1 A^k \mathcal{G} \mathcal{N} f\| \|\mathcal{N}_1 A^k \mathcal{G} \mathcal{N} u\| \\ &\quad + \beta_1 \|\mathcal{G} \mathcal{N} f\| \|\mathcal{G} \mathcal{N} u\| + \beta_2 \|f\| \|u\| \}. \end{aligned}$$

Hence from the lemma 2.1 we have

$$(2.12) \quad \frac{d}{dt} [(L_k u, u)^{1/2}] \leq \gamma_k (L_k u, u)^{1/2} + (L_k u, u)^{1/2},$$

where γ_k is a constant and $k \geq 2$. When $k=1$, instead of the norm $(L_k u, u)$ we use

$$(2.6)' \quad (\tilde{L}_1 u, u) \equiv (\mathcal{N}_1 A \mathcal{G} \mathcal{N} u, \mathcal{N}_1 A \mathcal{G} \mathcal{N} u) + \beta_1 (\mathcal{G} \mathcal{N} u, \mathcal{G} \mathcal{N} u) \\ + \beta_2 ((1+A)^{-1} u, (1+A)^{-1} u).$$

Then we can also obtain an inequality, similar to (2.12).

Integrate (2.12), then we have

$$(2.13) \quad (L_k u, u)^{1/2}(t) \leq e^{\gamma_k(t-t_0)} (L_k u, u)^{1/2}(t_0) + \int_{t_0}^t e^{\gamma_k(t-s)} (L_k f, f)^{1/2}(s) ds.$$

By virtue of the Lemma 2.1 and (2.13) we have

Theorem 2.1. *Assume the condition (C·A). Then we have the energy estimate:*

$$(2.14) \quad \|u(t)\|_{k-1} \leq c_k \|u(t_0)\|_k + \int_{t_0}^t \|f(s)\|_k ds,$$

for the solution $u(x, t)$ of (1.1) belonging to $\mathcal{E}_t^1(\mathcal{D}_{I^2}^k)$, where c_k is some constant and $k=1, 2, \dots$.

§ 3. Condition (C·A) and the Influence Domain

We will represent the condition (C·A) more explicitly. For this purpose let us calculate the symbol of $\mathcal{B}_1(x, t; D)$.

$$(3.1) \quad \sigma(B_1(x, t; D)) = \text{principal symbol of } \{i(\mathcal{N} \mathcal{A} - \mathcal{D} \mathcal{N}) + \mathcal{N}'_t + \mathcal{N} \mathcal{B}\} \mathcal{M} \\ = \left\{ \sum_{j=1}^l \frac{\partial N}{\partial \xi_j} \frac{\partial A}{\partial x_j} - \sum_{j=1}^l \frac{\partial D}{\partial \xi_j} \frac{\partial N}{\partial x_j} + \frac{\partial N}{\partial t} + NB \right\} M,$$

where $M = M(x, t; \xi)$ is the inverse matrix of $N(x, t; \xi)$ and $\mathcal{M} = \mathcal{M}$

$(x, t; D)$ is a p.d.op. with the symbol $M(x, t; \xi)$.

Now we denote by E the m -dimensional Euclidean space and by E^* the dual space of E , and we denote $\langle \xi, x \rangle = \xi_1 x_1 + \xi_2 x_2 + \dots + \xi_m x_m$, where $x = (x_1, x_2, \dots, x_m) \in E$ and $\xi = (\xi_1, \xi_2, \dots, \xi_m) \in E^*$.

Let $R_i(x, t; \xi) \in E$ (resp. $L_i(x, t; \xi) \in E^*$) be an eigenvector of $A(x, t; \xi)$ (resp. ${}^t A(x, t; \xi)$) corresponding to $\lambda_i(x, t; \xi)$, $(i=1, 2, \dots, s)$. Then N and M have the following forms:

$$N = {}^t(L_1^1, L_1, \dots, L_s^1, L_s, L_{s+1}, \dots, L_{m-s}),$$

$$M = (R_1, R_1^1, \dots, R_s, R_s^1, R_{s+1}, \dots, R_{m-s}),$$

where L_i^1 and R_i^1 are generalized eigenvectors $(i=1, 2, \dots, s)$. From the structures of D, N and M we can easily see

Proposition 3.1. The condition (C·A) is equivalent to the following condition:

$$(C \cdot B) \quad C_i(x, t; \xi) \equiv \left\langle L_i(x, t; \xi), \left(\frac{\partial}{\partial t} - \sum_{j=1}^l A_j \frac{\partial}{\partial x_j} - B \right) R_i(x, t; \xi) \right\rangle \\ + \sum_{j=1}^l \frac{\partial \lambda_i}{\partial x_j} \left\langle L_i(x, t; \xi), \frac{\partial}{\partial \xi_j} R_i(x, t; \xi) \right\rangle \equiv 0,$$

for $(x, t; \xi) \in \mathcal{Q} \times \mathbf{R}_\xi^l$, $(i=1, 2, \dots, s)$.

Remark. This condition is independent of the choice of eigenvectors.

In fact, take $\sigma_R(x, t; \xi)R(x, t; \xi)$ and $\sigma_L(x, t; \xi)L(x, t; \xi)$ in (C·B) instead of $R(x, t; \xi)$ and $L(x, t; \xi)$ respectively, where we omit the index i , and $\sigma_R(x, t; \xi)$ and $\sigma_L(x, t; \xi)$ are scalar functions. Then under the condition (C·B), the right hand side of (C·B) turns into

$$\sigma_L \left(\frac{\partial \sigma_R}{\partial t} - \sum_j \frac{\partial \lambda}{\partial x_j} \frac{\partial \sigma_R}{\partial \xi_j} \right) \langle L, R \rangle - \sigma_L \sum_j \frac{\partial \sigma_R}{\partial x_j} \langle L, A_j R \rangle.$$

On the other hand $\langle L, R \rangle \equiv 0$ because we can write $R = (\lambda - A)R^1$. And also $\langle L, A_j R \rangle \equiv 0$. In fact, from the identity:

$$(3 \cdot 2) \quad (\lambda(x, t; \xi) - \sum_j A_j(x, t) \xi_j) R(x, t; \xi) \equiv 0,$$

we have $\left(\frac{\partial \lambda}{\partial \xi_j} - A_j \right) R + (\lambda - A) \frac{\partial}{\partial \xi_j} R \equiv 0$. Taking the scalar product

with $L \in E^*$ we have $\langle L, A_j R \rangle = 0$.

Next, we take the following so-called "space-like" transformation:

$$(3.3) \quad \begin{cases} x_j' = x_j, & (1 \leq j \leq l), \\ t' = \varphi(x, t), \end{cases}$$

which satisfies $\varphi_i - \lambda_i(x, t; \varphi_x) \neq 0$, for $i = 1, 2, \dots, m - s$.

By (3.3), the equation (2.4) is transformed into

$$(3.4) \quad \left\{ \frac{\partial}{\partial t'} - (\varphi_i - A\varphi_x)^{-1} \sum A_j \frac{\partial}{\partial x_j'} - (\varphi_i - A\varphi_x)^{-1} B \right\} u(x, t) = f_1(x, t),$$

where $f_1(x, t) = (\varphi_i - A\varphi_x)^{-1} f(x, t)$ and $A\varphi_x = \sum_{j=1}^l A_j(x, t) \frac{\partial \varphi}{\partial x_j}$.

Let $\mu_i(x, t; \xi)$ be an eigenvalue of $(\varphi_i - A\varphi_x)^{-1} \sum A_j \xi_j$. At first we note

Lemma 3.1. (3.4) also satisfies (2.1) ~ (2.3).

Proof. Consider the identity:

$$(3.5) \quad \det(\tau - (\varphi_i - A\varphi_x)^{-1} A\xi) = \det(\varphi_i - A\varphi_x)^{-1} \det(\varphi_i \tau - A(\xi + \varphi_x \tau)) \\ = \det(\varphi_i - A\varphi_x)^{-1} \prod_i (\varphi_i \tau - \lambda_i(x, t; \xi + \varphi_x \tau))$$

Denote $\psi_i(\tau) = \varphi_i \tau - \lambda_i(x, t; \xi + \varphi_x \tau)$ and fix $(x, t; \xi)$. Then from the fact that (3.3) is space-like $\psi_i(\tau) = 0$ has at least one real root τ_i . Hence from (3.5) we have

$$(3.6) \quad \det(\varphi_i - A\varphi_x) \prod_i (\tau - \mu_i) = \prod_i (\tau - \tau_i) f_i(\tau),$$

where we have written $\psi_i(\tau) = (\tau - \tau_i) f_i(\tau)$. Now, since $\det(\varphi_i - A\varphi_x) \neq 0$ we can see $\mu_i = \tau_i$. This means

$$(3.7) \quad \varphi_i \mu_i - \lambda_i(x, t; \xi + \varphi_x \mu_i) = 0, \quad (i = 1, 2, \dots, m - s).$$

(2.1) ~ (2.3) follow easily from (3.7). Q.E.D.

Now we can prove the invariance of the condition (C·B) under the transformation (3.3), namely we have

Proposition 3.2. Assume (C·B), then we have

$$(3.8) \quad \left\langle \tilde{L}_i(x, t; \xi), \left(\frac{\partial}{\partial t'} - (\varphi_t - A\varphi_x)^{-1} \sum \frac{\partial}{\partial x_j} - (\varphi_t - A\varphi_x)^{-1} B \right) \right. \\ \left. \times \tilde{R}_i(x, t; \xi) \right\rangle + \sum_{j=1}^l \frac{\partial \mu_i}{\partial x_j} \left\langle \tilde{L}_i(x, t; \xi), \frac{\partial}{\partial \xi_j} \tilde{R}_i(x, t; \xi) \right\rangle \equiv 0,$$

for $(x, t; \xi) \in \mathcal{Q} \times (\mathbf{R}_\xi^l \setminus \{0\})$, $(i=1, 2, \dots, s)$, where $\tilde{R}_i(x, t; \xi)$ (resp. $\tilde{L}_i(x, t; \xi)$) is the eigenvector of $(\varphi_t - A\varphi_x)^{-1} A\xi$ (resp. ${}^t((\varphi_t - A\varphi_x)^{-1} A\xi)$) corresponding to $\mu_i(x, t; \xi)$.

Here we remark that $\mu_i(x, t; \xi)$ is also smooth with respect to $(x, t; \xi) \in \mathcal{Q} \times (\mathbf{R}_\xi^l \setminus \{0\})$, $(i=1, 2, \dots, m-s)$. In fact from Lemma 3.1 $\psi_i(\tau) = 0$ has only one real root μ_i . Therefore we have

$$(3.9) \quad \psi_i'(\mu_i) = f_i(\mu_i) \neq 0.$$

From now on we omit for simplicity the index i in the condition (C·B). Now we will prove (3·8). At first we note that the eigenvectors $\tilde{R}(x, t; \xi)$ and $\tilde{L}(x, t; \xi)$ are given in the form:

$$(3.10) \quad \begin{cases} \tilde{R}(x, t; \xi) = R(x, t; \xi + \mu\varphi_x) \\ \tilde{L}(x, t; \xi) = {}^t(\varphi_t - A\varphi_x)L(x, t; \xi + \mu\varphi_x). \end{cases}$$

In fact, consider the equation:

$$(\mu(x, t; \xi) - (\varphi_t - A\varphi_x)^{-1} A\xi) \tilde{R}(x, t; \xi) = 0,$$

which is equivalent to

$$(\varphi_t \mu - A(\xi + \varphi_x \mu)) \tilde{R}(x, t; \xi) = 0.$$

Then by virtue of (3·7) and the definition of $R(x, t; \xi)$ we can take $\tilde{R}(x, t; \xi) = R(x, t; \xi + \mu\varphi_x)$. And the second equality in (3·10) will be shown similarly.

Next, in preparation we shall calculate the derivatives of $\mu(x, t; \xi)$. By the differentiation of (3·7) with respect to t , we have

$$\mu_t \varphi_t + \mu \varphi_{tt} - \lambda_t - \sum \lambda_{\xi_j} (\mu \varphi_{x_j})_t = 0.$$

Let us define an operator ∂ by

$$\partial(f) = f_t - \sum \lambda_{\xi_j}(x, t; \xi + \mu\varphi_x) f_{x_j}.$$

Then denoting $\delta = \partial(\varphi)$ we have

$$(3.11) \quad \delta\mu_t + \mu\partial(\varphi_t) - \lambda_t = 0.$$

In the same way we obtain

$$(3.12) \quad \delta\mu_{x_j} + \mu\partial(\varphi_{x_j}) - \lambda_{x_j} = 0, \quad (j=1, 2, \dots, l),$$

$$(3.13) \quad \delta\mu_{\xi_j} - \lambda_{\xi_j} = 0, \quad (j=1, 2, \dots, l).$$

Here we remark that $\delta \neq 0$ follows from (3.9).

Now we return to (3.8). In the first term of (3.6) we can easily see

$$\frac{\partial}{\partial t'} - (\varphi_t - A\varphi_x)^{-1} \sum A_j \frac{\partial}{\partial x_j'} = (\varphi_t - A\varphi_x)^{-1} \left(\frac{\partial}{\partial t} - \sum A_j \frac{\partial}{\partial x_j} \right).$$

Hence noting (3.10) we have

$$\begin{aligned} \left(\frac{\partial}{\partial t} - \sum A_j \frac{\partial}{\partial x_j} - B \right) \tilde{R} &= (R_t - \sum_j A_j R_{x_j} - BR) \\ &\quad + \sum_\alpha ((\mu\varphi_{x_\alpha})_t - \sum_j A_j (\mu\varphi_{x_\alpha})_{x_j}) R_{\xi_\alpha}. \end{aligned}$$

And this can be written as follows:

$$\begin{aligned} (3.14) \quad &\left(\frac{\partial}{\partial t} - \sum_j A_j \frac{\partial}{\partial x_j} - B \right) \tilde{R} \\ &= (R_t - \sum_j A_j R_{x_j} - BR) + \sum_\alpha \partial(\mu\varphi_{x_\alpha}) R_{\xi_\alpha} \\ &\quad + \sum_{\alpha, j} (\lambda_{\xi_j} - A_j) (\mu\varphi_{x_\alpha})_{x_j} R_{\xi_\alpha}. \end{aligned}$$

As for the second term of (3.8), noting the transformation (3.3), the definition of ∂ and (3.13) we have

$$\begin{aligned} \sum_j \mu_{x_j'} \tilde{R}_{\xi_j} &= \sum_j (\mu_{x_j} - \varphi_{x_j} \varphi_t^{-1} \mu_t) (R_{\xi_j} - \delta^{-1} \lambda_{\xi_j} \sum_\alpha \varphi_{x_\alpha} R_{\xi_\alpha}) \\ &= \sum_j \mu_{x_j} R_{\xi_j} - \delta^{-1} \partial(\mu) \sum_\alpha \varphi_{x_\alpha} R_{\xi_\alpha}. \end{aligned}$$

Moreover from (3.12) this becomes

$$\sum_j \mu_{x_j'} \tilde{R}_{\xi_j} = \delta^{-1} \sum_j \lambda_{x_j} R_{\xi_j} - \delta^{-1} \sum_\alpha \partial(\mu\varphi_{x_\alpha}) R_{\xi_\alpha}.$$

Then we have

$$\begin{aligned} (3.15) \quad \langle \tilde{L}, \sum_j \mu_{x_j'} \tilde{R}_{\xi_j} \rangle &= \langle L, (\varphi_t - A\varphi_x) \sum_j \mu_{x_j'} \tilde{R}_{\xi_j} \rangle \\ &= \langle L, \sum_j \lambda_{x_j} R_{\xi_j} \rangle - \langle L, \sum_\alpha \partial(\mu\varphi_{x_\alpha}) R_{\xi_\alpha} \rangle \end{aligned}$$

$$\begin{aligned}
 & + \delta^{-1} \langle L, \sum_j (\lambda_{\xi_j} - A_j) \varphi_{x_j} (\sum_\alpha \lambda_{x_\alpha} R_{\xi_\alpha} \\
 & - \sum_\alpha \partial (\mu \varphi_{x_\alpha}) R_{\xi_\alpha}) \rangle,
 \end{aligned}$$

where we use $(\varphi_t - A\varphi_x) = \delta + \sum_j (\lambda_{\xi_j} - A_j) \varphi_{x_j}$.

From (3.14) and (3.15) the left-hand side of (3.6) is equal to

$$\begin{aligned}
 & \langle L, R_t - \sum_j A_j R_{x_j} - BR + \sum_j \lambda_{x_j} R_{\xi_j} \rangle \\
 & + \langle L, \sum_{\alpha, j} (\lambda_{\xi_j} - A_j) (\mu \varphi_{x_\alpha})_{x_j} R_{\xi_\alpha} + \delta^{-1} \langle L, \sum_j (\lambda_{\xi_j} - A_j) \varphi_{x_j} \\
 & \times (\sum_\alpha \lambda_{x_\alpha} R_{\xi_\alpha} - \sum_\alpha \partial (\mu \varphi_{x_\alpha}) R_{\xi_\alpha}) \rangle.
 \end{aligned}$$

Therefore our purpose will be accomplished if we prove the followings:

$$\begin{aligned}
 (3.16) \quad & \sum_{\alpha, j} (\lambda_{\xi_j} - A_j) (\mu \varphi_{x_\alpha})_{x_j} R_{\xi_\alpha} + \delta^{-1} \sum_j (\lambda_{\xi_j} - A_j) \varphi_{x_j} \\
 & \times (\sum_\alpha \lambda_{x_\alpha} R_{\xi_\alpha} - \sum_\alpha \partial (\mu \varphi_{x_\alpha}) R_{\xi_\alpha}) \simeq 0,
 \end{aligned}$$

where for a vector $V \in E$, $V \simeq 0$ means $\langle L, V \rangle = 0$.

Now, noting (3.16) we shall give the following

Lemma 3.2. Put $V_{ij} = (\lambda_{\xi_i} - A_i) R_{\xi_j}$, then we have

$$(3.17) \quad V_{ij} + V_{ji} \simeq 0.$$

Proof. Differentiate (3.2) with respect to ξ_i and ξ_j , then we have

$$\lambda_{\xi_i \xi_j} R + (\lambda_{\xi_i} - A_i) R_{\xi_j} + (\lambda_{\xi_j} - A_j) R_{\xi_i} + (\lambda - A) R_{\xi_i \xi_j} = 0.$$

Taking the scalar product with L we have (3.17). Q.E.D.

From (3.17) we can give the followings:

$$(3.18) \quad \sum_{\alpha, j} (\lambda_{\xi_j} - A_j) \varphi_{x_j x_\alpha} R_{\xi_\alpha} \simeq 0,$$

$$(3.19) \quad \sum_{\alpha, j} (\lambda_{\xi_j} - A_j) \varphi_{x_j} \varphi_{x_\alpha} R_{\xi_\alpha} \simeq 0,$$

$$(3.20) \quad \sum_{\alpha, j} (\lambda_{\xi_j} - A_j) \lambda_{x_j} \varphi_{x_\alpha} R_{\xi_\alpha} + \sum_{\alpha, j} (\lambda_{\xi_j} - A_j) \varphi_{x_j} \lambda_{x_\alpha} R_{\xi_\alpha} \simeq 0.$$

$$(3.21) \quad \sum_{\alpha, j} (\lambda_{\xi_j} - A_j) \partial (\varphi_{x_j}) \varphi_{x_\alpha} R_{\xi_\alpha} + \sum_{\alpha, j} (\lambda_{\xi_j} - A_j) \varphi_{x_j} \partial (\varphi_{x_\alpha}) R_{\xi_\alpha} \simeq 0.$$

Now, noting (3.12) and the above relations we can easily see (3.16).

Thus we have proved the Proposition 3.2.

From the energy estimate (2·14) and the Proposition 3.2 we have

Theorem 3.1. *The solution of the Cauchy problem:*

$$M[u(x, t)] = 0 \text{ with } u(x, t_0) = u_0(x)$$

has its support in $C(\lambda_{\max}, t_0; \text{supp}[u_0(x)])$, where we denote $C(\lambda, s; F) = \bigcup_{y \in F} \{(x, t); |x - y| \leq \lambda |t - s|, t \geq s\}$ and $\lambda_{\max} = \sup_{\substack{(x, t) \in \mathcal{Q}, |\xi| = 1 \\ i=1, 2, \dots, m-s}} |\lambda_i(x, t; \xi)|$.

§ 4. Sufficiency of the Condition (C·A)

In this section we shall show the existence of the solution for the Cauchy problem (1·1). For this purpose we use a localization.

By \mathcal{Q}_i denote a certain open set which include $\{(x, t) \in \mathbb{R}_x^l \times [t_i, t_{i+1}]\}$, where $t_0 \leq t_i < t_{i+1} \leq T$. And let $\{\mathcal{Q}_i^{(j)}\}$, ($j=1, 2, \dots$) be an open covering which is locally finite, such that $C(\lambda_{\max}, t_i; \mathcal{Q}_i^{(j)} \cap \{t = t_i\}) \cap \mathcal{Q}_i \subset \mathcal{Q}_i^{(j)}$. We remark that we can take $\mathcal{Q}_i^{(j)}$ in such a way that for any point, the number of $\mathcal{Q}_i^{(j)}$ which contain it is less than certain constant. We shall define the sizes later. Next let $\{\alpha_i^{(j)}(x, t)\}$, ($j=1, 2, \dots$), be a partition of unity on \mathcal{Q}_i which is subordinate to $\{\mathcal{Q}_i^{(j)}\}$, and define

$$\begin{cases} f_i^{(j)}(x, t) = \alpha_i^{(j)}(x, t)f(x, t) \\ u_0^{(j)}(x) = \alpha_i^{(j)}(x, t_i)u_0(x). \end{cases}$$

Now let $u_i^{(j)}(x, t)$ be the solution of the Cauchy problem:

$$(4.1) \quad \begin{cases} M[u_i^{(j)}(x, t)] = f_i^{(j)}(x, t), \\ u_i^{(j)}(x, t_i) = u_0^{(j)}(x). \end{cases}$$

Since $\text{supp}[u_i^{(j)}(x, t)]$ is contained in $\mathcal{Q}_i^{(j)}$ from Theorem 3.1 we can see that in \mathcal{Q}_i $u_i^{(j)}$ also satisfies

$$(4.2) \quad \begin{cases} \widetilde{M}_i^{(j)}[u_i^{(j)}(x, t)] = f_i^{(j)}(x, t), \\ u_i^{(j)}(x, t_i) = u_0^{(j)}(x), \end{cases}$$

where $\widetilde{M}_i^{(j)}$ is another operator which is modified outside $\mathcal{Q}_i^{(j)}$ in the following way. First we deform $A_j(x, t)$ in such a way that they remain

constant outside some open set which includes $\overline{\Omega_i^{(j)}}$. And take size of $\Omega_i^{(j)}$ sufficiently small so that the p.d.op. $\mathcal{N}(x, t; D)$ stated in § 2 is invertible in the space $\mathcal{D}_{L^2}^k$. Next we can deform $B(x, t)$ as a p.d.op. in such a way that the condition (C·B) is still valid for the system (4·2). In fact we can obtain a required B so that with the deformed A and the corresponding N, M and D , it should make the value of the right-hand side of (3·1) invariant. Here we remark that for this $\widetilde{M}_i^{(j)}$, the energy estimate is also obtained and we can take the corresponding constant c_k so as not to depend on i and j .

Hereafter we omit the indices i, j and we denote the modified coefficients merely by $A_j(x, t)$ or $B(x, t)$. The existence of the solution for (4·2) is shown as follows. To solve (4·2) is now equivalent to solving the Cauch problem for the equations:

$$(4.3) \quad \left\{ \frac{\partial}{\partial t} - i\mathcal{D}(x, t; D) - \mathcal{B}_1(x, t; D) \right\} v(x, t) = g(x, t),$$

where $\mathcal{B}_1(x, t; D)$ is a p.d.op. of order 0, $v(x, t) = \mathcal{N}u(x, t)$ and $g(x, t) = \mathcal{N}f(x, t)$. Next this is equivalent further to the equations:

$$(4.4) \quad \left\{ \frac{\partial}{\partial t} - i\mathcal{D}_1(x, t; D) - \mathcal{B}_2(x, t; D) \right\} w(x, t) = h(x, t),$$

where $\mathcal{B}_2(x, t; D)$ is a p.d.op. of order 0, $w(x, t) = \mathcal{J}v(x, t)$ and $h(x, t) = \mathcal{J}g(x, t)$. Remark that $\mathcal{D}_1(x, t; D)$ is diagonalizable. Hence we can solve the Cauchy problem (4·2). Then from Theorem 3.1 we can see the existence of the solution for (4·1).

Next, on account of Theorem 2.1 and Theorem 3.1 we have a global solution by the superposition, namely we have

Theorem 4. 1. *Suppose the condition (C·A), then for the given initial data $u_0(x) \in \mathcal{D}_{L^2}^k$ and the right-hand side $f(x, t) \in \mathcal{E}_t^0(\mathcal{D}_{L^2}^k)$, ($k=1, 2, \dots$), there exists a unique solution $u(x, t)$ of (1·1) belonging to $\mathcal{E}_t^0(\mathcal{D}_{L^2}^{k-1})$ and it satisfies the inequality (2·14).*

§ 5. Necessity of the Condition (C·A)

Here we shall show the following

Theorem 5.1. *The condition (C·A) is necessary for the uniformly-well-posedness of the Cauchy problem (1·1).*

For the proof of this theorem we use the idea employed in [4] and [7], so we show the outline. We suppose that the Cauchy problem (1·1) is well-posed and that at least one of $C_i(x, t; \xi)$ in (C·B) is not identically zero. Then we can show that these two hypotheses induce a contradiction.

At first let us deny the condition (C·B), then without a loss of generality we can suppose (see [7])

$$(5.1) \quad \begin{cases} C_i(x, t; \xi) \equiv 0 & \text{on } \Omega_\varepsilon = U_\varepsilon \times [0, \varepsilon] \times U_\varepsilon(\xi_0), \quad (|\xi_0| = 1), \\ & \text{for } i = 1, 2, \dots, s_1, \\ C_i(x, t; \xi) \neq 0 & \text{on } \Omega_\varepsilon, \quad \text{for } i = s_1 + 1, \dots, s, \end{cases}$$

where $s_1 < s$, $U_\varepsilon = \{x; |x| \leq \varepsilon\}$ and $U_\varepsilon(\xi_0) = \{\xi; |\xi - \xi_0| \leq \varepsilon\}$.

Remark. For $i = s_1 + 1, \dots, s$ we can suppose

$$(5.2) \quad |C_i(x, t; \xi)| \geq \delta_0 (> 0) \quad \text{for } (x, t; \xi) \in \Omega_\varepsilon.$$

For simplicity we only consider the case where $s_1 = 0$. We can also treat the general case in a similar way.

Now consider the equations:

$$(5.3) \quad \left\{ \frac{\partial}{\partial t} - i\mathcal{A}(x, t; D) - B(x, t) \right\} u(x, t) = 0.$$

Take a function $\zeta(x) \in C_0^\infty$ such that $\zeta(x) \equiv 1$ on $U_{2\varepsilon/3}$ and $\zeta(x) \equiv 0$ outside U_ε , and apply it to (5·3), then we have

$$(5.4) \quad \left\{ \frac{\partial}{\partial t} - i\mathcal{A}(x, t; D) - B(x, t) \right\} (\zeta(x)u) = \sum_{j=1}^l A_j \zeta_j(x)u,$$

where $\zeta_j(x) = \frac{\partial}{\partial x_j} \zeta(x)$.

Next, to (5·4) apply $\mathcal{N}(x, t; D)$ stated in § 2, then we have

$$(5.5) \quad \left\{ \frac{\partial}{\partial t} - i\mathcal{D}(x, t; D) - \mathcal{B}_1(x, t; D) \right\} (\mathcal{N}\zeta(x)u) \\ = \sum_{j=1}^l \mathcal{N}A_j \zeta_j(x)u + \mathcal{E}(x, t; D)\zeta(x)u,$$

where $\mathcal{B}_1(x, t; D)$ and $\mathcal{E}(x, t; D)$ are the p.d.o.p.s, homogeneous of order 0 and -1 respectively. Let $\beta(x)$ belong to C_0^∞ such that $\beta(0) \neq 0$ and $\text{supp}[\beta(x)] \subset u_{\varepsilon/\beta}$, then (5.5) can be written as follows:

$$(5.6) \quad \left\{ \frac{\partial}{\partial t} - i\mathcal{D}(x, t; D) - \mathcal{B}_1(x, t; D) \right\} (\beta(x)\mathcal{N}\zeta(x)u) \\ = i[\beta(x), \mathcal{D}](\mathcal{N}\zeta u) + [\beta(x), \mathcal{B}_1](\mathcal{N}\zeta u) \\ + \beta \sum \mathcal{N}A_j \zeta_j u + \beta \mathcal{E} \zeta u.$$

Moreover take $\hat{\alpha}(\xi) \in C_0^\infty(\mathbf{R}_\xi^l)$, $(0 \leq \hat{\alpha}(\xi) \leq 1)$ such that $\text{supp}[\hat{\alpha}(\xi)] \subset U_{\varepsilon/\beta}(\xi_0)$, and $\hat{\alpha}(\xi) \equiv 1$ on a neighborhood of ξ_0 . Put $\hat{\alpha}_n(\xi) = \hat{\alpha}\left(\frac{\xi}{n}\right)$ and define $\widehat{\alpha_n(D)u} = \alpha_n(\xi)\hat{u}(\xi)$. Then from (5.6) we have

$$(5.7) \quad \left\{ \frac{\partial}{\partial t} - i\mathcal{D}(x, t; D) - \mathcal{B}_1(x, t; D) \right\} (\alpha_n \beta(x)\mathcal{N}\zeta(x)u) \\ = i[\alpha_n, \mathcal{D}](\beta \mathcal{N}\zeta u) + [\alpha_n, \mathcal{B}_1](\beta \mathcal{N}\zeta u) + i\alpha_n[\beta, \mathcal{D}](\mathcal{N}\zeta u) \\ + \alpha_n[\beta, \mathcal{B}_1](\mathcal{N}\zeta u) + \sum \alpha_n \beta \mathcal{N}A_j \zeta_j u + \alpha_n \beta \mathcal{E} \zeta u.$$

Next define a p.d.o.p. $\tilde{\mathcal{B}}_1(x, t; D)$, homogeneous of order 0 such that

$$\sigma(\tilde{\mathcal{B}}_1)(x, t; \xi) = \sigma(\mathcal{B}_1)(\vartheta(x), t; \vartheta_1(\xi)),$$

where ϑ and ϑ_1 are the mappings similar to those in [7].

Remark. From the definition of $\vartheta(x)$ and $\vartheta_1(\xi)$ and the hypothesis (5.1) we have

$$(5.8) \quad |\tilde{b}_{2i, 2i-1}^{(1)}(x, t; \xi)| \geq \delta_0 \quad (> 0) \\ \text{for } (x, t; \xi) \in \mathbf{R}_x^l \times [0, \varepsilon] \times \mathbf{R}_\xi^l \setminus \{0\}, \quad (i=1, 2, \dots, s),$$

where

$$\sigma(\tilde{\mathcal{B}}_1)(x, t; \xi) = \begin{bmatrix} \tilde{b}_{1,1}^{(1)}(x, t; \xi) & \dots & \tilde{b}_{1,m}^{(1)}(x, t; \xi) \\ & \dots & \\ \tilde{b}_{m,1}^{(1)}(x, t; \xi) & \dots & \tilde{b}_{m,m}^{(1)}(x, t; \xi) \end{bmatrix}.$$

Then we can rewrite (5.7) in the form:

$$(5.9) \quad \left\{ \frac{\partial}{\partial t} - i\mathcal{D}(x, t; D) - \tilde{\mathcal{B}}_1(x, t; D) \right\} (\alpha_n \beta(x)\mathcal{N}\zeta(x)u) = f(x, t),$$

where $f(x, t) = (\mathcal{B}_1 - \tilde{\mathcal{B}}_1) (\alpha_n \beta \mathcal{N} \zeta u) + i[\alpha_n, \mathcal{D}] (\beta \mathcal{N} \zeta u)$
 $+ [\alpha_n, \mathcal{B}_1] (\beta \mathcal{N} \zeta u) + i\alpha_n [\beta, \mathcal{D}] (\mathcal{N} \zeta u) + \alpha_n [\beta, \mathcal{B}_1] (\mathcal{N} \zeta u)$
 $+ \sum \alpha_n \beta \mathcal{N} A_j \zeta_j u + \alpha_n \beta \mathcal{E} \zeta u.$

Now we want to diagonalize $(i\mathcal{D} + \mathcal{B}_1)$ in a sense. For this purpose we define

$$\mathcal{J}(D) = \left(\begin{array}{cc|ccc} (1+A)^{-1/2} & 0 & & & \\ 0 & 1 & & & 0 \\ \hline & & \ddots & & \\ & & & \begin{array}{cc} (1+A)^{-1/2} & 0 \\ 0 & 1 \end{array} & \\ \hline & 0 & & & \begin{array}{c} 1 \\ \vdots \\ 1 \end{array} \end{array} \right)$$

and apply this to (5.9), then we have

$$(5.10) \quad \left\{ \frac{\partial}{\partial t} - i\mathcal{D}_1(x, t; D) - \mathcal{B}_2(x, t; D) A^{1/2} - \mathcal{B}_3(x, t; D) \right\} \\ \times (\mathcal{J} \alpha_n \beta(x) \mathcal{N} \zeta(x) u) = \mathcal{J} f,$$

where $\mathcal{B}_3(x, t; D)$ is a p.d.op. of order 0 and \mathcal{D}_1 and \mathcal{B}_2 have the following structure:

$$\mathcal{D}_1 = \left(\begin{array}{cccc|cccc} \lambda_1 & 0 & & & & & & \\ 0 & \lambda_1 & & & & & & \\ \hline 0 & 0 & \lambda_2 & 0 & & & & \\ * & 0 & 0 & \lambda_2 & & & & 0 \\ \hline & & \dots & & \ddots & & & \\ \hline 0 & 0 & 0 & 0 & \dots & \lambda_s & 0 & \\ * & 0 & * & 0 & \dots & 0 & \lambda_s & \\ \hline * & 0 & * & 0 & \dots & * & 0 & \lambda_{s+1} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots & \vdots & 0 \\ * & 0 & * & 0 & \dots & * & 0 & 0 \quad \lambda_{m-s} \end{array} \right),$$

where $*$ is a p.d.op, homogeneous of order $\frac{1}{2}$, and

$$\mathcal{B}_2 = \left(\begin{array}{cc|cccc|c} 0 & a_1 & 0 & 0 & \cdots & 0 & 0 & \\ \mathcal{L}_{2,1}^{(2)} & 0 & * & 0 & \cdots & * & 0 & \\ \hline & & 0 & a_2 & \cdots & 0 & 0 & 0 \\ & & \mathcal{L}_{4,3}^{(2)} & 0 & \cdots & * & 0 & \\ & & & & \ddots & & & \\ & & & & & \cdots & & \\ & & & & & & 0 & a_s \\ & & & & & & \mathcal{L}_{2s,2s-1}^{(2)} & 0 \\ \hline & & & & & & & 0 \\ \hline & & & & & & & 0 \end{array} \right),$$

which is homogeneous of order 0.

Noting the structure of \mathcal{D}_1 , we can construct a diagonalizator $N_1(x, t; \xi)$ of $\sigma(\mathcal{D}_1)(x, t; \xi)$. Let $\mathcal{N}_1(x, t; D)$ be the p.d.op. whose symbol is $N_1(x, t; \xi)$.

Remark. We can write $\mathcal{N}_1(x, t; D) = I + \mathcal{N}_1^0(x, t; D)$, where $\mathcal{N}_1^0(x, t; D)$ is a p.d.op. of order $-\frac{1}{2}$.

Apply $\mathcal{N}_1(x, t; D)$ to (5.10), then we have

$$(5.11) \quad \left\{ \frac{\partial}{\partial t} - i\mathcal{D}_2(x, t; D) - \mathcal{B}_2(x, t; D)A^{1/2} \right\} (\mathcal{N}_1 \mathcal{J} \alpha_n \beta(x) \mathcal{N} \zeta(x) u) = \mathcal{B}_4 \mathcal{J} \alpha_n \beta \mathcal{N} \zeta u + \mathcal{N}_1 \mathcal{J} f,$$

where $\mathcal{D}_2(x, t; D)$ is a diagonal matrix whose symbol is pure imaginary and $\mathcal{B}_4 = \mathcal{B}_4(x, t; D)$ is of order 0.

By $\mathcal{D}_3(x, t; D)$ denote $i\mathcal{D}_2(x, t; D) + \mathcal{B}_2(x, t; D)A^{1/2}$. Then the roots of $\det(\tau I - D_3(x, t; \xi)) = 0$ are given in the form:

$$\begin{cases} \mu_{2i-1}(x, t; \xi) = \sqrt{-1} \lambda_i(x, t; \xi) + \nu_i(x, t; \xi) |\xi|^{1/2} \\ \mu_{2i}(x, t; \xi) = \sqrt{-1} \lambda_i(x, t; \xi) - \nu_i(x, t; \xi) |\xi|^{1/2}, \quad (i=1, 2, \dots, s), \\ \mu_j(x, t; \xi) = \sqrt{-1} \lambda_{j-s}(x, t; \xi), \quad (j=2s+1, \dots, m), \end{cases}$$

where $\nu_i(x, t; \xi)$ is homogeneous of degree 0 in ξ and satisfies

$$(5.12) \quad \text{Re} \nu_i(x, t; \xi) \geq \delta' (> 0),$$

for $(x, t; \xi) \in \mathbf{R}_x^l \times [0, \varepsilon] \times \mathbf{R}_\xi^l, (i=1, 2, \dots, s)$.

Now we can diagonalize $D_3(x, t; \xi)$. Denote the diagonalizator of

$D_s(x, t; \xi)$ by $N_2(x, t; \xi)$ and define a p.d.op. $\mathcal{N}_2(x, t; D)$ with the symbol $N_2(x, t; \xi)$.

As in [7], for $w(x, t) = {}^t(w_1, w_2, \dots, w_m)$, we define

$$(5.13) \quad S_n(t; w) = \exp \left\{ -\delta \sqrt{n} t \left(\sum_{i=1}^s \|w_{2i-1}\|^2 - \sum_{i=1}^s \|w_{2i}\|^2 - \sum_{j=2s+1}^m \|w_j\|^2 \right) \right\},$$

where δ is a suitable positive constant. And also define

$$(5.14) \quad \Theta_n(\nu, \kappa) u = \sqrt{n}^{|\nu| - |\kappa|} \mathcal{N}_2 \mathcal{N}_1 \mathcal{G} \alpha_n^{(\omega)} \beta_{(\kappa)}(x) \mathcal{N} \zeta(x) u,$$

where $\sigma(\alpha_n^{(\omega)}) = \left(\frac{\partial}{\partial \xi}\right)^\nu \alpha_n(\xi)$ and $\beta_{(\kappa)}(x) = \left(\frac{\partial}{\partial x}\right)^\kappa \beta(x)$, ν and κ being multi-indices.

Taking $\zeta_1(x) \in C_0^\infty$ such that $\zeta_1(x) \equiv 1$ on U and $\text{supp}[\zeta_1(x)] \subset U_{2\epsilon}$, we have

Proposition 5. 1.

$$(5.15) \quad \exp \{ \delta \sqrt{n} t \} S_n'(t; \Theta_n(0, 0) u) \geq \delta'' \sqrt{n} \|\Theta_n(0, 0) u\|^2 - c_1 \sqrt{n} \sum_{1 \leq |\nu| + |\kappa| \leq k} \|\Theta_n(\nu, \kappa) u\|^2 - c_2 n^{-k} \|\zeta_1(x) u\|^2,$$

where k is a positive integer we can take sufficiently large if necessary and δ'' , c_1 , c_2 , are positive constants, independent of $u(x, t)$ and n .

Moreover considering $\alpha_n^{(\omega)}$ and $\beta_{(\kappa)}(x)$ in (5.9) instead of α_n and $\beta(x)$ we have

Proposition 5. 2.

$$(5.16) \quad \exp \{ \delta \sqrt{n} t \} S_n'(t; \Theta_n(\nu, \kappa) u) \geq \delta'' \sqrt{n} \|\Theta_n(\nu, \kappa) u\|^2 - c_1(\nu, \kappa) \sqrt{n} \sum_{(\nu', \kappa') \in \mathcal{B}(\nu, \kappa)} \|\Theta_n(\nu', \kappa') u\|^2 - c_2(\nu, \kappa) n^{-k} \|\zeta_1(x) u\|^2,$$

where $c_1(\nu, \kappa)$ and $c_2(\nu, \kappa)$ are positive constants and $E(\nu, \kappa; k) = \{(\nu', \kappa'); \nu' \geq \nu, \kappa' \geq \kappa, |\nu| + |\kappa| + 1 \leq |\nu'| + |\kappa'| \leq k\}$.

Now we can show the contradiction. Let $\hat{\psi}(\xi)$ be a C^∞ -function whose support is located in a small neighborhood of ξ_0 ; On the support of $\hat{\psi}(\xi)$, $\hat{\alpha}(\xi) \equiv 1$. And define $\psi_n(x) = \exp(in \xi_0 x) \psi(x)$, where $\psi(x) = \mathcal{F}^{-1}[\hat{\psi}(\xi)]$.

Next, we define $u_n(x, t)$ as the solution of

$$(5.17) \quad M[u_n(x, t)] = 0 \quad \text{with}$$

$$u_n(x, 0) = \mathcal{M}(x, 0; D) \mathcal{G}(D)^{-1} \mathcal{M}_2(x, 0; D) \begin{bmatrix} \psi_n(x) \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where $\mathcal{M}_2(x, t; D)$ is the p.d.op. of order 0 with the symbol $M_2(x, t; \xi)$, which is the inverse matrix of $N_2(x, t; \xi)$.

Because we have supposed that (1.1) is \mathcal{E} -well-posed, we have

$$(5.18) \quad \|\zeta_1(x) u_n(x, t)\| \leq c_3 n^p, \quad \text{for } t \in [0, \varepsilon],$$

where c_3 is a constant, independent of n , and p is some integer. Put $k=2p$ in (5.16), then we have

$$(5.19) \quad \exp\{\delta \sqrt{n} t\} S_n'(t; \Theta_n(\nu, \kappa) u_n) \geq \delta'' \sqrt{n} \|\Theta_n(\nu, \kappa) u_n\|^2 - c_1(\nu, \kappa) \sqrt{n} \sum_{(\nu', \kappa')} \|\Theta_n(\nu', \kappa') u_n\|^2 - O(1).$$

Next, denote $\Phi_n(t) = \sum_{0 \leq |\nu| + |\kappa| \leq k} M^{|\nu| + |\kappa|} S_n(t; \Theta_n(\nu, \kappa) u_n)$, where M is some large constant. Then we have

$$(5.20) \quad \Phi_n'(t) \geq \frac{\delta''}{2} \sqrt{n} \Phi_n(t) - O(1).$$

By the way, if we see the initial data $u_n(x, 0)$ in (5.17), we have

Proposition 5.3. *There is a positive constant a such that*

$$(5.21) \quad \Phi_n(0) \geq a, \quad \text{for large } n.$$

Therefore, integrating (5.20) we have

$$(5.22) \quad \Phi_n(t) \geq \frac{a}{2} \exp\left(\frac{\delta''}{2} \sqrt{n} t\right).$$

On the other hand taking account of (5.18) and the definition of $\Phi_n(t)$ we can easily see

$$(5.23) \quad \Phi_n(t) \leq c_4 n^{2p} \exp\{-\delta \sqrt{n} t\}, \quad \text{for } t \in [0, \varepsilon],$$

where c_4 is a positive constant.

(5.22) and (5.23) cannot be compatible unless $t=0$. The proof of Theorem 5.1 is thus complete.

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