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Stability and Convergence of a Finite Element Method for Solving the Stefan Problem[1)]

By

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abstract

A finite element method based on the time dependent basis functions is presented for solving a one phase Stefan problem for the heat equation in one space dimension. The stability and the convergence of the method are studied, and a numerical example is given.

§ 1. One Phase Stefan Problem in One Space Dimension

We consider the following one phase Stefan problem in one space dimension. The main equation describing the dynamics of the system is the heat equation

$$(1\cdot 1)\qquad \frac{\partial u}{\partial t}=\sigma\frac{\partial^2 u}{\partial x^2}\quad \text{in}\quad 0<x<s(t),\quad 0<t\leqq T,$$

associated with a free boundary condition given below. $s(t)$ denotes the position of the free boundary. σ is assumed to be a positive constant, and T is an arbitrarily fixed positive number. At the boundary $x=0$ we assume a Dirichlet type boundary condition and at $x=s(t)$ we assume $u=0$:

$$(1\cdot 2)\qquad \begin{cases} u(0,t)=g(t) \\ u(s(t),t)=0 \end{cases}\quad \text{for}\quad 0\leqq t\leqq T.$$

To the initial condition we assign

$$(1\cdot 3)\qquad u(x,0)=f(x)\geqq 0\ \text{ for }\ 0\leqq x\leqq b,\ \ b=s(0)>0.$$

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1) Part of this work was accomplished while the author was visiting at the Courant Institute of Mathematical Sciences, New York University during 1974-75 academic year.

The free boundary moves according to the following equation called the Stefan condition:

$$(1\cdot4)\qquad \frac{ds}{dt} = -\kappa u_x(s(t), t) \quad \text{for} \quad 0 < t \leqq T,$$

where κ is assumed to be a positive constant. Furthermore for the initial data we make

Assumption A. $0 \leqq f(x) \leqq B(b-x)$,

where B is a positive constant.

It has been shown by Cannon and Hill [2] that the Stefan problem $(1\cdot1)\sim(1\cdot4)$ has a unique solution under Assumption A. Many approximating methods have been presented for solving the one phase Stefan problem in one dimension numerically. Landau [6] applied a variable transformation in order to change the varying interval $0 \leqq x \leqq s(t)$ into a fixed interval and employed the finite difference method. Douglas and Gallie [3] and Nogi [9] proposed finite difference methods in which an equi-distant space partition is employed and the time variable is discretized in such a way that the free boundary always coincides with a mesh point. Kawarada and Natori [5] combined the penalty method and the finite difference method. Bonnerot and Jamet [1] partitioned the space-time domain into quadrilateral elements and applied the two dimensional finite element method.

In the preceding paper [7] we presented a finite element method (FEM) for the problem $(1\cdot1)\sim(1\cdot4)$ based on the time dependent basis functions. In the present paper we shall study the stability and the convergence of the method.

§ 2. Application of the Finite Element Method and the Scheme

Consider the domain $0 \leqq x \leqq s(t)$ at time t. We partition $0 \leqq x \leqq s(t)$ into n subintervals in accordance with some rule in such a way that the end point of partition always coincides with the free boundary, and denote each node as x_j:

$$(2\cdot1)\qquad 0 = x_0 < x_1 < \cdots\cdots < x_n = s(t).$$

Although $s(t)$ is an unknown function of t which should be determined simultaneously together with $u(x, t)$, we compute $s(t)$ and $u(x, t)$ alternatively in the actual process of computation by means of a similar technique to the idea of "retarding the argument" by Cannon and Hill [2], and hence we write $s_n(t)$ instead of $s(t)$ in order to show explicitly that it is an approximation.

We construct piecewise linear basis functions $\{\phi_j\}$ for FEM as shown in Fig. 1:

$$(2\cdot2)\qquad \phi_j(x,t)=\begin{cases}\dfrac{x-x_{j-1}}{x_j-x_{j-1}}; & x_{j-1}<x\leqq x_j\\[2mm] \dfrac{x_{j+1}-x}{x_{j+1}-x_j}; & x_j<x\leqq x_{j+1}\\[2mm] 0 & ;\ \text{otherwise.}\end{cases}$$

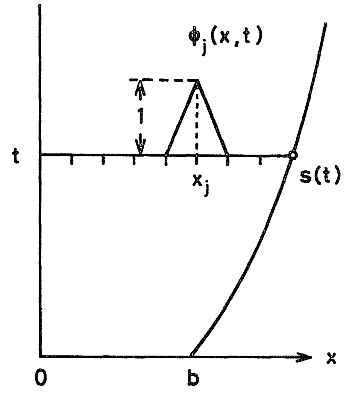

Fig. 1. The basis function $\phi_j(x, t)$.

For ϕ_0 and ϕ_n we take the components of $(2\cdot2)$ in $0\leqq x\leqq x_1$ and $x_{n-1}<x\leqq x_n$, respectively. $\phi_j(x, t)$ depends on time t through $s_n(t)$, and its derivatives with respect to x and t are given as

$$(2\cdot3)\qquad \frac{\partial\phi_j}{\partial x}=\begin{cases}\dfrac{1}{x_j-x_{j-1}}; & x_{j-1}<x\leqq x_j\\[2mm] -\dfrac{1}{x_{j+1}-x_j}; & x_j<x\leqq x_{j+1}\\[2mm] 0 & ;\ \text{otherwise,}\end{cases}$$

$$(2\cdot4)\qquad \frac{\partial\phi_j}{\partial t}=\begin{cases} -\dfrac{(x-x_{j-1})\dot{x}_j+(x_j-x)\dot{x}_{j-1}}{(x_j-x_{j-1})^2}; & x_{j-1}<x\leqq x_j \\ \dfrac{(x-x_j)\dot{x}_{j+1}+(x_{j+1}-x)\dot{x}_j}{(x_{j+1}-x_j)^2}; & x_j<x\leqq x_{j+1} \\ 0 & ;\ \text{otherwise,} \end{cases}$$

where

$$\dot{x}_j=\frac{dx_j}{dt}.$$

Henceforth we partition $0\leqq x\leqq s_n(t)$ into n equal subintervals for simplicity:

$$(2\cdot5)\qquad x_j=jh_n(t),\quad h_n(t)=\frac{1}{n}s_n(t).$$

Now we apply the Galerkin method based on the basis functions $\{\phi_j\}$ just constructed above. We expand the approximate solution $u_n(x, t)$ of $(1\cdot1)\sim(1\cdot4)$ in terms of a linear combination of $\phi_j(x, t)$'s:

$$(2\cdot6)\qquad u_n(x, t)=\sum_{j=0}^{n}a_j(t)\phi_j(x, t),$$

where

$$(2\cdot7)\qquad a_0(t)=g(t),\quad a_n(t)=0$$

in accordance with the boundary condition $(1\cdot2)$. Then we substitute $(2\cdot6)$ into u of $(1\cdot1)$, multiply ϕ_i, and integrate over $0<x<s_n(t)$. Then we have a system of ordinary differential equations

$$(2\cdot8)\qquad M(t)\frac{d\boldsymbol{a}(t)}{dt}=-(\sigma K(t)+N(t))\boldsymbol{a}(t),$$

where $\boldsymbol{a}(t)$ is an $(n+1)$-dimensional vector defined by

$$(2\cdot9)\qquad \boldsymbol{a}(t)=\begin{pmatrix} a_0(t) \\ a_1(t) \\ \vdots \\ a_n(t) \end{pmatrix}.$$

The first and the last elements of $(2\cdot9)$ are known functions of t as seen from $(2\cdot7)$. M, K and N are time dependent $(n-1)\times(n+1)$ matrices, the elements of which are given as follows for $i=1, 2, \cdots, n-1$;

$j=0, 1, \cdots, n$:

$$M_{ij}=\int_0^{s_n(t)}\phi_i\phi_j dx \qquad ; \text{ mass matrix} \tag{2·10}$$

$$K_{ij}=\int_0^{s_n(t)}\frac{\partial\phi_i}{\partial x}\times\frac{\partial\phi_j}{\partial x}dx; \text{ stiffness matrix} \tag{2·11}$$

$$N_{ij}=\int_0^{s_n(t)}\phi_i\times\frac{\partial\phi_j}{\partial t}dx \quad ; \text{ velocity matrix.} \tag{2·12}$$

Since the matrix N corresponds to the apparent velocity of the nodes, we gave it a name "velocity matrix". In usual FEM, we take away the first and the last columns from M and K, and call thus obtained square $(n-1)\times(n-1)$ matrices the mass matrix and the stiffness matrix, respectively.

The above definition of M is for the consistent mass system. For the lumped mass system characteristic functions

$$\psi_j(x,t)=\begin{cases}1; & \frac{1}{2}(x_{j-1}+x_j)<x\leqq\frac{1}{2}(x_j+x_{j+1})\\ 0; & \text{otherwise}\end{cases} \tag{2·13}$$

are used instead of $\phi_j(x,t)$ in the definition of M, i.e.

$$M_{ij}=\int_0^{s_n(t)}\psi_i\psi_j dx\,. \tag{2·14}$$

The explicit forms of the elements of the matrices are as follows:

Lumped mass system

$$M_{ij}=\begin{cases}\frac{1}{2}(x_{i+1}-x_{i-1})=h_n; & j=i\\ 0 & ; j\neq i\end{cases} \tag{2·15}$$

$$K_{ij}=\begin{cases}-\frac{1}{x_i-x_{i-1}}=-\frac{1}{h_n} & ; j=i-1\\ \frac{1}{x_i-x_{i-1}}+\frac{1}{x_{i+1}-x_i}=\frac{2}{h_n}; & j=i\\ -\frac{1}{x_{i+1}-x_i}=-\frac{1}{h_n} & ; j=i+1\\ 0 & ; \text{otherwise}\end{cases} \tag{2·16}$$

$$(2\cdot 17)\qquad N_{ij}=\begin{cases} \dfrac{1}{3}\dot{x}_i+\dfrac{1}{6}\dot{x}_{i-1}=\dfrac{1}{6}(3i-1)\dfrac{dh_n}{dt} & ;\ j=i-1 \\ \dfrac{1}{6}(\dot{x}_{i+1}-\dot{x}_{i-1})=\dfrac{1}{3}\times\dfrac{dh_n}{dt} & ;\ j=i \\ -\dfrac{1}{6}\dot{x}_{i+1}-\dfrac{1}{3}\dot{x}_i=-\dfrac{1}{6}(3i+1)\dfrac{dh_n}{dt}; & j=i+1 \\ 0 & ;\ \text{otherwise.} \end{cases}$$

Consistent mass system

In this case only the mass matrix should be changed as follows:

$$(2\cdot 18)\qquad M_{ij}=\begin{cases} \dfrac{1}{6}(x_i-x_{i-1})=\dfrac{1}{6}h_n & ;\ j=i-1 \\ \dfrac{1}{3}(x_{i+1}-x_{i-1})=\dfrac{2}{3}h_n; & j=i \\ \dfrac{1}{6}(x_{i+1}-x_i)=\dfrac{1}{6}h_n & ;\ j=i+1 \\ 0 & ;\ \text{otherwise.} \end{cases}$$

In the next step we discretize the time variable t, i.e. we partition $0\leqq t\leqq T$ into m equal subintervals with a constant mesh size Δt:

$$(2\cdot 19)\qquad \Delta t=\frac{T}{m},\quad t_k=k\Delta t,\quad k=0,\ 1,\cdots,\ m\,.$$

We replace the time derivative of $\boldsymbol{a}(t)$ by the time difference:

$$(2\cdot 20)\qquad \frac{d\boldsymbol{a}(k\Delta t)}{dt}\doteqdot\frac{\boldsymbol{a}(k\Delta t)-\boldsymbol{a}((k-1)\Delta t)}{\Delta t}$$

and we write

$$(2\cdot 21)\qquad a_j^k=a_j(k\Delta t)\,.$$

In this way we have a system of linear equations with respect to $\boldsymbol{a}(k\Delta t)$ corresponding to $(2\cdot 8)$. If we employ the values at $t=(k-1)\Delta t$ in $\boldsymbol{a}(t)$ of the right hand side of $(2\cdot 8)$, we have a forward difference scheme. If we employ, on the other hand, the values at $t=k\Delta t$, we have a backward one. In actual computation we can mix them in the ratio $\theta:1-\theta$ by introducing a parameter θ, $0\leqq\theta\leqq 1$, as will be shown later.

We compute the increment Δs_n of $s_n(t)$ from $t=(k-1)\Delta t$ to $t=k\Delta t$ by approximating the right hand side of (1·4) by the gradient of the approximate solution $u_n(x, t)$ at the free boundary $x=s_n(t)$, and by replacing the left hand side of (1·4) by the time difference $\Delta s_n/\Delta t$. For the computation of the velocity matrix N, we employ an approximation

$$\frac{dh_n}{dt} \doteqdot \frac{1}{n} \times \frac{\Delta s_n}{\Delta t}. \tag{2·22}$$

We summarize here the whole scheme obtained in the above procedure. θ is fixed to a value between 0 and 1 throughout the computation.

Initial routine

$$\begin{cases} a_j^0 = f(x_j), \quad j=0, 1, \cdots, n \\ \Delta s_n(\Delta t) = \kappa \dfrac{a_{n-1}^0}{h_n(0)} \Delta t \\ s_n(\Delta t) = b + \Delta s_n(\Delta t) \end{cases} \tag{2·23}$$

General routine

Repeat the following process for $k=1, 2, \cdots, m$. Compute M, K and N at $t=k\Delta t$ using the values of $s_n(k\Delta t)$ and $\Delta s_n(k\Delta t)$, and solve the following simultaneous linear equations for $\boldsymbol{a}(k\Delta t)$:

$$\begin{aligned} &\{M+\theta\Delta t(\sigma K+N)\}\boldsymbol{a}(k\Delta t) \\ &\quad = \{M-(1-\theta)\Delta t(\sigma K+N)\}\boldsymbol{a}((k-1)\Delta t). \end{aligned} \tag{2·24}$$

Then compute $\Delta s_n((k+1)\Delta t)$ and $s_n((k+1)\Delta t)$ according to the following equations using the known data $\boldsymbol{a}((k-1)\Delta t)$ and $\boldsymbol{a}(k\Delta t)$:

$$\Delta s_n((k+1)\Delta t) = \frac{\kappa}{2}\left\{\frac{a_{n-1}^{k-1}}{h_n((k-1)\Delta t)} + \frac{a_{n-1}^{k}}{h_n(k\Delta t)}\right\}\Delta t \tag{2·25}$$

$$s_n((k+1)\Delta t) = s_n(k\Delta t) + \Delta s_n((k+1)\Delta t). \tag{2·26}$$

The reason why we adopted the mean value of the data at $t=(k-1)\cdot\Delta t$ and $k\Delta t$ in (2·25) is in order to make $s_n(t)\in C^1$ as seen later (see (4·9)).

§ 3. Stability

In this section we discuss the stability of the scheme given in § 2. For simplicity we use the following notations:

$$\alpha_k \equiv \frac{\sigma n^2 \Delta t}{s_n^2(k\Delta t)} + \frac{\Delta s_n(k\Delta t)}{6 s_n(k\Delta t)} \tag{3·1}$$

$$\beta_k \equiv \frac{\Delta s_n(k\Delta t)}{2 s_n(k\Delta t)}. \tag{3·2}$$

For the moment we confine ourselves to the case of the lumped mass system. The stability of the scheme of the consistent mass system will be referred to at the end of this section.

The lumped mass scheme can be explicitly written as follows:

$$\begin{aligned} &-\theta(\alpha_k - i\beta_k) a_{j-1}^k + \{1+2\theta\alpha_k\} a_j^k - \theta(\alpha_k + j\beta_k) a_{j+1}^k \\ &= (1-\theta)(\alpha_k - j\beta_k) a_{j-1}^{k-1} + \{1-2(1-\theta)\alpha_k\} a_j^{k-1} \\ &\quad + (1-\theta)(\alpha_k + j\beta_k) a_{j+1}^{k-1}, \\ &j=1,2,\cdots,n-1;\ k=1,2,\cdots,m, \end{aligned} \tag{3·3}$$

where $a_0^k = g(k\Delta t)$ and $a_n^k = 0$ are known. For the later convenience we introduce the following operator P_L:

$$\begin{aligned} &P_L(k,j;\Delta t, s_n(k\Delta t), \Delta s_n(k\Delta t);\theta) w_j^k = P_L(k,j) w_j^k \\ &\equiv -\theta(\alpha_k - j\beta_k) w_{j-1}^k + \{1+2\theta\alpha_k\} w_j^k - \theta(\alpha_k + j\beta_k) w_{j+1}^k \\ &\quad - (1-\theta)(\alpha_k - j\beta_k) w_{j-1}^{k-1} - \{1-2(1-\theta)\alpha_k\} w_j^{k-1} \\ &\quad - (1-\theta)(\alpha_k + j\beta_k) w_{j+1}^{k-1}. \end{aligned} \tag{3·4}$$

Evidently the scheme (3·3) is written as $P_L(k,j) a_j^k = 0$.

Lemma 1 (*Lumped mass system*). *If*

$$n|\beta_k| \leqq \alpha_k \leqq \frac{1}{2(1-\theta)}, \quad 0 \leqq p_j^k,\ j=0,1,\cdots,n-1;\ k=1,2,\cdots,m, \tag{3·5}$$

then the scheme

$$P_L(k,j) w_j^k = p_j^k,\ j=1,2,\cdots,n-1 \tag{3·6}$$

satisfies the following maximum principle locally at each $k=1,2,\cdots,$

m:

$$(3\cdot 7)\qquad \min(w_0{}^k, w_n{}^k, w_{\min}^{k-1}) \leqq w_j{}^k \leqq \max(w_0{}^k, w_n{}^k, w_{\max}^{k-1}) + p_j{}^k, \quad j=0, 1, \cdots, n,$$

where

$$(3\cdot 8)\qquad w_{\min}^{k-1} = \min_{0\leqq j\leqq n} w_j^{k-1} \quad \text{and} \quad w_{\max}^{k-1} = \max_{0\leqq j\leqq n} w_j^{k-1}.$$

Proof. The left inequality in (3·7) is trivial if $w_j{}^k$ attains the minimum at $j=0$ or at $j=n$. Suppose $w_j{}^k$ attains the minimum at $j=M$ $(M\neq 0, n)$. From (3·5) it is evident that $\alpha_k \pm j|\beta_k| > 0$ and $1-2(1-\theta)\alpha_k \geqq 0$, so that we have

$$\begin{aligned}\{1+2\theta\alpha_k\} w_M{}^k &= \theta(\alpha_k - j\beta_k) w_{M-1}^k + \theta(\alpha_k + j\beta_k) w_{M+1}^k \\ &+ (1-\theta)(\alpha_k - j\beta_k) w_{M-1}^{k-1} + \{1-2(1-\theta)\alpha_k\} w_M{}^{k-1} \\ &+ (1-\theta)(\alpha_k + j\beta_k) w_{M+1}^{k-1} + p_j{}^k \geqq 2\theta\alpha_k w_M{}^k + w_{\min}^{k-1}.\end{aligned}$$

Hence the left inequality in (3·7) is valid.

Similarly, the right inequality in (3·7) is trivial if $w_j{}^k$ attains the maximum at $j=0$ or at $j=n$. Suppose $w_j{}^k$ attains the maximum at $j=M'$ $(M'\neq 0, n)$. Then we have

$$\{1+2\theta\alpha_k\} w_{M'}{}^k \leqq 2\theta\alpha_k w_{M'}{}^k + w_{\max}^{k-1} + p_j{}^k,$$

and so the right inequality in (3·7) also holds. Q.E.D.

Now we introduce the following quantities:

$$\left.\begin{aligned} &(3\cdot 9) && A \equiv \max\left(\frac{1}{b}\max_{0\leqq t\leqq T} g(t), B\right) \\ &(3\cdot 10) && l \equiv b + \kappa A T \\ &(3\cdot 11) && \lambda_b \equiv \frac{\sigma \Delta t}{h_n{}^2(0)} = \frac{\sigma n^2 \Delta t}{b^2} \\ &(3\cdot 12) && \lambda_l \equiv \frac{\sigma n^2 \Delta t}{l^2} \end{aligned}\right\}$$

In addition to Assumption A, we make the following ones:

Assumption B. $\lambda_b\left(1+\frac{\kappa bA}{6\sigma n^2}\right)\leqq\frac{1}{2(1-\theta)}$

Assumption C.
$$\begin{cases} A\leqq\dfrac{2\sigma n}{\kappa l} \\ A\leqq\dfrac{b}{(1-\theta)\kappa\Delta t} . \end{cases}$$

Lemma 2 (*lumped mass system*). *Under Assumptions A, B and C, we have*

$$(3\cdot13)\quad 0\leqq\frac{a_j^k}{\left(1-\frac{j}{n}\right)s_n(k\Delta t)}\leqq A;\ j=0,1,\cdots,n-1;\ k=0,1,\cdots,m .$$

Proof. By definition we have

$$(3\cdot14)\qquad P_L(k,j)\,a_j^k=0 .$$

We define (see Fig. 2)

$$(3\cdot15)\qquad d_j^k\equiv A\left(1-\frac{j}{n}\right)s_n(k\Delta t)-a_j^k .$$

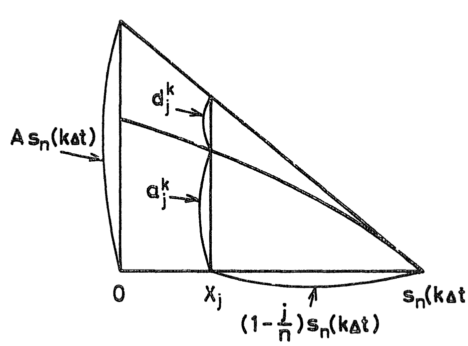

Fig. 2.

It is easy to see that d_j^k satisfies

$$(3\cdot16)\qquad P_L(k,j)\,d_j^k=p_j^k ,$$

where

$$(3\cdot 17)\qquad p_j{}^k = A\Delta s_n(k\Delta t)\left\{1-(1-\theta)\frac{j}{n}\times\frac{\Delta s_n(k\Delta t)}{s_n(k\Delta t)}\right\}.$$

In place of (3·13) we shall prove

$$\left.\begin{array}{ll}(3\cdot 18) & 0<b\leqq s_n(\Delta t)\leqq s_n(2\Delta t)\leqq\cdots\leqq s_n(k\Delta t)\\ (3\cdot 19) & 0\leqq a_j{}^k,\quad 0\leqq d_j{}^k,\quad j=0,1,\cdots,n-1\end{array}\right\}$$

by induction with respect to k. It would be clear that (3·18) and (3·19) imply (3·13).

For $k=0$, (3·18) and (3·19) are trivial because of $b=s_n(0)>0$, (1·3) and Assumption A.

Suppose that (3·18) and (3·19) hold for 0, 1, ⋯, $k-1$. Since $s_n(k\Delta t)\geqq 0$ from (2·23) and (2·25), (3·18) follows from (2·26). Now we claim that

$$(3\cdot 20)\qquad \frac{\Delta s_n(k\Delta t)}{\Delta t}\leqq\kappa A\,.$$

When $k=1$, this is evident from (2·23). When $k\geqq 2$, we put $j=n-1$ in (3·13) which is assumed to hold for $k-2$ and $k-1$. Then (3·20) follows from (2·25). Hence we have

$$(3\cdot 21)\qquad p_j{}^k\geqq A\Delta s_n(k\Delta t)\left\{1-(1-\theta)\frac{\kappa A}{b}\Delta t\right\}\geqq 0$$

by the second inequality of Assumption C. Here we apply Lemma 1 to (3·14) and (3·16). What is left to be done is to show $n\beta_k<\alpha_k\leqq\dfrac{1}{2(1-\theta)}$. Note that $s_n(k\Delta t)\leqq b+\kappa Ak\Delta t\leqq l$ from (3·20). Then

$$(3\cdot 22)\qquad \alpha_k-n\beta_k=\frac{\Delta t}{s_n(k\Delta t)}\left\{\frac{\sigma n^2}{s_n(k\Delta t)}-\frac{3n-1}{6}\times\frac{\Delta s_n(k\Delta t)}{\Delta t}\right\}$$
$$\geqq\frac{\kappa n\Delta t}{2l}\left(\frac{2\sigma n}{\kappa l}-A\right)\geqq 0$$

follows from the first inequality of Assumption C. Furthermore from (3·20) and Assumption B we have

$$(3\cdot 23)\qquad \frac{1}{2(1-\theta)}-\alpha_k\geqq\frac{\kappa\Delta t}{6b}\left[\frac{6\sigma n^2}{\kappa b}\left\{\frac{1}{2(1-\theta)\lambda_b}-1\right\}-A\right]\geqq 0\,.$$

All the inequalities appearing in the assumptions of Lemma 1 are guaranteed by (3·21), (3·22) and (3·23), so that we can apply it.

First note that $a_0{}^k=g(k\Delta t)\geqq 0$, $a_n{}^k=0$, and that $a_j{}^{k-1}\geqq 0$ by the assumption of induction. Then apply Lemma 1 to (3·14), so that we have $a_j{}^k\geqq 0$, $j=1, 2, \cdots, n-1$. Next note that

$$d_0{}^k=As_n(k\Delta t)-a_0{}^k\geqq Ab-g(k\Delta t)\geqq 0\,,\ d_n{}^k=0\,,$$

and that $d_j{}^{k-1}\geqq 0$ by the assumption of induction. Then the application of Lemma 1 results in $d_j{}^k\geqq 0$, $j=1, 2, \cdots, n-1$ by (3·21), verifying (3·19) with k increased one. Q.E.D.

Lemma 2 says that under Assumptions A,B and C the assumption (3·5) of Lemma 1 is satisfied. Therefore we conclude that at each time step our scheme satisfies the maximum principle in the same sense as in Lemma 1. Hence for stability we have

Theorem 1 (*Lumped mass system*). *Under Assumptions A, B and C, the scheme*

$$P_L(k,j)\,a_j{}^k=0\,,\ j=1, 2, \cdots, n-1 \tag{3·24}$$

is stable in the sense that at each time step $k=1, 2, \cdots, m$ (3·24) satisfies the following maximum principle:

$$0\leqq a_j{}^k\leqq \max(a_0{}^k,\ a_{\max}^{k-1}),\ j=1, 2, \cdots, n-1\,. \tag{3·25}$$

In the similar way we can show that Theorem 1 holds for the scheme of the consistent mass system

$$\begin{aligned}\{1-6\theta(\alpha_k-j\beta_k)\}\,a_{j-1}^k+\{4+12\theta\alpha_k\}\,a_j{}^k+\{1-6\theta(\alpha_k+j\beta_k)\}\,a_{j+1}^k\\ =\{1+6(1-\theta)(\alpha_k-j\beta_k)\}\,a_{j-1}^{k-1}+\{4-12(1-\theta)\alpha_k\}\,a_j{}^{k-1}\\ +\{1+6(1-\theta)(\alpha_k+j\beta_k)\}\,a_{j+1}^{k-1}\,,\\ j=1, 2, \cdots, n-1;\ k=1, 2, \cdots, m,\end{aligned} \tag{3·26}$$

if we replace Assumptions B and C by the following ones:

Assumption B.
$$\begin{cases}\lambda_b\left(1+\dfrac{\kappa bA}{6\sigma n^2}\right)\leqq\dfrac{1}{3(1-\theta)}\\[2mm] \dfrac{1}{6\theta}\leqq\lambda_l\left(1-\dfrac{\kappa lA}{2\sigma n}\right)\end{cases}$$

Assumption C. $A \leqq \frac{b}{(1-\theta)\kappa \Delta t}$.

Since $\lambda_l < \lambda_b$, θ must satisfy $1/3 < \theta$.

Assumption B is always essential both in the lumped and the consistent mass system, while Assumption C becomes trivial as $\Delta t \to 0$ and $n \to \infty$. Note that Assumption B corresponds to the stability condition given by Fujii [4] in FEM for the normal heat equation with a fixed boundary.

§ 4. Convergence

This section is concerned with the convergence of the scheme, i.e. we shall show here that $u_n(x, t)$ of $(2\cdot 6)$ converges to the solution of $(1\cdot 1) \sim (1\cdot 4)$ as $\Delta t \to 0$ and $n \to \infty$. Henceforth we shall confine ourselves to the lumped mass system with $\theta = 1$, i.e. to the scheme

$$(4\cdot 1) \qquad -(\alpha_k - j\beta_k) a_{j-1}^k + (1+2\alpha_k) a_j^k - (\alpha_k + j\beta_k) a_{j+1}^k = a_j^{k-1}.$$

First we assume that the limit $\Delta t \to 0$ or $n \to \infty$ is taken under the following constraint condition:

Assumption D. $\lambda_b = \frac{\sigma n^2 \Delta t}{b^2} = \text{constant}.$

Furthermore for the initial and the boundary data we make

Assumption E. $\begin{cases} f(x) \in C^2(x), & g(t) \in C^1(t) \\ f(0) = g(0), & \frac{dg}{dt}(0) = \sigma \frac{d^2 f}{dx^2}(0). \end{cases}$

If we put $j = n-1$ in $(3\cdot 13)$ we have

$$(4\cdot 2) \qquad 0 \leqq \frac{a_{n-1}^k}{h_n(k\Delta t)} \leqq A.$$

From this inequality, $(3\cdot 20)$ and Assumption D, we have estimates for α_k and β_k:

$$(4\cdot3)\qquad \beta_k \leqq \frac{\kappa A}{2b}\Delta t$$

$$(4\cdot4)\qquad n\beta_k \leqq \frac{\kappa A\sqrt{\lambda_b}}{2\sqrt{\sigma}}\Delta t^{1/2}$$

$$(4\cdot5)\qquad \frac{b^2}{l^2}\lambda_b \leqq \alpha_k \leqq \lambda_b + \frac{\kappa A}{6b}\Delta t\,.$$

We extend the approximate solution $u_n(x, t)$, which is given only at the discrete points $t=k\Delta t$, to that given also at intermediate values of t, i.e. at $(k-1)\Delta t<t\leqq k\Delta t$, by interpolating the gradient of $u_n(x, t)$ in the following fashion. First we define $z_n(t)$ which corresponds to the gradient of $u_n(x, t)$ at $x=s_n(t)$ by linear interpolation:

$$(4\cdot6)\qquad z_n(t)=\kappa\frac{a_{n-1}^{k-1}}{h_n((k-1)\Delta t)}+\kappa\left\{\frac{a_{n-1}^{k}}{h_n(k\Delta t)}-\frac{a_{n-1}^{k-1}}{h_n((k-1)\Delta t)}\right\}\frac{t-(k-1)\Delta t}{\Delta t},$$

$$(k-1)\Delta t<t\leqq k\Delta t\,,\ k=1, 2, \cdots, m.$$

Next we define $s_n(t)$ at $(k-1)\Delta t<t\leqq k\Delta t$ based on the similar idea to that of retarding the argument [2], i.e.

$$(4\cdot7)\qquad s_n(t)=\begin{cases} b+\kappa\dfrac{a_{n-1}^{0}}{h_n(0)}t;\ 0\leqq t\leqq \Delta t \\ s_n((k-1)\Delta t)+\displaystyle\int_{(k-1)\Delta t}^{t} z_n(\tau-\Delta t)\,d\tau \\ \qquad ;\ (k-1)\Delta t<t\leqq k\Delta t,\ k=2, 3, \cdots, m\,. \end{cases}$$

If we put $t=k\Delta t$, we have

$$(4\cdot8)\qquad \int_{(k-1)\Delta t}^{k\Delta t} z_n(\tau-\Delta t)\,d\tau=\Delta s_n(k\Delta t),\quad k=2, 3, \cdots, m$$

which is consistent with $(2\cdot25)$. It is clear from this definition that $s_n(t)$ is differentiable. In addition to that, the derivative of $s_n(t)$

$$(4\cdot9)\qquad \frac{ds_n}{dt}(t)=\begin{cases} \kappa\dfrac{a_{n-1}^{0}}{h_n(0)},\quad 0\leqq t\leqq \Delta t \\ z_n(t-\Delta t),\quad \Delta t<t\leqq T \end{cases}$$

is continuous on $0\leqq t\leqq T$ because of the continuity of $z_n(t)$. Since $s_n(t)$ is defined at every t, we can construct the basis functions $\phi_j(x, t)$ for any t by dividing the interval $0\leqq x\leqq s_n(t)$ into n equal subintervals.

By linear interpolation of the gradient of $u_n(x, t)$ in each interval $(j-1)\times h_n(t)<x\leqq jh_n(t)$ based on the values at $t=(k-1)\Delta t$ and $k\Delta t$, we have the extended solution $u_n(x, t)$ for any value of x and t.

By the definition (4·6) and from (4·2) we have

Lemma 3 (*lumped and consistent mass system*). *Under Assumptions A, B and C,*

$$0\leqq\frac{ds_n(t)}{dt}\leqq\kappa A\,. \tag{4·10}$$

From this Lemma we see that $\{s_n(t)\}$ is uniformly bounded on $0\leqq t\leqq T$, i.e.

$$b\leqq s_n(t)\leqq b+\kappa AT=l \text{ for } 0\leqq t\leqq T \tag{4·11}$$

and equi-continuous, so that we can select a subsequence from $\{s_n(t)\}$ that converges, i.e., if we write this subsequence as $\{s_n(t)\}$ again, we have for any $\varepsilon>0$

$$|s_n(t)-s_\infty(t)|<\varepsilon\,, \tag{4·12}$$

where $s_\infty(t)$ is a limit function.

We consider next a solution $u(x, t)$ of the heat equation $(1·1)\sim(1·3)$ in which the boundary $s(t)$ is supposed to be given and fixed as $s_\infty(t)$. The present purpose is to show that $u_n(x, t)$ converges to $u(x, t)$ uniformly as $\Delta t\to0$ $(n\to\infty)$ in $0<t\leqq T$, $0\leqq x<s_\infty(t)$. For that object we introduce an auxiliary function $v_n(x, t)$ which is a solution of $(1·1)\sim(1·3)$ having a fixed boundary $s_n(t)$ instead of $s_\infty(t)$. Note that $v_n(x, t)$ and $u(x, t)$ exist because $s_n(t)$ and $s_\infty(t)$ are uniformly Lipschitz continuous functions as seen from (4·10) (see e.g. [10]).

We compare first $u(x, t)$ with $v_n(x, t)$, and secondly $v_n(x, t)$ with $u_n(x, t)$. For the first step we use

Lemma 4 (Cannon and Hill [2]). *Let $s(t)$ be a monotonic non-decreasing function, and $u(x, t)$ be a solution of* $(1·1)\sim(1·3)$. *Then under Assumption A*

$$0\leqq\rho^{-1}u(s(t)-\rho, t)\leqq A \tag{4·13}$$

for all $0 \leqq t \leqq T,\ 0 < \rho < b.$

Lemma 2 is nothing but a FEM version of this Lemma 4.

We extend u and v_n in such a way that they are identically equal to zero outside the boundaries $s_\infty(t)$ and $s_n(t)$, respectively. Note that u and v_n have common initial and boundary data. For any $\varepsilon > 0$ let $\rho = \varepsilon / A$. For sufficiently large n we have

$$|s_n(t) - s_\infty(t)| \leqq \rho$$

from (4·12). Then using Lemma 4 and from the maximum principle [2], we have

$$|v_n(x, t) - u(x, t)| \leqq A\rho = \varepsilon \tag{4·14}$$

in $0 \leqq x \leqq \max(s_n(t), s_\infty(t))$, $0 \leqq t \leqq T$, which shows the convergence of v_n to u.

In the second step we compare $v_n(x, t)$ with $u_n(x, t)$. In order to prove that $|u_n(x, t) - v_n(x, t)| < \varepsilon$ in $0 \leqq x < s_\infty(t)$, $0 < t \leqq T$ for sufficiently large n, we shall show it in the domain

$$D_\delta \equiv \{(x, t) \mid 0 \leqq x \leqq s_\infty(t) - \delta,\ 0 \leqq t \leqq T\} \tag{4·15}$$

for any arbitrarily small $\delta > 0$. We define

$$\delta' = \min\left(\delta, \frac{\varepsilon}{A}\right), \tag{4·16}$$

and prove that $|u_n - v_n| < \varepsilon$ in $D_{\delta'}$ since the inequality $|u_n - v_n| < \varepsilon$ in $D_{\delta'}$ implies that in D_δ. Let n_0 be a sufficiently large integer. Then there exist such δ_1, $0 < \delta_1 < \delta'$, and $J < n$ for any $n > n_0$ that

$$s_\infty(k\Delta t) - \delta' < x_J{}^k < s_\infty(k\Delta t) - \delta_1, \quad x_J{}^k = \frac{J}{n} s_n(k\Delta t) \tag{4·17}$$

$$s_n(k\Delta t) - \delta' < x_J{}^k < s_n(k\Delta t) - \delta_1, \tag{4·18}$$

for all $k = 0, 1, \cdots, m$ (see Fig. 3). What we plan to do is to show that for sufficiently large n

$$|u_n(x_j{}^k, k\Delta t) - v_n(x_j{}^k, k\Delta t)| < \varepsilon \tag{4·19}$$

for $x_j{}^k = \dfrac{j}{n} s_n(k\Delta t)$; $j = 0, 1, \cdots, J$; $k = 1, 2, \cdots, m$. Note that $\dfrac{\partial v_n}{\partial t}$, $\dfrac{\partial^2 v_n}{\partial x^2}$ and

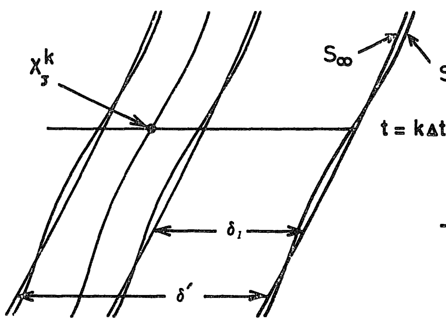

Fig. 3.

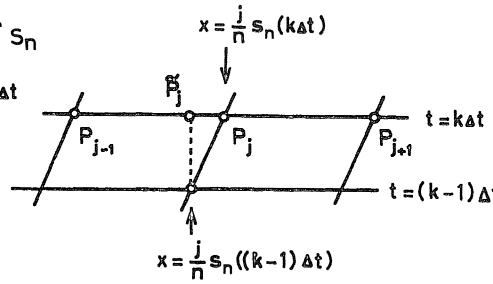

Fig. 4.

$\frac{\partial u}{\partial t}$, $\frac{\partial^2 u}{\partial x^2}$, where u is the limit function of v_n, are uniformly continuous in D_{δ_1} if we fix δ_1 (see e.g. [10]).

Let v_j^k be the value of v_n at the node $P_j\left(x=\frac{j}{n}s_n(k\Delta t),\ t=k\Delta t\right)$, i.e.

$$v_j^k \equiv v_n(x_j^k, k\Delta t), \tag{4·20}$$

and define the difference

$$\varepsilon_j^k \equiv a_j^k - v_j^k . \tag{4·21}$$

Since the initial and the boundary data are common, we have

$$\varepsilon_0^k = 0 , \quad \varepsilon_j^0 = 0 . \tag{4·22}$$

For the difference in the neighborhood of the free boundary we use Lemmas 2 and 4, and obtain an estimate

$$|\varepsilon_J^k| \leqq \max(u_n(x_J^k, k\Delta t), v_n(x_J^k, k\Delta t)) < \varepsilon \tag{4·23}$$

from (4·16) and (4·18).

It is easy to check that ε_j^k satisfies

$$P_L(k, j; \Delta t, s_n, \Delta s_n; 1)\varepsilon_j^k = -P_L(k, j; \Delta t, s_n, \Delta s_n; 1)v_j^k . \tag{4·24}$$

We denote the point $x=\frac{j}{n}s_n((k-1)\Delta t)$, $t=k\Delta t$ as $\tilde{P}_j$ as is shown in Fig. 4, and put

$$\tilde{v}_j^k \equiv j\frac{\Delta s_n}{2s_n}v_{j-1}^k + v_j^k - j\frac{\Delta s_n}{2s_n}v_{j+1}^k \tag{4·25}$$

as an approximate value of v_j^k. $\tilde{v}_j^k$ is the mean value of the linear interpolation of $v_n(x, t)$ based on the abscissas P_{j-1} and P_j and that based on P_j and P_{j+1} in Fig. 4. The difference between v_n at $\tilde{P}_j$ and $\tilde{v}_j^k$ is given as

$$(4\cdot26)\qquad v_n(\tilde{P}_j)-\tilde{v}_j^{\,k}=\frac{1}{2}\left\{\frac{1}{2!}\left(\frac{j}{n}\Delta s_n\right)\left(\frac{s_n-\Delta s_n}{n}\right)\overline{\frac{\partial^2 u}{\partial x^2}}+\frac{1}{2!}\left(\frac{j}{n}\Delta s_n\right)\left(\frac{s_n+\Delta s_n}{n}\right)\overline{\overline{\frac{\partial^2 u}{\partial x^2}}}\right\}$$

by the error formula for the Lagrange interpolation, where $\overline{\frac{\partial^2 u}{\partial x^2}}$ and $\overline{\overline{\frac{\partial^2 u}{\partial x^2}}}$ are values of $\frac{\partial^2 u}{\partial x^2}$ at certain points in the interval of interpolation and are uniformly bounded in D_{δ_1}. On the other hand, since $\Delta s_n=O(\Delta t)$ by (3·2) and (4·3), we have for any $\varepsilon>0$

$$(4\cdot27)\qquad |v_n(\tilde{P}_j)-\tilde{v}_j^{\,k}|<\varepsilon\Delta t$$

for sufficiently large n. Furthermore, if we write the right hand side of (4·24) explicitly, we have

$$(4\cdot28)\qquad -P_L(k,j;\Delta t,s_n,\Delta s_n;1)\,\varepsilon_j^{\,k}$$
$$=\Delta t\left\{\frac{\tilde{v}_j^{\,k}-v_j^{\,k-1}}{\Delta t}-\sigma\left(1+\frac{s_n}{6\sigma n^2}\times\frac{\Delta s_n}{\Delta t}\right)\frac{v_{j-1}^k-2v_j^{\,k}+v_{j+1}^k}{h_n^{\,2}(k\Delta t)}\right\}.$$

Note that $\frac{\partial v_n}{\partial t}$ and $\frac{\partial^2 v_n}{\partial x^2}$ are uniformly continuous in D_{δ_1}, so that for sufficiently large n we have

$$\left|\frac{\partial v_n}{\partial t}(\tilde{P}_j)-\frac{v_n(\tilde{P}_j)-v_j^{\,k-1}}{\Delta t}\right|<\varepsilon$$

and

$$\left|\frac{\partial^2 v_n}{\partial x^2}(\tilde{P}_j)-\frac{v_{j-1}^k-2v_j^{\,k}+v_{j+1}^k}{h_n^{\,2}(k\Delta t)}\right|<\varepsilon$$

for any $\varepsilon>0$. Since v_n satisfies (1·1), we have from (4·27)

$$(4\cdot29)\qquad |P_L(k,j;\Delta t,s_n,\Delta s_n;1)\,\varepsilon_j^{\,k}|<\varepsilon\Delta t$$

for sufficiently large n. Therefore by Lemma 1 together with (4·22) and (4·23) we have

$$(4\cdot30)\qquad |\varepsilon_j^{\,k}|=|v_n(x_j^{\,k},\Delta t)-u_n(x_j^{\,k},\Delta t)|<\varepsilon,\ j=0,1,\cdots,J,$$

showing that $|u_n(x,t)-v_n(x,t)|<\varepsilon$ in $0\leqq x<s_\infty(t)$, $0<t\leqq T$ as $\Delta t\to0$

$(n\to\infty)$.

From (4·14) and (4·30) we conclude that the approximate solution $u_n(x,t)$ converges to $u(x,t)$ in $0\leqq x<s_\infty(t)$, $0<t\leqq T$ as $\Delta t\to 0$ $(n\to\infty)$.

What is left to be proved is that $u(x,t)$, which is the solution of $(1\cdot1)\sim(1\cdot3)$ with the boundary $s_\infty(t)$, satisfies the Stefan condition (1·4). For that purpose we define the second difference of a_j^k:

$$c_j^k=\frac{a_{j-1}^k-2a_j^k+a_{j+1}^k}{h_n^2(k\Delta t)},\quad j=1,2,\cdots,n-1. \tag{4·31}$$

We need the following two lemmas.

Lemma 5 (*lumped mass system with* $\theta=1$). *Under Assumptions A, B, C and E,* $\{c_j^k\}$ *is uniformly bounded, i.e.*

$$|c_j^k|\leqq M,\ j=1,2,\cdots,n-1;\ k=1,2,\cdots,m. \tag{4·32}$$

Proof. We extend the definition of the scheme (4·1), which was originally defined only for $1\leqq j\leqq n-1$, to $j=0$ and n, and define a_{-1}^k and a_{n+1}^k consistently. By this extension c_0^k and c_n^k can also be defined.

Putting $j=0$ in (4·1) and dividing it by Δt, we have

$$\frac{a_0^k-a_0^{k-1}}{\Delta t}=\frac{s_n^2(k\Delta t)}{n^2\Delta t}\alpha_k c_0^k,$$

the left hand side of which is uniformly bounded by Assumption *E*. As to the right hand side we have from Assumption *D* and (4·5)

$$\frac{s_n^2(k\Delta t)}{n^2\Delta t}\alpha_k\geqq\frac{b^2}{l^2}\sigma, \tag{4·33}$$

so that c_0^k is uniformly bounded:

$$|c_0^k|\leqq M_1. \tag{4·34}$$

Putting $j=n$ in (4·1) and dividing it by $h_n(k\Delta t)$, we have

$$(\alpha_k-n\beta_k)\frac{a_{n-1}^k}{h_n(k\Delta t)}+(\alpha_k+n\beta_k)\frac{a_{n+1}^k}{h_n(k\Delta t)}=0$$

since $a_n^k=a_n^{k-1}=0$. $a_{n-1}^k/h_n(k\Delta t)$ is uniformly bounded from (4·2), and hence we see that $a_{n+1}^k/h_n(k\Delta t)$ is also uniformly bounded from $b^2\lambda_b/l^2\leqq\alpha_k+n\beta_k$ and $\alpha_k-n\beta_k\leqq\alpha_k-\beta_k/3\leqq\lambda_b$. If we put $j=n$ again in (4·1) and divide it by $h_n^2(k\Delta t)$, we have

$$\alpha_k c_n^k - \frac{n\beta_k}{h_n(k\Delta t)}\left\{\frac{a_{n-1}^k}{h_n(k\Delta t)} - \frac{a_{n+1}^k}{h_n(k\Delta t)}\right\} = 0$$

which shows, by $n\beta_k/\{\alpha_k h_n(k\Delta t)\} \leqq \kappa A l^2/(2\sigma b^2)$, that c_n^k is also uniformly bounded:

$$(4\cdot 35) \qquad |c_n^k| \leqq M_2 .$$

Furthermore, from Assumption E, we have

$$(4\cdot 36) \qquad \max_{1\leqq j\leqq n-1} |c_j^0| \leqq M_3 .$$

Now if we compute the identity

$$(4\cdot 37) \qquad \{P_L(k,j-1)a_{j-1}^k - 2P_L(k,j)a_j^k + P_L(k,j+1)a_{j+1}^k\}/h_n^2(k\Delta t) = 0$$

using (4·1), we obtain the following scheme similar to (4·1) satisfied by $\{c_j^k\}$:

$$(4\cdot 38) \qquad -\{\alpha_k - (j-1)\beta_k\}c_{j-1}^k + \{1+2(\alpha_k-\beta_k)\}c_j^k - \{\alpha_k + (j+1)\beta_k\}c_{j+1}^k$$
$$= (1-2\beta_k)^2 c_j^{k-1}, \; j=1, 2, \cdots, n-1 \; ; \; k=1, 2, \cdots, m .$$

Let

$$(4\cdot 39) \qquad c_M^k = \max_{0\leqq j\leqq n} |c_j^k|, \quad k=1, 2, \cdots, m .$$

If $c_M^k \neq |c_0^k|$, $|c_n^k|$, we have

$$(4\cdot 40) \qquad c_M^k \leqq \frac{(1-2\beta_k)^2}{1-4\beta_k} c_M^{k-1}$$

from (4·38) under Assumption A, B and C. For sufficiently small Δt, there exists $\mu>0$ such that

$$(4\cdot 41) \qquad \frac{(1-2\beta_k)^2}{1-4\beta_k} \leqq 1 + \mu\Delta t .$$

Hence, if we take into account the case where $|c_j^k|$ attains the maximum at $j=0$ or $j=n$, we have

$$(4\cdot 42) \qquad c_M^k \leqq \max(|c_0^k|, |c_n^k|, (1+\mu\Delta t)c_M^{k-1})$$
$$\leqq (1+\mu\Delta t)\max(|c_0^k|, |c_n^k|, c_M^{k-1})$$
$$\leqq e^{\mu\Delta t}\max(|c_0^k|, |c_n^k|, c_M^{k-1}).$$

Putting $t=k\Delta t$, we conclude from (4·34), (4·35) and (4·36) that

$$(4\cdot 43)\qquad |c_j^k|\leqq e^{\mu t}\max(M_1, M_2, M_3)\leqq e^{\mu T}\max(M_1, M_2, M_3).$$

Q.E.D.

Lemma 6 (*lumped mass system with* $\theta=1$). *Under Assumptions A, B, C and E,*

$$(4\cdot 44)\qquad |z_n(t)-z_n(t-\Delta t)|\leqq M_4\Delta t^{1/2}.$$

Proof. Putting $j=n-1$ in (4·1) and dividing it by Δt, we have

$$(4\cdot 45)\qquad \alpha_k\frac{h_n^2(k\Delta t)}{\Delta t}c_{n-1}^k-\frac{(n-1)\beta_k a_{n-2}^k+a_{n-1}^k-a_{n-1}^{k-1}}{\Delta t}=0.$$

Since c_{n-1}^k is uniformly bounded and

$$(4\cdot 46)\qquad \alpha_k\frac{h_n^2(k\Delta t)}{\Delta t}\leqq\frac{l^2}{b^2}\sigma+\frac{\kappa A\sigma l^2}{6\lambda_b b^3}\Delta t,$$

the first term of the left hand side of (4·45) is uniformly bounded, so that

$$|(n-1)\beta_k a_{n-2}^k+a_{n-1}^k-a_{n-1}^{k-1}|\leqq M\Delta t.$$

Dividing the both sides by $h_n(k\Delta t)$, we have

$$(4\cdot 47)\qquad \left|2(n-1)\beta_k\frac{a_{n-2}^k}{2h_n(k\Delta t)}+\left\{\frac{a_{n-1}^k}{h_n(k\Delta t)}-\frac{a_{n-1}^{k-1}}{h_n((k-1)\Delta t)}\right\}\right.$$
$$\left.+a_{n-1}^{k-1}\left\{\frac{1}{h_n((k-1)\Delta t)}-\frac{1}{h_n(k\Delta t)}\right\}\right|\leqq M\frac{\Delta t}{h_n(k\Delta t)}.$$

If we put $j=2$ in (3·13) of Lemma 2 and use (4·4), we have

$$\left|2(n-1)\beta_k\frac{a_{n-2}^k}{2h_n(k\Delta t)}\right|\leqq\frac{\kappa A^2\sqrt{\lambda_b}}{\sqrt{\sigma}}\Delta t^{1/2}.$$

From (4·3) we have

$$\left|\frac{1}{h_n((k-1)\Delta t)}-\frac{1}{h_n(k\Delta t)}\right|=\left|\frac{2n\beta_k}{s_n(k\Delta t)-\Delta s_n(k\Delta t)}\right|\leqq\frac{\kappa A\sqrt{\lambda_b}}{\sqrt{\sigma}b}\Delta t^{1/2}.$$

As to the right hand side of (4·47), we have an estimate

$$\frac{\Delta t}{h_n(k\Delta t)}=\sqrt{\frac{n^2\Delta t}{s_n^2(k\Delta t)}}\Delta t^{1/2}\leqq\sqrt{\frac{\lambda_b}{\sigma}}\Delta t^{1/2},$$

so that for sufficiently large n we obtain from (4·47)

$$(4\cdot48)\qquad \left|\frac{a_{n-1}^{k}}{h_n(k\Delta t)}-\frac{a_{n-1}^{k-1}}{h_n((k-1)\Delta t)}\right|\leqq M'\Delta t^{1/2}.$$

This inequality in combination with the definition (4·6) gives (4·44).

Q.E.D.

It is easy to see from Lemma 5 and from the manner in which we extended $u_n(x, t)$ from $t=k\Delta t$ to intermediate values of t that

$$\lim_{x\to s_n(t)}\frac{\partial}{\partial x}u_n(x, t)\quad\text{and}\quad\lim_{x\to s_\infty(t)}\frac{\partial}{\partial x}u(x, t)$$

exist (cf. Lemma 1 of [2]). Furthermore from (4·6)

$$(4\cdot49)\qquad \lim_{x\to s_n(t)}-\kappa\frac{\partial u_n}{\partial t}(x, t)=z_n(t)\quad(\text{uniformly in } 0<t\leqq T),$$

so that we have

$$(4\cdot50)\qquad \lim_{n\to\infty}z_n(t)=-\kappa\frac{\partial u}{\partial x}(s_\infty(t), t)\quad(\text{uniformly in } 0<t\leqq T).$$

By the definition (4·7), on the other hand, we have

$$\begin{aligned}(4\cdot51)\qquad s_n(t)&=b+\kappa\frac{a_{n-1}^{0}}{h_n(0)}\Delta t+\int_{\Delta t}^{t}z_n(\tau-\Delta t)\,d\tau\\ &=b+\kappa\frac{a_{n-1}^{0}}{h_n(0)}\Delta t+\int_{\Delta t}^{t}z_n(\tau)\,d\tau\\ &\qquad+\int_{\Delta t}^{t}\{z_n(\tau-\Delta t)-z_n(\tau)\}\,d\tau,\quad \Delta t\leqq t\leqq T.\end{aligned}$$

If we let $\Delta t\to0$ in (4·51), we have

$$(4\cdot52)\qquad s_\infty(t)=b-\kappa\int_0^t\frac{\partial u}{\partial x}(s_\infty(\tau), \tau)\,d\tau$$

from Lemma 6 and (4·50), and so we conclude

$$(4\cdot53)\qquad \frac{ds_\infty(t)}{dt}=-\kappa\frac{\partial u}{\partial x}(s_\infty(t), t).$$

We proved the convergence of the scheme so far, but we also proved the existence of the solution of (1·1)$\sim$(1·4) under Assumptions A and E.

Cannon and Hill [2] proved the uniqueness of the solution of $(1\cdot1)\sim(1\cdot4)$, so that we have the following convergence

Theorem 2 (*lumped mass system with* $\theta=1$). *Under Assumption A, B, C, D and E, the approximate solution obtained by* $(2\cdot23)\sim(2\cdot26)$ *converges to the unique solution of the Stefan problem* $(1\cdot1)\sim(1\cdot4)$ *as* $\Delta t\to0$ $(n\to\infty)$.

§ 5. Numerical Example

We shall show here a numerical result of the application of the present scheme to the following model problem:

$$(5\cdot1)\qquad \frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2},\quad 0<x<s(t),\quad 0<t\leqq1$$

$$(5\cdot2)\qquad u(0,t)=g(t)=1-\frac{1}{2}t,\quad 0\leqq t\leqq1$$

$$(5\cdot3)\qquad u(x,0)=f(x)=1-x,\quad b=s(0)=1$$

$$(5\cdot4)\qquad \frac{ds}{dt}=-u_x(s(t),t).$$

The actual computation was carried out using the lumped mass system with $\theta=1$. We employed $n=8, 16, 32, 64$ and $m=4n^2$, i.e. $\Delta t=1/(4n^2)$. In this example, $A=1$, $l=2$, $\lambda_b=1/16$ and $\lambda_l=1/4$, so that all Assumptions A, B, C, D and E are satisfied except $\frac{dg}{dt}(0)=\frac{d^2f}{dx^2}(0)$. Fig. 5 represents the solution $u_n(x,t)$ for $n=16$, $m=1024$, and Fig. 6 represents $s_n(t)$.

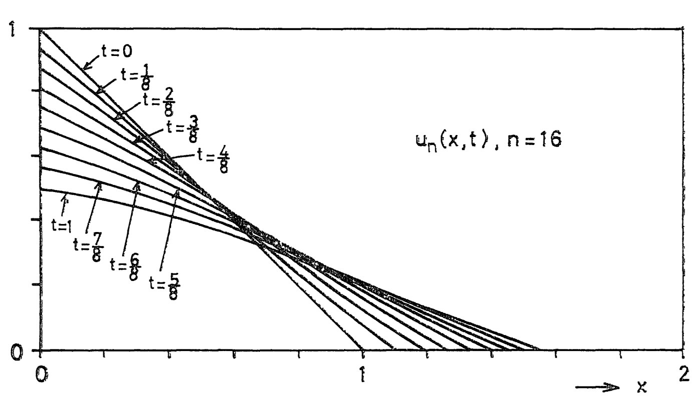

Fig. 5. Approximate solution $u_n(x,t)$.

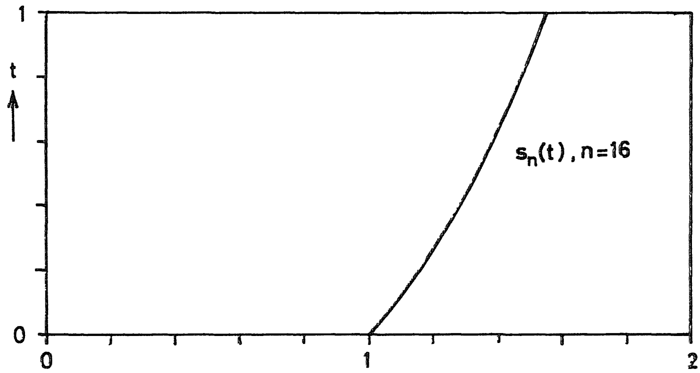

Fig. 6. The change of the free boundary $s_n(t)$.

In order to see the speed of convergence, we show the differences between the results for $n=8$, 16, 32 and that for $n=64$ in Table 1. Both the rate of convergence of $s_n(t)$ and that of $u_n(x, t)$ seem to be approximately of the order of $1/n$.

Table 1. The rate of convergence. $u_n(j, 1)$ and $u_{64}(j, 1)$ are the values at $x = \frac{j}{n}s_n(1)$ and $t=1$, and Δt is equal to $1/m$.

n	m	$\max_{0\leq j\leq n} \lvert u_n(j, 1)-u_{64}(j, 1)\rvert$	$\max_{0\leq k\leq m} \lvert s_n(k\Delta t)-s_{64}(k\Delta t)\rvert$
8	256	3.34×10^{-4}	15.4×10^{-3}
16	1024	1.55×10^{-4}	6.41×10^{-3}
32	4096	0.539×10^{-4}	2.11×10^{-3}

Finally we note that the present idea is easy to apply to the two phase problem or to problems in higher space dimension [8].

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