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On a Non-Linear Semi-Group Attached to Stochastic Optimal Control

By

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§ 1. Introduction

In [6] we introduced a non-linear semi-group attached to the stochastic control of diffusion type, by the following way. Let Γ be a σ-compact subset of $\boldsymbol{R}^k$, called by a control region. Let a triple (Ω, B, U) be an admissible system where Ω is a probability space, B is an n-dimensional Brownian motion on Ω and U is a Γ-valued B-non-anticipative process on Ω. For an admissible system (Ω, B, U) we consider the following n-dimensional stochastic differential equation

$$dX(t) = \alpha(X(t), U(t))\,dB(t) + \gamma(X(t), U(t))\,dt \tag{1}$$

where $\alpha(x, u)$ is a symmetric $n\times n$-matrix and $\gamma(x, u)$ an n-vector. Under the condition of smoothness and boundness of the coefficients α and γ, there exists a unique solution X, which is called the response for U.

By C we denote the Banach lattice of all bounded and uniformly continuous functions on $\boldsymbol{R}^n$ endowed with the usual supremum norm and the usual order. Let $c(x, u)$ be non-negative and $f(x, u)$ real. We assume that both c and f are smooth and bounded. For any $\phi\in C$ we define Q_t by

$$Q_t\phi(x) = \sup_{\text{adm. syst.}} E_x \int_0^t \exp\left\{-\int_0^s c(X(\theta), U(\theta))\,d\theta\right\} \times f(X(s), U(s))\,ds + \exp\left\{-\int_0^t c(X(\theta), U(\theta))\,d\theta\right\}\phi(X(t)), \tag{2}$$

where X is the response for U, starting at $X(0)=x$. Then Q_t is a strongly continuous non-linear semi-group on C, which is contractive and

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monotone. Moreover the generator G of Q_t is given by

$$G\phi=\sup_{u\in\Gamma}\,[A^u\phi+f^u] \tag{3}$$

$$A^u\phi(x)=\frac{1}{2}\sum_{ij}\alpha^2(x,u)_{ij}\frac{\partial^2\phi}{\partial x_i\partial x_j}(x) + \sum_i \gamma_i(x,u)\frac{\partial\phi}{\partial x_i}(x)-c(x,u)\phi(x) \tag{4}$$

for ϕ whose first and second derivatives are in C. The right side of (3) can be found in the famous Bellman equation, [2], [4]. Furthermore the least Q_t-excessive majorant has a close relation to the optimal stopping problem, [3], [4].

In this note we shall discuss a similar problem in a more general set-up. Let A^u be the generator of a Markov process. We seek a semi-group of operators acting on $L_\infty(\boldsymbol{R}^n,\mu)$ whose generator is an extension of $G\phi=\sup_u(A^u\phi+f^u)$. Such a semi-group (with generator G) will be obtained as the envelope of the semi-groups

$$T_t^u\phi=P_t^u\phi+\int_0^t P_\theta^u f^u d\theta \quad u\in\Gamma$$

whose generators are

$$G^u\phi=A^u\phi+f^u\,,\quad u\in\Gamma$$

respectively, as we can image from the fact that G is the envelope of $G^u, u\in\Gamma$. In fact we will prove the following theorem in § 3.

Theorem 1. *Let A^u be the generator of positive contractive and strongly continuous linear semi-group P_t^u on $L_\infty(\boldsymbol{R}^n,\mu)$. We assume the following conditions* (A1)$\sim$(A3).

(A1) *If $\phi_n\in L_\infty(\boldsymbol{R}^n,\mu)$ is an increasing sequence tending to $\phi\in L_\infty(\boldsymbol{R}^n,\mu)$ $\mu-a.e.$, then $P_t^u\phi_n$ increases and tends to $P_t^u\phi$ $\mu-a.e.$ for every $u\in\Gamma$ and every $t\geq 0$.*

(A2) *Let $D(A^u)$ denote the domain of the generator A^u. The subset D of $L_\infty(\boldsymbol{R}^n,\mu)$ defined by*

$$D=\{\phi\in\bigcap_u D(A^u)\,;\,\sup_u\|A^u\phi\|<\infty\}$$

is strongly dense in $L_\infty(\boldsymbol{R}^n, \mu)$.

(A3) $$\sup_u \|f^u\| < \infty .$$

Then there exists a unique non-linear semi-group S_t *on* $L_\infty(\boldsymbol{R}^n, \mu)$ *satisfying the following conditions* (0) ~ (vi):

(0) *semi-group property*: $S_0 = identity$, $S_{t+\theta}\phi = S_t(S_\theta\phi) = S_\theta(S_t\phi)$,

(i) *monotone*: $S_t\phi \leq S_t\psi$, *whenever* $\phi \leq \psi$,

(ii) *contractive*: $\|S_t\phi - S_t\psi\| \leq \|\phi - \psi\|$,

(iii) *strongly continuous*: $\|S_t\phi - S_\theta\phi\| \to 0$, *as* $t \to \theta$,

(iv) $P_t^u\phi + \int_0^t P_\theta^u f^u d\theta \leq S_t\phi$, *for* $\forall t$ *and* u, *where the integral stands for the Bochner integral*,

(v) *the generator* G *of* S_t *is expressed by*

(5) $$G\phi = \sup_u [A^u\phi + f^u] \quad for \quad \phi \in D(G) \cap D ,$$

(vi) *minimum*: *if* $\tilde{S}_t$ *is a non-linear semi-group with* (i) ~ (iv), *then*

$$S_t\phi \leq \tilde{S}_t\phi .$$

In § 4, we shall show the existence of the least S_t-excessive function.

Theorem 2. *Suppose that there exists a positive* c *such that* $|P_t^u| \leq e^{-ct}$ *for any* u. *Then, for any* $g \in L_\infty(\boldsymbol{R}^n, \mu)$, *there exists a unique* $v \in L_\infty(\boldsymbol{R}^n, \mu)$ *such that*

(i) S_t*-excessive majorant of* g: $g \leq v$ *and* $S_t v \leq v$ $\forall t \geq 0$

(ii) *least*: *if* V *is an* S_t*-excessive majorant of* g, *then* $v \leq V$.

In § 5 we will mention two simple examples as applications of our results. Since we formulate control problems in terms of non-linear semi-groups on $L_\infty(\boldsymbol{R}^n, \mu)$ in this note, the stochastic control of diffusion type does not lie in our framework, but some optimal controls can be treated in our way, as we shall see in § 5.

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§ 2. Preliminaries

Let μ be a σ-finite measure on $\boldsymbol{R}^n$. Let $L(\equiv L_\infty(\boldsymbol{R}^n, \mu))$ denote the set of all Borel measurable, essential bounded functions, defined μ-a.e. on $\boldsymbol{R}^n$. L becomes a complete Banach lattice by the usual norm and partial order, [cf. 7], i.e.

$$\|\phi\| \equiv \operatorname*{ess.sup.}_{x \in \boldsymbol{R}^n} |\phi(x)|$$

and "$\phi \leq \psi$" is defined by "$\phi(x) \leq \psi(x)$, μ-*a.e.*" A subset $\{\phi_\alpha\}$ of L is said to be O-bounded, if there exist $\underline{\psi}$ and $\overline{\psi}$ in L such that

$$\underline{\psi} \leq \psi_\alpha \leq \overline{\psi}, \quad \forall \alpha .$$

Hence a subset $\{\psi_\alpha\}$ of L is O-bounded, if and only if "$\sup_\alpha \|\psi_\alpha\| < \infty$". When $\psi_n \in L$ increasingly tends to $\psi \in L$, we say $\psi = O_i - \lim_n \psi_n$. Hence, if $\psi = O_i - \lim \psi_n$, then "$\sup_u \|\psi_n\| < \infty$". In this note we often use the following well-known facts,

Proposition 1. *For any O-bounded set* $\{\psi_\alpha\}$ *of L there uniquely exist* ψ^+ *and* ψ^- *in L such that*

(i) $\psi_\alpha \leq \psi^+$, $\forall \alpha$

(ii) *if* ψ *satisfies* "$\psi_\alpha \leq \psi$, $\forall \alpha$", *then* $\psi^+ \leq \psi$,

and

(i)′ $\psi^- \leq \psi_\alpha$, $\forall \alpha$

(ii)′ *if* ψ *satisfies* "$\psi \leq \psi_\alpha$, $\forall \alpha$", *then* $\psi \leq \psi^-$,

$\sup \psi_\alpha$ *and* $\inf \psi_\alpha$ *are denoted by* ψ^+ *and* ψ^- *respectively.*

Moreover,

$$\inf(\phi_\alpha - \psi_\alpha) \leq \sup \phi_\alpha - \sup \psi_\alpha \leq \sup(\phi_\alpha - \psi_\alpha).$$

$$\|\sup \phi_\alpha - \sup \psi_\alpha\| \leq \sup \|\phi_\alpha - \psi_\alpha\| .$$

Let $T_t\phi$ *be strongly continuous in t. Then* $T_t\phi$ *has a* (t, x)*-Borel measurable version which is continuous in t.*

Proof. Let $\{r_i\}$ be countable and dense in $[0, \infty)$ and $\Phi(r_i, \cdot)$ a Borel measurable version of $T_{r_i}\phi$. Then the set Σ of $\{x \in \boldsymbol{R}^n;\ |\Phi(r_i, x) - \Phi(r_j, x)| \leq \|T_{r_j}\phi - T_{r_i}\phi\|\ \forall ij\}$ is μ-full. On the other hand, for any positives ε and l, there exists a positive δ such that

$$\|T_t\phi - T_\theta\phi\| < \varepsilon \quad \text{whenever} \quad |t-\theta| < \delta \quad \text{and} \quad 0 \leq t,\ \theta \leq l.$$

Hence, for $x \in \Sigma$, $\Phi(r_i, x)$ is uniformly continuous on $\{r_i\} \subset [0, l]$. Thus, $\Phi(\cdot, x)$ can be extended to a continuous function $\tilde{\Phi}(\cdot, x)$ on $[0, l]$. Letting l tend to ∞, we get our wanted version $\tilde{\Phi}$.

The Bochner integral $\int_0^t T_\theta \phi d\theta$ can understood as the usual Rieman integral $\int_0^t \tilde{\Phi}(\theta, x) d\theta$.

Let P_t be a positive, contractive and strongly continuous linear semi-group on L. Define T_t for $f \in L$ by

$$T_t\phi = P_t\phi + \int_0^t P_\theta f d\theta, \quad \phi \in L. \tag{1}$$

Then T_t is a mapping from L into L and has the following properties

(T0) semi-group property: $T_0\phi = \phi$, $T_{t+\theta}\phi = T_t(T_\theta\phi) = T_\theta(T_t\phi)$,

(T1) monotone: $T_t\phi \leq T_t\psi$ whenever $\phi \leq \psi$,

(T2) contractive; $\|T_t\phi - T_t\psi\| \leq \|\phi - \phi\|$

(T3) strongly continuous: $\|T_t\phi - T_\theta\phi\| \to 0$ as $t \to \theta$

(T4) the generator G of T_t: Let A be the generator of P_t. Then $D(G) = D(A)$ and

$$G\phi = A\phi + f \tag{2}$$

$$\text{(T5)} \qquad T_t\phi - \phi = \int_0^t P_\theta G\phi d\theta \quad \forall \phi \in D(G).$$

Proof. Since (T1), (T2) and (T3) are obvious, we shall only show (T0), (T4) and (T5).

(T0). $T_{t+\theta}\phi = P_{t+\theta}\phi + \int_0^{t+\theta} P_s f ds = P_\theta(P_t\phi) + \int_\theta^{t+\theta} P_s f ds + \int_0^\theta P_s f ds$

$$= P_\theta\left(P_t\phi + \int_0^t P_s f ds\right) + \int_0^\theta P_s f ds = P_\theta(T_t\phi) + \int_0^\theta P_s f ds = T_\theta(T_t\phi).$$

(T4). For $\varepsilon > 0$, there exists a positive δ such that $\|P_\theta f - f\| < \varepsilon$ for $\theta < \delta$. Hence

$$\left\|\frac{1}{t}\int_0^t P_\theta f d\theta - f\right\| = \left\|\frac{1}{t}\int_0^t (P_\theta f - f) d\theta\right\|$$

$$\leq \frac{1}{t}\int_0^t \|P_\theta f - f\| d\theta < \varepsilon \quad \text{for} \quad t < \delta.$$

Therefore $\lim_{t\downarrow 0}\frac{1}{t}(T_t\phi-\phi)$ exists if and only if $\lim_{t\downarrow 0}\frac{1}{t}(P_t\phi-\phi)$ exists. Moreover (2) is valid.

(T5). For any $\phi\in D(A)$, we have

$$T_t\phi-\phi=P_t\phi-\phi+\int_0^t P_\theta f d\theta$$

$$=\int_0^t P_\theta A\phi d\theta+\int_0^t P_\theta f d\theta=\int_0^t P_\theta(A\phi+f)\,d\theta\,.$$

Proposition 2. *Suppose* (A1) *and* (A3). *If* $\phi=O_i-\lim\phi_n$ *then*

(3) $$\sup_u T_t^u\phi=O_i-\lim_n\sup_u T_t^u\phi_n.$$

Proof. Since T_t^u satisfies (T1) and (T2), we have $T_t^u\phi_n\leq T_t^u\phi_{n+1}$ and

(4) $$\|T_t^u\phi_n\|\leq\|T_t^u\phi_n-T_t^uO\|+\|T_t^uO\|\leq\|\phi_n\|+\sup\|f^u\|t\,.$$

Thus $\sup_n T_t^u\phi_n$ is increasing as $n\to\infty$ and the set $\{\sup_u T_t^u\phi_n, n=1, 2, \cdots\}$ is O-bounded. Therefore

(5) $$O_i-\lim_n\sup_u T_t^u\phi_n\leq\sup_u T_t^u\phi\,.$$

On the other hand, from (A1) we can derive, for any u

(6) $$T_t^u\phi=O_i-\lim_n T_t^u\phi_n\leq O_i-\lim_n\sup_u T_t^u\phi_n\,.$$

By (5) and (6) we conclude Proposition 2.

§ 3. Proof of Theorem 1

We shall construct our required semi-group S_t. Define $J=J(N)$ by

(1) $$J\phi=\sup_u T^u_{1/2^N}\phi\,,\ \ \phi\in L\,.$$

Then J is a mapping from L into L. Define J^k by

$$J^{k+1}\phi=J(J^k\phi)\quad\text{and}\quad J^0\phi=\phi\,.$$

Lemma 1. J^k *has the following properties,*

(J0) $J^{k+l}\phi=J^k(J^l\phi)=J^l(J^k\phi)$,

(J1) *monotone*: $J^k\phi\leq J^k\psi$ *whenever* $\phi\leq\psi$, (J2) *contractive*: $\|J^k\phi - J^k\psi\| \leq \|\phi - \psi\|$

(J3) $$\|J^k\phi - \phi\| \leq \frac{k}{2^N} (\sup_u \|A^u\phi\| + \sup_u \|f^u\|) \quad for \quad \phi \in D,$$

(J4) $$T^u_{k/2^N}\phi \leq J^k\phi \,,$$

(J5) $$J^k\phi = O_i - \lim_n J^k\phi_n \quad if \quad \phi = O_i - \lim_n \phi_n \,.$$

Proof. Since T_t^u is monotone, we have

$$J\phi \leq J\psi \text{ whenever } \phi \leq \psi.$$

Hence we can show (J1) by induction.

Put $\Delta = \frac{1}{2^N}$. The following evaluation is clear,

$$\|J\phi - J\psi\| = \|\sup_u T_\Delta^u\phi - \sup_u T_\Delta^u\psi\| \leq \sup_u \|T_\Delta^u\phi - T_\Delta^u\psi\| \leq \|\phi - \psi\| \,.$$

Thus if we assume that (J2) holds for k, then

$$\|J^{k+1}\phi - J^{k+1}\psi\| = \|J(J^k\phi) - J(J^k\psi)\| \leq \|J^k\phi - J^k\psi\| \leq \|\phi - \psi\|$$

namely (J2) holds for $k+1$.

Put $K(\phi) = \sup_u \|A^u\phi\| + \sup_u \|f^u\|$. Recalling (T5) we have, for $\phi \in D$

$$T_\Delta^u\phi - \phi = \int_0^\Delta P_\theta^u A^u\phi d\theta + \int_0^\Delta P_\theta^u f^u d\theta \,.$$

So

$$\|J\phi - \phi\| \leq \sup_u \|T_\Delta^u\phi - \phi\| \leq \Delta K(\phi) \,.$$

Therefore by (J2) we see

$$\|J^k\phi - \phi\| \leq \sum_{j=1}^k \|J^j\phi - J^{j-1}\phi\| = \sum_{j=1}^k \|J^{j-1}(J\phi) - J^{j-1}\phi\|$$
$$\leq k\|J\phi - \phi\| \leq k\Delta \cdot K(\phi) \,.$$

This completes the proof of (J3).

By the definition of J we get

$$T_\Delta^u\psi \leq J\psi \quad \forall \psi \in L \,.$$

Hence, if we assume that (J4) holds for k, then

$$T^u_{(k+1)\Delta}\phi = T_\Delta^u(T^u_{k\Delta}\phi) \leq T_\Delta^u(J^k\phi) \leq J(J^k\phi) = J^{k+1}\phi \,,$$

namely (J4) holds for $k+1$.

For $k=1$, (J5) is Proposition 2 in § 2. If (J5) holds for k, then

$$J^{k+1}\phi=J(J^{k}\phi)=J(O_i-\lim J^{k}\phi_n)=O_i-\lim J(J^{k}\phi_n)=O_i-\lim J^{k+1}\phi_n\,.$$

Therefore we get (J5).

Put $S_t^{(N)}\phi=J^{k}(N)\phi$ for $t=\dfrac{k}{2^N}$, $k=0, 1, 2, \cdots$.

Lemma 2. *$S_t^{(N)}$ is increasing as $N\to\infty$, i.e.*

$$S_t^{(N)}\phi\leq S_t^{(N+1)}\phi \quad for \quad t=\frac{k}{2^N}\,. \tag{2}$$

Proof. Put $\Delta=1/2^{N+1}$. Recalling (T0) and (T1), we have

$$T^u_{2\Delta}\phi=T_\Delta{}^u(T_\Delta{}^u\phi)\leq T_\Delta{}^u(S_\Delta{}^{(N+1)}\phi). \tag{3}$$

Taking the supremum of both sides, we get

$$S^{(N)}_{2\Delta}\phi\leq S_\Delta{}^{(N+1)}(S_\Delta{}^{(N+1)}\phi)=S^{(N+1)}_{2\Delta}\phi\,, \tag{4}$$

namely (2) is valid for $k=1$. If (2) holds for k, then

$$\begin{aligned} S^{(N)}_{2(k+1)\Delta}\phi&=S^{(N)}_{2\Delta}(S^{(N)}_{2k\Delta}\phi)\leq S^{(N)}_{2\Delta}(S^{(N+1)}_{2k\Delta}\phi)\\ &\leq S^{(N+1)}_{2\Delta}(S^{(N+1)}_{2k\Delta}\phi)=S^{(N+1)}_{2(k+1)\Delta}\phi\,. \end{aligned} \tag{5}$$

This completes the proof of Lemma 2.

Hereafter we put $h=\sup_u\|f^u\|$. By virtue of (J2), putting $\Delta=\dfrac{1}{2^N}$ and $t=k\Delta$ we have

$$\|S_t^{(N)}\phi\|\leq\|S_t^{(N)}\phi-S_t^{(N)}O\|+\|S_t^{(N)}O\|\leq\|\phi\|+\|S_t^{(N)}O\| \tag{6}$$

and

$$\|S_\Delta{}^{(N)}O\|\leq\sup_u\|\int_0^\Delta P_\theta{}^u f^u d\theta\|\leq\Delta h\,.$$

Suppose $\|S^{(N)}_{k\Delta}O\|\leq k\Delta h$. Then

$$\begin{aligned} \|S^{(N)}_{(k+1)\Delta}O\|&=\|S^{(N)}_\Delta(S^{(N)}_{k\Delta}O)\|\leq\sup_u\|T_\Delta{}^u(S^{(N)}_{k\Delta}O)\|\\ &\leq\|S^{(N)}_{k\Delta}O\|+\Delta h\leq(k+1)\Delta h\,. \end{aligned} \tag{7}$$

Hence we have

(8) $$\|S_t^{(N)}\phi\| \leq \|\phi\| + th\,.$$

This implies that, for any fixed binary $t=\frac{j}{2^l}$, the set $\{S_t^{(N)}\phi, N\geq l\}$ is O-bounded. So we can define S_t by

(9) $$S_t\phi = O_i - \lim_n S_t^{(n)}\phi \quad \text{for binary} \quad t.$$

S_t has the following properties:

Lemma 3. *For binary t and θ,*

(S0) $S_0\phi=\phi$,

(S1) *monotone*: $S_t\phi \leq S_t\psi$, *whenever* $\phi\leq\psi$,

(S2) *contractive*: $\|S_t\phi - S_t\psi\| \leq \|\phi-\psi\|$

(S3) $\|S_t\phi - S_\theta\psi\| \leq |t-\theta| K(\phi)$ *for* $\phi\in D$,

(S4) $T_t^u\phi \leq S_t\phi$.

Proof. From the definition of S_t and Lemma 1, these properties are clear. We shall only show (S3). Put $t=\frac{i}{2^l}$ and $\theta=\frac{j}{2^l}$, $(j\leq i)$. For any $N\geq l$, we have

$$\|S_t^{(N)}\phi - S_\theta^{(N)}\phi\| = \|S_\theta^{(N)}(S_{t-\theta}^{(N)}\phi) - S_\theta^{(N)}\phi\| \leq \|S_{t-\theta}^{(N)}\phi - \phi\| \leq |t-\theta| K(\phi).$$

Since $S_t^{(N)}\phi - S_\theta^{(N)}\phi$ converges to $S_t\phi - S_\theta\phi$ μ-a.e. as $N\to\infty$, we get

$$\|S_t\phi - S_\theta\phi\| \leq \varliminf_{N\to\infty} \|S_t^{(N)}\phi - S_\theta^{(N)}\phi\| \leq |t-\theta| K(\phi).$$

Using (S3) we can define $S_t\phi$, $t\geq 0$, by

(10) $$S_t\phi = \lim S_{t_l}\phi\,,\ \phi\in D\,,$$

where $\{t_l\}$ is a sequence of binary times approximating t. (S3) implies that the left side of (10) does not depend on the special choice of $\{t_l\}$. Moreover (S1) $\sim$ (S4) hold.

Lemma 3′. *For $\theta, t\geq 0$ and $\psi, \phi\in D$,*

(S1)′ *monotone*: $S_t\phi \leq S_t\psi$ *whenever* $\phi\leq\psi$,

(S2)′ *contractive*: $\|S_t\phi - S_t\psi\| \leq \|\phi-\psi\|$,

(S3)′ $\|S_t\phi - S_\theta\phi\| \leq |t-\theta| K(\phi)$,

(S4)′ $T_t^u\phi \leq S_t\phi$.

Recalling (A2) and (S2)′, we can extend S_t on L by

(11) $$S_t\phi=\lim S_t\phi_n\,,\ \phi\in L\,,$$

where $\{\phi_n\}$ is a sequence of functions in D approximating ϕ.

Proposition 3. *S_t has the following properties*

(i) *monotone*: $S_t\phi\leq S_t\psi$ *whenever* $\phi\leq\psi$,

(ii) *contractive*: $\|S_t\phi-S_t\psi\|\leq\|\phi-\psi\|$,

(iii) *strongly continuous*: $\|S_t\phi-S_\theta\phi\|\to 0$ *as* $t\to\theta$,

(iv) $T_t^u\phi\leq S_t\phi$.

Proof. First we shall show (ii). Take $\phi_n\in D$ and $\psi_n\in D$ approximating ϕ and ψ respectively. Hence

$$\|S_t\phi-S_t\psi\|\leq\lim_n\|S_t\phi_n-S_t\psi_n\|\leq\lim_n\|\phi_n-\psi_n\|=\|\phi-\psi\|.$$

(i). For $\varepsilon>0$, we take an approximation $\phi_n(\varepsilon)\in D$ to $\phi-\varepsilon$. Let $\psi_n\in D$ approximate ψ. Then, for large n.

$$\phi_n(\varepsilon)\leq\psi_n\,.$$

Hence, by (S1)′,

$$S_t\phi_n(\varepsilon)\leq S_t\psi_n\quad\text{for large } n.$$

Therefore tending n to ∞ we have

$$S_t(\phi-\varepsilon)\leq S_t\psi\,.$$

On the other hand $\phi-\varepsilon$ converges to ϕ, so (ii) implies $S_t\phi=\lim_{\varepsilon\downarrow 0}S_t(\phi-\varepsilon)$. Hence

$$S_t\phi\leq S_t\psi\,.$$

(iii). For $\varepsilon>0$, we take $\psi\in D$ such that $\|\phi-\psi\|<\varepsilon$. Then we have

$$\begin{aligned}\|S_t\phi-S_\theta\phi\|&\leq\|S_t\phi-S_t\psi\|+\|S_t\psi-S_\theta\psi\|+\|S_\theta\psi-S_\theta\phi\|\\&<2\varepsilon+\|S_t\psi-S_\theta\psi\|\leq 2\varepsilon+|t-\theta|\,K(\psi)\,.\end{aligned}$$

Hence there exists a small positive $\delta=\delta(\phi,\varepsilon)$ such that $\|S_t\phi-S_\theta\phi\|<3\varepsilon$ whenever $|t-\theta|<\delta$.

(iv). By (S4)′ we have $T_t^u\phi_n\leq S_t\phi_n$ where $\phi_n\in D$ tends to ϕ. Letting n tend to ∞, we get (iv).

Proposition 4. *S_t is a semi-group on L.*

Proof. Let t and θ be binary, say $t=\frac{i}{2^l}$ and $\theta=\frac{j}{2^l}$. For $N\geq l$, we have

(12) $$S_{t+\theta}^{(N)}\phi=S_\theta^{(N)}(S_t^{(N)}\phi)\leq S_\theta^{(N)}(S_t\phi),$$

(13) $$S_\theta(S_t\phi)=O_i-\lim_N S_\theta^{(N)}(S_t\phi),$$

and

(14) $$S_{\theta+t}\phi=O_i-\lim_N S_{\theta+t}^{(N)}\phi .$$

Hence

(15) $$S_{\theta+t}\phi\leq O_i-\lim_N S_\theta^{(N)}(S_t\phi)=S_\theta(S_t\phi).$$

On the other hand, for $l\leq n\leq N$, we see

$$S_\theta^{(n)}(S_t^{(N)}\phi)\leq S_\theta^{(N)}(S_t^{(N)}\phi)=S_{\theta+t}^{(N)}\phi\leq S_{\theta+t}\phi$$

and recalling (J5) of Lemma 1 we have

$$S_\theta^{(n)}(S_t\phi)=O_i-\lim_N S_\theta^{(n)}(S_t^{(N)}\phi).$$

Therefore, for $n\geq l$,

$$S_\theta^{(n)}(S_t\phi)\leq S_{\theta+t}\phi .$$

Tending n to ∞, we get

(16) $$S_\theta(S_t\phi)\leq S_{\theta+t}\phi .$$

From (15) and (16) we have

(17) $$S_\theta(S_t\phi)=S_{\theta+t}\phi \quad \text{for binary } t \text{ and } \theta .$$

Let t_n be a binary approximation to t. Then for any binary θ,

$$S_\theta(S_{t_n}\phi)=S_{\theta+t_n}\phi .$$

So appealing to (ii) and (iii) we get

(18) $$S_\theta(S_t\phi)=S_{\theta+t}\phi \quad \text{for binary } \theta.$$

Again by virtue of (iii) we obtain the semi-group property of S_t.

Let G be the generator of S_t, namely

$$G\phi=\lim_{t\downarrow 0}\frac{1}{t}(S_t\phi-\phi)$$

and

$$D(G)=\left\{\phi\in L, \lim_{t\downarrow 0}\frac{1}{t}(S_t\phi-\phi) \text{ exists}\right\}.$$

Proposition 5.

(19) $$G\phi=\sup_u(A^u\phi+f^u) \quad \textit{for} \quad \phi\in D(G)\cap D.$$

Moreover, if $f^u\in D(A^u)$ *and* $\sup_u \|A^u f^u\|<\infty$, *then*

(20) $$D(G)\supset\{\phi\in D, A^u\phi\in D(A^u) \quad \textit{for} \quad \forall u$$
$$\textit{and} \quad \sup_u\|A^u(A^u\phi)\|<\infty\}, \ (\textit{say } \Theta).$$

Proof. In the case $f^u\equiv 0$ for any u, we denote S_t by Λ_t. Put $A\phi\equiv\sup_u G^u\phi=\sup_u(A^u\phi+f^u)$ and $\Delta=\frac{1}{2^N}$. Recalling (T5) we have for $\phi\in D$

(21) $$S_\Delta^{(N)}\phi-\phi=\sup_u(T_\Delta^u\phi-\phi)=\sup_u\int_0^\Delta P_\theta^u G^u\phi d\theta$$
$$\leq\sup_u\int_0^\Delta P_\theta^u A\phi d\theta\leq\int_0^\Delta \Lambda_\theta A\phi d\theta.$$

Moreover

(22) $$S_{2\Delta}^{(N)}\phi-S_\Delta^{(N)}\phi=\sup_u T_\Delta^u(S_\Delta^{(N)}\phi)-\sup_u T_\Delta^u\phi$$
$$\leq\sup_u[T_\Delta^u(S_\Delta^{(N)}\phi)-T_\Delta^u\phi]=\sup_u[P_\Delta^u(S_\Delta^{(N)}\phi)-P_\Delta^u\phi]$$
$$=\sup_u[P_\Delta^u(S_\Delta^{(N)}\phi-\phi)]=\Lambda_\Delta(S_\Delta^{(N)}\phi-\phi)$$
$$\leq\Lambda_\Delta\left(\int_0^\Delta\Lambda_\theta A\phi d\theta\right)=\int_0^\Delta\Lambda_{\Delta+\theta}A\phi d\theta=\int_\Delta^{2\Delta}\Lambda_\theta A\phi d\theta.$$

Suppose $S_{k\Delta}^{(N)}\phi-S_{(k-1)\Delta}^{(N)}\phi\leq\int_{(k-1)\Delta}^{k\Delta}\Lambda_\theta A\phi d\theta$. Then, by the similar calculation, we see

$$S^{(N)}_{(k+1)\Delta}\phi - S^{(N)}_{k\Delta}\phi \le \Lambda_\Delta (S^{(N)}_{k\Delta}\phi - S^{(N)}_{(k-1)\Delta}\phi) < \int_{k\Delta}^{(k+1)\Delta} \Lambda_\theta A\phi d\theta .$$

Hence taking the summation for k we get

$$S_t^{(N)}\phi - \phi \le \int_0^t \Lambda_\theta A\phi d\theta \quad \text{for} \quad t = \frac{i}{2^N} . \tag{23}$$

Tending N to ∞ we have

$$S_t\phi - \phi \le \int_0^t \Lambda_\theta A\phi d\theta \quad \text{for binary } t \text{ and } \phi \in D. \tag{24}$$

Since the both sides of (24) are continuous in t, (24) holds for any $t \ge 0$. Furthermore

$$\frac{1}{t}(S_t\phi - \phi) \le \frac{1}{t}\int_0^t \Lambda_\theta A\phi d\theta \le \|A\phi\| 1 , \tag{25}$$

where 1 is the unit in L. On the other hand, by virtue of (T5) and (iv) of Proposition 3, we have

$$\frac{1}{t}(S_t\phi - \phi) \ge \frac{1}{t}(T_t^u - \phi) = \frac{1}{t}\int_0^t P_\theta^u G^u \phi d\theta \ge -\|G^u\phi\| 1 . \tag{26}$$

Therefore the set $\left\{\frac{1}{t}(S_t\phi - \phi), t > 0\right\}$ is O-bounded. Hence $\inf_{\theta>0} \sup_{t>\theta} \frac{1}{t} \times (S_t\phi - \phi)$, i.e. $O - \overline{\lim}_{t\downarrow 0} \frac{1}{t}(S_t\phi - \phi)$ exists, and $\sup_{\theta>0} \inf_{t>\theta} \frac{1}{t}(S_t\phi - \phi)$, i.e. $O - \underline{\lim}_{t\downarrow 0} \frac{1}{t}(S_t\phi - \phi)$, exists. Since

$$\lim_{t\downarrow 0} \frac{1}{t}\int_0^t \Lambda_\theta A\phi d\theta = A\phi , \tag{27}$$

and

$$\lim_{t\downarrow 0} \frac{1}{t}\int_0^t P_\theta^u G^u \phi d\theta = G^u\phi , \tag{28}$$

we have by (25), (26), (27) and (28),

$$O - \overline{\lim}_{t\downarrow 0} \frac{1}{t}(S_t\phi - \phi) \le A\phi \tag{29}$$

and

$$O - \underline{\lim}_{t\downarrow 0} \frac{1}{t}(S_t\phi - \phi) \ge G^u\phi \qquad \forall u . \tag{30}$$

Hence

$$(31)\qquad O-\varliminf_{t\downarrow 0}\frac{1}{t}(S_t\phi-\phi)\geq\sup_u G^u\phi=A\phi.$$

From (29) and (31) we have

$$O-\varlimsup_{t\downarrow 0}\frac{1}{t}(S_t\phi-\phi)=O-\varliminf_{t\downarrow 0}\frac{1}{t}(S_t\phi-\phi)=A\phi.$$

Thus, for $\phi\in D(G)\cap D$, we have

$$G\phi=\lim_{t\downarrow 0}\frac{1}{t}(S_t\phi-\phi)=O-\lim_{t\downarrow 0}\frac{1}{t}(S_t\phi-\phi)=A\phi.$$

Next we shall show (20). From (25)

$$\frac{1}{t}(S_t\phi-\phi)-A\phi\leq\frac{1}{t}\int_0^t\Lambda_\theta A\phi d\theta-A\phi.$$

By (27) the right side converges to 0 as $t\to 0$. Hence, for $\varepsilon>0$, there exists a positive $\delta=\delta(\varepsilon)$, such that

$$(32)\qquad \operatorname*{ess.\,sup.}_x\left[\frac{1}{t}(S_t\phi-\phi)(x)-A\phi(x)\right]<\varepsilon\quad\text{for}\quad t\in(0,\delta).$$

On the other hand, by (26) we have

$$(33)\qquad \frac{1}{t}(S_t\phi-\phi)-A\phi\geq\sup_u\frac{1}{t}\int_0^t P_\theta{}^u G^u\phi d\theta-A\phi$$

$$=\sup_u\frac{1}{t}\int_0^t P_\theta{}^u G^u\phi d\theta-\sup_u G^u\phi\geq\inf_u\left[\frac{1}{t}\int_0^t P_\theta{}^u G^u\phi d\theta-G^u\phi\right].$$

For $\phi\in\Theta$, we have $G^u\phi\in D(A^u)$ and

$$P_\theta{}^u G^u\phi-G^u\phi=\int_0^\theta P_s{}^u A^u G^u\phi ds.$$

Thus

$$\frac{1}{t}\int_0^t P_\theta{}^u G^u\phi d\theta-G^u\phi=\frac{1}{t}\int_0^t\left(\int_0^\theta P_s{}^u A^u G^u\phi ds\right)d\theta.$$

So we have

$$(34)\qquad \left\|\frac{1}{t}\int_0^t P_\theta{}^u G^u\phi d\theta-G^u\phi\right\|\leq\|A^u G^u\phi\|t\leq\|A^u(A^u\phi)+A^u f^u\|t.$$

Therefore by (33) and (34) we have

(35) $$\operatorname*{ess.inf.}_{x}\left[\frac{1}{t}(S_t\phi-\phi)-A\phi\right]\geq-\sup_u\|A^u(A^u\phi)+A^uf^u\|t.$$

Hence (32) and (35) complete the proof of (20).

Remark 1. If $S_t\phi$ is differentiable in $t>0$ and $S_t\phi$ belongs to D, then

$$\begin{cases}\dfrac{d}{dt}S_t\phi=\sup_u(A^uS_t\phi+f^u),\ t>0,\\ S_0\phi=\phi.\end{cases}$$

This is the so-called Bellman equation. So S_t is called a Bellman semi-group.

Remark 2. If each A^u is a bounded operator on L and

(36) $$\sup_u\|A^u\|<\infty,$$

then $\sup_u\|A^uf^u\|<\infty$ and $\Theta=L$. Moreover $S_t\phi$ is differentiable in t and satisfies the Bellman equation.

Proof. Since A^u is a bounded linear operator on L,

$$P_t^u=\sum_{k=0}^{\infty}\frac{1}{k!}(tA^u)^k=\exp tA^u$$

and $D(A^u)=L$. Hence $f^u\in D(A^u)$ and $\sup_u\|A^uf^u\|\leq\sup\|A^u\|h<\infty$. Moreover $\sup_u\|A^u\phi\|<\infty$, for any $\phi\in L$. Thus $D=L$. Since $\sup_u\|A^u(A^u\phi)\|\leq(\sup_u\|A^u\|)^2\|\phi\|$, we have $\Theta=L$.

For the proof of the latter half, we apply the same method as for linear semi-groups. Since $D(G)\supset\Theta=L$, the right derivative of $S_t\phi$,

$$\frac{d^+}{dt}S_t\phi=\lim_{\theta\downarrow0}\frac{1}{\theta}(S_{t+\theta}\phi-S_t\phi)$$

exists and, by $\Theta=L$,

$$\frac{d^+}{dt}S_t\phi=\sup_u(A^uS_t\phi+f^u)=AS_t\phi.$$

Hence, for any $F\in L'$, we have

$$F(AS_t\phi)=F\Big(\frac{d^+}{dt}S_t\phi\Big)=\lim_{\theta\downarrow 0}\frac{1}{\theta}(F(S_{t+\theta}\phi)-F(S_t\phi))=\frac{d^+}{dt}F(S_t\phi).$$

On the other hand by (36) we get

$$\begin{aligned}\|AS_t\phi-AS_\theta\phi\|&\le\sup_u\|A^uS_t\phi-A^uS_\theta\phi\|\\&=\sup_u\|A^u(S_t\phi-S_\theta\phi)\|\le(\sup_u\|A^u\|)\|S_t\phi-S_\theta\phi\|.\end{aligned}$$

Therefore $AS_t\phi$ is continuous in t. So $F(AS_t\phi)$ is a real continuous function of t, namely the right derivative of $F(S_t\phi)$ is continuous. Therefore $F(S_t\phi)$ is differentiable and its derivative $\frac{dF(S_t\phi)}{dt}$ is continuous. Therefore

$$(37)\qquad \begin{aligned}F(S_t\phi-\phi)&=F(S_t\phi)-F(\phi)=\int_0^t\frac{d}{d\theta}F(S_\theta\phi)\,d\theta\\&=\int_0^tF(AS_\theta\phi)\,d\theta=F\Big(\int_0^tAS_\theta\phi d\theta\Big).\end{aligned}$$

Since F is arbitrary, (37) implies

$$(38)\qquad S_t\phi-\phi=\int_0^tAS_\theta\phi d\theta\,.$$

By the continuity of $AS_\theta\phi$, (38) implies the differentiability of $S_t\phi$. Therefore by Remark 1 $S_t\phi$ satisfies the Bellman equation. In fact the operator S_t thus obtained is identical with e^{tA} in the sense of [1].

Proposition 6. *If $\tilde{S}_t$ is a semi-group on L satisfying the condition* (i)$\sim$(iv), *then for any $t\ge 0$ and $\phi\in L$,*

$$S_t\phi\le\tilde{S}_t\phi\,.$$

Proof. Putting $\Delta=\frac{1}{2^N}$, we have

$$(39)\qquad S_\Delta^{(N)}\phi=\sup_u T_\Delta^u\phi\le\tilde{S}_\Delta\phi\,,\quad \forall\phi\in L\,.$$

Suppose

$$(40)\qquad S_{k\Delta}^{(N)}\phi\le\tilde{S}_{k\Delta}\phi\,.$$

Then

$$S_{(k+1)\Delta}^{(N)}\phi=S_\Delta^{(N)}(S_{k\Delta}^{(N)}\phi)\le S_\Delta^{(N)}(\tilde{S}_{k\Delta}\phi)\le\tilde{S}_\Delta(\tilde{S}_{k\Delta}\phi)=\tilde{S}_{(k+1)\Delta}\phi\,.$$

Hence, for any k, we have (40).

This implies for any binary t

$$S_t^{(n)}\phi\leq\tilde{S}_t\phi \quad \text{for large } n.$$

Therefore for binary t

$$S_t\phi=O_i-\lim_n S_t^{(n)}\phi\leq\tilde{S}_t\phi.$$

Since the both sides are continuous in t, we complete the proof of Proposition 6.

For any constant $c\geq0$, we replace P_t^u by $e^{-ct}P_t^u$. Then we can easily show the following,

Corollary. *Theorem* 1 *is still valid, when we replace* (iv) *and* (5) *respectively by*

(iv)′ $$e^{-ct}P_t^u+\int_0^t e^{-c\theta}P_\theta^u\phi d\theta\leq S_t\phi,$$

and

(v)′ $$G\phi=\sup_u(A^u\phi-c\phi+f^u), \quad \text{for} \quad \phi\in D(G)\cap D.$$

For positive c, we denote the semi-group of Corollary by $\tilde{S}_t$.

Proposition 7. *There exists a unique* $v\in L$ *such that*

$$\lim_{t\uparrow\infty}\tilde{S}_t\phi=v \quad \textit{for any} \quad \phi\in L.$$

Proof. Using $e^{-ct}P_t^u$ instead of P_t^u, we define $\tilde{J}(N)$ and $\tilde{S}^{(N)}$ by the similar way. Then, putting $\tilde{J}=\tilde{J}(N)$ and $\varDelta=\frac{1}{2^N}$, we have

$$\|\tilde{J}\phi-\tilde{J}\psi\|\leq\sup_u\|e^{-c\varDelta}P_\varDelta^u\phi-e^{-c\varDelta}P_\varDelta\psi\|\leq e^{-c\varDelta}\|\phi-\psi\|.$$

Moreover we can show (41) by the induction,

(41) $$\|\tilde{J}^k\phi-\tilde{J}^k\psi\|\leq e^{-ck\varDelta}\|\phi-\psi\|.$$

On the other hand we can easily see the following inequality

$$\|\tilde{J}\phi\| \leq e^{-c\Delta}\|\phi\| + \frac{h}{c}(1-e^{-c\Delta})$$

and moreover, we have (42) by the induction,

$$\|\tilde{J}^k\phi\| \leq e^{-ck\Delta}\|\phi\| + \frac{h}{c}(1-e^{-ck\Delta}). \tag{42}$$

(41) and (42) mean, for $t=\frac{k}{2^N}$,

$$\|\tilde{S}_t^{(N)}\phi - \tilde{S}_t^{(N)}\psi\| \leq e^{-ct}\|\phi-\psi\|$$

and

$$\|\tilde{S}_t^{(N)}\phi\| \leq e^{-ct}\|\phi\| + \frac{h}{c}(1-e^{-ct}).$$

Therefore, for binary t, we have (43) and (44),

$$\|\tilde{S}_t\phi - \tilde{S}_t\psi\| \leq \underline{\lim}_N \|\tilde{S}_t^{(N)}\phi - \tilde{S}_t^{(N)}\psi\| \leq e^{-ct}\|\phi-\psi\| \tag{43}$$

and

$$\|\tilde{S}_t\phi\| \leq e^{-ct}\|\phi\| + \frac{h}{c}(1-e^{-ct}) \leq \|\phi\| + \frac{h}{c}. \tag{44}$$

Since the both sides of the above inequalities (43) and (44) are continuous in t, we have

$$\|\tilde{S}_{t+\theta}\phi - \tilde{S}_t\phi\| = \|\tilde{S}_t(\tilde{S}_\theta\phi) - \tilde{S}_t\phi\| \leq e^{-ct}\left(2\|\phi\| + \frac{h}{c}\right). \tag{45}$$

Hence there exists $\lim_{t\to\infty} \tilde{S}_t\phi$, say v_ϕ. By virtue of (43), we can see that v_ϕ does not depend on ϕ.

Corollary. *$\tilde{S}_t v = v$ for any $t \geq 0$, and if v belongs to D, then* $\sup_u (A^u v - cv + f^u) = 0$.

§ 4. Proof of Theorem 2

For any $\lambda \geq 0$ and $g \in L$ we define

$$T_t^{u\lambda g}\phi = e^{-\lambda t}P_t^u\phi + \int_0^t e^{-\lambda\theta}P_\theta^u(f^u + \lambda g)\,d\theta. \tag{1}$$

Then we have

$$\|T_t^{u\lambda g}\phi\| \leq e^{-(\lambda+c)t}\|\phi\| + \frac{h}{c}(1-e^{-ct}) + (1-e^{-\lambda t})\|g\| \tag{2}$$

and its generator $G^{u\lambda g}$ is as follows

$$G^{u\lambda g}\phi = A^u\phi - \lambda\phi + f^u + \lambda g\ .$$

For simplicity we omit g in $T_t^{u\lambda g}$ and $G^{u\lambda g}$ for the moment, if any confusion does not occur. In order to prove Theorem 2, we apply the same method as [4], namely we take $\Gamma\times[0,\infty)$ for the control region. Appealing to (2), we can define $J=J(N)$ by

$$J\phi = \sup_{u\lambda} T^{u\lambda}_{1/2^N}\phi\ ,\ \ \phi\in L$$

and

$$J^{k+1}\phi = J(J^k\phi)\ ,\ \ J^0\phi=\phi\ .$$

Then Lemma 1 is easy.

Lemma 1. *Putting* $\Delta=\dfrac{1}{2^N}$, *we have*

(J0) $J^{k+l}\phi = J^k(J^l\phi) = J^l(J^k\phi)$,

(J1) $J^k\phi\le J^k\psi$ *whenever* $\phi\le\psi$,

(J2) $\|J^k\phi - J^k\psi\|\le e^{-ck\Delta}\|\phi-\psi\|$,

(J3) $\|J^k\phi\|\le e^{-ck\Delta}\|\phi\| + \dfrac{h}{c}(1-e^{-ck\Delta}) + \|g\|$,

(J4) $\phi = O_i-\lim\limits_n \phi_n$ *implies* $J^k\phi = O_i - \lim\limits_n J^k\phi_n$.

(J5) $g\le J\phi$.

Proof. We show (J3) by the induction. For $k=1$, (J3) comes from (2). Suppose (J3) holds for k. Then we have

$$\|J^{k+1}\phi\| = \|J(J^k\phi)\|\le \sup_{u\lambda}\|T_\Delta^{u\lambda}(J^k\phi)\|\ . \tag{3}$$

Recalling (2) we see

$$\begin{aligned}\|T_\Delta^{u\lambda}(J^k\phi)\| &\le e^{-(\lambda+c)\Delta}\|J^k\phi\| + \frac{h}{c}(1-e^{-c\Delta}) + (1-e^{-\lambda\Delta})\|g\| \\ &\le e^{-(\lambda+c)\Delta}\Big(e^{-ck\Delta}\|\phi\| + \frac{h}{c}(1-e^{-ck\Delta}) + \|g\|\Big) + \frac{h}{c}(1-e^{-c\Delta}) + (1-e^{-\lambda\Delta})\|g\| \\ &\le e^{-c(k+1)\Delta}\|\phi\| + \frac{h}{c}(e^{-c\Delta} - e^{-c(k+1)\Delta} + 1 - e^{-c\Delta}) + \|g\|\,.\end{aligned} \tag{4}$$

From (3) and (4) we have (J3) for $k+1$.

We have, for any $u\in\Gamma$ and $t>0$,

$$g=\lim_{\lambda\to\infty} T_t^{u\lambda}\phi\,.$$

Hence (J5) is valid.

Define $S_t^{(N)}$ by $S_t^{(N)}\phi=J^k(N)\phi$ for $t=\frac{k}{2^N}$. Then $S_t^{(N)}\phi$ is increasing as $N\to\infty$. Moreover we have

Lemma 2. *If $\phi\leq g$, then $S_t^{(N)}\phi$ is increasing as $t\to\infty$.*

Proof. Putting $\Delta=1/2^N$, we get by (J5)

(5) $$\phi\leq g\leq S_\Delta^{(N)}\phi\,.$$

Hence, by (J1),

(6) $$\phi\leq S_\Delta^{(N)}\phi\leq S_{2\Delta}^{(N)}\phi\leq\cdots S_{k\Delta}^{(N)}\phi\leq S_{(k+1)}^{(N)}\phi\,.$$

(J3) means the following (7).

(7) $$\|S_t^{(N)}\phi\|\leq e^{-ct}\|\phi\|+\frac{h}{c}(1-e^{-ct})+\|g\|\,.$$

Therefore, for binary t, the set $\{S_t^{(N)}\phi,\ N\ \textit{large}\}$ if O-bounded. Hence we can define S_t by

$$S_t\phi=O_i-\lim_N S_t^{(N)}\phi \quad \text{for binary } t.$$

From (J4) we can again see, for binary t,

(8) $$S_t\phi=O_i-\lim_n S_t\phi_n \quad \text{if} \quad \phi=O_i-\lim_n \phi_n\,.$$

Therefore we can derive the semi-group property on binary parameter.

$$S_{t+\theta}\phi=S_t(S_\theta\phi)=S_\theta(S_t\phi) \quad \text{for binary } t \text{ and } \theta\,.$$

Again, by (7), we have

(9) $$\|S_t\phi\|\leq e^{-ct}\|\phi\|+\frac{h}{c}(1-e^{-ct})+\|g\|\,.$$

Hence the set $\{S_t\phi,\ \textit{binary}\ t\}$ is also O-bounded.

Lemma 3. *If $\phi\leq g$, then $S_t\phi$ is increasing in t and $O_i-\lim_t S_t\phi$ exists, say v_ϕ. Moreover*

$$g \leq v_\phi \, . \tag{10}$$

Proof. By Lemma 2 we have for $t<\theta$,

$$\mathbf{S}_t\phi = O_i - \lim_N \mathbf{S}_t^{(N)}\phi \leq O_i - \lim_N \mathbf{S}_\theta^{(N)}\phi = \mathbf{S}_\theta\phi \, .$$

Hence $\mathbf{S}_t\phi$ is increasing as binary $t\to\infty$. (10) is clear by (J5). For simplicity we put $v=v_\phi$ if any confusion does not occur.

Lemma 4. *v is $\mathbf{S}_t$-invariance, i.e.*

$$\mathbf{S}_t v = v \text{ for binary } t. \tag{11}$$

Proof. By the definition of v and (8),

$$\mathbf{S}_t v = \mathbf{S}_t (O_i - \lim_\theta \mathbf{S}_\theta\phi) = O_i - \lim_\theta \mathbf{S}_{t+\theta}\phi = v \, .$$

Proposition 8. *v is an S_t-excessive majorant of g, i.e. $v\geq g$ and*

$$S_t v \leq v \, , \ \forall t \geq 0 \, . \tag{12}$$

Proof. By the definitions of $\mathbf{S}_t$ and S_t, we have

$$S_t\psi \leq \mathbf{S}_t\psi \quad \forall \text{ binary } t \text{ and } \psi \in L \, .$$

Hence by Lemma 4

$$S_t v \leq \mathbf{S}_t v = v \, .$$

Namely we get (12) for binary t. Since $S_t v$ is continuous in t, (12) is valid for any t. Recalling (10) we complete the proof.

Proposition 9. *For any $\phi\leq g$, v_ϕ is the least S_t-excessive majorant of g.*

Proof. Let V be an S_t-excessive majorant of g. Recalling the definitions of $T_t^{u\lambda V}$ and T_t^u, we have

$$T_t^{u\lambda V}\psi = e^{-\lambda t} P_t^u \psi + \int_0^t e^{-\lambda\theta} P_\theta^u (f^u + \lambda V) \, d\theta \tag{13}$$

and

$$T_t^{u00}\psi = P_t^u\psi + \int_0^t P_\theta^u f^u = T_t^u\psi\,. \tag{14}$$

Hence

$$\begin{aligned} T_t^{u\lambda V}V = e^{-\lambda t}T_t^u V + \int_0^t e^{-\lambda\theta}P_\theta^u f^u d\theta \\ +\lambda\int_0^t e^{-\lambda\theta}P_\theta^u g d\theta - e^{-\lambda t}\int_0^t P_\theta^u f^u d\theta\,, \end{aligned} \tag{15}$$

and, from (14), we see

$$\begin{aligned} \lambda\int_0^t e^{-\lambda\theta}P_\theta^u V d\theta &= \lambda\int_0^t e^{-\lambda\theta}\left(T_\theta^u V - \int_0^\theta P_s^u f^u ds\right)d\theta \\ &= \lambda\int_0^t e^{-\lambda\theta}T_\theta^u V d\theta - \int_0^t (e^{-\lambda s} - e^{-\lambda t})P_s^u f^u ds\,. \end{aligned} \tag{16}$$

Therefore, by (15) and (16) we have

$$T_t^{u\lambda V}V = e^{-\lambda t}T_t^u V + \lambda\int_0^t e^{-\lambda\theta}T_\theta^u V d\theta\,. \tag{17}$$

Since "$T_t^u\psi \leq S_t\psi$" and V is S_t-excessive, we have

$$e^{-\lambda t}T_t^u V \leq e^{-\lambda t}S_t V \leq e^{-\lambda t}V\,. \tag{18}$$

Combining (18) with (17) we can see

$$T_t^{u\lambda V}V \leq e^{-\lambda t}V + \lambda\int_0^t e^{-\lambda\theta}V d\theta = V\,. \tag{19}$$

Hence we have, denoting $J(N)$ for $T_t^{u\lambda V}$ by $\tilde{J}(N)$,

$$\tilde{J}(N)V \leq V \quad \text{and} \quad \tilde{J}^k(N)V \leq V\,. \tag{20}$$

This tells us the following inequality,

$$\tilde{\mathbf{S}}_t V \leq V \quad \text{for binary } t\,. \tag{21}$$

Appealing to "$g \leq V$" and the definition of $T_t^{u\lambda g}$, we have

$$T_t^{u\lambda g}\psi \leq T_t^{u\lambda V}\psi \quad \forall\psi \in L\,.$$

Hence

$$J(N)\psi \leq \tilde{J}(N)\psi \quad \text{and} \quad \mathbf{S}_t\psi \leq \tilde{\mathbf{S}}_t\psi\,.$$

So, by (21), we have for binary t,

$$S_t\phi \leq S_t g \leq S_t V \leq \tilde{S}_t V \leq V .$$

Tending t to ∞, we can derive

$$v \leq V .$$

Corollary. $v_\phi = v_g \quad \forall \phi \leq g$.

Proof. Since the least S_t-excessive majorant of g is unique, $v_\phi = v_g$.

§ 5. Examples

We will show two simple examples of control problems related Markov processes with exponential holding times, [cf. 5].

Example 1. Let $A^u = (a^u(i, j))$ be an $l \times l$-matrix. Suppose $a^u(i, j) \geq 0$ for $i \neq j$ and $\sum_{j=1}^{l} a^u(i, j) = 0$. Then A^u is the generator of the transition semi-group $P_t^u = (P_t^u(i, j)) = e^{tA}$.

Put $\mu\{i\} = 1$, $i = 1, \cdots, l$ and $\mu(R^1 - \{1, 2, \cdots, l\}) = 0$. Then A^u becomes a bounded linear operator on $L = L_\infty(R^1, \mu)$ and P_t^u a positive contractive and continuous semi-group on L. Assume

(1) $$\sup_u |a^u(i, j)| < \infty, \quad \forall ij = 1, \cdots, l.$$

Thus $\sup_u \|A^u\| < \infty$. Let $\sup_u |f^u(i)| < \infty$ for $i = 1, \cdots, l$. Then we can construct Bellman semi-group S_t for $\{A^u, f^u\}$. Moreover, for $\phi \in L$, $S_t\phi$ is a solution of the following Bellman equation,

$$\begin{cases} \dfrac{dS_t\phi(i)}{dt} = \sup_u \left[\sum_{j=1}^{l} a^u(i, j) S_t\phi(j) + f^u(i) \right], \ i = 1, \cdots, l, \\ S_0\phi(i) = \phi(i) . \end{cases}$$

Example 2. Let X^u be a 1-dimensional Lévy process of pure jump type with finite Lévy measure n^u

$$X(t) = x + \int_{R^1} \int_0^t z N^u(dsdz)$$

and $EN^u(dsdz) = dsn^u(dz)$. Thus every point of R^1 is an exponential holding point.

Suppose that n^u has the density, say $n^u(dz)=n^u(z)dz$. Put $Y^u(t)=\int_{R^1}\int_0^t zN^u(dsdz)$. We denote its i-th jump time by τ_i^u, $\tau_0^u=0$, and $Y^u(\tau_i^u)-Y^u(\tau_{i-1}^u)$ by ζ_i^u. For simplicity we skip the suffix u if any confusion does not occur. We have the following well-known facts,

(i) $\tau_i-\tau_{i-1}$, $i=1,2,\cdots$, ζ_i, $i=1,2,\cdots$ are independent.

(ii) $P(\tau_i-\tau_{i-1}>t)=e^{-t\lambda}$ where $\lambda=n(R^1)$.

(iii) $P(\zeta_i\in A)=\dfrac{n(A)}{\lambda}=\dfrac{1}{\lambda}\int_A n(z)\,dz.$

Hence

$$(2)\quad P(Y(t)\in A)=\chi_A(O)P(\tau_1<t)+\sum_{i=1}^{\infty}P(Y(\tau_i)\in A)P(\tau_i\leq t<\tau_{i+1})$$

$$=\chi_A(O)e^{-t\lambda}+\sum_{i=1}^{\infty}P(\zeta_1+\cdots+\zeta_i\in A)P(\tau_i\leq t\leq\tau_{i+1})$$

$$=\chi_A(O)e^{-t\lambda}+m(A,t).$$

By virtue of (i) and (iii) the measure $m(\cdot,t)$ is absolutely continuous w.r. to the Lebesgue measure μ. Suppose $\phi=\psi$ $\mu-$a.e. Then, for any x where "$\psi(x)=\phi(x)$" holds, we see

$$P_t\phi(x)=E_x\phi(X(t))=E\phi(x+Y(t))$$

$$=\phi(x)e^{-\lambda t}+\int\phi(x+y)\,m'(y,t)\,dy$$

$$=\phi(x)e^{-\lambda t}+\int\phi(y)\,m'(y-x,t)\,dy$$

$$=\psi(x)e^{-\lambda t}+\int\psi(y)\,m'(y-x,t)\,dy=P_t\psi(x).$$

Hence the transition semi-group P_t can act on $L=L_\infty(R^1,\mu)$. On the other hand we have

$$|P_t\phi(x)-\phi(x)|\leq|\phi(x)|\,(1-e^{-t\lambda})+\|\phi\|P(\tau_1\leq t)$$

$$\leq 2\|\phi\|(1-e^{-t\lambda})\to 0\quad\text{as}\quad t\downarrow 0\,.$$

So P_t is strongly continuous.

Thus P_t^u is a positive contractive and strongly continuous linear semi-group on L whose generator A^u is

$$A^u\phi(x)=\int_{R^1}(\phi(x+y)-\phi(x))n^u(y)\,dy\,,\quad \phi\in L\,.$$

Since $\|A^u\phi\|\leq 2\|\phi\|\lambda^u$, this example 2 satisfies the condition (36) of Remark 2, if

$$\sup_u \lambda^u<\infty\,. \tag{3}$$

Therefore, for $f^u\in L$ with (A3), we have a solution, $V(t,x)=S_t\phi(x)$, of Bellman equation

$$\begin{cases} \dfrac{\partial V(t,x)}{\partial t}=\sup_u\left[\displaystyle\int_{R^1}(V(t,x+y)-V(t,x))n^u(y)\,dy+f^u(x)\right]\text{a.e. } \forall t>0\,, \\ V(0,x)=\phi(x)\,. \end{cases}$$

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