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Linear Transformation of Quasi-Invariant Measures

By

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Introduction

In harmonic analysis of a real separable Hilbert space H, we often wish to require a nice measure μ, whose measure theoretical structure is closely connected with the topological structure of H. In this direction, we have already known that an important measure is not a measure lying on H but rather a continuous cylindrical measure lying on a nuclear extension of H. Moreover it will be turned out that if μ is also H-quasi-invariant, then the convergence of linear functionals in μ is identical with the strong convergence in H, (see Theorem 2.1). Therefore H-continuous (cylindrical) and H-quasi-invariant measures are regarded as nice measures and are worth special interest. From now on, realizing H as l^2, we shall consider these measures on $\mathbf{R}^\infty$, $\mathbf{R}_0^\infty \subset l^2 \subset \mathbf{R}^\infty$. $\mathbf{R}_0^\infty$ is the set of all $x=(x_1,\dots, x_n,\dots) \in \mathbf{R}^\infty$ such that $x_n=0$ except finite numbers of n. The general description for $\mathbf{R}_0^\infty$-quasi-invariant measures was given by Skorohod. In [13] he characterized them in terms of a partial independence of sub-σ-fields. But this result does not directly lead a classification of l^2-continuous and l^2-quasi-invariant measures. In above classification, we identify μ and μ' if these measures are equivalent with each other. So it is desirable to have a concept which is invariant on the equivalence classes. One of these concepts is the set Λ_μ of admissible linear operators on l^2, (see Definition 3.1). It seems to the author that Λ_μ is a natural concept and plays an effective role in this problem. (It will be turned out in Theorem 3.2 that

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for a measure μ of Gauss type, the correspondence $\mu \rightarrow \Lambda_\mu$ is one to one up to a trivial relation.) Therefore in this paper we shall consider the transformations of μ which arise from linear operators on l^2, and shall investigate the basic facts for Λ_μ.

§1. General Description for Quasi-Invariant Measures

Throughout this paper, we shall only consider probability measures which are defined on the usual σ-field $\mathscr{B}(\mathbf{R}^\infty)$. The set of all probability measures on $\mathscr{B}(\mathbf{R}^\infty)$ will be denoted by $M(\mathbf{R}^\infty)$. Let $\mu \in M(\mathbf{R}^\infty)$ and $t \in \mathbf{R}^\infty$. We define the transformed measure $\mu_t \in M(\mathbf{R}^\infty)$ by $\mu_t(A) = \mu(A-t)$ for all $A \in \mathscr{B}(\mathbf{R}^\infty)$.

Definition 1.1. *$\mu \in M(\mathbf{R}^\infty)$ is called t-quasi-invariant or t is admissible translation for μ, if and only if μ_t is equivalent with μ ($\mu_t \simeq \mu$). The set of all such t will be denoted by T_μ. If $\Phi \subset T_\mu$, or $\Phi = T_\mu$ holds, we say that μ is Φ-quasi-invariant or strictly-Φ-quasi-invariant respectively.*

Definition 1.2. *Let $\mu \in M(\mathbf{R}^\infty)$ be Φ-quasi-invariant. If the following condition is satisfied, we say that μ is Φ-ergodic.*

For any Φ-quasi-invariant measure μ', the relation $\mu' \lesssim \mu$ implies $\mu' = 0$ or $\mu' \simeq \mu$. ($\lesssim$ means the relation of absolute continuity.)

Several equivalent versions of Definition 1.2 are stated in [15].

Let $p_{n,m}$ $(n > m)$ be the projection from $\mathbf{R}^\infty$ to $\mathbf{R}^{n-m}$, $x = (x_1, \ldots, x_{m+1}, \ldots, x_n, \ldots) \rightarrow (x_{m+1}, \ldots, x_n)$, and $\mu_{n,m}$ be the image measure of μ by the map $p_{n,m}$, $\mu_{n,m} = p_{n,m}\mu$. Especially we shall write p_n (μ_n) instead of $p_{n,0}$ ($\mu_{n,0}$) respectively. If each $\mu_{n,m}$ is absolutely continuous with the Lebesgue measure on $\mathbf{R}^{n-m}$, then using density function $f_{n,m}$, we shall write $\mu = \{f_{n,m}\}$.

Proposition 1.1. *Let $\mu \in M(\mathbf{R}^\infty)$ be $\mathbf{R}_0^\infty$-quasi-invariant. Then each $\mu_{n,m}$ is equivalent with the Lebesgue measure on $\mathbf{R}^{n-m}$.*

Proof. Since any quasi-invariant σ-finite measure on any finite-dimensional Euclid space is equivalent to Lebesgue measure, we shall show that each $\mu_{n,m}$ is $\mathbf{R}^{n-m}$-quasi-invariant. Suppose that $\mu_{n,m}(A)=0$ for some $A\in\mathscr{B}(\mathbf{R}^{n-m})$ and that $t=(t_{m+1},\dots,t_n)\in\mathbf{R}^{n-m}$. Then putting $y=(0,\dots,0,t_{m+1},\dots,t_n,0,\dots)\in\mathbf{R}_0^\infty$, we have $p_{n,m}^{-1}(A-t)=p_{n,m}^{-1}(A)-y$, and from the $\mathbf{R}_0^\infty$-quasi-invariance of μ, $\mu_{n,m}(A-t)=\mu(p_{n,m}^{-1}(A)-y)=0$. In a similar way, $\mu_{n,m}(A-t)=0$ implies $\mu_{n,m}(A)=0$. Q.E.D.

The converse assertion of the above proposition does not hold in general. We shall give a counter-example for it after the following theorem.

Theorem 1.1. *Let $\mu,\mu^1\in M(\mathbf{R}^\infty)$ and assume that $\mu_n\gtrsim\mu_n^1$ for all n. Then*

(a) $\dfrac{d\mu_n^1}{d\mu_n}(p_n(x))$ *converges to some function $\rho(x)$ for μ-a.e.x.*

(b) *for the Lebesgue decomposition of μ^1 in terms of μ, $\rho(x)$ is the density function of its absolutely continuous part.*

Especially, in order that $\mu\gtrsim\mu^1$, it is necessary and sufficient that

(c) $\mu_n\gtrsim\mu_n^1$

(d) $\left\{\dfrac{d\mu_n^1}{d\mu_n}(p_n(x))\right\}$ *forms a Cauchy sequence in $L_\mu^1(\mathbf{R}^\infty)$.*

Proof. Since $\left\{\dfrac{d\mu_n^1}{d\mu_n}(p_n(x))\right\}$ forms a non-negative martingale with respect to $(\mathscr{B}_n)$, where $\mathscr{B}_n$ is the minimal σ-field with which $p_n(x)$ is measurable, so (a) is assured by a martingale convergence theorem, (for example see [7]). Let $\mu^1(A)=\int_A F(x)d\mu(x)+s(A)$, $A\in\mathscr{B}(\mathbf{R}^\infty)$ be the Lebesgue decomposition, in which $s\in M(\mathbf{R}^\infty)$ is singular with μ. We denote the conditional expectation of F to $\mathscr{B}_n$ by $E[F|\mathscr{B}_n]$ for each n. Then for any $A\in\mathscr{B}_n$, $\int_A E[F|\mathscr{B}_n](x)d\mu(x)=\int_A F(x)d\mu(x)\leqq\mu^1(A)=\int_A\dfrac{d\mu_n^1}{d\mu_n}(p_n(x))d\mu(x)$. Hence $E[F|\mathscr{B}_n](x)\leqq\dfrac{d\mu_n^1}{d\mu_n}(p_n(x))$ and letting $n\to\infty$,

$$F(x)\leqq\rho(x)\qquad\text{for }\mu\text{-a.e.x.}\tag{1}$$

On the other hand, for any $A\in\mathscr{B}_n$ and for any n, $\int_A\rho(x)d\mu(x)\leqq\underline{\lim}_n\int_A\dfrac{d\mu_n^1}{d\mu_n}(p_n(x))d\mu(x)=\mu^1(A)$, and therefore the same inequality holds for any $A\in\mathscr{B}(\mathbf{R}^\infty)$. We take a set $B\in\mathscr{B}(\mathbf{R}^\infty)$ such that $\mu(B)=1$ and $s(B)=0$. Then $\int_B(F(x)-\rho(x))d\mu(x)\geqq 0$, so from (1) we conclude that $F(x)=\rho(x)$ for μ-a.e.x.

If $\mu\gtrsim\mu^1$, then clearly (c) holds and $\int\rho(x)d\mu(x)=1$. Since $\frac{d\mu_n^1}{d\mu_n}(p_n(x))$ is non-negative modulo μ-null sets, by the well known theorem, $\int\left|\rho(x)-\frac{d\mu_n^1}{d\mu_n}(p_n(x))\right|d\mu(x)\to 0$, which assures (d). Conversely, if (c) and (d) hold, then $\int\rho(x)d\mu(x)=1$, therefore singular part must vanish. Q.E.D.

Counter-Example

We start from the class S_n of all skew-symmetrical matrices acting on $\mathbf{R}^n$. Naturally S_n may be identified with $\mathbf{R}^{k_n}$ ($k_n=n(n-1)/2$) under the correspondence,

$$X=\begin{pmatrix} 0, x_1, x_2, x_4\cdots \\ -x_1, 0, x_3, \quad\vdots \\ -x_2, \quad 0, \quad\vdots \\ \vdots \qquad \ddots\ x_{k_n} \\ \cdots\cdots\cdots -x_{k_n}, 0\end{pmatrix}\longleftrightarrow(x_1, x_2,\dots, x_{k_n})=x.$$

Now we shall define a measure μ_{k_n} on $\mathscr{B}(\mathbf{R}^{k_n})$ such that, $d\mu_{k_n}(x)=\gamma_n\det(I+X)^{-(n-1)}dx$, where dx is the volume element of Lebesgue measure on $\mathscr{B}(\mathbf{R}^{k_n})$, $X\in S_n$ is the corresponding matrix to x and γ_n is the normalizing constant such as $\mu_{k_n}(\mathbf{R}^{k_n})=1$. μ_{k_n} is identified with the image measure of the normalized Haar measure λ_n on $SO(n)$ by the Cayley transformation. That is, for a bounded measurable function f,

$$\int_{\mathbf{R}^{k_n}}f(x)d\mu_{k_n}(x)=\int_{U\in SO(n)}f((I-U)(I+U)^{-1})d\lambda_n(U). \tag{2}$$

For the projection p_m^n from $\mathbf{R}^n$ to $\mathbf{R}^m$, we see that $p_{k_m}^{k_n}\mu_{k_n}=\mu_{k_m}$ $(n>m)$. (For these facts, see [8].) For $k_{n-1}<j<k_n$ we define the measure μ_j on $\mathscr{B}(\mathbf{R}^j)$ such that $\mu_j=p_j^{k_n}\mu_{k_n}$ and obtain a consistent sequence in the sense of Kolmogorov. Therefore a unique $\mu\in M(\mathbf{R}^\infty)$ exists such that $p_n\mu=\mu_n$ for all n. Now we shall show that $t=(u, 0, 0,\dots)\in\mathbf{R}_0^\infty$ is

not admissible for μ. We put $\rho(x)=\lim_n \frac{d(\mu_t)_{k_n}}{d\mu_{k_n}}(p_{k_n}(x))$ and let $X_{k_n}\in S_n$ be the corresponding matrix to $p_{k_n}(x)$. Then,

$$\frac{d(\mu_t)_{k_n}}{d\mu_{k_n}}(p_{k_n}(x))=\left\{\frac{\det(I+X_{k_n})}{\det(I+X_{k_n}-T_{k_n})}\right\}^{n-1}, \text{ where } T_{k_n}=\overbrace{\begin{pmatrix} 0, u, & & 0 \\ -u, 0, & & \\ & & \ddots & \\ 0 & & & 0 \end{pmatrix}}^{n}\Bigg\} n .$$

Putting $a_{i,j}^{(n)}$ for the (i, j) entry of $(I+X_{k_n})^{-1}$, we have

$$\det(I+X_{k_n})\det(I+X_{k_n}-T_{k_n})^{-1}=\{1+g_n(u, X_{k_n})\}^{-1} \quad \text{and}$$

$$g_n(u, X_{k_n})=(a_{1,2}^{(n)}-a_{2,1}^{(n)})u+(a_{1,1}^{(n)}a_{2,2}^{(n)}-a_{1,2}^{(n)}a_{2,1}^{(n)})u^2.$$

Since $(I-X_{k_n})(I+X_{k_n})^{-1}=U_n\in SO(n)$ and $(I+X_{k_n})^{-1}=2^{-1}(I+U_n)$, so $|g_n(u, X_{k_n})|\leqq 2(|u|+u^2)$ for all n and u. From now on we shall assume that $\mu_t\simeq\mu$ and shall derive a contradiction. Then it follows that $0<\rho(x)<\infty$, for μ-a.e.x, therefore, $\overline{\lim_n}(n-1)|g_n(u, X_{k_n})|\leqq C\,\overline{\lim_n}(n-1)|\log(1+g_n(u, X_{k_n}))|=C|\log\rho(x)|<\infty$, for μ-a.e.x. (C is some constant depending on μ.) Thus $\lim_n g_n(u, X_{k_n})=0$ for μ-a.e.x and

$$\lim_n \int g_n(u, X_{k_n})d\mu(x)=0 \tag{3}$$

in virtue of Lebesgue's convergence theorem. On the other hand, using (2) we can easily show that $\int g_n(u, X_{k_n})d\mu(x)=u^2/4$, for all n and u. It contradicts with (3).

We shall introduce Kakutani's metric d on $M(\mathbf{R}^\infty)$,

$$d(\mu_1, \mu_2)=\left\{\int\left|\sqrt{\frac{d\mu_1}{d\lambda}(x)}-\sqrt{\frac{d\mu_2}{d\lambda}(x)}\right|^2 d\lambda(x)\right\}^{1/2},$$

where $\lambda\in M(\mathbf{R}^\infty)$ is taken such that $\lambda\gtrsim\mu_i$ $(i=1, 2)$. d does not depend on a particular choice of λ. For fixed $\mu\in M(\mathbf{R}^\infty)$, using one to one correspondence $t\to\mu_t$, we shall induce d to the set T_μ, $d(t_1, t_2)=d(\mu_{t_1}, \mu_{t_2})$ for $t_i\in T_\mu$ $(i=1, 2)$. It is clear that $d(t, 0)=d(-t, 0)$ and $d(t_1, t_2)=d(t_1-t_2, 0)$. Following theorems are due to [1].

Theorem 1.2. *(T_μ, d) is a complete metric space and the natural* *injection of* T_μ *into* $\mathbf{R}^\infty$ *is continuous.*

Theorem 1.3. *Let* Φ *be a complete metric linear topological subspace of* $\mathbf{R}^\infty$, *and be continuously imbedded into* $\mathbf{R}^\infty$. *If* $\Phi\subset T_\mu$, *then the natural injection* $\Phi\to T_\mu$ *is continuous.*

Proofs are omitted.

Remark. *Under the assumption of Theorem* 1.3, *for any bounded measurable function* F,

$$\int |F(x+t)-F(x)|d\mu(x)\longrightarrow 0 \qquad (t\longrightarrow 0 \text{ in } \Phi).$$

Proposition 1.2. *Let* Φ *be of the same meaning as in Theorem* 1.3. *Assume that* $\Phi\subset T_\mu$ *and* Φ *contains* $\mathbf{R}_0^\infty$ *densely. Then for a quasi-invariant measure* μ, $\mathbf{R}_0^\infty$*-ergodicity is equivalent to* Φ*-ergodicity.*

Proof. Since $\mathbf{R}_0^\infty\subset\Phi$, so $\mathbf{R}_0^\infty$-ergodicity is always stronger than Φ-ergodicity. We shall prove the converse relation. For it, it will be sufficient that

(*) For any $B\in\mathscr{B}(\mathbf{R}^\infty)$, the relation $\mu((B-y)\ominus B)=0$ for any $y\in\mathbf{R}_0^\infty$ implies $\mu(B)=0$ or $\mu(B)=1$.

Now in virtue of Remark after Theorem 1.3, $\mu((A-y)\ominus A)$ is a continuous function of $y\in\Phi$ for each fixed $A\in\mathscr{B}(\mathbf{R}^\infty)$, so $\mu((B-y)\ominus B)=0$ for any $y\in\Phi$, because $\mathbf{R}_0^\infty$ is dense in Φ. Therefore from Φ-ergodicity we have $\mu(B)=0$ or 1. Q.E.D.

Let $\mu\in M(\mathbf{R}^\infty)$ be $\mathbf{R}_0^\infty$-quasi-invariant. Then $\int\sqrt{\frac{d\mu_y}{d\mu}}(x)d\mu(x)$ is a positive definite function of $y\in\mathbf{R}_0^\infty$ and continuous with the inductive limit topology of $\mathbf{R}_0^\infty$. Therefore there exists a unique $\nu\in M(\mathbf{R}^\infty)$ (which is called the adjoint measure of μ) such that,

$$\hat{\nu}(y)=\int\exp(2\pi ix(y))d\nu(x)=\int\sqrt{\frac{d\mu_y}{d\mu}}(x)d\mu(x)$$

for all $y=(y_1,\dots,y_n,\dots)\in\mathbf{R}_0^\infty$. $x(y)$ means $\sum_{n=1}^\infty y_nx_n$.

Proposition 1.3. *Let* $\mu\in M(\mathbf{R}^\infty)$ *be* $\mathbf{R}_0^\infty$*-quasi-invariant. Let* Φ *be* *a complete metric linear topological subspace of* $\mathbf{R}^\infty$ *such that*

(a) Φ *is continuously imbedded into* $\mathbf{R}^\infty$

(b) Φ *contains* $\mathbf{R}_0^\infty$ *densely.*

Then in order that $T_\mu \supset \Phi$, *it is necessary and sufficient that* $\hat{\nu}(y)$ *is continuous with the induced topology from* Φ.

Proof. The necessity is an immediate consequence of Theorem 1.3. For the sufficiency, let $y \in \Phi$ and $\{y_n\} \subset \mathbf{R}_0^\infty$ such that $y_n \to y$ $(n \to \infty)$ in Φ. By the assumption, $d(y_n, y_m) = 2\{1 - \hat{\nu}(y_n - y_m)\} \to 0$ $(n, m \to \infty)$, which shows $\{y_n\}$ forms a Cauchy sequence in T_μ. From Theorem 1.2, there exists $t \in T_\mu$ such that $d(y_n, t) \to 0$ $(n \to \infty)$. Since the both injections of Φ and T_μ into $\mathbf{R}^\infty$ are continuous, so $y = t$. Q.E.D.

For a sequence $a = \{a_n\}$, we set $H_a = \{x = (x_1, \dots, x_n, \dots) \in \mathbf{R}^\infty \mid \sum_{n=1}^\infty a_n^2 x_n^2 < \infty\}$.

Proposition 1.4. *Let* $\mu \in M(\mathbf{R}^\infty)$. *Then there exists a positive sequence* $a = \{a_n\}$ *such that* $\mu(H_a) = 1$.

Proof. Since $\mathbf{R}^\infty$ is a Polish space, we can take a compact set $K_n \subset \mathbf{R}^\infty$ such that $\mu(K_n) > 1 - 1/n$ for each n. Without loss of generality, we may assume that $\{K_n\}$ is increasing. We take a positive sequence $a = \{a_n\}$ such that, $\sum_{n=1}^\infty a_n^2 \int_{K_n} x_n^2 d\mu(x) < \infty$. Then for any N,

$$\int_{K_N} \left(\sum_{n=1}^\infty a_n^2 x_n^2\right) d\mu(x) \leqq \sum_{n=1}^N a_n^2 \int_{K_N} x_n^2 d\mu(x) + \sum_{n=N+1}^\infty a_n^2 \int_{K_n} x_n^2 d\mu(x) < \infty.$$

It follows easily $\mu(H_a) = 1$. Q.E.D.

Proposition 1.5. *Let* $\mu \in M(\mathbf{R}^\infty)$ *be* $\mathbf{R}_0^\infty$*-quasi-invariant. Then there exists some sequence* $a = \{a_n\}$ *such that* $T_\mu \supset H_a$.

Proof. Let ν be the adjoint measure of μ. Applying Proposition 1.4 for ν, $\nu(H_b) = 1$ for some positive sequence $b = \{b_n\}$. We put $a_n = b_n^{-1}$ for each n and $a = \{a_n\}$. Since for any $y = (y_1, \dots, y_n, \dots) \in \mathbf{R}_0^\infty$ and for any $R > 0$,

$$|1 - \hat{\nu}(y)| \leqq \int |1 - \exp(2\pi i \textstyle\sum_{n=1}^\infty y_n a_n b_n x_n)| d\nu(x)$$

$$\leqq 2\int_{\sum b_n^2 x_n^2 \geqq R^2} d\nu(x) + 2\pi \int_{\sum b_n^2 x_n^2 < R^2} |\textstyle\sum_{n=1}^{\infty} y_n a_n b_n x_n| d\nu(x)$$

$$\leqq 2\nu(\textstyle\sum_{n=1}^{\infty} b_n^2 x_n^2 \geqq R^2) + 2\pi R(\sum_{n=1}^{\infty} a_n^2 y_n^2)^{1/2},$$

so $\hat{\nu}(y)$ is continuous with the natural Hilbertian topology of H_a. Hence $T_\mu \supset H_a$ from Proposition 1.3. Q.E.D.

Let g_c be a one-dimensional Gaussian measure with mean 0 and variance c^2, and G_α $(\alpha = \{\alpha_n\}) \in M(\mathbf{R}^\infty)$ be the product-measure of $\{g_{\alpha_n}\}$. It is easy that $G_\alpha(H_a) = 1$ for any sequence $a = \{a_n\}$ such that $\sum_{n=1}^{\infty} a_n^2 \alpha_n^2 < \infty$. Now let $\mu \in M(\mathbf{R}^\infty)$ be a $\mathbf{R}_0^\infty$-quasi-invariant measure. We take a sequence $a = \{a_n\}$ assured by Proposition 1.4 such that $T_\mu \supset H_a$ and take a positive sequence $\alpha = \{\alpha_n\}$ such that $\sum_{n=1}^{\infty} a_n^2 \alpha_n^2 < \infty$. Then $\mu * G_\alpha$ (convoluted measure by μ and G_α) is equivalent with μ, because $G_\alpha(T_\mu) = 1$. Conversely, for any positive sequence $\alpha = \{\alpha_n\}$ and for any $\mu^1 \in M(\mathbf{R}^\infty)$, $\mu^1 * G_\alpha$ is $\mathbf{R}_0^\infty$-quasi-invariant, because G_α is $\mathbf{R}_0^\infty$-quasi-invariant. Thus,

Theorem 1.4. *In order that $\mu \in M(\mathbf{R}^\infty)$ is $\mathbf{R}_0^\infty$-quasi-invariant, it is necessary and sufficient that there exist some $\mu^1 \in M(\mathbf{R}^\infty)$ and a positive sequence $\alpha = \{\alpha_n\}$ such that $\mu \simeq \mu^1 * G_\alpha$.*

Let $\mathscr{B}^n$ be the minimal σ-field with which all the functions $p_{j,n}(x)$ $(j \geqq n+1)$ are measurable and put $\mathscr{B}_\infty = \bigcap_{n=1}^{\infty} \mathscr{B}^n$. We say that $\mu \in M(\mathbf{R}^\infty)$ is tail-trivial if μ takes only the value 0 or 1 on $\mathscr{B}_\infty$.

Theorem 1.5. *In order that $\mu \in M(\mathbf{R}^\infty)$ is $\mathbf{R}_0^\infty$-quasi-invariant and $\mathbf{R}_0^\infty$-ergodic, it is necessary and sufficient that there exist a tail-trivial measure μ^1 and $\alpha = \{\alpha_n\}$ as in Theorem* 1.5.

Proof. In general, tail-trivial condition is equivalent to $\mathbf{R}_0^\infty$-ergodicity for a measure with $\mathbf{R}_0^\infty$-quasi-invariance. See, [13]. Therefore the necessity part follows from preceding arguments to Theorem 1.5. For the sufficiency, we have only to check that $\mu^1 * G_\alpha$ is tail-trivial. Let $A \in \mathscr{B}_\infty$. Since G_α is $\mathbf{R}_0^\infty$-ergodic (assured by 0–1 law) so $G_\alpha(A - x)$ takes only the value 0 or 1 as a function of $x \in \mathbf{R}^\infty$. Moreover, a set $E = \{x \in \mathbf{R}^\infty | G_\alpha(A - x) = 1\}$ belongs to $\mathscr{B}_\infty$. Therefore $\mu^1 * G_\alpha(A) = \mu^1(E) = 0$

or 1. Q.E.D.

§2. l^2-Quasi-Invariant and l^2-Continuous Measure and Its Linear Transformations

Let $\mu\in M(\mathbf{R}^\infty)$. We say that μ is l^2-continuous if its Fourier-Bochner transformation, $\hat{\mu}(y)=\int\exp(2\pi ix(y))d\mu(x)$ is a continuous function of $y\in\mathbf{R}_0^\infty$ with the induced topology from l^2. Since for any $\varepsilon>0$ and for any $y\in\mathbf{R}_0^\infty$,

$$\mu(x|\,|x(y)|>\varepsilon)\leqq e^\varepsilon(e^\varepsilon-1)^{-1}\int(1-\exp(-|x(y)|)d\mu(x)$$

$$=e^\varepsilon(e^\varepsilon-1)^{-1}\int(1-\exp(iux(y))\pi^{-1}(1+u^2)^{-1}dud\mu(x)$$

$$\leqq e^\varepsilon(e^\varepsilon-1)^{-1}\int_{|u|\leqq R}|1-\hat{\mu}(uy)|\pi^{-1}(1+u^2)^{-1}du$$

$$+4e^\varepsilon(e^\varepsilon-1)^{-1}\pi^{-1}(\pi/2-\tan^{-1}R),$$

so $\mu(x|\,|x(y)|>\varepsilon)\to0$ as $\|y\|=(\sum_{n=1}^\infty y_n^2)^{1/2}\to0$. Therefore for any $h\in l^2$, we can define $x(h)$, taking a limit (in the sense of convergence in μ) of $\{x(h_n)\}$ such that $\{h_n\}\subset\mathbf{R}_0^\infty$ and $h_n\to h$ in l^2. We shall denote the set of all μ-measurable real-valued functions by $\mathrm{Mes}(\mathbf{R}^\infty,\mu,\mathbf{R}^1)$.

Theorem 2.1. *Let $\mu\in M(\mathbf{R}^\infty)$ be an l^2-continuous and l^2-quasi-invariant measure. (in abbreviation, l^2-c.q. measure) Then the map $h\in l^2\to x(h)\in\mathrm{Mes}(\mathbf{R}^\infty,\mu,\mathbf{R}^1)$ equipped with the topology of convergence in μ is a homeomorphic operator.*

Proof. The continuity of the map follows from above arguments. We shall prove the inverse continuity. Let $\{h_n\}\subset l^2$ and $x(h_n)\to0$ in μ. It follows that $1-\int\exp(-|x(h_n)|)d\mu(x)\to0\ (n\to\infty)$, and therefore for an appropriate subsequence $\{n_j\}$, $\sum_{j=1}^\infty\left\{1-\int\exp(-|x(h_{n_j})|)\,d\mu(x)\right\}<\infty$, which yields $\sum_{j=1}^\infty|x(h_{n_j})|<\infty$ for μ-a.e.x. It follows from the l^2-quasi-invariance that for any $h\in l^2$, $\sum_{j=1}^\infty|(x\pm h)(h_{n_j})|<\infty$ for μ-a.e.x. and therefore $\sum_{j=1}^\infty|h(h_{n_j})|<\infty$. In this step, we put $s_1=h_{n_1}$ and put inductively $s_j = h_{n_j}$ or $-h_{n_j}$ as $\|s_1+\cdots+s_j\|^2 \geqq \sum_{n=1}^{j}\|s_n\|^2$ will be satisfied. And we put $S_n = s_1+\cdots+s_n$. Then for any $h \in l^2$, $|h(S_n)-h(S_m)| \leqq \sum_{j=n+1}^{m}|h(s_j)| \to 0$ $(n, m \to \infty)$, which derives that $\{\|S_n\|\}$ is bounded. Therefore $\|h_{n_j}\| = \|s_j\| \to 0$ $(j \to \infty)$. It follows easily that $h_n \to 0$ $(n \to \infty)$.

Q. E. D.

Proposition 2.1. *Let $\mu, \mu^1 \in M(\mathbf{R}^\infty)$ and $\mu \gtrsim \mu^1$. If μ is l^2-continuous, then μ^1 is also l^2-continuous.*

Proof. We put $A_n = \left\{x \,\middle|\, \frac{d\mu^1}{d\mu}(x) \leqq n\right\}$. Then $\mu(A_n^c) \to 0$, therefore $\mu^1(A_n^c) \to 0$ $(n \to \infty)$. Now for $y \in \mathbf{R}_0^\infty$,

$$\left|1-\int \exp(ix(y))d\mu^1(x)\right| \leqq \int_{A_n} |1-\exp(ix(y))| \frac{d\mu^1}{d\mu}(x)d\mu(x)+2\mu^1(A_n^c)$$

$$\leqq n\int |1-\exp(ix(y))|d\mu(x)+2\mu^1(A_n^c)$$

$$\leqq \sqrt{2}\,n \left|\int (1-\exp(ix(y))d\mu(x)|\right|^{1/2} + 2\mu^1(A_n^c).$$

Q. E. D.

It shows that $\widehat{\mu^1}(y)$ is continuous with $\|y\|$.

Proposition 2.2. *Let $\mu \in M(\mathbf{R}^\infty)$ be l^2-continuous. Then $T_\mu \subset l^2$.*

Proof. Let $t \in T_\mu$. Then $\widehat{\mu_t}(y)$ is continuous in virtue of Proposition 2.1. Since $\widehat{\mu_t}(y) = \exp(it(y))\hat{\mu}(y)$ and $|1-\exp(it(y))| = |\hat{\mu}(y)-\widehat{\mu_t}(y)|\,|\hat{\mu}(y)|^{-1}$, so $\exp(it(y))$ (equivalently, $t(y)$) is a continuous function of $\|y\|$. Consequently, $t \in l^2$. Q. E. D.

Let μ be an l^2-continuous measure on $\mathscr{B}(\mathbf{R}^\infty)$ and S be a linear operator (not necessarily bounded) on l^2. Then the function $\hat{\mu}(Sy)$ of $y \in \mathbf{R}_0^\infty$ is positive definite and continuous with a inductive limit topology of $\mathbf{R}_0^\infty$. Therefore a unique $\mu_S \in M(\mathbf{R}^\infty)$ corresponds to $\hat{\mu}(Sy)$ through the Fourier-Bochner transformation.

Proposition 2.3. *Let S be a linear operator on l^2. Assume that μ is l^2-c.q. Then in order that μ_S is l^2-continuous, it is necessary and* *sufficient that there exists a bounded operator* $\tilde{S}$ *on* l^2 *such that* $\tilde{S}|\mathbf{R}_0^\infty = S|\mathbf{R}_0^\infty$.

Proof. Clearly the existence of such $\tilde{S}$ implies the continuity of μ_S. Conversely, suppose that μ_S is l^2-continuous and that $\{y_n\} \subset \mathbf{R}_0^\infty$, $y_n \to 0$ in l^2. Then $\widehat{\mu_S}(uy_n) = \hat{\mu}(uSy_n) \to 1$ $(n \to \infty)$ for all $u \in \mathbf{R}^1$. Therefore $\{Sy_n\}$ converges to 0 in μ, consequently $Sy_n \to 0$ $(n \to \infty)$ in l^2 by Theorem 2.1. It shows that $S|\mathbf{R}_0^\infty$ is continuous with respect to the induced topology from l^2, therefore it can be extended to a bounded operator $\tilde{S}$ on l^2. Q.E.D.

Theorem 2.2. *Let* $\mu, \mu^1 \in M(\mathbf{R}^\infty)$. *Assume that* μ *is* l^2-*continuous and that* S *is a bounded operator on* l^2. *Then if* $\mu \gtrsim \mu^1$, *we have* $\mu_S \gtrsim \mu_S^1$.

Proof. By Proposition 2.1, μ^1 is also l^2-continuous, so μ_S^1 has a meaning. Let $\mathscr{B}$ be the minimal σ-field with which all the functions $x(Sh)$, $h \in l^2$ are measurable. And let D be the set of all trigonometric polynomials of a type of $\sum_{j=1}^n \alpha_j \exp(ix(Sh_j))$, where $\alpha_j \in \mathbf{C}$, $h_j \in l^2$ and n is arbitrary but finite. The $L_\mu^2(\mathbf{R}^\infty)$-closure of D (denoted by $\bar{D}$) consists with all $\mathscr{B}$-measurable square summable functions with μ. We shall denote the conditional expectation of $X \in L_\mu^1(\mathbf{R}^\infty)$ to $\mathscr{B}$ by $E[X|\mathscr{B}]$. A map U defined on D such that $\sum_{j=1}^n \alpha_j \exp(ix(Sh_j)) \to \sum_{j=1}^n \alpha_j \exp(ix(h_j))$ is an isometric operator from $D \subset L_\mu^2(\mathbf{R}^\infty)$ into $L_{\mu_S}^2(\mathbf{R}^\infty)$. So it can be extended to $\bar{D}$ with the same property. We put $U\left(\sqrt{E\left[\frac{d\mu^1}{d\mu}\middle|\mathscr{B}\right]}\right) = Y \in L_{\mu_S}^2(\mathbf{R}^\infty)$. Since for any $X \in \bar{D}$ and for any $h \in l^2$, $U(\exp(ix(Sh))X) = \exp(ix(h))U(X)$, so

$$\int \exp(ix(h))\,|Y(x)|^2 d\mu_S(x) = \int \exp(ix(Sh)) E\left[\frac{d\mu^1}{d\mu}\middle|\mathscr{B}\right](x)\, d\mu(x)$$

$$= \int \exp(ix(Sh)) d\mu^1(x) = \int \exp(ix(h)) d\mu_S^1(x).$$

Thus, $d\mu_S^1(x) = |Y(x)|^2 d\mu_S(x)$. Q.E.D.

Theorem 2.3. *Let* $\mu \in M(\mathbf{R}^\infty)$ *be* l^2-*c.q. Then for any bounded operator* S *on* l^2, $T_{\mu_S} = S^* l^2$. *Moreover if* μ *is* l^2-*ergodic, then* μ_S *is*

S^*l^2-ergodic.

Proof. Let $h\in l^2$. Since $(\mu_h)_S=(\mu_S)_{S^*h}$, so $(\mu_S)_{S^*h}\simeq\mu_S$ in virtue of Theorem 2.2, which shows $S^*l^2\subset T_{\mu_S}$. We shall prove the converse relation in a similar method with in Theorem 2.2. Let D_1 be a set of all trigonometric polynomials of a type of $\sum_{j=1}^n \alpha_j \exp(ix(h_j))$, where $\alpha_j\in\mathbf{C}$, $h_j\in l^2$ and n is arbitrary but finite. A map U_1 defined on D_1 such that $\sum_{j=1}^n\alpha_j\exp(ix(h_j))\to\sum_{j=1}^n\alpha_j\exp(ix(Sh_j))$ is an isometric operator from $D_1\subset L^2_{\mu_S}(\mathbf{R}^\infty)$ into $L^2_\mu(\mathbf{R}^\infty)$, and it can be extended to the whole space $L^2_{\mu_S}(\mathbf{R}^\infty)$, because D_1 is dense in $L^2_{\mu_S}(\mathbf{R}^\infty)$. Now let $t\in T_{\mu_S}$. Putting $U_1\left(\sqrt{\frac{d(\mu_S)_t}{d\mu}}\right)=X_t$, we have for $h\in l^2$,

$$\int\exp(ix(Sh))|X_t|^2d\mu(x)=\int\exp(ix(h))d(\mu_S)_t(x)=\exp(it(h))\int\exp(ix(Sh))d\mu(x).$$

Since $X_t^2d\mu(x)\lesssim d\mu(x)$, so from Proposition 2.1, $\exp(it(h))$ is a continuous function of $\|Sh\|$, therefore the same holds for $t(h)$. It follows that there exists a suitable constant $K>0$ such that $|t(h)|\leqq K\|Sh\|$ for any $h\in l^2$. Consequently, $t\in S^*l^2$.

For the ergodicity, it will be sufficient that a function $X\in L^2_{\mu_S}(\mathbf{R}^\infty)$ which satisfies for any $h\in l^2$, $X(x)=X(x-S^*h)$ for μ_S-a.e.x is a constant function for μ_S-a.e.x. First we shall state a following general consideration. Let $Z(x)\in L^2_{\mu_S}(\mathbf{R}^\infty)$ such that for some $h\in l^2$, $Z(x-S^*h)=Z_h(x)\in L^2_{\mu_S}(\mathbf{R}^\infty)$. We put $U_1(Z)=W$ and $U_1(Z_h)=W_h$. Then for any $\varepsilon>0$, there exists trigonometric polynomial such that,

$$\|Z_h(x)-\textstyle\sum_{j=1}^n\alpha_j\exp(ix(h_j))\|_{\mu_S}<\varepsilon \quad\text{and}$$

$$\|Z_h(x)-\textstyle\sum_{j=1}^n\alpha_j\exp(ix(h_j))\|_{(\mu_S)_{S^*h}}<\varepsilon.$$

Therefore,

$$\|W_h(x)-\textstyle\sum_{j=1}^n\alpha_j\exp(ix(Sh_j))\|_\mu<\varepsilon \quad\text{and}$$

$$\|W(x)-\textstyle\sum_{j=1}^n\alpha_j\exp(ih(Sh_j))\exp(ix(Sh_j))\|_\mu<\varepsilon.$$

If necessary, taking a subsequence, we may assume that the above two trigonometric polynomials converge to $W_h(x)$ and to $W(x)$ for μ-a.e.x respectively. From the l^2-quasi-invariance, it follows that $W_h(x+h)=$

$W(x)$ for μ-a.e.x. Returning to X, we put $U_1(X)=Y$. Since $X(x)=X_h(x)$ for μ_S-a.e.x, so for any $h\in l^2$, $Y(x+h)=Y(x)$ for μ-a.e.x. Consequently, $Y(x)=\text{const}$ for μ-a.e.x in virtue of l^2-ergodicity of μ, which derives that $X(x)=\text{const}$ for μ_S-a.e.x. Q.E.D.

§3. Admissible Linear Transformations

Definition 3.1. *Let $\mu\in M(\mathbf{R}^\infty)$ be l^2-c.q., and S be a bounded operator on l^2. We say that S is admissible for μ, if $\mu_S\simeq\mu$. We denote the set of all such S by Λ_μ.*

Proposition 3.1. *Let $\mu, \mu^1\in M(\mathbf{R}^\infty)$ be l^2-c.q. Then,*

(a) *$\mu\simeq\mu^1$ implies $\Lambda_\mu=\Lambda_{\mu_1}$.*

(b) *if T is a homeomorphic operator on l^2, then $\Lambda_{\mu_T}=T^{-1}\Lambda_\mu T$. Especially in the case of $T=\alpha I$, (I is an identity operator on l^2 and $\alpha\neq 0\in\mathbf{R}^1$) $\Lambda_{\mu_T}=\Lambda_\mu$.*

(c) *$S_1, S_2\in\Lambda_\mu$ implies $S_1\cdot S_2\in\Lambda_\mu$. If S has a bounded inverse, $S\in\Lambda_\mu$ implies $S^{-1}\in\Lambda_\mu$.*

(d) *$S\in\Lambda_\mu$ implies S^* is onto. Hence S is a homeomorphism from l^2 to a closed subspace of l^2.*

Proof. (a) and (d) are immediate consequences of Theorem 2.2 and of Theorem 2.3 respectively. (b) and (c) follow from the fact $(\mu_S)_T=\mu_{ST}$.

In this section we shall study Λ_μ first for a measure of Gauss type and later for a general μ.

1. A Measure of Gauss Type

We say that $G_V\in M(\mathbf{R}^\infty)$ is a measure of Gauss type if its Fourier-Bochner transformation has a following form.

$$\widehat{G_V}(e)=\int\exp(ix(e))dG_V(x)=\exp(-\|Ve\|^2/2),$$

where V is a bounded operator on l^2, such that V^* is onto. This definition is particular comparing with a usual definition, because we shall demand that G_V is l^2-c.q. Actually G_V is l^2-c.q., which is assured by Theorem 2.3 and by the fact that G_I is l^2-c.q. One more remark is that, since $G_V = G_{\sqrt{V^*V}}$, so without loss of generality we may assume that V is a positive definite Hermitian homeomorpfic operator.

Theorem 3.1. *Let G_V be a measure of Gauss type. Then $\Lambda_{G_V} = \{S | S^*$ is onto, and $V^*V-(VS)^*VS$ is a Hilbert-Shmidt operator.$\}$.*

This result is due to [2]. We omit the proof.

Theorem 3.2. *Let $G_V, G_W \in M(\mathbf{R}^\infty)$ be measures of Gauss type. Then for $\Lambda_{G_V} = \Lambda_{G_W}$, it is necessary and sufficient that $G_V \simeq G_{\alpha W}$ for a some positive constant α.*

Proof. The sufficiency follows from Proposition 3.1. Let $\Lambda_{G_V} = \Lambda_{G_W}$. V, W may be assumed as homeomorphic operators. Then from (b) in Proposition 3.1, $\Lambda_{G_{VW^{-1}}} = \Lambda_{G_I}$. Taking $S = \sqrt{(VW^{-1})^* VW^{-1}}$ in place of VW^{-1}, we have $\Lambda_{G_S} = \Lambda_{G_I}$. From it we can derive that there exists some positive constant α such that $S - \alpha I$ is a Hilbert-Shmidt operator. (We shall prove it in a subsequent lemma.) Consequently from Theorem 3.1, we have $G_S \simeq G_{\alpha I}$, equivalently $G_V \simeq G_{\alpha W}$. Q.E.D.

Since any isometric operator belongs to Λ_{G_I}, so for the remainder part of the above proof it will be sufficient to assure the following fact.

Lemma 3.1. *Let S be a Hermitian bounded operator on l^2. Assume that for any isometric operator U, $S - U^*SU$ is a Hilbert-Shmidt operator. Then we conclude that there exists some real constant α such that $S - \alpha I$ is a Hilbert-Shmidt operator.*

Proof. Let $\{E_\lambda\}$ be the resolution of unity of S, $S = \int \lambda dE_\lambda$. We shall denote the set of all continuous spectrums (of all point spectrums) of S by $C(S)$ $(P(S))$ respectively. We divide the proof into five steps. (I) For any $\lambda \in C(S)$ and for $\forall p < \lambda < \forall q$, the dimension of Range$(E_p - E_q)$ is infinite.

Proof is derived from the Hermitian property of S.

(II) $C(S)$ consists of at most single point.

Suppose the contrary case, and let $\lambda_1, \lambda_2 \in C(S)$, $\lambda_1 < \lambda_2$. Taking $p_i, q_i\ (i=1, 2)$ such that $p_1 < \lambda_1 < q_1 < p_2 < \lambda_2 < q_2$, we set $M_i = \mathrm{Range}\,(E_{q_i} - E_{p_i})$. Then M_1 and M_2 are mutually orthogonal and their dimensions are infinite. Take an orthogonal operator U on l^2 such that $UM_1 = M_2$ and $UM_1^\perp = M_2^\perp$. Then for any $m \in M_1$, $<Sm, m> \leqq q_1\|m\|^2$ and $<SUm, Um> \geqq p_2\|m\|^2$, therefore $<(U^*SU-S)m, m> \geqq (p_2-q_1)\|m\|^2$. ($<\cdot, \cdot>$ means the scalar product in l^2.) It contradicts with the assumption of a Hilbert-Schmidt operator.

(III) If λ is an accumulation point of $P(S)$ (that is, whose any neighbourhood meets infinitely many points of $P(S)$) and $C(S) \neq \emptyset$, then $C(S) = \{\lambda\}$. The set of all accumulation points of $P(S)$ consists of at most single point.

Proof is carried out in a similar way with in (II).

(IV) We put $\alpha = \lambda$, in the case of (A) $C(S) = \{\lambda\}$, (B) $C(S) = \emptyset$ and λ is an accumulation point of $P(S)$. If $C(S) = \emptyset$ and $P(S)$ consists of only finitely many elements, then there exists a unique $\lambda \in P(S)$ such that the eigen-vector space corresponding to λ has an infinite dimension. In this case putting $\alpha = \lambda$, $S - \alpha I$ becomes a finite-rank operator. So we shall consider the problem in the case of (A) or (B). Put $T = S - \alpha I$. Then the continuous spectrum of T (if it exists) is origin and $P(T)$ accumulates only to origin. Let $\{F_\lambda\}$ be the resolution of unity of T, $T = \int_a^b \lambda dF_\lambda$, and let $\{\varepsilon_n\}$ ($\{\eta_n\}$) be a decreasing (increasing) sequence which converges to 0 respectively. We put $T_n = \int_a^{\eta_n} \lambda dF_\lambda + \int_{\varepsilon_n}^b \lambda dF_\lambda$. Then T_n is a compact Hermitian operator and for any $h \in l^2$, $|<(T-T_n)h, h>| \leqq (|\eta_n| + \varepsilon_n)\|h\|^2 \to 0$ $(n \to \infty)$. Since for any bounded Hermitian operator H, $\sup_{\|h\| \leqq 1} |<Hh, h>| = \|H\|$, it follows that T is also a compact operator.

(V) Let $\lambda_n\ (\neq 0)$ be an eigen-value of T and h_n be a corresponding unit eigen-vector of T, $Th = \sum_{n=1}^\infty \lambda_n <h, h_n> h_n$, $\lim_n \lambda_n = 0$. We take a subsequence $\{n_j\}$ such that $\sum_{j=1}^\infty |\lambda_{n_j}|^2 < \infty$, and define an isometric operator U such that $Uh_j = h_{n_j}$ for all j and $U|\ker T = $ identity. Then from the assumption, $\infty > \sum_{j=1}^\infty \|(U^*TU - T)h_j\|^2 = \sum_{j=1}^\infty |\lambda_{n_j} - \lambda_j|^2$, which is equivalent to $\sum_{j=1}^\infty \lambda_j^2 < \infty$. Q.E.D.

Theorem 3.3. *Let S be an Hermitian operator on l^2. Assume that $S \in \Lambda_{G_V}$ for some G_V. Then $I-S^2$ is a Hilbert-Shmidt operator.*

Proof. Since V^*V-SV^*VS is a Hilbert-Shmidt operator, the proof follows from the following lemma.

Lemma 3.2. *Let P be a positive definite Hermitian homeomorphic operator on l^2. Assume that T is a bounded operator on l^2 and that $P-TPT$ is a Hilbert-Shmidt operator. Then $I-T^2$ is a Hilbert-Shmidt operator.*

Proof. First we shall prove for any $n>0$,

$$(**) \qquad P^{2n+1}-TP^{2n+1}T \text{ is a Hilbert-Shmidt operator.}$$

Inductively, we shall assume that (**) holds for $1 \leqq j \leqq n-1$. Multiplying P^2 by $P^{2n-1}-TP^{2n-1}T$, we have $P^{2n+1}-P^2TP^{2n-1}T$ is a Hilbert-Shmidt operator. On the other hand, since both $TP^2-TPTPT$ and $P^2T-TPTPT$ are Hilbert-Shmidt operators, so the same holds for TP^2-P^2T. Substituting TP^2 for P^2T in $P^2TP^{2n-1}T$, we can assure that (**) holds for n. Let $\{E_\lambda\}$ be the resolution of unity of P, $P=\int_a^b \lambda dE_\lambda$. Without loss of generality we can assume that $0<a<b<1$. Approximating $\lambda^{1/2n+1}$ by polynomials of λ on the interval $[-b, b]$, for any $\varepsilon>0$, there exist m and $a_j\ (j=1, 2,\dots, m) \in \mathbf{R}^1$ such that $\|P^{1/2n+1}-\sum_{j=1}^m a_j P^{2j+1}\|<\varepsilon$ for each fixed n. Generally speaking, for a bounded operator B_1, B_2, $\|B_1-TB_1T-(B_2-TB_2T)\| \leqq (1+\|T\|^2)\|B_1-B_2\|$, so $P^{1/2n+1}-TP^{1/2n+1}T$ is a compact operator. Since,

$$\|(I-P^{1/n})x\|^2=\int_a^b (1-\lambda^{1/n})^2 d<E_\lambda x, x> \leqq (1-a^{1/n})^2\|x\|^2,$$

it follows that by the same argument as in above $I-T^2$ is a compact operator. Let $(T^2-I)h=\sum_{n=1}^\infty \lambda_n <h, h_n> g_n$ be the spectre decomposition, where $\{h_n\}$ and $\{g_n\}$ are orthonomal systems respectively, and $\lambda_n>0$, $\lim_n \lambda_n=0$. Then,

$$<T^2PT^2h_n, g_n> = <P(\lambda_n g_n+h_n), \lambda_n h_n+g_n>$$

$$=(1+\lambda_n^2)<Ph_n, g_n> + \lambda_n\{<Ph_n, h_n> + <Pg_n, g_n>\}.$$

Since $\{<T^2PT^2h_n, g_n> - <Ph_n, g_n>\} \in l^2$ and $<Ph, h> \geqq a\|h\|^2$ for any $h \in l^2$, it follows that $\{\lambda_n\} \in l^2$, which shows that $I - T^2$ is a Hilbert-Shmidt operator. Q.E.D.

Corollary. *Let G_V be a measure of Gauss type. Then for a positive definite Hermitian homeomorphic operator S, following conditions are equivalent.*

(a) *$I - S$ is a Hilbert-Shmidt operator.*

(b) *$S \in \Lambda_{G_V}$.*

2. General μ

First we shall state the following fact comparing with Corollary of Theorem 3.3. Let S be an arbitrary homeomorphic operator on l^2, and let $\mu \in M(\mathbf{R}^\infty)$ be an l^2-c.q. measure. We put

$$\mu^S(A) = \sum_{n=0}^{\infty} 1/2^{n+2}\{\mu_{S^n}(A) + \mu_{S^{-n}}(A)\} \qquad \text{for} \quad A \in \mathscr{B}(\mathbf{R}^\infty).$$

Then, (a) μ^S is l^2-c.q., (b) $S \in \Lambda_{\mu^S}$.

Therefore some l^2-c.q. measures have an arbitrarily given homeomorphic operator as an admissible element. However if we confine our consideration to l^2-ergodic measures, we can generalize Theorem 3.3 as follows.

Theorem 3.4. *Let S be an Hermitian bounded operator on l^2. Then in order that $S \in \Lambda_\mu$ for some l^2-continuous, l^2-quasi-invariant and l^2-ergodic (in an abbreviation, l^2-c.q.e.) measure $\mu \in M(\mathbf{R}^\infty)$, it is necessary that $I - S^2$ is a compact operator.*

Proof is derived from following lemmas.

Lemma 3.3. *Let T be an Hermitian bounded operator on l^2. And let $m \in M(\mathbf{R}^\infty)$ be l^2-c.q. and $\mu \in M(\mathbf{R}^\infty)$ be l^2-continuous. Then for $m*\mu \simeq m*\mu_T$, it is sufficient that $I - T$ is a Hilbert-Shmidt operator.*

Proof. Let $Th = \sum_{n=1}^{\infty}(1+\lambda_n)<h, h_n>h_n$ be the spectre decomposition of T, where $\{h_n\}$ is c.o.n.s. in l^2 and $\sum_{n=1}^{\infty}\lambda_n^2 < \infty$. We set

$e_n=(0, 0,\dots, 0, \overset{n}{\overbrace{1}}, 0,\dots)\in \mathbf{R}^\infty$ for each n, and take an orthogonal operator on l^2 such that $Ue_n=h_n$. Then $U^*TUh=\sum_{n=1}^\infty(1+\lambda_n)<h, e_n>e_n$, and $m*\mu\simeq m*\mu_T$ is equivalent to $m_U*\mu_U\simeq m_U*(\mu_U)_{U^*TU}$. Therefore substituting m_U for m, μ_U for μ and U^*TU for T, we may prove it in the case of $h_n=e_n$. In this case T can be extended naturally to $\mathbf{R}^\infty$. Now in virtue of the l^2-continuity, $\sum_{n=1}^\infty\lambda_n^2x_n^2<\infty$, for μ-a.e.x. We shall denote the above set by $H_\lambda(\subset\mathbf{R}^\infty)$. Since for any $x=(x_1,\dots, x_n,\dots)\in H_\lambda$, $\sum_{n=1}^\infty\{Tx(e_n)-x(e_n)\}^2=\sum_{n=1}^\infty\lambda_n^2x_n^2<\infty$, so $m*\mu(A)=\int_{H_\lambda}m(A-x)d\mu(x)=0$ implies $\int_{H_\lambda}m(A-Tx+Tx-x)d\mu(x)=0$, equivalently $m(A-Tx)=0$ for μ-a.e.x. It yields $m(A-x)=0$ for μ_T-a.e.x and therefore $m*\mu_T(A)=0$. The converse relation is shown in a similar way, so $m*\mu_T\simeq m*\mu$. Q.E.D.

Lemma 3.4. *Let $\mu^1, \mu^2\in M(\mathbf{R}^\infty)$ be $\mathbf{R}_0^\infty$-quasi-invariant, and $\mu^i=\{f_{n,m}^i\}$ $(i=1, 2)$. Suppose that μ^1 is $\mathbf{R}_0^\infty$-ergodic and that $\mu^1\gtrsim\mu^2$. Then $\int_{R^{n-m}}|f_{n,m}^1(x)-f_{n,m}^2(x)|dx\to 0$ $(n\geqq m\to\infty)$.*

Proof. Since $\left\{E\left[\frac{d\mu^2}{d\mu^1}\,\middle|\,\mathscr{B}^m\right]\right\}$ forms an inverse martingale with respect to $(\mathscr{B}^m)$ and μ^1 is tail-trivial,

$$\int\left|E\left[\frac{d\mu^2}{d\mu^1}\,\middle|\,\mathscr{B}^m\right](x)-1\right|d\mu^1(x)\longrightarrow 0 \qquad (m\longrightarrow\infty).$$

On the other hand, for a fixed m,

$$\int\left|E\left[\frac{d\mu^2}{d\mu^1}\,\middle|\,\mathscr{B}^m\right](x)-\frac{f_{n,m}^2}{f_{n,m}^1}(p_{n,m}(x))\right|d\mu(x)\longrightarrow 0 \qquad (n\longrightarrow\infty),\ \text{and}$$

$\int\left|1-\frac{f_{n,m}^2}{f_{n,m}^1}(p_{n,m}(x))\right|d\mu(x)$ is a decreasing sequence of m $(\leqq n)$. From these results we have the desired conclusion. Q.E.D.

Lemma 3.5. *Let $\mu\in M(\mathbf{R}^\infty)$ be l^2-c.q. and let $f_n(u)$ be the density function of $\mu_{n,n-1}=p_{n,n-1}\mu$ with one-dimensional Lebesgue measure du. Then $\{f_n\}$ forms a totally bounded set of $L_{du}^1(\mathbf{R}^1)$.*

Proof. By Theorem 1.3, $\int\left|1-\frac{d\mu_t}{d\mu}(x)\right|d\mu(x)$ is a continuous function of $t\in l^2$. Since $\int|f_n(u-a)-f_n(u)|du\leqq\int\left|1-\frac{d\mu_t}{d\mu}(x)\right|d\mu(x)$ for any $t=(0,\dots, 0, \overset{n}{\overbrace{a}}, 0,\dots)$, $a\in\mathbf{R}^1$, so $\left\{\int|f_n(u-a)-f_n(u)|du\right\}$ is a family of equi-continuous functions of a. On the other hand, l^2-continuity assures that for any given $\varepsilon>0$, there exists R not depending on n such that, $\int_{|u|>R} f_n(u)du<\varepsilon$. It follows from an exercise in p.p. 458 of [14] that $\{f_n\}$ is a totally bounded set. Q.E.D.

Proof of Theorem 3.4. We put $T=S^2$, then T is a positive definite Hermitian homeomorphic operator. According to [6], there exists equivalence operator E such that E^*TE has a complete set of eigenvectors in l^2. Equivalence operator means that, (a) it is one to one onto, bounded and therefore has a bounded inverse, (b) $I-E^*E$ is a Hilbert-Shmidt operator. Since $\mu_T \simeq \mu$, so $\mu_{TE} \simeq \mu_E$ and $\mu_{(E^*)^{-1}E^*TE} \simeq \mu_{(E^*)^{-1}E^*E}$. We put $\mu_{(E^*)^{-1}}=\mu^1$. Then by Lemma 3.3, for a measure $G_I=G$ of Gauss type, $G*\mu^1_{E^*TE} \simeq G*\mu^1_{E^*E} \simeq G*\mu^1$. For the spectre decomposition of E^*TE, using the same argument as in the proof of Lemma 3.3 and using the rotational invariance of G, we may assume that $E^*TEh=\sum_{n=1}^{\infty}\lambda_n<h, e_n>e_n$, where $e_n=(0,\dots, 0, \underset{n}{\overset{\smile}{1}}, 0,\dots)$ and $c_1 \leqq \forall \lambda_n \leqq c_2$ for some positive constants c_1, c_2. Let $f_n(u)$ be a density function of $p_{n,n-1}\mu^1$ with du. Since the density function of $p_{n,n-1}\mu^1_{E^*TE}$ is $\lambda_n^{-1}f_n(\lambda_n^{-1}u)$, so from the l^2-ergodicity due to Theorem 2.3 and from Proposition 1.2, Theorem 1.5 and Lemma 3.4,

$$\int|f_n(u)*(\sqrt{2\pi})^{-1}\exp(-u^2/2)-\lambda_n^{-1}f_n(\lambda_n^{-1}u)*(\sqrt{2\pi})^{-1}\exp(-u^2/2)|du \longrightarrow 0$$

$$(n \longrightarrow \infty).$$

Especially, $\exp(-v^2/2)|\hat{f}_n(v)-\hat{f}_n(\lambda_n v)| \to 0$ $(n\to\infty)$, where $\hat{f}_n(v)=\int\exp(iuv)f_n(u)du$.

Let λ be an arbitrary limiting point of $\{\lambda_n\}$. Then by the compactness of $\{f_n\}$ assured by Lemma 3.5, there exists $f\in L^1_{du}(\mathbf{R}^1)$, $\int|f(u)|\,du=1$ such that $\hat{f}(\lambda v)=\hat{f}(v)$ for all $v\in\mathbf{R}^1$. Since for any positive integer n, $\hat{f}(\lambda^n v)=\hat{f}(v)$, so in the case of $\lambda>1$, we have $\hat{f}(v)=0$ for any $v\neq 0$, and in the case of $c_2 \leqq \lambda<1$, we have $\hat{f}(v)=1$ for any $v\in\mathbf{R}^1$. Therefore $\lambda=1$ and it follows that $\lim_n \lambda_n=1$, which shows the compactness of E^*TE-I. As $E^*(I-T)E=E^*E-I+I-E^*TE$, so $I-T$ is a compact operator. Q.E.D.

Generally speaking, in Theorem 3.4 we cannot replace a compact operator with a Hilbert-Shmidt operator.

Example 3.1. Let dx be the volume element of Lebesgue measure on $\mathscr{B}(\mathbf{R}^n)$ and we put $\|x\|=\sqrt{x_1^2+\cdots+x_n^2}$ for $x=(x_1,\dots,x_n)\in\mathbf{R}^n$. Then for an integer $k>-n$,

$$\int\|x\|^k\exp(-\|x\|^2)dx=\pi^{n/2}\Gamma((n+k)/2)\Gamma(n/2)^{-1}.$$

We put $\gamma_n=\pi^{n/2}\Gamma((n+k)/2)\Gamma(n/2)^{-1}$, $v_n=\sqrt{2n(n+k)^{-1}}$ and form a measure $\mu_{n,k}$ on $\mathscr{B}(\mathbf{R}^n)$ such that

$$d\mu_{n,k}(x)=(\gamma_n v_n^n)^{-1}\|v_n^{-1}x\|^k\exp(-\|v_n^{-1}x\|^2)dx.$$

Then some calculations derive that

$$\int x_j^2 d\mu_{n,k}(x)=1,\qquad \text{for}\quad 1\leqq\forall j\leqq n. \tag{4}$$

$$1-\int\sqrt{\frac{d(\mu_{n,k})_t}{d\mu_{n,k}}}(x)d\mu_{n,k}(x)=1-\frac{\Omega_{n-1}}{\sqrt{n}\,\gamma_n}\exp(-\delta^2/n)\int_0^\infty dr\int_{-\infty}^\infty r^{n+k-2}\exp\left(-r^2-\frac{u^2}{n}\right)\left\{1+\frac{(u+\delta)^2}{nr^2}\right\}^{k/4}\left\{1+\frac{(u-\delta)^2}{nr^2}\right\}^{k/4}du, \tag{5}$$

for all $t\in\mathbf{R}^n$, where $\Omega_{n-1}=2\pi^{(n-1)/2}\Gamma((n-1)/2)^{-1}$ and $\delta=\sqrt{8^{-1}(n+k)}\|t\|$.

We shall estimate the value (5) as $n\to\infty$. In this step, we select and fix $\beta>2$ and put $k=\beta-n$. We shall write $\mu_{n,\beta}$ instead of $\mu_{n,k}$. We put

$$I_{n,\delta}=1-\exp(\delta^2/n)\int\sqrt{\frac{d(\mu_{n,\beta})_t}{d\mu_{n,\beta}}}(x)d\mu_{n,\beta}(x),$$

and divide it into two terms, $I_{n,\delta}=J_{n,\delta}+K_n$.

$$J_{n,\delta}=\frac{\Omega_{n-1}}{\sqrt{n}\,\gamma_n}\int_0^\infty dr\int_{-\infty}^\infty r^{\beta-2}\exp\left(-r^2-\frac{u^2}{n}\right)\left\{\exp\left(-\frac{u^2}{2r^2}\right)-\left(1+\frac{(u+\delta)^2}{nr^2}\right)^{\frac{\beta-n}{4}}\left(1+\frac{(u-\delta)^2}{nr^2}\right)^{\frac{\beta-n}{4}}\right\}du,$$

$$K_n=\frac{\Omega_{n-1}}{\sqrt{n}\,\gamma_n}\int_0^\infty dr\int_{-\infty}^\infty r^{\beta-2}\exp\left(-r^2-\frac{u^2}{n}\right)\left\{\left(1+\frac{u^2}{nr^2}\right)^{\frac{\beta-n}{2}}-\exp\left(-\frac{u^2}{2r^2}\right)\right\}du.$$

Then,

$$J_{n,\delta} \leqq \frac{\Omega_{n-1}}{\sqrt{n}\,\gamma_n} \int_0^\infty dr \int_{-\infty}^\infty r^{\beta-2} \exp(-r^2) \left\{ \exp\left(-\frac{u^2}{2r^2}\right) - \exp\left(-\frac{u^2+\delta^2}{2r^2}\right) \right\} du$$

$$\leqq \frac{\sqrt{2\pi}\,\Omega_{n-1}}{2\sqrt{n}\,\gamma_n} \delta^2 \int_0^\infty r^{\beta-3} \exp(-r^2) dr, \quad \text{and}$$

$$K_n = \frac{\Omega_{n-1}}{\sqrt{n}\,\gamma_n} \int_0^\infty dr \int_{-\infty}^\infty r^{\beta-1} \exp(-r^2) \left| \left(1+\frac{u^2}{n}\right)^{\frac{\beta-n}{2}} - \exp\left(-\frac{u^2}{2}\right) \right| du \longrightarrow 0 \quad (n \longrightarrow \infty).$$

Since $\frac{\Omega_{n-1}}{\sqrt{n}\,\gamma_n} \to \frac{2}{\sqrt{2\pi}\,\Gamma(2^{-1}\beta)}$ $(n\to\infty)$, so it follows that

$$1 - \int \sqrt{\frac{d(\mu_{n,\beta})_t}{d\mu_{n,\beta}}}(x)\, d\mu_{n,\beta}(x) \leqq c\|t\|^2 + \varepsilon_n, \tag{6}$$

for some universal constant c and for some positive sequence $\{\varepsilon_n\}$ which converges to 0.

Now let λ be any positive constant and consider a map $x \in \mathbf{R}^n \to \lambda x \in \mathbf{R}^n$. We shall denote the image measure of $\mu_{n,\beta}$ by this map by $\mu_{n,\beta}^\lambda$. Then after some calculations,

$$1 - \int \sqrt{\frac{d\mu_{n,\beta}^\lambda}{d\mu_{n,\beta}}}(x)\, d\mu_{n,\beta}(x) = 1 - \left(\frac{2\lambda}{1+\lambda^2}\right)^{\frac{\beta}{2}}. \tag{7}$$

Lastly we shall choose a subsequence $\{n_j\}$ such that $\sum_{j=1}^\infty \varepsilon_{n_j} < \infty$, and put $m_0 = 0$, $m_j = n_1 + \cdots + n_j$. Let $\mu_\beta \in M(\mathbf{R}^\infty)$ be the product-measure of $\{\mu_{n_j,\beta}\}$ such that $p_{m_j, m_{j-1}} \mu_\beta = \mu_{n_j,\beta}$ $(j = 1, 2, \ldots)$. Then μ_β has following properties.

(a) μ_β is l^2-continuous in virtue of (4) and of the symmetry of each $\mu_{n,\beta}$.

(b) μ_β is l^2-quasi-invariant and l^2-ergodic in virtue of (6).

(c) Let $\{a_j\}$ be a positive sequence such that $\sum_{j=1}^\infty (1-a_j)^2 < \infty$.

Then for a sequence $\{\lambda_n\}$ such that $\lambda_n = a_j$ for $m_{j-1} < n \leqq m_j$, we obtain $T_\lambda \in \Lambda_{\mu_\beta}$ in virtue of (7), where $T_\lambda h = \sum_{n=1}^\infty \lambda_n < h, e_n > e_n$ and $e_n = (\overbrace{0, 0, \ldots, 0, 1}^{n}, 0, \ldots)$.

Since $\sum_{n=1}^\infty (1-\lambda_n)^2 = \sum_{j=1}^\infty n_j (1-a_j)^2$, so T_λ is not necessarily a Hilbert-Shmidt operator.

Lastly comparing with corollary of Theorem 3.3, we shall give an example of ergodic measure μ, for which the implication (a) $\Longrightarrow$ (b) in the same corollary does not hold.

Example 3.2. We put $a_n=1+1/2+\cdots+1/n$ and $b_n=(a_n+a_{n+1})/2$. Let $f(u)$ be a function defined on $\mathbf{R}^1$ such that,

$$f(u)=\begin{cases} u^2, & |u|\leqq 1/2. \\ (1-|u|)^2, & 1/2<|u|\leqq 1. \\ (|u|-a_n)^2/\log^2(n+1), & a_n<|u|\leqq b_n. \\ (a_{n+1}-|u|)^2/\log^2(n+1), & b_n<|u|\leqq a_{n+1}. \end{cases}$$

Then $\sqrt{f}(u)$ is an even and absolutely continuous function, and from an elementary calculations,

$$\int_{-\infty}^{\infty} f(u)du=c<\infty,\ \int_{-\infty}^{\infty} u^2 f(u)du<\infty \text{ and } \int_{-\infty}^{\infty}\left|\frac{d\sqrt{f}}{du}(u)\right|^2 du<\infty.$$

Putting $f(u)/c=F(u)$, we form the product-measure $\mu\in M(\mathbf{R}^\infty)$ of one-dimensional measures $\{F(u)du\}$. Then from the above properties, μ is l^2-c.q.e., see [10]. Let $\lambda=\{\lambda_n\}$ be a positive sequence and from it we form T_λ as before. Then for $T_\lambda\in\Lambda_\mu$, it is necessary and sufficient that

$$\sum_{n=1}^{\infty}\left\{1-2\sqrt{\lambda_n}\int_0^{\infty}\sqrt{F(\lambda_n u)}\sqrt{F(u)}\,du\right\}<\infty, \tag{8}$$

because $\sqrt{F}$ is an even function. Changing the variable u to e^v and putting $2F(e^v)e^v=H(v)$, (8) is equivalent to

$$\sum_{n=1}^{\infty}\left\{1-\int_{-\infty}^{\infty}\sqrt{H(v+c_n)}\sqrt{H(v)}\,dv\right\}<\infty, \tag{9}$$

where $c_n=\log\lambda_n$. Therefore if $T_\lambda\in\Lambda_u$ for any positive sequence $\lambda=\{\lambda_n\}$ such that $\sum_{n=1}^{\infty}(1-\lambda_n)^2<\infty$, then (9) must be satisfied for all $\{c_n\}\in l^2$, which is equivalent to

$$\int_{-\infty}^{\infty}\left|\frac{d\sqrt{H}}{dv}(v)\right|^2 dv<\infty. \tag{10}$$

Since $\frac{d\sqrt{H}}{dv}(v)=\sqrt{2}\left\{2^{-1}\exp(v/2)\sqrt{F}(e^v)+\exp(3v/2)\frac{d\sqrt{F}}{dv}(e^v)\right\}$, and $\int_{-\infty}^{\infty} e^v F(e^v)dv=1/2$, so (10) is equivalent to

$$\int_{-\infty}^{\infty} e^{3v}\left|\frac{d\sqrt{F}}{dv}(e^v)\right|^2 dv=\int_0^{\infty} u^2\left|\frac{d\sqrt{F}}{du}(u)\right|^2 du<\infty.$$

However, $\int_{a_n}^{b_n} u^2\left|\frac{d\sqrt{f}}{du}(u)\right|^2 du \geqq \frac{a_n^2}{2(n+1)\log^2(n+1)}$, so $\int_{-\infty}^{\infty}\left|\frac{d\sqrt{H}}{dv}(v)\right|^2 dv=\infty$.

It shows that $T\notin\Lambda_\mu$ for some positive definite Hermitian homeomorphic operator T such that $I-T$ is a Hilbert-Shmidt operator.

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