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Dual Action on a von Neumann Algebra and Takesaki's Duality for a Locally Compact Group

By

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Abstract

We define a dual action β of a locally compact group G on a von Neumann algebra N and a crossed dual product $N \otimes_\beta^d G$. Then the Takesaki's duality is generalized in terms of these definitions as follows:

$$(M \otimes_\alpha G) \otimes_{\hat{\alpha}}^d G \sim M \otimes B(L^2(G)),$$

where $\hat{\alpha}$ is the dual action dual to a given action α, and

$$(N \otimes_\beta^d G) \otimes_{\hat{\beta}} G \sim N \otimes B(L^2(G)),$$

where $\hat{\beta}$ is the action dual to a given dual action β. As an application

$$M \otimes_\alpha G \sim M^\alpha \otimes B(L^2(G))$$

whenever $1 \otimes L^\infty(G) \subset M$ and $\alpha(1 \otimes f) = 1 \otimes \varepsilon f$.

Introduction

The main purpose of this paper is to generalize the Takesaki's duality of crossed products for locally compact abelian groups to that for a non abelian one [18, 13].

To see the situation more precisely we shall prepare some results which are necessary for Takesaki's duality. We first notice that a necessary and sufficient condition for an isomorphism α of M into $M \otimes L^\infty(G)$ to be induced from an action is that α satisfies the commutative diagram:

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$$\begin{array}{ccc} M & \xrightarrow{\alpha} & M\otimes L^\infty(G) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha\otimes\iota} \\ M\otimes L^\infty(G) & \xrightarrow[\iota\otimes\delta]{} & M\otimes L^\infty(G)\otimes L^\infty(G), \end{array}$$

where ι is the identity automorphism and δ is given by (1.5), (Theorem 2.1). Let $R(G)$ be the von Neumann algebra generated by the regular representation of G on $L^2(G)$. The crossed product $M\otimes_\alpha G$ is then defined as the von Neumann algebra generated by $\alpha(M)$ and $1\otimes R(G)$. Here, we denote $M\otimes_\alpha G$ by N. Then the same diagram holds for the action $\hat{\alpha}$ dual to α of the dual group $\hat{G}$ on N:

$$\begin{array}{ccc} N & \xrightarrow{\hat{\alpha}} & N\otimes L^\infty(\hat{G}) \\ {\scriptstyle\hat{\alpha}}\downarrow & & \downarrow{\scriptstyle\hat{\alpha}\otimes\iota} \\ N\otimes L^\infty(\hat{G}) & \xrightarrow[\iota\otimes\delta]{} & N\otimes L^\infty(\hat{G})\otimes L^\infty(\hat{G}). \end{array}$$

Making use of these α and $\hat{\alpha}$, we can state Takesaki's duality as follows:

$$(M\otimes_\alpha G)\otimes_{\hat{\alpha}}\hat{G}\sim M\otimes B(L^2(G)),$$

where $A\sim B$ means that A is isomorphic to B. Let F be the Fourier transformation of $L^2(\hat{G})$ onto $L^2(G)$ and β a mapping of N into $N\otimes R(G)$ defined by $\beta\equiv(\mathrm{Ad}\,1\otimes F)\circ\hat{\alpha}$. Let λ be the regular representation of G on $L^2(G)$ and γ a mapping given by (1.5). Since

$$(\mathrm{Ad}\,F\otimes F)\circ\delta=\gamma\circ\mathrm{Ad}\,F$$

by (1.8), β satisfies

$$(*)\qquad \begin{array}{ccc} N & \xrightarrow{\beta} & N\otimes R(G) \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta\otimes\iota} \\ N\otimes R(G) & \xrightarrow[\iota\otimes\gamma]{} & N\otimes R(G)\otimes R(G), \end{array}$$

and $\hat{\alpha}$ coincides with β up to the spatial isomorphism $\mathrm{Ad}\,1\otimes F$. We shall call an isomorphism satisfying the commutative diagram $(*)$ a *dual action*. β is then a dual action which is dual to α. By using a dual action we shall define a *crossed dual product* $N\otimes_\beta^d G$ of N by G as the von Neumann algebra

$$(**)\qquad \{\beta(N),\ 1\otimes R(G)\}''\quad (=\mathrm{Ad}\,1\otimes F(N\otimes_{\hat{\alpha}}\hat{G})),$$

(Definition 2.2). Then the Takesaki's duality is restated as follows:

$$(M\otimes_{\alpha}G)\otimes_{\beta}^{d}G\sim M\otimes B(L^{2}(G)).$$

Changing the roles of $\{\alpha, G\}$ and $\{\hat{\alpha}, \hat{G}\}$ and applying the Fourier transformation, we have

$$(N\otimes_{\beta}^{d}G)\otimes_{\hat{\beta}}G\sim N\otimes B(L^{2}(G)),$$

where $\hat{\beta}$ is the action dual to β, (Theorem 2.3).

Since Theorem 2.1 holds for a general locally compact group as well, the diagram (*) and the crossed dual product (**) have their meanings even when G is not necessarily abelian. Therefore our generalizations are obtained in the same forms as above in Theorems 3.1 and 7.1. The contents of this paper is the following:

Here, the reader who wants to know directly the Takesaki's duality of the second type, can skip Sections 5 and 6, which are prepared only for Corollaries 7.4, 7.5 and Section 8. In Section 8 we shall give a sufficient condition for a crossed product $M\otimes_{\alpha}G$ and a crossed dual product $N\otimes_{\beta}^{d}G$ to be factorized into $M^{\alpha}\otimes B(L^{2}(G))$ and $N^{\beta}\otimes B(L^{2}(G))$, respectively, by using the idea of Landstad, [9], (Theorems 8.4 and 8.2).

Recently, Roberts [14] has obtained interesting results which have close connection with ours.

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1. Preliminary

Let G be a locally compact group, dt the right invariant Haar measure and $I(f)$ the integral

$$\int f(t)dt.$$

The modular function Δ satisfies $\Delta(t)I(f)=I(f_t)$ for $f\in L^1(G)$, where $f_t(s)\equiv f(ts)$. Let λ be the right regular representation of G on $L^2(G)$, $\lambda(f)$ for f in $L^1(G)$ the integral

$$\int f(t)\lambda(t)dt$$

and $R(G)$ the von Neumann algebra generated by $\lambda(G)$ or $\lambda(L^1(G))$.

When G is abelian, the spectrum $\hat{G}$ of $L^1(G)$ becomes a locally compact abelian group and the spectrum of $L^1(\hat{G})$ is isomorphic to G. For ξ, η in $L^2(G)$ and f in $L^1(G)$, using Plancherel theorem, we have

$$\begin{aligned} \omega_{\xi,\eta}(\lambda(f)) &= \int_G f(t)\omega_{\xi,\eta}(\lambda(t))dt \\ &= \int_G f(t)(\tilde{\eta}*\xi)(t)dt \qquad (1.1) \\ &= \int_{\hat{G}} \hat{f}(\zeta)\hat{\xi}(\zeta)\bar{\hat{\eta}}(\zeta)d\zeta, \end{aligned}$$

where $*$ denotes the convolution, $\tilde{\eta}(t)\equiv\overline{\eta(t^{-1})}$ and $d\zeta$ denotes the Haar measure on $\hat{G}$ associated with dt. We identify $R(G)$ and $R(G)_*$ with $L^\infty(\hat{G})$ and $L^1(\hat{G})$, respectively, through the correspondence in (1.1):

$$\begin{aligned} &\lambda(f)\in R(G) \longleftrightarrow \hat{f}\in L^\infty(\hat{G}) \\ &\omega_{\xi,\eta}\in R(G)_* \longleftrightarrow \hat{\xi}\bar{\hat{\eta}}\in L^1(\hat{G}). \end{aligned} \qquad (1.2)$$

Then the duality between G and $\hat{G}$ is expressed by λ as the following diagram:

$$\begin{aligned} &L^1(G) \xrightarrow{\lambda} R(G) \\ &L^\infty(G) \xleftarrow[\lambda_*]{} R(G)_*, \end{aligned} \qquad (1.3)$$

where λ_* denotes the dual of λ to $R(G)_*$. λ is therefore considered as the Fourier transformation through the above identification (1.2).

For a non abelian G, the duality theorem for G by Eymard, Takesaki and Saito [5, 17, 15] has the same diagram as (1.3). Let $K(G)$ denote the set of all continuous functions with compact carrier on G and $B(G)$ the set of all limits relative to compact convergence of finite linear combinations of functions of positive type on G with respect to λ. Define a norm of g in $B(G)$ by

$$\sup\{|(f|g)| : f \in L^1(G),\ \|\lambda(f)\| \leq 1\} < \infty. \tag{1.4}$$

$B(G)$ is a commutative Banach algebra. The mapping $f \mapsto \tilde{f}$ gives an involution in $B(G)$. Denote the closed linear span of

$$\{\tilde{g}*f : f, g \in K(G)\} \quad ((\tilde{g}*f)(t) \equiv \int \tilde{g}(ts^{-1})f(s)ds)$$

in $B(G)$ by $A(G)$, which is called the Fourier algebra of G. It is known that $A(G)$ is a regular, semi-simple, abelian and involutive Banach algebra and coincides with the set of all $\tilde{\eta}*\xi$ with ξ, η in $L^2(G)$. Since for ξ and η in $L^2(G)$

$$(\lambda(s)\xi|\eta) = (\tilde{\eta}*\xi)(s)$$

as in (1.1), $\lambda_*\omega_{\xi,\eta} = \tilde{\eta}*\xi$ and $A(G) = \lambda_* R(G)_*$. Therefore $R(G)_*$ has the same algebraic structure as $A(G)$. Denote the product of ϕ and ψ in $R(G)_*$ by $\phi\psi$, that is, $\lambda_*(\phi\psi) = (\lambda_*\phi)(\lambda_*\psi)$. For a non zero y in $R(G)$ the following two conditions are equivalent:

(i) $y = \lambda(t)$ for some $t \in G$; and

(ii) $\langle y, \phi\psi \rangle = \langle y, \phi \rangle \langle y, \psi \rangle$ for all $\phi, \psi \in R(G)_*$.

In what follows we shall identify the spectrum (the set of characters) of $A(G)$ with the original G through λ.

Now we define two mappings

$$\delta : L^\infty(G) \longrightarrow L^\infty(G) \otimes L^\infty(G)$$

$$\gamma : R(G) \longrightarrow R(G) \otimes R(G)$$

by

$$(\delta f)(s, t) \equiv f(st), \quad \gamma\lambda(s) \equiv \lambda(s) \otimes \lambda(s). \tag{1.5}$$

Then by (ii) we have

$$<\gamma\lambda(t), \phi\otimes\psi> = <\lambda(t), \phi\psi> . \tag{1.6}$$

In case of an abelian G, if f is an element of $L^1(G)$,

$$\begin{aligned} <\gamma\lambda(f), \phi\otimes\psi> &= \int_G f(t)<\gamma\lambda(t), \phi\otimes\psi>dt \\ &= \int_G f(t)(\lambda_*\phi)(t)(\lambda_*\psi)(t)dt \\ &= \int_{\hat{G}} \hat{f}(\zeta)(\lambda_*\phi)\hat{}*(\lambda_*\psi)\hat{}(\zeta)d\zeta \\ &= \int_{\hat{G}}\int_{\hat{G}} \hat{f}(\zeta\zeta')(\lambda_*\phi)\hat{}(\zeta)(\lambda_*\psi)\hat{}(\zeta')d\zeta d\zeta' \\ &= <\delta\hat{f}, (\lambda_*\phi)\hat{}\otimes(\lambda_*\psi)\hat{}> , \end{aligned} \tag{1.7}$$

where $d\zeta$ and $d\zeta'$ are the Haar measure on G associated with dt. Therefore, under the identification of $R(G)$ with $L^\infty(\hat{G})$ as in (1.2), we have

$$\gamma\lambda(f) = \delta\hat{f}, \tag{1.8}$$

with which we combine the argument in Introduction, we shall define a crossed dual product in Section 2.

The following four unitary operators on $L^2(G)\otimes L^2(G)$ play important roles in our paper:

$$\begin{aligned} &(W\xi)(s, t) \equiv \xi(s, ts), \quad (W'\xi)(s, t) \equiv \Delta(s)^{1/2}\xi(s, s^{-1}t) \\ &(V\xi)(s, t) \equiv \xi(st, t), \quad (V'\xi)(s, t) \equiv \Delta(t)^{1/2}\xi(t^{-1}s, t). \end{aligned} \tag{1.9}$$

Let 1 (resp. 1_G) be the identity operator on a Hilbert space $\mathscr{H}$ or $\mathscr{K}$ (resp. $L^2(G)$). Let λ' denote the left regular representation of G on $L^2(G)$:

$$(\lambda'(s)\xi)(t) \equiv \Delta(s)^{1/2}\xi(s^{-1}t), \qquad \xi \in L^2(G)$$

and

$$\begin{aligned} &\lambda_1(t) \equiv 1\otimes\lambda(t), \quad \lambda_2(t) \equiv 1\otimes 1_G\otimes\lambda(t) \\ &\lambda'_1(t) \equiv 1\otimes\lambda'(t), \quad \lambda'_2(t) \equiv 1\otimes 1_G\otimes\lambda'(t). \end{aligned} \tag{1.10}$$

When a measure μ converges to a Dirac measure ε_e at the unit e of G in the dual space of $C(G)$ with the compact convergence topology, we say simply that μ converges to ε_e in this paper. For example, let $\mathscr{U}$ be a compact symmetric neighbourhood of e, $\{\mathscr{V}\}$ a fundamental system of compact symmeteric neighbourhoods of e satisfying $\mathscr{V}^2 \subset \mathscr{U}$ and $g_{\mathscr{V}} \equiv \tilde{\chi}_{\mathscr{V}} * \chi_{\mathscr{V}} / \|\tilde{\chi}_{\mathscr{V}} * \chi_{\mathscr{V}}\|_1$, where $\chi_{\mathscr{V}}$ denotes the indicator function of $\mathscr{V}$. Then $g_{\mathscr{V}} \in P(G) \cap K(G)_+$ and a measure $g_{\mathscr{V}}(t)dt$ converges to ε_e, where $P(G) \equiv \lambda_* R(G)_*^+$.

2. Dual Action β and Crossed Dual Product

In this section we shall define a dual action and a crossed dual product for our later Sections.

Let M be a von Neumann algebra on a Hilbert space $\mathscr{H}$ and $\mathrm{Aut}\, M$ the automorphism group of M. By an action of G on M we mean a homomorphism $\sigma: t \in G \mapsto \sigma_t \in \mathrm{Aut}\, M$ such that for each x in M the mapping $t \in G \mapsto \sigma_t(x) \in M$ is σ-strongly* continuous. Let $\{\pi_\sigma, \lambda_1\}$ be a covariant representation of $\{M, \sigma\}$ on $\mathscr{H} \otimes L^2(G)$ defined by

$$(2.1) \qquad \begin{aligned} (\pi_\sigma(x)\xi)(s) &\equiv \sigma_s(x)\xi(s) \\ (\lambda_1(r)\xi)(s) &\equiv \xi(sr), \end{aligned}$$

for $\xi \in \mathscr{H} \otimes L^2(G)$. The crossed product $M \otimes_\sigma G$ of M by G is the von Neumann algebra generated by $\pi_\sigma(M)$ and $\lambda_1(G)$.

Since $M \otimes L^\infty(G)$ is isomorphic to the set $L^\infty(G, M)$ of all essentially bounded M-valued σ-weakly measurable functions on G by [12, 16], $\pi_\sigma(x)$ is identified with a function $s \mapsto \sigma_s(x)$ in $L^\infty(G, M)$.

Theorem 2.1. *A necessary and sufficient condition that a mapping α of M into $M \otimes L^\infty(G)$ be induced by an action σ with*

$$(\alpha(x)\xi)(s) = \sigma_s(x)\xi(s)$$

is that α be an isomorphism which satisfies

$$(2.2) \qquad (\alpha \otimes \iota) \circ \alpha = (\iota \otimes \delta) \circ \alpha .$$

Proof. Necessity. Since α is known to be an isomorphism, we have only to show (2.2). Since $\alpha(x)$ is an essentially bounded σ-weakly measurable function

$$s \in G \longmapsto \sigma_s(x) \in M,$$

$(\alpha\otimes\iota)\alpha(x)$ and $(\iota\otimes\delta)\alpha(x)$ correspond respectively essentially bounded σ-weakly measurable functions

$$(s, t) \in G\times G \longmapsto \sigma_s(\sigma_t(x)) \in M$$

and

$$(s, t) \in G\times G \longmapsto \sigma_{st}(x) \in M.$$

Since σ is an action by hypothesis, these two functions coincide and hence (2.2) follows.

Sufficiency. We shall begin by showing that $\tau_r \equiv \mathrm{Ad}\,\lambda_1(r) \restriction \alpha(M)$, $r \in G$ is an action on $\alpha(M)$. Put $L \equiv \alpha(M)$ and $\bar{\delta} \equiv \iota\otimes\delta$. Since

$$\begin{aligned}\bar{\delta}(L) &= (\alpha\otimes\iota)\alpha(M) \qquad\qquad \text{(By (2.2))}\\ &\subset (\alpha\otimes\iota)(M\otimes L^{\infty}(G)) = L\otimes L^{\infty}(G)\end{aligned}$$

and since

$$|<\bar{\delta}(z), \omega'\otimes g>| \leq \|z\|\,\|\omega'\|\,\|g\|_1$$

for each $z \in L$, $\omega' \in L_*$ and $g \in L^1(G)$, we can define a bounded linear operator $\bar{\delta}_g$ on L by

$$<\bar{\delta}_g(z), \omega'> = <\bar{\delta}(z), \omega'\otimes g>.$$

If $y \in L$, then for any ω defined by vectors in $\mathscr{H}$ and $f, g \in K(G)$ we have

$$\begin{aligned}<\bar{\delta}_g(y), \omega\otimes f> &= \iint \omega(y(st))f(s)g(t)dsdt\\ &= \int \omega(y(s))f*g(s)ds.\end{aligned}$$

Making the measure $g(t)dt$ converge to the Dirac measure ε_r at $r \in G$, we know that the right hand side converges to

$$\int \omega(y(s))f(sr^{-1})ds = \int \omega(y(sr))f(s)ds$$

$$= \langle \tau_r(y),\ \omega \otimes f \rangle.$$

In the above convergence we may assume that $\|g\|_1 \leq 1$ and hence $\|\bar{\delta}_g(y)\| \leq \|y\|$. Further, $\omega \otimes f$ are total in the predual of L. Indeed, since the convex hull of all $\omega \otimes f$ is weakly dense in L_*, it is also norm dense by the Hahn-Banach's separation theorem. Therefore, $\bar{\delta}_g(y)$ converges σ-weakly to $\tau_r(y)$. Since $\bar{\delta}_g(y) \in L$ from the above, $\tau_r(y) \in L$. Since $\tau_r^{-1} = \tau_{r^{-1}}$, $\tau_r(L) = L$. Since $\mathrm{Ad}\,\lambda_1(r)$ is an isomorphism and $r \mapsto \mathrm{Ad}\,\lambda_1(r)(z)$ is σ-strongly* continuous for each $z \in M \otimes L^\infty(G)$, its restriction τ_r to L is an action of G on L.

Now we define an action σ of G on M by

$$\sigma_s \equiv \alpha^{-1} \circ \tau_s \circ \alpha. \tag{2.3}$$

We shall show $\alpha = \pi_\sigma$. For this we define two bounded linear operators α_g and $\sigma(g)$ on M for $g \in L^1(G)$ by

$$\langle \alpha_g(x),\ \omega \rangle = \langle \alpha(x),\ \omega \otimes g \rangle \tag{2.4}$$

for $x \in M$ and $\omega \in M_*$, and

$$\sigma(g) \equiv \int g(s)\sigma_s ds.$$

If $\omega \in M_*$ and $f \in L^1(G)$, then

$$\begin{aligned}
&\langle \alpha(\sigma(g)x),\ \omega \otimes f \rangle \\
&= \int g(t) \langle \alpha \circ \sigma_t(x),\ \omega \otimes f \rangle dt \\
&= \int g(t) \langle \tau_t \circ \alpha(x),\ \omega \otimes f \rangle dt \qquad \text{(By (2.3))} \\
&= \int g(t) \langle \alpha(x),\ \omega \otimes_{t^{-1}} f \rangle dt \\
&= \langle \alpha(x),\ \omega \otimes (f * g) \rangle \qquad (2.5) \\
&= \langle (\iota \otimes \delta)\alpha(x),\ \omega \otimes f \otimes g \rangle \qquad \text{(By (1.7))}
\end{aligned}$$

$$= <(\alpha\otimes\iota)\alpha(x),\ \omega\otimes f\otimes g> \qquad \text{(By (2.2))}$$

$$= <\alpha(x),\ \alpha_*(\omega\otimes f)\otimes g>$$

$$= <\alpha(\alpha_g(x)),\ \omega\otimes f>,$$

where ${}_rf(s)\equiv f(sr)$ and α_* is the dual mapping of α. Since ω and f are arbitrary and α is an isomorphism, we have $\sigma(g)x=\alpha_g(x)$. Therefore

$$<\pi_\sigma(x),\ \omega\otimes g> = <\sigma(g)x,\ \omega>$$

$$= <\alpha_g(x),\ \omega> = <\alpha(x),\ \omega\otimes g> .$$

Since ω and g are arbitrary, $\alpha(x)=\pi_\sigma(x)$. Q. E. D.

From this theorem we can identify an isomorphism of M into $M\otimes L^\infty(G)$ satisfying (2.2) with an action of G on M. Therefore we shall use the same letter for them.

Definition 2.2. A *dual action* β of G on N is an isomorphism of a von Neumann algebra N into $N\otimes R(G)$ satisfying

$$(2.6) \qquad (\beta\otimes\iota)\circ\beta=(\iota\otimes\gamma)\circ\beta\,.$$

A *crossed dual product* of N by G with respect to β is the von Neumann algebra generated by $\beta(N)$ and $1\otimes L^\infty(G)$, which is denoted by $N\otimes_\beta^d G$.

Theorem 2.3. (i) *Let* σ^W *be an isomorphism of* $B(\mathscr{H}\otimes L^2(G))$ *into* $B(\mathscr{H}\otimes L^2(G)\otimes L^2(G))$ *defined by*

$$(2.7) \qquad \sigma^W(y)\equiv \mathrm{Ad}\,1\otimes W^*(y\otimes 1_G)\,.$$

If α *is an action of* G *on* M, *then* $\hat{\alpha}\equiv\sigma^W\restriction M\otimes_\alpha G$ *is a dual action of* G *on* $M\otimes_\alpha G$.

(ii) *Let* N *be a von Neumann algebra on a Hilbert space* $\mathscr{K}$ *and* $\sigma^{V'}$ *an isomorphism of* $B(\mathscr{K}\otimes L^2(G))$ *into* $B(\mathscr{K}\otimes L^2(G)\otimes L^2(G))$ *defined by*

$$(2.8) \qquad \sigma^{V'}(z)\equiv \mathrm{Ad}\,1\otimes V'(z\otimes 1_G)\,.$$

If β is a dual action of G on N, then $\hat{\beta}\equiv\sigma^{V'}\upharpoonright N\otimes_{\beta}^{d}G$ is an action of G on $N\otimes_{\beta}^{d}G$.

Proof. (i) If $x\in M$ and $\xi\in\mathscr{H}\otimes L^2(G)\otimes L^2(G)$, then

$$((1\otimes W)^*(\alpha(x)\otimes 1_G)(1\otimes W)\xi)(s, t)$$
$$=((\alpha(x)\otimes 1_G)(1\otimes W)\xi)(s, ts^{-1})$$
$$=\alpha_s(x)((1\otimes W)\xi)(s, ts^{-1})$$
$$=\alpha_s(x)\xi(s, t)=((\alpha(x)\otimes 1_G)\xi)(s, t)$$

and

$$((1\otimes W)^*(\lambda_1(r)\otimes 1_G)(1\otimes W)\xi)(s, t)$$
$$=((\lambda_1(r)\otimes 1_G)(1\otimes W)\xi)(s, ts^{-1})$$
$$=((1\otimes W)\xi)(sr, ts^{-1})=\xi(sr, tr)$$
$$=((\lambda_1(r)\otimes\lambda(r))\xi)(s, t).$$

Therefore

$$\hat{\alpha}(\alpha(x))=\alpha(x)\otimes 1_G \quad\text{and}\quad \hat{\alpha}(\lambda_1(r))=\lambda_1(r)\otimes\lambda(r). \tag{2.9}$$

Since $M\otimes_{\alpha}G$ is generated by $\alpha(M)$ and $\lambda_1(G)$, $\hat{\alpha}$ is a mapping of $M\otimes_{\alpha}G$ into $(M\otimes_{\alpha}G)\otimes R(G)$. It is clear that $\hat{\alpha}$ is an isomorphism. Since

$$((\hat{\alpha}\otimes\iota)\circ\hat{\alpha})\alpha(x)=(\hat{\alpha}\otimes\iota)(\alpha(x)\otimes 1_G)$$
$$=\alpha(x)\otimes 1_G\otimes 1_G=(\iota\otimes\gamma)(\alpha(x)\otimes 1_G)$$
$$=((\iota\otimes\gamma)\circ\hat{\alpha})\alpha(x)$$

and

$$((\hat{\alpha}\otimes\iota)\circ\hat{\alpha})\lambda_1(r)=(\hat{\alpha}\otimes\iota)(\lambda_1(r)\otimes\lambda(r))$$
$$=\lambda_1(r)\otimes\lambda(r)\otimes\lambda(r)=(\iota\otimes\gamma)(\lambda_1(r)\otimes\lambda(r))$$
$$=((\iota\otimes\gamma)\circ\hat{\alpha})\lambda_1(r),$$

(2.6) holds for $M\otimes_{\alpha}G$ and $\hat{\alpha}$.

(ii) The argument will proceed similarly as (i). For each f in $L^\infty(G)$ we define $T_1(f)$ and εf by

$$T_1(f) \equiv 1 \otimes f \tag{2.10}$$

on $\mathscr{K} \otimes L^2(G)$ and

$$(\varepsilon f)(s, t) \equiv f(t^{-1}s). \tag{2.11}$$

Since $\beta(N) \subset N \otimes R(G)$, it follows from (1.9) that $[\beta(y) \otimes 1_G, 1 \otimes V'] = 0$ for all y in N. Since

$$\begin{aligned}
&((1 \otimes V')(T_1(f) \otimes 1_G)(1 \otimes V')^* \xi)(s, t) \\
&= \Delta(t)^{1/2}((T_1(f) \otimes 1_G)(1 \otimes V')^* \xi)(t^{-1}s, t) \\
&= \Delta(t)^{1/2} f(t^{-1}s)((1 \otimes V')^* \xi)(t^{-1}s, t) \\
&= f(t^{-1}s)\xi(s, t) = ((\varepsilon f)\xi)(s, t)
\end{aligned}$$

for $\xi \in \mathscr{K} \otimes L^2(G) \otimes L^2(G)$, we have

$$\hat{\beta}(\beta(y)) = \beta(y) \otimes 1_G \quad \text{and} \quad \hat{\beta}(T_1(f)) = 1 \otimes \varepsilon f \tag{2.12}$$

for all $y \in N$ and $f \in L^\infty(G)$. Since

$$\begin{aligned}
&((\hat{\beta} \otimes \iota)(1 \otimes \varepsilon f))(s, t, r) = (1 \otimes \varepsilon f)(t^{-1}s, r) \\
&= (1 \otimes f)(r^{-1}t^{-1}s) = (1 \otimes \varepsilon f)(s, tr) \\
&= ((\iota \otimes \delta)(1 \otimes \varepsilon f))(s, t, r),
\end{aligned}$$

we have

$$\begin{aligned}
&((\hat{\beta} \otimes \iota) \circ \hat{\beta})(T_1(f)) = (\hat{\beta} \otimes \iota)(1 \otimes \varepsilon f) \\
&= (\iota \otimes \delta)(1 \otimes \varepsilon f) = ((\iota \otimes \delta) \circ \hat{\beta})(T_1(f)).
\end{aligned}$$

Moreover, since

$$\begin{aligned}
&((\hat{\beta} \otimes \iota) \circ \hat{\beta})(\beta(y)) = (\hat{\beta} \otimes \iota)(\beta(y) \otimes 1_G) \\
&= \beta(y) \otimes 1_G \otimes 1_G = (\iota \otimes \delta)(\beta(y) \otimes 1_G) \\
&= ((\iota \otimes \delta) \circ \hat{\beta})(\beta(y))
\end{aligned}$$

for all $y \in N$, (2.2) holds for $N \otimes_{\hat{\beta}}^d G$ and $\hat{\beta}$. Q.E.D.

Definition 2.4. A dual action $\hat{\alpha}$ (resp. an action $\hat{\beta}$) in Theorem 2.3 is said to *be dual to* α (resp. β).

Let α^j be an action of G on M_j $(j=1, 2)$. When an isomorphism ρ of M_1 onto M_2 satisfies

$$(\rho \otimes \iota) \circ \alpha^1 = \alpha^2 \circ \rho \qquad (\text{or } \rho \circ \alpha_t^1 = \alpha_t^2 \circ \rho)$$

$\{M_1, \alpha^1\}$ and $\{M_2, \alpha^2\}$ are said to be equivalent. In this case, $M_1 \otimes_{\alpha^1} G$ is isomorphic to $M_2 \otimes_{\alpha^2} G$.

Definition 2.5. Let β_j be a dual action of G on N_j $(j=1, 2)$. $\{N_1, \beta_1\}$ and $\{N_2, \beta_2\}$ are said to be *equivalent* if there is an isomorphism ρ of N_1 onto N_2 satisfying

$$(\rho \otimes \iota) \circ \beta_1 = \beta_2 \circ \rho .$$

Of course, $N_1 \otimes_{\beta_1}^d G$ is isomorphic to $N_2 \otimes_{\beta_2}^d G$.

3. Duality for Crossed Product by α

We are now ready to show the following duality theorem for crossed products of von Neumann algebras by a locally compact group.

Theorem 3.1. *Let M be a von Neumann algebra on a Hilbert space $\mathscr{H}$ and σ an action of G on M. Let $\alpha \equiv \pi_\sigma$, $\beta \equiv \hat{\alpha}$, $\tilde{\alpha} \equiv \hat{\beta}$ and $\tilde{\sigma}$ the action associated with $\tilde{\alpha}$ as in Theorem 2.1. Then $(M \otimes_\alpha G) \otimes_\beta^d G$ is isomorphic to $M \otimes B(L^2(G))$ and the isomorphism transforms the action $\tilde{\sigma}$ on the former into the action $\sigma \otimes \mathrm{Ad}\, \lambda'$ on the latter.*

Proof. Let $\mathscr{H}_1 \equiv \mathscr{H} \otimes L^2(G)$ and $\mathscr{H}_2 \equiv \mathscr{H} \otimes L^2(G) \otimes L^2(G)$. Using (2.9) and (2.10), we set

$$\Lambda(r) \equiv \lambda_1(r) \otimes \lambda(r) \quad \text{and} \quad T_2(f) \equiv 1 \otimes 1_G \otimes f$$

for f in $L^\infty(G)$. Let $N \equiv M \otimes_\alpha G$ and $D \equiv N \otimes_\beta^d G$. N is generated by $\alpha(M)$ and $1 \otimes R(G)$ on $\mathscr{H}_1$ and D is generated by $\beta(N)$ and $1_N \otimes L^\infty(G)$

on $\mathscr{H}_2$. Therefore by (2.9) D is generated by $\alpha(M)\otimes 1_G$, $\Lambda(G)$ and $T_2(L^\infty(G))$. Since

$$(3.1)\qquad \begin{aligned}(\Lambda(r)T_2(f)\xi)(s,t) &= (T_2(f)\xi)(sr,tr)\\ &= f(tr)\xi(sr,tr) = f(tr)(\Lambda(r)\xi)(s,t)\\ &= (T_2({}_rf)\Lambda(r)\xi)(s,t),\end{aligned}$$

Λ and T_2 satisfy the commutation relation in the sense of Mackey, [10]. Therefore the von Neumann algebra B generated by $\Lambda(G)$ and $T_2(L^\infty(G))$ is isomorphic to $B(L^2(G))$, and hence D is isomorphic to $(D\cap B')\otimes B$. Put

$$\pi(x) \equiv \mathrm{Ad}\, 1\otimes V^*(\alpha(x)\otimes 1_G)$$

on M. Then π is an isomorphism of M into $B(\mathscr{H}\otimes L^2(G)\otimes L^2(G))$ and satisfies

$$(\pi(x)\xi)(s,t) = \sigma_{st^{-1}}(x)\xi(s,t).$$

Since

$$\begin{aligned}(\pi(x)T_2(f)\xi)(s,t) &= \sigma_{st^{-1}}(x)(T_2(f)\xi)(s,t)\\ &= \sigma_{st^{-1}}(x)f(t)\xi(s,t) = f(t)\sigma_{st^{-1}}(x)\xi(s,t)\\ &= (T_2(f)\pi(x)\xi)(s,t)\end{aligned}$$

and

$$\begin{aligned}(\pi(x)\Lambda(r)\xi)(s,t) &= \sigma_{st^{-1}}(x)(\Lambda(r)\xi)(s,t)\\ &= \sigma_{sr(tr)^{-1}}(x)\xi(sr,tr) = (\pi(x)\xi)(sr,tr)\\ &= (\Lambda(r)\pi(x)\xi)(s,t),\end{aligned}$$

we have $\pi(M)\subset B'$.

Let $K(G\times G,\mathscr{H})$ be the set of all continuous functions on $G\times G$ with compact carriers and with values in $\mathscr{H}$. For each f and g in $K(G)$ with $g\geq 0$ and $\|g\|_1 = 1$ we put

$$(3.2)\qquad x_{f,g} \equiv \int f(r)(\alpha(\sigma_r(x))\otimes 1_G)T_2(g_r)dr,$$

where $g_r(s) \equiv g(rs)$. Then $x_{f,g} \in D$. For any ξ and η in $K(G \times G, \mathscr{H})$ we have

$$(x_{f,g}\xi|\eta) = \int f(r)\,((\alpha(\sigma_r(x)) \otimes 1_G) T_2(g_r)\xi|\eta)dr$$

$$= \iiint f(r)g(rt)\,(\sigma_{sr}(x)\xi(s, t)|\eta(s, t))dsdtdr$$

$$= \iiint f(rt^{-1})g(r)\,(\sigma_{srt^{-1}}(x)\xi(s, t)|\eta(s, t))drdsdt.$$

Since $r \mapsto f(rt^{-1})\,(\sigma_{srt^{-1}}(x)\xi(s, t)|\eta(s, t))$ belongs to $K(G)$, when the measure $g(r)dr$ converges to the Dirac measure ε_e at the unit e of G, the right hand side converges to

$$\iint f(t^{-1})\,(\sigma_{st^{-1}}(x)\xi(s, t)|\eta(s, t))dsdt,$$

and this converges to $(\pi(x)\xi|\eta)$ as f converges to the constant 1 function uniformly on each compact subset of G. Since $K(G \times G, \mathscr{H})$ is dense in $\mathscr{H}_2$, and since $\|x_{f,g}\| \le \|f\|\,\|x\|\,\|g\|_1$, $x_{f,g}$ converges weakly to $\pi(x)$ and hence $\pi(M) \subset D$, namely, $\pi(M) \subset D \cap B'$.

Next we shall show that D is generated by $\pi(M)$ and B. For each f and g in $K(G)$ we put

$$y_{f,g} \equiv \int f(r)\pi(\sigma_r^{-1}(y))T_2(g_r)dr.$$

Then $y_{f,g} \in (\pi(M) \cup B)''$. For each ξ and η in $K(G \times G, \mathscr{H})$ we have

$$(y_{f,g}\xi|\eta) = \int f(r)\,(\pi(\sigma_r^{-1}(y))T_2(g_r)\xi|\eta)dr$$

$$= \iiint f(r)g(rt)\,(\sigma_{st^{-1}r^{-1}}(y)\xi(s, t)|\eta(s, t))dsdtdr$$

$$= \iiint f(rt^{-1})g(r)\,(\sigma_{sr^{-1}}(y)\xi(s, t)|\eta(s, t))drdsdt.$$

By the same reason as above, when the measure $g(r)dr$ converges to ε_e, the right hand side converges to

$$\iint f(t^{-1})\,(\sigma_s(y)\xi(s, t)|\eta(s, t))dsdt,$$

which converges to $((\alpha(y)\otimes 1_G)\xi|\eta)$ as f tends to 1 in an appropriate sense. Thus $\alpha(M)\otimes 1_G\subset(\pi(M)\cup B)''$, and hence $D\subset(\pi(M)\cup B)''$. Since the converse inclusion is obtained in the above, $D=(\pi(M)\cup B)''$. Therefore D is isomorphic to $\pi(M)\otimes B$.

By Theorem 2.1 $\tilde{\alpha}$ and $\tilde{\sigma}$ satisfies

$$(\tilde{\alpha}(z)\xi)(r)=\tilde{\sigma}_r(z)\xi(r) \qquad z\in D$$

for $\xi\in\mathscr{H}_2\otimes L^2(G)$. Since we know from (2.12) that

$$\tilde{\alpha}(\beta(y))=\beta(y)\otimes 1_G \quad \text{and} \quad \tilde{\alpha}(T_2(f))=1\otimes 1_G\otimes\varepsilon f,$$

we have

$$\begin{aligned}\tilde{\sigma}_r(\alpha(x)\otimes 1_G)\xi(r)&=((\alpha(x)\otimes 1_G\otimes 1_G)\xi)(r)\\&=(\alpha(x)\otimes 1_G)\xi(r)\end{aligned} \tag{3.3}$$

$$\tilde{\sigma}_r(\Lambda(s))\xi(r)=((\Lambda(s)\otimes 1_G)\xi)(r)=\Lambda(s)\xi(r)$$

and

$$\begin{aligned}\tilde{\sigma}_r(T_2(f))\xi(r)&=((1\otimes 1_G\otimes\varepsilon f)\xi)(r)\\&=T_2(f_{r^{-1}})\xi(r)\,.\end{aligned} \tag{3.4}$$

Since $[\Lambda(r), \lambda_2'(r')]=0$ for all $r, r'\in G$, it follows from (3.4) that

$$\tilde{\sigma}_r=\operatorname{Ad}\lambda_2'(r) \tag{3.5}$$

on B.

We apply (3.3) and (3.4) to $x_{f,g}$ defined by (3.2). Then

$$\begin{aligned}&\tilde{\sigma}_a(x_{f,g})\xi(s, t)\\&=\int f(r)(\alpha(\sigma_r(x))\otimes 1_G)\tilde{\sigma}_a(T_2(g_r))\xi(s, t)dr\\&=\int f(r)g(ra^{-1}t)\sigma_{sr}(x)\xi(s, t)dr\\&=\int f(rt^{-1}a)g(r)\sigma_{srt^{-1}}(\sigma_a(x))\xi(s, t)dr\end{aligned}$$

When $g(r)dr$ converges to ε_e, $(\tilde{\sigma}_a(x_{f,g})\xi|\eta)$ converges to

$$\iint f(t^{-1}a)(\sigma_{st^{-1}}(\sigma_a(x))\xi(s,t)|\eta(s,t))dsdt$$

$$=\iint f(t^{-1}a)(\pi(\sigma_a(x))\xi(s,t)|\eta(s,t))dsdt.$$

Therefore, if $g(r)dr$ converges to ε_e and then f to 1, then $x_{f,g}$ converges to $\pi(x)$ as before and hence

$$\tilde{\sigma}_a(\pi(x))=\pi(\sigma_a(x)) \qquad x\in M. \tag{3.6}$$

Combining (3.5) and (3.6), we have

$$\rho\circ\tilde{\sigma}_a=(\sigma_a\otimes\mathrm{Ad}\,\lambda_1'(a))\circ\rho$$

for all $a\in G$, where ρ is the isomorphism of D onto $M\otimes B$ obtained before. Q.E.D.

4. Some Technical Lemmas for β

Let N be a von Neumann algebra on a Hilbert space $\mathscr{K}$ and β a dual action of G on N. For any ϕ in $R(G)_*$ and ω in N_* we define linear mappings β_ϕ on N and Φ_ω of N into $R(G)$ by

$$\begin{aligned}&<\beta_\phi(x),\omega'>=<\beta(x),\omega'\otimes\phi>\\&<\Phi_\omega(x),\phi'>=<\beta(x),\omega\otimes\phi'>\end{aligned} \tag{4.1}$$

for all $x\in N$, $\omega'\in N_*$ and $\phi'\in R(G)_*$. Let β_* denote the mapping of $(N\otimes R(G))_*$ onto N_* defined by

$$<x,\beta_*(\omega\otimes\phi)>=<\beta(x),\omega\otimes\phi> \tag{4.2}$$

for all $\omega\in N_*$ and $\phi\in R(G)_*$.

Since γ is a dual action of G on $R(G)$, γ_ϕ and γ_* are defined by (4.1) and (4.2).

Lemma 4.1. *Let* ϕ *and* ψ *be elements in* $R(G)_*$.

(i) $\beta_{\phi\psi}=\beta_\phi\beta_\psi$.

(ii) $\gamma_\psi(\Phi_\omega(x))=\Phi_\omega(\beta_\psi(x))$ *for* $\omega\in N_*$ *and* $x\in N$.

(iii) $<\beta(\beta_\psi(x)),\ \omega\otimes\phi>=<\gamma\Phi_\omega(x),\ \psi\otimes\phi>$.

Proof. (i) If $\omega\in N_*$ and $x\in N$, then

$$\begin{aligned}(4.3)\qquad &<\beta_{\phi\psi}(x),\ \omega>=<\beta(x),\ \omega\otimes\phi\psi>\\ &=<(\iota\otimes\gamma)\beta(x),\ \omega\otimes\phi\otimes\psi> \qquad \text{(By (1.6))}\\ &=<(\beta\otimes\iota)\beta(x),\ \omega\otimes\phi\otimes\psi> \qquad \text{(By (2.6))}\\ &=<\beta(x),\ \beta_*(\omega\otimes\phi)\otimes\psi>\\ &=<\beta_\psi(x),\ \beta_*(\omega\otimes\phi)>\\ &=<\beta(\beta_\psi(x)),\ \omega\otimes\phi>\\ &=<\beta_\phi(\beta_\psi(x)),\ \omega>.\end{aligned}$$

(ii) If $\phi\in R(G)_*$, then

$$\begin{aligned}(4.4)\qquad &<\gamma_\psi\Phi_\omega(x),\ \phi>=<\gamma\Phi_\omega(x),\ \phi\otimes\psi>\\ &=<\Phi_\omega(x),\ \phi\psi>=<\beta(x),\ \omega\otimes\phi\psi>\\ &=<\beta(\beta_\psi(x)),\ \omega\otimes\phi> \qquad \text{(By (4.3))}\\ &=<\Phi_\omega(\beta_\psi(x)),\ \phi>.\end{aligned}$$

(iii) From (4.4) we have

$$\begin{aligned}&<\beta(\beta_\psi(x)),\ \omega\otimes\phi>=<\Phi_\omega(x),\ \psi\phi>\\ &=<\gamma\Phi_\omega(x),\ \psi\otimes\phi>.\end{aligned}$$

Q.E.D.

The following lemma is an immediate consequence of [11, Theorem 5, Chapter 3]. For the sake of completeness we shall give a direct proof.

Lemma 4.2. *Let L be a von Neumann algebra and* $t\mapsto u(t)$ *a weakly continuous unitary representation of G in L. If* ϕ *is an ele-*

ment of L_*, *then* $t \mapsto u(t)^*\phi$ (*or* $u(t)\phi$, $\phi u(t)$, $\phi u(t)^*$) *is continuous* (*in norm*).

By means of this lemma we know that the functions in the above lemma are Bochner integrable on every compact subset of G and their integrals exist in L_*.

Proof. For each f in $L^1(G)$ we denote by $u(f)$ the integral

$$\int f(t)u(t)dt\,.$$

Let L_0 be the set of all $u(f)^*\psi$ with $f \in K(G)$ and $\psi \in L_*$. Since $u(f)^*\psi$ converges weakly to ψ as $f(t)dt$ tends to ε_e, L_0 is weakly dense in L_* and hence it is total in L_* in norm by the Hahn-Banach's separation theorem.

For any ϕ in L_0 of the form $u(f)^*\psi$ we have

$$\begin{aligned}\|(u(t)^* - u(s)^*)\phi\| &\leqq \|u(f)(u(t) - u(s))\|\,\|\psi\| \\ &\leqq \|{}_{t^{-1}}f - {}_{s^{-1}}f\|_1 \|\psi\|\,.\end{aligned}$$

Since L_0 is total in L_* in norm, $t \mapsto u(t)^*\phi$ is continuous for all ϕ in L_*.

As for the remaining functions $t \mapsto u(t)\phi$, $\phi u(t)$ and $\phi u(t)^*$ we can give their proofs in a similar way. Q.E.D.

As $<\lambda(s), \lambda(r)^*\phi> = <\lambda(sr^{-1}), \phi>$, we have

$$\lambda_*(\lambda(r)^*\phi) = \mathrm{Ad}\,\lambda(r)^*(\lambda_*\phi)\,.$$

The following two lemmas are crucial from the technical point of view. In particular, Lemma 4.3 plays a role of Fourier expansion.

Lemma 4.3. *Let* ϕ *be elements in* $R(G)_*$ *satisfying* $\lambda_*\phi \in K(G)_+$, $\|\Delta\lambda_*\phi\|_1 = 1$ *and* $<\lambda(r)^*, \phi>dr$ *tends to* ε_e.

(i) *If* ψ *is an element in* $R(G)_*$ *with* $\lambda_*\psi \in L^1(G)$, *the integral*

$$\int <\lambda(r), \psi> \lambda(r)^*\phi dr \tag{4.5}$$

exists in $R(G)_*$, *is bounded by* $\|\psi\|$ *and converges weakly to* ψ.

(ii) *If* $x=\beta_\rho(x)$ *for some* $\lambda_*\rho\in K(G)$, *the integral*

$$\int\beta_{\lambda(r)*\phi}(x)\otimes\lambda(r)dr \tag{4.6}$$

exists in $N\otimes_{\gamma^*}R(G)$, *is bounded by* $\|x\|$ *and converges weakly* to* $\beta(x)$, *where* γ^* *is the dual norm of* γ-*norm.*

Proof. (i) We denote (4.5) by ψ_ϕ. By Lemma 4.2, $r\mapsto\lambda(r)^*\phi$ is continuous. Since $\lambda_*\psi\in L^1(G)$ by assumption and $\|\lambda(r)^*\phi\|=\|\phi\|$, the function $r\mapsto<\lambda(r), \psi>(\lambda(r)^*\phi)$ is Bochner integrable and hence ψ_ϕ exists in $R(G)_*$. If $f\in L^1(G)$, then

$$\begin{aligned}
&\int<\lambda(r)^*, \phi><\lambda(r)\lambda(f), \psi>dr\\
&=\int<\lambda(r)^*, \phi>\int f(s)<\lambda(rs), \psi>dsdr\\
&=\iint<\lambda(sr^{-1}), \phi>f(s)ds<\lambda(r), \psi>dr\\
&=\int<\lambda(f), \lambda(r)^*\phi><\lambda(r), \psi>dr\\
&=<\lambda(f), \psi_\phi>\ .
\end{aligned}\tag{4.7}$$

The integral on the left hand side

$$\int<\lambda(r)^*, \phi>(\psi\lambda(r))dr \tag{4.8}$$

exists in $R(G)_*$ by a similar reason as above and hence coincides with ψ_ϕ by (4.7). Therefore

$$<x, \psi_\phi>=\int<\lambda(r)^*, \phi><\lambda(r)x, \psi>dr$$

for all $x\in R(G)$. Here, since $r\mapsto<\lambda(r)x, \psi>$ is continuous, if $<\lambda(r)^*, \phi>dr$ converges to ε_e, then ψ_ϕ converges weakly to ψ.

From (4.8) and the assumption $\|\Delta\lambda_*\phi\|_1=1$ it follows that the norm of ψ_ϕ is majorized by $\|\psi\|$.

(ii) Since $\operatorname{car}\lambda_*((\lambda(r)^*\phi)\rho)\subset(\operatorname{car}\lambda_*\phi)r\cap\operatorname{car}\lambda_*\rho$, $r\mapsto\beta_{\lambda(r)^*\phi}(x)=\beta_{(\lambda(r)^*\phi)\rho}(x)$ has a compact carrier. Since $r\mapsto\beta_{\lambda(r)^*\phi}$ is continuous and $\|\beta_{\lambda(r)^*\phi}(x)\|\leq\|\phi\|\,\|x\|$, $r\mapsto\beta_{\lambda(r)^*\phi}(x)$ is Bochner integrable and hence (4.6) exists in $N\otimes R(G)$. If $\omega\in N_*$ and $\psi\in R(G)_*$ with $\lambda_*\psi\in K(G)$, then

$$
\begin{aligned}
&<\int\beta_{\lambda(r)^*\phi}(x)\otimes\lambda(r)dr,\ \omega\otimes\psi>\\
&=\int<\beta_{\lambda(r)^*\phi}(x),\ \omega><\lambda(r),\ \psi>dr \qquad (4.9)\\
&=\int<\beta(x),\ \omega\otimes\lambda(r)^*\phi><\lambda(r),\ \psi>dr,
\end{aligned}
$$

which converges to $<\beta(x), \omega\otimes\psi>$ by (i). Since the set of all $\psi\in R(G)_*$ satisfying $\lambda_*\psi\in K(G)$ is weakly dense, it is dense in $R(G)_*$. Since the absolute value of the right hand side of (4.9) is majorized by $\|x\|\,\|\omega\|\,\|\psi\|$ by (i), γ^*-norm of (4.6) is bounded by $\|x\|$. Therefore (4.6) converges weakly* to $\beta(x)$. Q.E.D.

The above (ii) in Lemma 4.3 or the following remark can be used to prove (ii) in Theorem 7.1. However, we shall intend to utilize the former in this paper.

Remark. If x is of the form $\beta_\psi(y)$ for some $\psi\in R(G)_*$ with $\lambda_*\psi\in K(G)$, then (4.6) exists in $N\otimes R(G)$ and converges σ-weakly to $\beta(x)$. For this it suffices to show that (4.6) is uniformly bounded in ϕ. If we use the argument which will be done in Lemma 7.3, the integrals

$$F^0_{\phi,\psi}\equiv\int(\lambda(r)^*\phi)\psi\otimes((1_G\otimes\lambda(r))\omega')dr$$

$$G^0_\phi\equiv\int<\lambda(r)^*,\ \phi>(\omega'(1_G\otimes\lambda(r)))\circ\operatorname{Ad}W^*dr$$

exist as vector forms and satisfy

$$<\gamma\Phi_\omega(y)\otimes 1_G,\ F^0_{\phi,\psi}>=<\beta(\beta_\psi(y))\otimes 1_G,\ \omega\otimes G^0_\phi>.$$

Since

$$<(\beta\otimes\iota)\Big(\int\beta_{(\lambda(r)^*\phi)}(x)\otimes\lambda(r)dr\Big),\ \omega\otimes\omega'>$$

$$=<\gamma\Phi_\omega(y)\otimes 1_G,\ F^0_{\phi,\psi}>$$

$$=<\Big(\int<\lambda(r)^*,\ \phi>1\otimes 1_G\otimes\lambda(r)dr\Big)\mathrm{Ad}\,1\otimes W^*(\beta(x)\otimes 1_G),\ \omega\otimes\omega'>,$$

it follows from the right hand side that (4.6) is bounded by $\|x\|$ under the assumption of Lemma 4.3.

Lemma 4.4. *Let ϕ be an element of $R(G)_*$ satisfying $\lambda_*\phi\in K(G)_+$ and $\|\Delta\lambda_*\phi\|_1=1$.*

(i) *If ψ is an element in $R(G)_*$ with $\lambda_*\psi\in K(G)$, the integral*

$$\int(\lambda(r)^*\phi)\psi dr \tag{4.10}$$

exists in $R(G)_*$ and coincides with ψ.

(ii) *If $x=\beta_\rho(x)$ for some $\lambda_*\rho\in K(G)$, the integral*

$$\int\beta_{\lambda(r)^*\phi}(x)dr \tag{4,11}$$

exists in N and coincides with x.

Proof. (i) Since $\lambda_*\phi$, $\lambda_*\psi\in K(G)$ and $\mathrm{car}\,\lambda_*((\lambda(r)^*\phi)\psi)\subset(\mathrm{car}\,\lambda_*\phi)r\cap\mathrm{car}\,\lambda_*\psi$, $r\mapsto(\lambda(r)^*\phi)\psi$ has a compact carrier. Since $\|(\lambda(r)^*\phi)\psi\|\le\|\phi\|\,\|\psi\|$ and $r\mapsto(\lambda(r)^*\phi)\psi$ is continuous by Lemma 4.2, (4.10) exists in $R(G)_*$. Since

$$\int<\lambda(s),\ \lambda(r)^*\phi>dr=\int<\lambda(r)^*,\ \phi>dr=1$$

for all $s\in G$ by assumption, we have for any f in $L^1(G)$

$$<\lambda(f),\ \psi>=\int<\lambda(r)^*,\ \phi>dr<\lambda(f),\ \psi>$$

$$=\iint f(s)<\gamma\lambda(s),\ \lambda(r)^*\phi\otimes\psi>drds$$

$$=<\lambda(f),\ \int(\lambda(r)^*\phi)\psi dr>\ .$$

Since $\lambda(L^1(G))$ is σ-weakly dense in $R(G)$, ψ is given by (4.10).

(ii) By a similar reason as in the proof of (ii) in Lemma 4.3, (4.11) exists in N and is bounded. For any $\omega \in N_*$ and any $\psi \in R(G)_*$ with $\lambda_*\psi \in K(G)$

$$\begin{aligned} <\beta(x), \omega \otimes \psi> &= \int <\beta(x), \omega \otimes (\lambda(r)^* \phi)\psi> dr \\ &= <\beta\left(\int \beta_{\lambda(r)^*\phi}(x) dr\right), \omega \otimes \psi>. \qquad \text{(By (4.3))}. \end{aligned} \tag{4.12}$$

Since the set of $\psi \in R(G)_*$ with $\lambda_*\psi \in K(G)$ is dense in $R(G)_*$, the linear span of $\omega \otimes \psi$ with $\lambda_*\psi \in K(G)$ is dense in $(N \otimes R(G))_*$. Therefore (4.11) coincides with x. Q. E. D.

Lemma 4.5. *If $x \in N$, then x belongs to the von Neumann algebra generated by $\beta_\psi(x)$ with $\lambda_*\psi \in K(G)$.*

Proof. Let N_0 be the von Neumann algebra generated by $\beta_\psi(x)$ with $\lambda^*\psi \in K(G)$. If $\omega \in N_*$ annihilates on N_0, then

$$<\beta(x), \omega \otimes \psi> = <\beta_\psi(x), \omega> = 0 \tag{4.13}$$

for all ψ with $\lambda_*\psi \in K(G)$. Since the set of all ψ with $\lambda_*\psi \in K(G)$ is dense in $R(G)_*$, (4.13) holds for all $\psi \in R(G)_*$. Therefore $\beta(x)$ belongs to $N_0 \otimes R(G)$ by [19]. By considering $\beta_\psi(x)$ as x in (4.13), $\beta \restriction N_0$ is a dual action of G on N_0 and hence $x \in N_0$ by [22, Proposition II. 1.1]. Q. E. D.

5. Spectrum of β

The spectrum of an action of G on a C^*-algebra was investigated by the method of abstract harmonic analysis, [2]. We shall define the corresponding concept for a dual action of G on N by using the same ideas. When the set of x in N is trivial whose spectrum with respect to β is $\{e\}$, β is considered to be ergodic.

The basic theorem in Gelfand's theory for a commutative Banach algebra tells us that there is a bijection between the set of all maximal regular ideals m of $R(G)_*$ and the spectrum G satisfying

$$m^{\perp}=\boldsymbol{C}\lambda(t) \quad \text{and} \quad \{\lambda(t)\}^{\perp}=m \tag{5.1}$$

for $t\in G$. The Tauberian theorem is generalized by Eymard [5] as the following: if m is a closed ideal of $R(G)_*$ such that for any $t\in G$ there exists a $\phi\in m$ with $<\lambda(t), \phi>\neq 0$, then $m=R(G)_*$. Therefore every proper closed ideal is included in a maximal regular ideal.

Definition 5.1. For any ϕ in $R(G)_*$ let $\Gamma(\phi)$ denote the set of all $t\in G$ with $<\lambda(t), \phi>=0$. Let

$$\mathrm{sp}(\beta)\equiv \cap \{\Gamma(\phi): \beta_\phi=0\}$$

$$(\text{resp.}\ \mathrm{sp}_\beta(x)\equiv \cap \{\Gamma(\phi): \beta_\phi(x)=0\}\ \text{for}\ x\in N).$$

Let m_β (resp. m_x) denote the set of all $\phi\in R(G)_*$ with $\beta_\phi=0$ (resp. $\beta_\phi(x)=0$).

From (i) in Lemma 4.1 it follows that m_β and m_x are closed ideals of $R(G)_*$ and $\mathrm{sp}(\beta)$ (resp. $\mathrm{sp}_\beta(x)$) is the hull of m_β (resp. m_x). Besides, if $\mathrm{sp}_\beta(x)=\phi$, then $m_x=R(G)_*$ by Tauberian theorem and hence $x=0$.

Proposition 5.2. *For a non zero $x\in N$ and $t\in G$ the following four conditions are equivalent:*

(i) $\mathrm{sp}_\beta(x)=\{t\}$;

(ii) $\beta(x)=x\otimes\lambda(t)$;

(iii) $\beta_\phi(x)=<\lambda(t), \phi>x$ *for all* $\phi\in R(G)_*$; *and*

(iv) $\Phi_\omega(x)=<x, \omega>\lambda(t)$ *for all* $\omega\in N_*$.

Proof. The equivalence among conditions (ii), (iii) and (iv) is immediate from (4.1). It suffices to show the implications (iii)⇒(i) and (i)⇒(ii).

(iii)⇒(i) Suppose the condition (iii). Since $\beta_\phi(x)=0$ implies $<\lambda(t), \phi>=0$ for all $\phi\in R(G)_*$, we have $t\in\mathrm{sp}_\beta(x)$. For any $s\in G$ with $s\neq t$ we can select a ψ in $R(G)_*$ satisfying $<\lambda(t), \psi>=0$ and $<\lambda(s), \psi>\neq 0$. Since $<\lambda(t), \psi>=0$ implies $\beta_\psi(x)=0$ by assumption, s does not belong to $\mathrm{sp}_\beta(x)$. Since s is arbitrary with $s\neq t$, the condition (i) is obtained.

(i)⇒(ii) Suppose the condition (i). Let m_t denote the maximal regular ideal associated with $t\in G$ by (5.1). Since m_x is included in

m_t but not in m_s with $s \neq t$, m_x is primary. By using [5, (4.10)], $m_x = m_t$ and hence

$$<\Phi_\omega(x), \phi> = <\beta_\phi(x), \omega> = 0$$

for all $\phi \in m_t$. Since m_t is a maximal regular ideal, $\Phi_\omega(x) = \mu(\omega)\lambda(t)$ for some $\mu(\omega) \in \boldsymbol{C}$. From the linearity of Φ_ω in $\omega \in N_*$, and $\|\Phi_\omega(x)\| \leq \|x\| \|\omega\|$ it follows that μ is a bounded linear form on N_*, namely, we have a y in N with $\Phi_\omega(x) = <y, \omega>\lambda(t)$. Since for any ω in N_* and any ϕ in $R(G)_*$

$$<\beta(x), \omega \otimes \phi> = <\Phi_\omega(x), \phi>$$

$$= <y, \omega><\lambda(t), \phi> = <y \otimes \lambda(t), \omega \otimes \phi>,$$

we have $\beta(x) = y \otimes \lambda(t)$. Since

$$\beta(y) \otimes \lambda(t) = (\beta \otimes \iota)(y \otimes \lambda(t)) = (\beta \otimes \iota)\beta(x)$$

$$= (\iota \otimes \gamma)\beta(x) = (\iota \otimes \gamma)(y \otimes \lambda(t)) = y \otimes \lambda(t) \otimes \lambda(t),$$

we have $(\beta(y) - y \otimes \lambda(t)) \otimes \lambda(t) = 0$ and hence $\beta(y) = y \otimes \lambda(t) = \beta(x)$. Therefore $x = y$ and hence $\beta(x) = x \otimes \lambda(t)$. Q.E.D.

Definition 5.3. Let N^β denote the set of all x in N with $\mathrm{sp}_\beta(x) = \{e\}$, and M^α the fixed point algebra of α_t for all $t \in G$.

The following proposition is not necessary for later use. For each element $y \in R(G)$ the carrier $\mathrm{supp}(y)$ of y is defined by Eymard [5] as $\mathrm{sp}_\gamma(y)$. We can describe $\mathrm{sp}_\beta(x)$ in terms of the carriers of $\Phi_\omega(x)$, $\omega \in N_*$.

Proposition 5.4. *$\mathrm{sp}_\beta(x)$ is the closure of the union of $\mathrm{sp}_\gamma(\Phi_\omega(x))$ for all ω in N_*.*

Proof. If $\omega \in N_*$ and $\psi \in R(G)_*$, then

$$<\gamma_\phi(\Phi_\omega(x)), \psi> = <\beta(\beta_\phi(x)), \omega \otimes \psi>$$

by (iii) in Lemma 4.1. Hence $\beta_\phi(x) = 0$ if and only if $\gamma_\phi \Phi_\omega(x) = 0$ for all ω in N_*. Therefore

$$m_x = \bigcap_{\omega \in N_*} \{\phi \in R(G)_* : \gamma_\phi \Phi_\omega(x) = 0\},$$

whose hull gives the desired result. Q.E.D.

6. Fixed Points of α and β

Let α be an action of G on M and $\hat{\alpha}$ the dual action of G on $M \otimes_\alpha G$ dual to α. A generalized conditional expectation of $M \otimes_\alpha G$ onto $(M \otimes_\alpha G)^{\hat{\alpha}}$ has been investigated by Landstad in his forthcomming paper, [9]. Using his results, we shall show that $(M \otimes_\alpha G)^{\hat{\alpha}} = \alpha(M)$. Similar argument for a dual action β of G on N has been developed by Haagerup, [6]. In the latter half of this section we shall give an independent argument in order to show $(N \otimes_\beta^d G)^{\hat{\beta}} = \beta(N)$.

The case of α.

$K(G)$ is a left Hilbert algebra with respect to a product $(f, g) \mapsto f * g$, an involution $f \mapsto \tilde{f}$ and an inner product $(f|g) \equiv I(f\bar{g})$. Using the left representation π of $K(G)$ we have

$$(6.1) \qquad (\pi(f)g)(t) = (f*g)(t) = (\lambda'(\Delta^{1/2} f)g)(t).$$

The modular conjugation J of $K(G)$ is of the form $(Jf)(t) = \Delta(t)^{1/2}\tilde{f}(t)$ and $\lambda'(t) = J\lambda(t)J$. Therefore $R(G)'$ is generated by $\lambda'(G)$ and it is the left von Neumann algebra of $K(G)$ by (6.1). The extension $\dot{\psi}$ over $\mathfrak{m}_\psi$ of the canonical weight ψ on $R(G)'_+$ associated with $K(G)$ is given by

$$\dot{\psi}(\pi(\tilde{g}*f)) = \psi(\pi(g)^*\pi(f)) = (f|g) = (\tilde{g}*f)(e),$$

where e denotes the unit of G. We denote by ω_e the weight on $R(G)$ defined by $\dot{\psi} \circ \mathrm{Ad}\, J$. Then

$$\omega_e(\lambda(\Delta^{1/2}(\tilde{g}*f))) = (\tilde{g}*f)(e).$$

Let F be a net in $R(G)_*^+$ such that $\lambda_* F \subset K(G)$ and

$$(6.2) \qquad \omega_e(x) = \sup\{<x, \phi> : \phi \in F\} \qquad x \in R(G)_+.$$

Let β be a dual action of G on N. If $y \in N_+$, then $\{\beta_\phi(y) : \phi \in F\}$ is an increasing net in N_+. Define a generalized conditional expectation

E_β for $y \in N_+$ by

$$E_\beta(y) \equiv \sup\{\beta_\phi(y) : \phi \in F\}.$$

Since $\omega \otimes \omega_e$ for $\omega \in N_*^+$ is a semi-finite normal weight on $N \otimes R(G)$, we have

$$<E_\beta(y), \omega> = \sup_\phi <\beta_\phi(y), \omega> = <\beta(y), \omega \otimes \omega_e>$$

for $y \in N_+$. Let $\mathfrak{n}_\beta$ be the set of all $x \in N$ such that

$$<\beta(x^*x), \omega \otimes \omega_e> \leq \mu_x \|\omega\| \qquad \omega \in N_*^+$$

for some $\mu_x > 0$. Since $x^*y^*yx \leq \|y\|^2 x^*x$, $\mathfrak{n}_\beta$ is a left ideal of N. Let $\mathfrak{m}_\beta = \mathfrak{n}_\beta^* \mathfrak{n}_\beta$ and $\dot{E}_\beta$ be the linear extension of E_β over $\mathfrak{m}_\beta$. Then E_β satisfies

(i) $E_\beta(x+y) = E_\beta(x) + E_\beta(y)$ $\qquad x, y \in N_+$

(ii) $E_\beta(\mu x) = \mu E_\beta(x)$ $\qquad \mu \geq 0$

(iii) $x_\iota \uparrow x$ implies $E_\beta(x_\iota) \uparrow E_\beta(x)$ $\qquad x_\iota \in N_+$

(iv) $\beta(\dot{E}_\beta(z)) = \dot{E}_\beta(z) \otimes 1_G$ $\qquad z \in \mathfrak{m}_\beta$

(v) $\dot{E}_\beta(b^*zb) = b^*\dot{E}_\beta(z)b$ $\qquad b \in N_\beta$,

where N_β denotes the set of all $x \in N$ with $\beta(x) = x \otimes 1_G$. For instance, (iv) is proved as follows: If $z \in \mathfrak{m}_\beta^+$, $\omega \in N_*^+$ and $\psi \in R(G)_*^+$, then

$$<\beta(E_\beta(z)), \omega \otimes \psi> = \sup_\phi <\beta(\beta_\phi(z)), \omega \otimes \psi>$$

$$= \sup_\phi <(\iota \otimes \gamma)\beta(z), \omega \otimes \phi \otimes \psi> \qquad \text{(By (4.3))}.$$

According to the choice of F in (6.2) we may assume that $<\lambda(t), \phi> dt$ converges to ε_e. Since

$$<(\iota \otimes \gamma)(x \otimes \lambda(f)), \omega \otimes \phi \otimes \psi>$$

$$= <x, \omega> \int f(t) <\lambda(t), \phi> <\lambda(t), \psi> dt,$$

the right hand side converges to

$$<x, \omega> f(e) <1_G, \psi> = <x \otimes \lambda(f), \omega \otimes \omega_e> <1_G, \psi>.$$

Now, let α be an action of G on M, $N\equiv M\otimes_{\alpha}G$ and $\beta\equiv\hat{\alpha}$. Using results of Landstad [9, Lemma 2.8 and Corollary 1.3], we know that

(a) if $f\in A(G)$, then $1_M\otimes\lambda(f)\in\mathfrak{m}_{\beta}$ and $\dot{E}_{\beta}(1_M\otimes\lambda(f))=<\lambda(f), \omega_e>1_N$

(b) $N_{\beta}\mathfrak{m}_{\beta}N_{\beta}\subset\mathfrak{m}_{\beta}$

(c) $\dot{E}_{\beta}(\mathfrak{m}_{\beta})=N_{\beta}$

(d) the mapping $y\in\mathfrak{m}_{\beta}\mapsto\dot{E}_{\beta}(b^*yb)$ is σ-weakly continuous for each $b\in\mathfrak{n}_{\beta}$.

For example, (a) is shown by

$$<\beta(1_M\otimes\lambda(f)), \omega\otimes\psi\otimes\omega_e>$$

$$=\int f(t)<1_M\otimes\gamma\lambda(t), \omega\otimes\psi\otimes\omega_e>dt \qquad \text{(By (2.9))}$$

$$=f(e)<1_M\otimes 1_G, \omega\otimes\psi>$$

$$=<<\lambda(f), \omega_e>1_N, \omega\otimes\psi>$$

for any $f\in A(G)$. Using these results we have the following proposition.

Proposition 6.1. *If α is an action of G on M, then $\alpha(M)=(M\otimes_{\alpha}G)^{\hat{\alpha}}$.*

Proof. Let $N\equiv M\otimes_{\alpha}G$ and $\beta\equiv\hat{\alpha}$. By virtue of Proposition 5.2, $N^{\beta}=N_{\beta}$. Since $\alpha(M)\subset N_{\beta}$ by (2.9), we have only to show the converse inclusion.

Let N_0 be the set of all

$$\int(1_M\otimes\lambda(t))\alpha(x(t))dt$$

with $t\mapsto x(t)$ in $K(G, M)$. The linear span N_1 of all y^*x with $x, y\in N_0$ is σ-weakly dense in N. Since the convex cone N_1^+ spanned by x^*x with $x\in N_0$ generates linearly N_1, N_1^+ is σ-weakly dense in N_+. Since $N_1\subset\mathfrak{m}_{\beta}$ by (a) and (b), N_1^+ is σ-weakly dense in $\mathfrak{m}_{\beta}^+$. It follows from (d) that $E_{\beta}((1_M\otimes\lambda(g)^*)N_1^+(1_M\otimes\lambda(g)))$ is σ-weakly dense in $E_{\beta}((1_M\otimes\lambda(g)^*)\mathfrak{m}_{\beta}^+(1_M\otimes\lambda(g)))$ for all $g\in K(g)$. Since E_{β} is normal by (iii)

$$\bigcup_{g\in K(G)} E_{\beta}((1_M\otimes\lambda(g)^*)\mathfrak{m}_{\beta}^+(1_M\otimes\lambda(g)))$$

is σ-weakly dense in $E_\beta(\mathfrak{m}_\beta^+)=N_\beta^+$(by (c)). Consequently,

$$\bigcup_{g\in K(G)} E_\beta((1_M\otimes\lambda(g)^*)N_1^+(1_M\otimes\lambda(g)))$$

is σ-weakly dense in N_β^+. Since $E_\beta((1_M\otimes\lambda(g)^*)N_1^+(1_M\otimes\lambda(g)))\subset\alpha(M)$ by (v) and (a), we have $N_\beta^+\subset\alpha(M)$ and hence $N_\beta\subset\alpha(M)$. Q.E.D.

The case of β.

Let's recall α_g defined by (2.4). Let I' be the left invariant Haar integral or $I'(f)\equiv I(\Delta f)$. Since I' is a semi-finite faithful normal weight on $L^\infty(G)$, there exists an increasing net $F\subset K(G)_+$ such that

(6.3) $$I'(f)=\sup\{I(fg): g\in F\},\qquad f\in L^\infty(G)_+\,.$$

If $y\in M_+$, then $\{\alpha_g(y): g\in F\}$ is an increasing net in M_+. Define E_α for $y\in M_+$ by

(6.4) $$E_\alpha(y)\equiv\sup\{\alpha_g(y): g\in F\}\,.$$

Since $\omega\otimes I'$ for $\omega\in M_*^+$ is a semi-finite normal weight on $M\otimes L^\infty(G)$, we have

(6.5) $$<E_\alpha(y),\,\omega>=\sup_g<\alpha(y),\,\omega\otimes g>=<\alpha(y),\,\omega\otimes\Delta>\,.$$

Since $E_\alpha(y)$ is not necessarily bounded, we shall take out the bounded part by considering the set $\mathfrak{n}_\alpha$ of all $x\in M$ such that

(6.6) $$<\alpha(x^*x),\,\omega\otimes\Delta>\le\mu_x\|\omega\|\qquad \omega\in M_*^+$$

for some $\mu_x>0$. Since $x^*y^*yx\le\|y\|^2x^*x$, $\mathfrak{n}_\alpha$ is a left ideal of M. Put $\mathfrak{m}_\alpha\equiv\mathfrak{n}_\alpha^*\mathfrak{n}_\alpha$. E_α is, by (6.4), linear and normal on M_+ and is extended canonically over $\mathfrak{m}_\alpha$, which is denoted by $\dot{E}_\alpha$.

Lemma 6.2. *Let α be an action of G on M and let M_α be the set of all $x\in M$ with $\alpha(x)=x\otimes1$. If E_α and $\mathfrak{m}_\alpha$ are defined as above, then*

(i) $E_\alpha(x+y)=E_\alpha(x)+E_\alpha(y)$, $\qquad x,\,y\in M_+$
(ii) $E_\alpha(\mu x)=\mu E_\alpha(x)$ $\qquad \mu\ge0$
(iii) $x_\iota\uparrow x$ implies $E_\alpha(x_\iota)\uparrow E_\alpha(x)$ $\qquad x_\iota\in M_+$

(iv) $\alpha(\dot{E}_\alpha(z))=\dot{E}_\alpha(z)\otimes 1_G$ $\qquad z\in \mathfrak{m}_\alpha$

(v) $\dot{E}_\alpha(b^*zb)=b^*\dot{E}_\alpha(z)b$ $\qquad b\in M_\alpha$.

Proof. (i), (ii) and (iii) are already shown in the above. We have only to prove (iv) and (v).

(iv) If $z\in \mathfrak{m}_\alpha^+$, $\omega\in M_*^+$ and $f\in L^1(G)_+$, then

$$<\alpha(E_\alpha(z)),\ \omega\otimes f> =\sup_g <\alpha(\alpha_g(z)),\ \omega\otimes f>$$

$$(6.7)\qquad =\sup_g <\alpha(z),\ \omega\otimes(f*g)> \qquad \text{(By (2.5))}$$

$$= <\alpha(z),\ \omega\otimes\Delta> <1_G, f> = <E_\alpha(z)\otimes 1_G,\ \omega\otimes f>.$$

Since $\omega\otimes f$ are total in $(M\otimes L^\infty(G))_*$, we have

$$\alpha(\dot{E}_\alpha(x))=\dot{E}_\alpha(x)\otimes 1_G$$

for $x\in\mathfrak{m}_\alpha$.

(v) If $b\in M_\alpha$, $x\in M_+$ and $\omega\in M_*^+$, then

$$<bE_\alpha(x)b^*,\ \omega> = <E_\alpha(x),\ b^*\omega b>$$

$$= <\alpha(x),\ (b^*\omega b)\otimes\Delta> = <(b\otimes 1)\alpha(x)(b^*\otimes 1),\ \omega\otimes\Delta>$$

$$= <\alpha(bxb^*),\ \omega\otimes\Delta>.$$

Q. E. D.

Lemma 6.3. *Let β be a dual action of G on N, $M\equiv N\otimes_\beta^d G$ and $\alpha\equiv\hat{\beta}$. Let $\mathfrak{m}_\alpha$ and M_α be as in Lemma 6.2.*

(i) *If $g\in L^1(G)\cap L^\infty(G)$, then $1_N\otimes g\in\mathfrak{m}_\alpha$ and $\dot{E}_\alpha(1_N\otimes g)=I(g)1_M$.*

(ii) $M_\alpha\mathfrak{m}_\alpha M_\alpha\subset\mathfrak{m}_\alpha$.

(iii) $\dot{E}_\alpha(\mathfrak{m}_\alpha)=M_\alpha$.

Proof. (i) If $\xi\in\mathscr{K}$ and $f\in L^2(G)$, then

$$<\alpha(1_N\otimes g),\ \omega_\xi\otimes\omega_f\otimes\Delta>$$

$$(6.8)\qquad = <1_N\otimes\varepsilon g,\ \omega_\xi\otimes|f|^2\otimes\Delta> \qquad \text{(By (2.12))}$$

$$= <1_N,\ \omega_\xi> \iint g(t^{-1}s)|f(s)|^2\Delta(t)dsdt$$

$$= <1_M,\ \omega_\xi\otimes\omega_f> I(g).$$

Since $\omega_\xi \otimes \omega_f$ are total in M_*, $1_N \otimes g \in \mathfrak{m}_\alpha$ and $\dot{E}_\alpha(1_N \otimes g) = I(g) 1_M$.

(ii) If $b \in M_\alpha$, $x \in \mathfrak{n}_\alpha$ and $\omega \in M_*$, then

$$< \alpha(b^* x^* x b), \omega \otimes \Delta >$$

$$= < (b^* \otimes 1_G) \alpha(x^* x)(b \otimes 1_G), \omega \otimes \Delta >$$

$$= < \alpha(x^* x), b \omega b^* \otimes \Delta >$$

$$\leq \mu_x \| b \omega b^* \| = \mu_x \| \omega \| \, \| b \|$$

by (6.6). Therefore $\mathfrak{n}_\alpha M_\alpha \subset \mathfrak{n}_\alpha$ and hence $M_\alpha \mathfrak{m}_\alpha M_\alpha \subset \mathfrak{m}_\alpha$.

(iii) Since $\dot{E}_\alpha(\mathfrak{m}_\alpha) \subset M_\alpha$ by (iv) in Lemma 6.2, it suffices to show the converse inclusion. If $x \in M_\alpha$ and $g \in L^1(G) \cap L^\infty(G)$, then $(1_N \otimes g)x \in \mathfrak{m}_\alpha$ by (ii) and

$$\dot{E}_\alpha((1_N \otimes g)x) = I(g)x$$

by (v) of Lemma 6.2 and (i). Thus $x \in \dot{E}_\alpha(\mathfrak{m}_\alpha)$. Q.E.D.

Proposition 6.4. *If β is a dual action of G on N, then $\beta(N) = (N \otimes_\beta^d G)^\beta$.*

Proof. Let $M \equiv N \otimes_\beta^d G$ and $\alpha \equiv \hat{\beta}$. It is known that $M^\alpha = M_\alpha$. Indeed, $M^\alpha \subset M_\alpha$ is clear. If $x \in M_\alpha$, then $\alpha_t(x) = x$ locally almost everywhere in $t \in G$. Since $s \mapsto \alpha_s(x)$ is σ-strongly* continuous, $\alpha_s(x) = x$ for all s. Thus $x \in M^\alpha$.

Since $\beta(N) \subset M_\alpha$ by (2.12), it suffices to show the converse inclusion. We first notice that the mapping $y \in \mathfrak{m}_\alpha \mapsto \dot{E}_\alpha((1_N \otimes g) y (1_N \otimes g)) \in M_\alpha$ is σ-weakly continuous for each $g \in F$, where F is a net in $K(G)_+$ defining I' given at (6.3). This is because

$$\| \alpha(1_N \otimes g)(\omega \otimes \Delta) \alpha(1_N \otimes g) \leq \mu_{1_N \otimes g} \| \omega \|$$

by (6.6) and

$$< \dot{E}_\alpha((1_N \otimes g) y (1_N \otimes g)), \omega > = < \alpha((1_N \otimes g) y (1_N \otimes g)), \omega \otimes \Delta >$$

$$= < \alpha(y), \alpha(1_N \otimes g)(\omega \otimes \Delta) \alpha(1_N \otimes g) > .$$

Let M_0 be the linear span of

$$\int \beta(y(t))(1_N \otimes_{t^{-1}} f)dt$$

with $t \mapsto y(t)$ in $K(G, M)$ and $f \in K(G)$. The linear span M_1 of all z^*y with $y, z \in M_0$ is σ-weakly dense in M. Since the convex cone M_1^+ spanned by y^*y with $y \in M_0$ generates linearly M_1, M_1^+ is σ-weakly dense in M_+. Since $M_1 \subset \mathfrak{m}_\alpha$ by (i) and (ii) of Lemma 6.3, M_1^+ is σ-weakly dense in $\mathfrak{m}_\alpha^+$. The σ-weak continuity shown in the above implies that $E_\alpha((1_N \otimes g)M_1^+(1_N \otimes g))$ is σ-weakly dense in $E_\alpha((1_N \otimes g)\mathfrak{m}_\alpha^+(1_N \otimes g))$ for all $g \in F$. Since E_α is normal by (iii) in Lemma 6.2,

$$\bigcup_{g \in F} E_\alpha((1_N \otimes g)\mathfrak{m}_\alpha^+(1_N \otimes g))$$

is σ-weakly dense in $E_\alpha(\mathfrak{m}_\alpha^+) = M_\alpha^+$, which is due to (iii) of Lemma 6.3. Consequently, since E_α is normal,

$$\bigcup_{g \in F} E_\alpha((1_N \otimes g)M_1^+(1_N \otimes g))$$

is σ-weakly dense in M_α^+. Since $E_\alpha((1_N \otimes g)M_1^+(1_N \otimes g))$ is included in $\beta(N)$ by Lemma 6.5 below, we have $M_\alpha^+ \subset \beta(N)$, namely, $M_\alpha \subset \beta(N)$.

Q. E. D.

Lemma 6.5. $E_\alpha((1_N \otimes g)\beta(x)(1_N \otimes g)) \in \beta(N)$.

Proof. Since $(1_N \otimes g)\beta(z)(1_N \otimes g) \in \mathfrak{m}_\alpha$ for all $z \in N$, we may assume that $\mathrm{sp}_\beta(x)$ is compact by Lemma 4.5. Denote by F the function on G:

$$r \longmapsto \int g(tr^{-1})g(t)\Delta(t)dt.$$

Then $F = \lambda_* \rho$ for some $\rho \in R(G)_*$. Therefore

$$\begin{aligned}
&< E_\alpha((1_N \otimes g)\beta(x)(1_N \otimes g)), \omega > \\
&= < \alpha((1_N \otimes g)\beta(x)(1_N \otimes g)), \omega \otimes \Delta > \\
&= \lim < \int \beta_{\lambda(r)*\phi}(x) \otimes < \lambda(r), \rho > \lambda(r)dr, \omega > \\
&= < \beta(\beta_\rho(x)), \omega > .
\end{aligned}$$

Q. E. D.

7. Duality for Crossed Dual Product by β

We shall show another duality theorem for crossed product, which is also a generalization of Takesaki's duality.

Theorem 7.1. *Let N be a von Neumann algebra on $\mathscr{K}$. Let β be a dual action of G on N, $\alpha\equiv\hat{\beta}$ and $\tilde{\beta}\equiv\hat{\alpha}$. Let π be a faithful representation of N on $\mathscr{K}\otimes L^2(G)\otimes L^2(G)$ defined by*

$$\pi(x)\equiv(1\otimes W')(\beta(x)\otimes 1_G)(1\otimes W')^* \tag{7.1}$$

for $x\in N$, where W' is defined by (1.9). Then

(i) *$(N\otimes_\beta^d G)\otimes_\alpha G$ is isomorphic to $N\otimes B(L^2(G))$ and the isomorphism transforms $\pi(x)$ in the former to $x\otimes 1_G$ in the latter; and*

(ii) *$U^*\tilde{\beta}(\pi(x))U=(\pi\otimes\iota)\beta(x)$,*

where U is defined on $\mathscr{K}\otimes L^2(G)\otimes L^2(G)\otimes L^2(G)$ by

$$(U\xi)(r,s,t)\equiv\xi(r,s,ts^{-2}r^2).$$

Before going into the proof we shall prepare the following lemmas.

Lemma 7.2. (i) *If y_r is defined on $L^2(G)\otimes L^2(G)$ by $\mathrm{Ad}\,W'(1_G\otimes\lambda'(r))$ or*

$$(y_r\xi)(s,t)\equiv\Delta(r)^{1/2}\xi(s,sr^{-1}s^{-1}t),$$

then y_r belongs to the von Neumann algebra generated by $\varepsilon L^\infty(G)$ and $1_G\otimes R(G)$, where $(\varepsilon f)(s,t)\equiv f(t^{-1}s)$.

(ii) $\mathrm{Ad}\,W'(\lambda(r)\otimes\lambda'(r))=\lambda(r)\otimes 1_G$.

Proof. (i) For each f and g in $K(G)$ we set

$$x_{f,g}\equiv\Delta(r)^{1/2}\int\varepsilon({}_{a^{-1}}g)(1_G\otimes\lambda(ar^{-1}a^{-1}))f(a)da.$$

We may assume that $g\in K(G)$ and $\|\Delta g\|_1=1$. If $\xi\in K(G\times G)$, then

$$\begin{aligned}&(x_{f,g}\xi)(s,t)\\&=\Delta(r)^{1/2}\int(\varepsilon({}_{a^{-1}}g)(1_G\otimes\lambda(ar^{-1}a^{-1}))\xi)(s,t)f(a)da\end{aligned}$$

$$=\Delta(r)^{1/2}\int g(t^{-1}sa^{-1})\xi(s,\ tar^{-1}a^{-1})f(a)da$$

$$=\int f(at^{-1}s)g(a^{-1})\Delta(r)^{1/2}\xi(s,\ tat^{-1}sr^{-1}s^{-1}ta^{-1})da\,.$$

Since $\|x_{f,g}\|\leq\Delta(r)^{1/2}\|f\|\ \|\Delta g\|_1$, when $(\Delta g)(a)da$ converges to ε_e, $x_{f,g}$ converges weakly to $(\varepsilon f)y_r$. Since $x_{f,g}$ is in the von Neumann algebra B_0 generated by $\varepsilon L^\infty(G)$ and $1_G\otimes R(G)$, so is $(\varepsilon f)y_r$. Since $1_G\otimes 1_G$ is in the weak closure of $\varepsilon K(G)$, y_r belongs to B_0.

(ii) If $\xi\in L^2(G)\otimes L^2(G)$, then

$$(W'(\lambda(r)\otimes\lambda'(r))W'^*\xi)(s,\ t)$$

$$=\Delta(sr)^{1/2}(W'^*\xi)(sr,\ r^{-1}s^{-1}t)$$

$$=\xi(sr,\ t)=((\lambda(r)\otimes 1_G)\xi)(s,\ t)\,.$$

Q. E. D.

Lemma 7.3. *Let ϕ and ψ be elements in $R(G)_*$ with $\lambda_*\phi, \lambda_*\psi\in K(G)$ and ω' an element in $B(L^2(G)\otimes L^2(G))_*$.*

(i) *The following four integrals exist as vector forms*

$$F^1_{\phi,\psi}\equiv\int(\lambda(r)^*\phi)\psi\otimes y_{r^{-1}}\omega' dr$$

$$F^2_{\phi,\psi}\equiv\int(\lambda(r)^*\phi)\psi\otimes\omega'(1_G\otimes\lambda'(r))dr$$

$$G^1_\phi\equiv\int<\lambda(r)^*,\ \phi>((1_G\otimes\lambda'(r)^*)(\omega'\circ\mathrm{Ad}\ W'))dr$$

$$G^2_\phi\equiv\int<\lambda(r)^*,\ \phi>((\omega'\circ\mathrm{Ad}\ W'^*)y_r)dr.$$

(ii) *If $\omega\in N_*$ and $y\in N$, then*

$$<\gamma\Phi_\omega(y)\otimes 1_G,\ F^j_{\phi,\psi}>=<\beta(\beta_\psi(y))\otimes 1_G,\ \omega\otimes G^j_\phi>\qquad (j=1,\ 2). \tag{7.2}$$

Proof. The proof for $F^2_{\phi,\psi}$ and G^2_ϕ proceeds similarly as that for $F^1_{\phi,\psi}$ and G^1_ϕ. We have only consider the latter.

(i) We first consider $F^1_{\phi,\psi}$. Since $r\mapsto\lambda(r)^*\phi$ and $r\mapsto y_{r^{-1}}\omega'$ are continuous by Lemma 4.2, $r\mapsto(\lambda(r)^*\phi)\psi\otimes y_{r^{-1}}\omega'$ is continuous. Since $\lambda_*\phi, \lambda_*\psi\in K(G)$ and $\operatorname{car}\lambda_*((\lambda(r)^*\phi)\psi)\subset(\operatorname{car}\lambda_*\phi)r\cap\operatorname{car}\lambda_*\psi$, $r\mapsto(\lambda(r)^*\phi)\psi\otimes y_{r^{-1}}\omega'$ has a compact carrier. Therefore $F^1_{\phi,\psi}$ is Bochner integrable and hence it is the norm limit of vector forms.

As for G^1_ϕ we have only notice that $\lambda_*\phi\in K(G)$ and $r\mapsto(1_G\otimes\lambda'(r)^*)(\omega'\circ\operatorname{Ad}W')$ is continuous.

(ii) We first show that

$$<\gamma z\otimes 1_G, F^1_{\phi,\psi}> = <\gamma_\psi z\otimes 1_G, G^1_\phi> \tag{7.3}$$

for all $z\in R(G)$. Since $F^1_{\phi,\psi}$ and G^1_ϕ are vector forms, it suffices to show (7.3) for all $\lambda(f)$ with $f\in L^1(G)$. Now, if $f\in L^1(G)$, then

$$\begin{aligned}
&<\gamma\lambda(f)\otimes 1_G, F^1_{\phi,\psi}>\\
&=\int f(s)<\lambda(s)\otimes\lambda(s)\otimes 1_G, F^1_{\phi,\psi}>ds\\
&=\iint f(s)<\lambda(s), \psi><\lambda(sr^{-1}), \phi><(\lambda(s)\otimes 1_G)y_{r^{-1}}, \omega'>drds\\
&=\iint f(s)<\lambda(s), \psi><\lambda(r)^*, \phi><(\lambda(s)\otimes 1_G)y_{s^{-1}r^{-1}}, \omega'>dsdr \qquad (7.4)\\
&=\iint f(s)<\lambda(s), \psi><\lambda(r)^*, \phi><\lambda(s)\otimes\lambda'(r)^*, \omega'\circ\operatorname{Ad}W'>dsdr\\
&=\int<\lambda(r)^*, \phi><\gamma_\psi\lambda(f)\otimes\lambda'(r)^*, \omega'\circ\operatorname{Ad}W'>dr\\
&=<\gamma_\psi\lambda(f)\otimes 1_G, G^1_\phi>,
\end{aligned}$$

where the third equality is due to the Fubini theorem and the right invariance of Haar measure, and the fourth equality follows from Lemma 7.2.

Now we replace z in (7.3) by $\Phi_\omega(y)$. Then we have

$$\begin{aligned}
<\gamma\Phi_\omega(y)\otimes 1_G, F^1_{\phi,\psi}> &= <\gamma_\psi\Phi_\omega(y)\otimes 1_G, G^1_\phi>\\
&=<\Phi_\omega(\beta_\psi(y))\otimes 1_G, G^1_\phi>\\
&=<\beta(\beta_\psi(y))\otimes 1_G, \omega\otimes G^1_\phi>,
\end{aligned}$$

where the second equality is due to (ii) of Lemma 4.1. Thus (7.2) for $j=1$ is proved. Q. E. D.

Proof of Theorem 7.1. Let $M\equiv N\otimes_{\beta}^{d}G$ and $D\equiv M\otimes_{\alpha}G$. Since M is generated by $\beta(N)$ and $1\otimes L^{\infty}(G)$ and since D is generated by $\alpha(M)$ and $1_{M}\otimes R(G)$, D is generated by

$$\beta(N)\otimes 1_{G},\quad 1\otimes\varepsilon L^{\infty}(G)\quad\text{and}\quad 1\otimes 1_{G}\otimes R(G)$$

by (2.12). Let $Q(f)\equiv 1\otimes\varepsilon f$ and $\lambda_{2}(r)\equiv 1\otimes 1_{G}\otimes\lambda(r)$ as in (1.10). Since

$$\lambda_{2}(r)Q(f)\lambda_{2}(r)^{*}=Q(f_{r^{-1}})$$

by direct culculation, Q and λ_{2} satisfy the commutation relation, [10]. Therefore the von Neumann algebra B generated by $Q(L^{\infty}(G))$ and $1\otimes 1_{G}\otimes R(G)$ is isomorphic to $B(L^{2}(G))$ and hence D is isomorphic to $(D\cap B')\otimes B$. It is clear that $\lambda_{2}(r)$ commutes with $1\otimes W'$ and $\beta(x)\otimes 1_{G}$ and hence $\pi(x)$ commutes with $\lambda_{2}(r)$. Since for any $z\in N$ and $r\in G$

$$((1\otimes W')(z\otimes\lambda(r)\otimes 1_{G})(1\otimes W'^{*})\xi)(s,t)=\Delta(r)^{-1/2}z\xi(sr,srs^{-1}t),$$

it follows that $\mathrm{Ad}\,1\otimes W'(z\otimes\lambda(r)\otimes 1_{G})$ commutes with $Q(f)$ and hence that $\pi(x)$ commutes with $Q(f)$. Therefore $\pi(N)\subset B'$.

Now, we shall show that $\pi(N)\subset D$. Choose $x\in N$ and $\phi,\psi\in R(G)_{*}$ with $\lambda_{*}\psi,\lambda_{*}\psi\in K(G)$. Since $r\mapsto\beta_{(\lambda(r)^{*}\phi)\psi}(x)$ has a compact carrier, the integral

$$\int(\beta(\beta_{(\lambda(r)^{*}\phi)\psi}(x))\otimes 1_{G})(1\otimes y_{r^{-1}})dr$$

exists for every $\phi\in R(G)_{*}$. We denote it by $x_{\phi,\psi}$. Then $x_{\phi,\psi}$ belongs to D by (i) in Lemma 7.2. For any ω' in $B(L^{2}(G)\otimes L^{2}(G))_{*}$ we have

$$\begin{aligned}
&<x_{\phi,\psi},\omega\otimes\omega'>\\
&=\int<\beta(\beta_{(\lambda(r)^{*}\phi)\psi}(x))\otimes 1_{G},\omega\otimes y_{r^{-1}}\omega'>dr\\
&=<\gamma\Phi_{\omega}(x)\otimes 1_{G},F_{\phi,\psi}^{1}>\\
&=<\beta(\beta_{\psi}(x))\otimes 1_{G},\omega\otimes G_{\phi}^{1}>\\
&=<\mathrm{Ad}\,1\otimes W'(\beta(\beta_{\psi}(x))\otimes\int<\lambda(r)^{*},\phi>\lambda'(r)^{*}dr),\omega\otimes\omega'>
\end{aligned}\tag{7.5}$$

where the second equality follows from (iii) of Lemma 4.1 and the third equality follows from Lemma 7.3. Since we may assume that $\lambda_*\phi \in K(G)_+$ and $\|\Delta\lambda_*\phi\|_1 = 1$, the norm of the first argument of the right hand side of (7.5) is majorized by $\|\beta_\psi(x)\|$. Further, since $\omega\otimes\omega'$ are total in the set of all vector forms, $x_{\phi,\psi}$ is bounded by $\|\beta_\psi(x)\|$. Since $\omega\otimes\omega'$ are total in D_*, (7.5) shows that $x_{\phi,\psi}$ converges σ-weakly to

$$\mathrm{Ad}\,1\otimes W'(\beta(\beta_\psi(x))\otimes 1_G) = \pi(\beta_\psi(x))$$

as $<\lambda(r)^*, \phi>dr$ tends to ε_e. Sincc $x_{\phi,\psi} \in D$, $\pi(\beta_\psi(x)) \in D$. Since x is in the von Neumann algebra generated by $\beta_\psi(x)$ with $\psi \in R(G)_*$ and $\lambda_*\psi \in K(G)$ by Lemma 4.5, $\pi(x)$ belongs to D.

Next, we shall show that $\beta(N)\otimes 1_G$ is included in $(\pi(N)\cup B)''$. For each $y \in N$ we denote by $y_{\phi,\psi}$ an element of the form

$$\int \beta(\beta_{(\lambda(r)^*\phi)\psi}(y))\otimes\lambda'(r)dr\,,$$

where ϕ, ψ are in $R(G)_*$ with $\lambda_*\phi$, $\lambda_*\psi \in K(G)$. Since $y_r = \mathrm{Ad}\,W'(1_G\otimes \lambda'(r))$ by (i) in Lemma 7.2, $\mathrm{Ad}\,1\otimes W'(y_{\phi,\psi})$ belongs to $(\pi(N)\cup B)''$. For any ω' in $B(L^2(G)\otimes L^2(G))_*$ we have

$$<y_{\phi,\psi}, \omega\otimes\omega'>$$

$$= \int <\beta(\beta_{(\lambda(r)^*\phi)\psi}(y))\otimes 1_G, \omega\otimes\omega'(1_G\otimes\lambda'(r))>dr$$

$$= <\gamma\Phi_\omega(y)\otimes 1_G, F^2_{\phi,\psi}> \tag{7.6}$$

$$= <\beta(\beta_\psi(y))\otimes 1_G, \omega\otimes G^2_\phi>$$

$$= <\mathrm{Ad}\,1\otimes W'^*(\beta(\beta_\psi(y))\otimes 1_G\int <\lambda(r)^*, \phi>1\otimes y_r dr), \omega\otimes\omega'>,$$

where the second and third equalities follow from Lemmas 4.1 and 7.3, respectively. Since we may assume that $\lambda_*\phi \in K(G)_+$ and $\|\Delta\lambda_*\phi\|_1 = 1$, $y_{\phi,\psi}$ is bounded by $\|\beta_\psi(y)\|$ by a similar reason as before. Since $\omega\otimes\omega'$ are total in the predual of $\mathrm{Ad}\,1\otimes W'^*(\pi(N)\cup B)''$, (7.6) shows that $y_{\phi,\psi}$ converges σ-weakly to

$$\mathrm{Ad}\,1\otimes W'^*(\beta(\beta_\psi(y))\otimes 1_G),$$

as $<\lambda(r)^*, \phi>dr$ tends to ε_e. Since $\mathrm{Ad}\,1\otimes W'(y_{\phi,\psi})$ belongs to $(\pi(N) \cup B)''$, $\beta(\beta_\psi(y))\otimes 1_G$ belongs to $(\pi(N)\cup B)''$ and hence $\beta(y)\otimes 1_G \in (\pi(N) \cup B)''$.

Consequently, we have shown both that $\pi(N)\cup B \subset D$ and that $(\beta(N)\otimes 1_G)\cup B\subset(\pi(N)\cup B)''$. Since D is generated by $\beta(N)\otimes 1_G$ and B, we have $D=(\pi(N)\cup B)''$. Since $\pi(N)\subset D\cap B'$, D is isomorphic to $\pi(N) \otimes B$.

(ii) Put $U_0 \equiv U^*(1\otimes 1_G\otimes W)^*(1\otimes W'\otimes 1_G)$. Then

$$(U_0\xi)(r, s, t)=\Delta(r)^{1/2}\xi(r, r^{-1}s, tr^{-2}s). \tag{7.7}$$

Here, by Lemma 4.5, we have only to show (ii) for x with compact $\mathrm{sp}_\beta(x)$. If $\phi\in R(G)_*$ and $\lambda_*\phi \in K(G)$, then for any $a\in G$

$$\Big(\Big(\int\beta_{\lambda(a)*\phi}(x)\otimes\lambda(a)\otimes 1_G\otimes 1_G da\Big)U_0^*\xi\Big)(r, r^{-1}s, tr^{-2}s)$$

$$=\int\beta_{\lambda(a)*\phi}(x)(U_0^*\xi)(ra, r^{-1}s, tr^{-2}s)da$$

$$=\int\beta_{\lambda(a)*\phi}(x)\Delta(ra)^{-1/2}\xi(ra, rar^{-1}s, ta)da \tag{7.8}$$

$$=\int\beta_{\lambda(a)*\phi}(x)((1\otimes W'\otimes 1_G)^*\xi)(ra, r^{-1}s, ta)da$$

$$=\Big(\Big(\int\beta_{\lambda(a)*\phi}(x)\otimes\lambda(a)\otimes 1_G\otimes\lambda(a)da\Big)(1\otimes W'\otimes 1_G)^*\xi\Big)(r, r^{-1}s, t).$$

Here we assume that ω_ξ belongs to the algebraic tensor product $N_*\odot R(G)_*\odot R(G)_*\odot R(G)_*$. By virtue of (ii) in Lemma 4.3 if $<\lambda(a)^*, \phi>da$ converges to ε_e, the left hand side of (7.8) multiplied by $\Delta(r)^{1/2}$ converges to

$$\Delta(r)^{1/2}((\beta(x)\otimes 1_G\otimes 1_G)U_0^*\xi)(r, r^{-1}s, tr^{-2}s)$$

$$=(U_0(\beta(x)\otimes 1_G\otimes 1_G)U_0^*\xi)(r, s, t) \qquad \text{(By (7.7))}$$

$$=(U^*\tilde{\beta}(\pi(x))U\xi)(r, s, t).$$

Define a unitary U_1 on $\mathscr{K}\otimes L^2(G)\otimes L^2(G)\otimes L^2(G)$ by

$$(U_1\xi)(r, s, t) \equiv \xi(r, t, s)$$

and put $U_2 \equiv (1 \otimes W' \otimes 1_G)U_1$. Then the right hand side of (7.8) multiplied by $\Delta(r)^{1/2}$ converges to

$$\Delta(r)^{1/2}(U_1((\iota\otimes\gamma)\beta(x)\otimes 1_G)U_1^*(1\otimes W'\otimes 1_G)^*\xi)(r, r^{-1}s, t)$$

$$=(U_2((\iota\otimes\gamma)\beta(x)\otimes 1_G)U_2^*\xi)(r, s, t)$$

$$=(U_2((\beta\otimes\iota)\beta(x)\otimes 1_G)U_2^*\xi)(r, s, t) \qquad \text{(By (2.4))}$$

$$=(((\pi\otimes\iota)\beta(x))\xi)(r, s, t).$$

Since the set of ξ considered there is dense in $\mathscr{K}\otimes L^2(G)\otimes L^2(G)\otimes L^2(G)$, we complete the proof of (ii).

Combining Proposition 6.4 and Theorem 7.1, we have the following corollary.

Corollary 7.4. *If M is of the form $N\otimes_\beta^d G$ for some N and a dual action β of G on N, then*

(i) *$M^\alpha = \beta(N)$ for $\alpha \equiv \hat{\beta}$; and*

(ii) *$M\otimes_\alpha G$ is isomorphic to $M^\alpha \otimes B(L^2(G))$.*

The existence of a pair $\{N, \beta\}$ in the above corollary is always assured by Theorem 3.1 whenever M is properly infinite and G is separable.

Combining Theorem 3.1 and Proposition 6.1, we have the following corollary.

Corollary 7.5. *If N is of the form $M\otimes_\alpha G$ for some M and an action α of G on M, then*

(i) *$N^\beta = \alpha(M)$ for $\beta \equiv \hat{\alpha}$; and*

(ii) *$N\otimes_\beta^d G$ is isomorphic to $N^\beta \otimes B(L^2(G))$.*

The existence of a pair of $\{M, \alpha\}$ in Corollary 7.5 is assured by Theorem 7.1 whenever N is properly infinite and G is separable.

8. Haga's Factorization of Crossed Product

In this section we shall establish a structure theorem of a crossed dual product corresponding to the Landstad's theorem which gives a necessary and sufficient condition for a given von Neumann algebra to be a crossed product with respect to a given locally compact group. Combining this theorem with the Takesaki's duality of second type, we can give a sufficient condition under which a Haga's factorization for a crossed product is possible.

Theorem 8.1 (Landstad [9]). *Let N be a von Neumann algebra and G a locally compact group. The following two conditions are equivalent*:

(i) *there exist a von Neumann algebra M and an action α of G on M satisfying $N \sim M \otimes_{\alpha} G$; and*

(ii) *there exist a weakly continuous unitary representation u of G in N and a dual action β of G on N satisfying $\beta(u(t)) = u(t) \otimes \lambda(t)$ for all $t \in G$.*

Proof. (i)⇒(ii) We may assume that $N = M \otimes_{\alpha} G$. If we put $\beta \equiv \hat{\alpha}$ and $u(t) \equiv \lambda_1(t)$, then (ii) follows from (2.9).

(ii)⇒(i) Let $\mathfrak{m}_{\beta}$ and N_{β} be as in Section 6. Let F be an increasing net given at (6.2), namely, $\omega_e = \sup\{\omega : \omega \in F\}$.

First we shall show that N is generated by N_{β} and $u(t)$, $t \in G$. If $\phi \in F$, $y \in \mathfrak{m}_{\beta}^{+}$, $\omega \in N_{*}^{+}$ and $\psi \in R(G)_{*}^{+}$, then

$$
\begin{aligned}
&< \beta(\beta_{\phi}(yu(t)^{*})u(t)),\ \omega \otimes \psi > \\
&= < \beta(\beta_{\phi}(yu(t)^{*})),\ u(t)\omega \otimes \lambda(t)\psi > \\
&= < \beta(yu(t)^{*}),\ u(t)\omega \otimes \phi(\lambda(t)\psi) > \qquad \text{(By (4.3))} \\
&= < \beta(y),\ \omega \otimes (\lambda(t)^{*}\phi)\psi > .
\end{aligned} \tag{8.1}
$$

Since we may assume that $< \lambda(t)^{*}, \phi > dt$ converges to ε_e,

$$< \int \beta(\dot{E}_{\beta}(yu(t)^{*})u(t))dt,\ \omega \otimes \psi > = < \beta(y),\ \omega \otimes \psi >$$

by (8.1) and hence

$$y=\int \dot{E}_\beta(yu(t)^*)u(t)dt\,.$$

Since $\mathfrak{m}_\beta^+$ is σ-weakly dense in N_+, N is generated by N_β and $u(t)$, $t\in G$ by the fact that $\dot{E}_\beta(\mathfrak{m}_\beta)=N_\beta$.

Next, we notice that $\mathrm{Ad}\,u(t)$ is an action of G on N_β. Indeed, if $x\in N_\beta$, then

$$\beta(u(t)xu(t)^*)=\mathrm{Ad}\,u(t)\otimes\lambda(t)\,(x\otimes 1_G)=u(t)xu(t)^*\otimes 1_G$$

and hence $\mathrm{Ad}\,u(t)\,(N_\beta)=N_\beta$.

Finally we shall show that N is isomorphic to $M\otimes_\alpha G$, where $M\equiv N_\beta$ and $\alpha(x)\equiv\mathrm{Ad}\,u(x\otimes 1_G)$ for $x\in M$. Since

$$(8.2)\qquad \begin{aligned}(u^*(1_M\otimes\lambda(t))u\xi)(s)&=u(s)^*(u\xi)(st)\\ &=u(t)\xi(st)=((u(t)\otimes\lambda(t))\xi)(s)=(\beta(u(t))\xi)(s)\end{aligned}$$

for ξ in $L^2(G,\mathscr{K})$ and

$$(8.3)\qquad u^*\alpha(x)u=x\otimes 1_G=\beta(x)$$

for x in $M\equiv N_\beta$, we have $M\otimes_\alpha G=\mathrm{Ad}\,u\circ\beta(N)$ and hence N is isomorphic to $M\otimes_\alpha G$. Q.E.D.

If we combine Corollary 7.5 and Theorem 8.1, we will have the following theorem.

Theorem 8.2. *Let β be a dual action of G on N. If there exists a weakly continuous unitary representation u of G in N satisfying $\beta(u(t))=u(t)\otimes\lambda(t)$ for all $t\in G$, then $N\otimes_\beta^d G$ is isomorphic to $N^\beta\otimes B(L^2(G))$.*

Proof. According to Theorem 8.1 there exists a von Neumann algebra M and an action α of G on M such that N is isomorphic to $M\otimes_\alpha G$. Put $\rho\equiv\mathrm{Ad}\,u\circ\beta$. Using (8.2) and (8.3), we have $\rho(u(t))=\lambda_1(t)$ and $\rho(x)=\alpha(x)$ for $x\in N_\beta$. Since

$$\begin{aligned}(\rho\otimes\iota)\beta(x)&=\rho(x)\otimes 1_G=\alpha(x)\otimes 1_G\\ &=\hat{\alpha}(\alpha(x))=\hat{\alpha}(\rho(x))\qquad\text{(By (2.9))}\end{aligned}$$

for $x\in N_\beta$ and

$$(\rho\otimes\iota)\beta(u(t))=(\rho\otimes\iota)(u(t)\otimes\lambda(t))=\lambda_1(t)\otimes\lambda(t)$$
$$=\hat{\alpha}(\lambda_1(t))=\hat{\alpha}(\rho(u(t))),$$

we know that $(\rho\otimes\iota)\circ\beta=\hat{\alpha}\circ\rho$ and hence that β is a dual action dual to α through the isomorphism ρ. Therefore by Corollary 7.5 we complete the proof.

Theorem 8.3. *Let M be a von Neumann algebra and G a locally compact group. The following three conditions are equivalent:*

(i) *there exist a von Neumann algebra N and a dual action β of G on N satisfying $M\sim N\otimes_\beta^d G$;*

(ii) *there exists an action α of G on M and a Hilbert space $\mathscr{K}$ such that $1_{\mathscr{K}}\otimes L^\infty(G)\subset M$ and $\alpha(1_{\mathscr{K}}\otimes f)=1_{\mathscr{K}}\otimes\varepsilon f$ (or $\alpha_t(1_{\mathscr{K}}\otimes f)=1_{\mathscr{K}}\otimes f_{t^{-1}}$ for all $t\in G$); and*

(iii) *(assume that M is standard) there exist an action α of G on M and a weakly continuous unitary representation v on a Hilbert space $\mathscr{K}$ such that $1_{\mathscr{K}}\otimes L^\infty(G)\subset M$ and $\alpha_t=\mathrm{Ad}\,v(t)\otimes\lambda'(t)\restriction M$ for all $t\in G$.*

Proof. (i)⇒(ii) We may assume that $M\equiv N\otimes_\beta^d G$. Put $\alpha\equiv\hat{\beta}$. Then $1_N\otimes L^\infty(G)\subset M$ and $\alpha(1_N\otimes f)=1_N\otimes\varepsilon f$ by (2.12). Therefore

$$\alpha_t(1_N\otimes f)\xi(t)=(\alpha(1_N\otimes f)\xi)(t)$$
$$=((1_N\otimes\varepsilon f)\xi)(t)=(1_N\otimes f_{t^{-1}})\xi(t)$$

and so $\alpha_t(1_N\otimes f)=1_N\otimes f_{t^{-1}}$ for all $t\in G$.

(ii)⇒(iii) We have only to show the existence of a weakly continuous unitary representation v on $\mathscr{K}$ such that $\alpha_t=\mathrm{Ad}\,v(t)\otimes\lambda'(t)$ on M.

Now we may assume that α_t is implemented by a weakly continuous unitary representation u of G on $\mathscr{K}\otimes L^2(G)$ by considering M to be standard. Since by (ii)

$$\mathrm{Ad}\,u(r)(1_{\mathscr{K}}\otimes f)=\alpha_r(1_{\mathscr{K}}\otimes f)$$
$$=1_{\mathscr{K}}\otimes f_{r^{-1}}=\mathrm{Ad}\,1_{\mathscr{K}}\otimes\lambda'(r)(1_{\mathscr{K}}\otimes f)$$

for all $f\in L^\infty(G)$, we have $(1_{\mathscr{K}}\otimes\lambda'(r))^*u(r)\in B(\mathscr{K})\otimes L^\infty(G)$. Therefore there is an essentially bounded weakly measurable function $r\in G\mapsto v(r)$ in $L^\infty(G, B(\mathscr{K}))$ such that $v(r)$ are unitaries on $\mathscr{K}$ for all $r\in G$ and $u(r)$

$=v(r)\otimes\lambda'(r)$. Since u is a representation of G, so is v. The continuity of unitary representation is immediate from measurability.

(iii)⇒(i) Let $\mathfrak{m}_\alpha$ and N_β be as in Section 6. Let F be an increasing net given at (6.3), namely $I'=\sup\{g: g\in F\}$.

First we shall show that M is generated by M_α and $1_{\mathscr{K}}\otimes L^\infty(G)$. Suppose that $y\in\mathfrak{m}_\alpha$. If $k\in K(G)$, $g\in F$, $\omega\in B(\mathscr{K})_*$, $h\in L^2(G)$ and $f\in L^1(G)$, then

$$(8.4)\quad \begin{aligned} &<\alpha((1_{\mathscr{K}}\otimes{}_{r^{-1}}k)\alpha_g((1_{\mathscr{K}}\otimes{}_{r^{-1}}k)y)),\ \omega\otimes\omega_h\otimes f>\\ &=<\alpha(\alpha_g((1_{\mathscr{K}}\otimes{}_{r^{-1}}k)y)),\ \omega\otimes\omega_{h\otimes 1_G,(\varepsilon_{r^{-1}}k)(h\otimes f)}>\\ &=<\alpha((1_{\mathscr{K}}\otimes{}_{r^{-1}}k)y),\ \omega\otimes\omega_{h\otimes 1_G,((\varepsilon_{r^{-1}}k)(h\otimes f))\hat{*}(1_G\otimes g)}>\\ &=<\alpha(y),\ \omega\otimes\omega_{h\otimes 1_G,l_r}>\end{aligned}$$

where $\hat{*}$ indicates the convolution product with respect to the second argument, $\omega_{h\otimes 1_G,K}(\lambda(a)\otimes H)\equiv I\otimes I(\overline{K}({}_a h\otimes H))$ for $H\in L^\infty(G)$ and

$$l_r\equiv(\varepsilon_{r^{-1}}k)\{((\varepsilon_{r^{-1}}k)(h\otimes f))\hat{*}(1_G\otimes g)\}.$$

Since

$$\begin{aligned} l_r(s,t)&=k(t^{-1}sr^{-1})\int((\varepsilon_{r^{-1}}k)(h\otimes f))(s,tb^{-1})g(b)db\\ &=k(t^{-1}sr^{-1})\int k(bt^{-1}sr^{-1})h(s)f(tb^{-1})g(b)db\\ &=k(t^{-1}sr^{-1})\int k(b)h(s)f(sr^{-1}b^{-1})g(brs^{-1}t)db,\end{aligned}$$

we have, by right invariance of Haar measure,

$$\begin{aligned}\int l_r(s,t)dr&=\int k(r^{-1})\int k(b)h(s)f(tr^{-1}b^{-1})g(br)dbdr\\ &=(h\otimes\iint k(r^{-1})k(br^{-1})g(b)_{b^{-1}}fdrdb)(s,t)\\ &=(h\otimes(f*((k*\check{k})g)))(s,t),\end{aligned}$$

where $\check{k}(r)\equiv k(r^{-1})$. Therefore

(8.5)
$$\int <\alpha((1_{\mathscr{H}}\otimes_{r^{-1}}k)\alpha_g((1_{\mathscr{H}}\otimes_{r^{-1}}k)y)),\ \omega\otimes\omega_h\otimes f> dr$$
$$= <\alpha(y),\ \omega\otimes\omega_h\otimes(f*((k*\check{k})g))>$$

by (8.4). If $g\in F$ converges to Δ in the compact convergence topology,

$$<\alpha\left(\int(1_{\mathscr{H}}\otimes_{r^{-1}}k)E_\alpha((1_{\mathscr{H}}\otimes_{r^{-1}}k)y)dr\right),\ \omega\otimes\omega_h\otimes f>$$
$$= <\alpha(y),\ \omega\otimes\omega_h\otimes(f*((k\otimes\check{k})\Delta))>$$

by (8.5). If $(k*\check{k})(r)dr$ converges to ε_e, then

(8.6)
$$\int(1_{\mathscr{H}}\otimes_{r^{-1}}k)E_\alpha((1_{\mathscr{H}}\otimes_{r^{-1}}k)y)dr$$

converges σ-weakly to y. Since the element at (8.6) belongs to the von Neumann algebra generated by M_α and $1_{\mathscr{H}}\otimes L^\infty(G)$, so does y. Since $\mathfrak{m}_\alpha$ is σ-weakly dense in M as in the proof of Proposition 6.4, M is generated by M_α and $1_{\mathscr{H}}\otimes L^\infty(G)$.

Now we define a unitary w on $\mathscr{H}\otimes L^2(G)\otimes L^2(G)$ by

$$(w\xi)(s,t)\equiv\Delta(ts^{-1})^{1/2}\xi(s,st^{-1})$$

and an isomorphism β by

$$\beta(x)\equiv\mathrm{Ad}\,w(x\otimes 1_G)$$

for all x in M_α. Then $(w^*\xi)(s,t)=\Delta(t)^{1/2}\xi(s,t^{-1}s)$. We shall show that β is an isomorphism of M_α into $M_\alpha\otimes R(G)$.

Since $w^*(1_M\otimes\lambda'(r))w=1_M\otimes\lambda(r)$ by

$$(w^*(1_M\otimes\lambda'(r))w\xi)(s,t)=\Delta(t)^{1/2}((1_M\otimes\lambda'(r))w\xi)(s,t^{-1}s)$$
$$=\Delta(tr)^{1/2}(w\xi)(s,r^{-1}t^{-1}s)=\xi(s,tr)=((1_M\otimes\lambda(r))\xi)(s,t),$$

we have, for any x in M_α,

$$w(x\otimes 1_G)w^*(1_M\otimes\lambda'(r))=w(x\otimes 1_G)(1_M\otimes\lambda(r))w^*$$
$$=w(1_M\otimes\lambda(r))(x\otimes 1_G)w^*=(1_M\otimes\lambda'(r))w(x\otimes 1_G)w^*$$

and hence

(8.7) $$w(x\otimes 1_G)w^* \in B(\mathscr{H}\otimes L^2(G))\otimes R(G).$$

Since $M'\subset B(\mathscr{H})\otimes L^\infty(G)$ by assumption (iii), if $y\in M'$, then y is an essentially bounded weakly measurable function $r\mapsto y(r)$ in $L^\infty(G, B(\mathscr{H}))$ and hence $[w, y\otimes 1_G]=0$, for

$$\begin{aligned}(w(y\otimes 1_G)\xi)(s, t) &= \Delta(ts^{-1})^{1/2}((y\otimes 1_G)\xi)(s, st^{-1})\\ &= \Delta(ts^{-1})^{1/2}y(s)\xi(s, st^{-1}) = y(s)(w\xi)(s, t)\\ &= ((y\otimes 1_G)w\xi)(s, t).\end{aligned}$$

Therefore, for any x in M_α,

$$[w(x\otimes 1_G)w^*, y\otimes 1_G]=0$$

and hence by (8.7)

(8.8) $$w(x\otimes 1_G)w^* \in M\otimes R(G).$$

Here we set $u(t)\equiv v(t)\otimes\lambda'(t)$ on $\mathscr{H}\otimes L^2(G)$. Since $w^*(u(r)\otimes 1_G)w=u(r)\otimes\lambda'(r)$ by

$$\begin{aligned}(w^*(u(r)\otimes 1_G)w\xi)(s, t) &= \Delta(t)^{1/2}((u(r)\otimes 1_G)w\xi)(s, t^{-1}s)\\ &= \Delta(tr)^{1/2}v(r)(w\xi)(r^{-1}s, t^{-1}s)\\ &= \Delta(r)v(r)\xi(r^{-1}s, r^{-1}t) = ((u(r)\otimes\lambda'(r))\xi)(s, t),\end{aligned}$$

we have, for any x in M_α,

(8.9) $$\begin{aligned}&(u(r)\otimes 1_G)(w(x\otimes 1_G)w^*)(u(r)\otimes 1_G)^*\\ &= w(u(r)\otimes\lambda'(r))(x\otimes 1_G)(u(r)\otimes\lambda'(r))^*w^*\\ &= w(\alpha_r(x)\otimes 1_G)w^* = w(x\otimes 1_G)w^*.\end{aligned}$$

Combining (8.8) and (8.9), we have

(8.10) $$w(x\otimes 1_G)w^* \in M_\alpha\otimes R(G),$$

which shows that β is an isomorphism of M_α into $M_\alpha\otimes R(G)$.

Next we shall show that β is a dual action of G on M_α. Define a unitary U_1 on $\mathscr{H}\otimes L^2(G)\otimes L^2(G)\otimes L^2(G)$ by

$$(U_1\xi)(r, s, t)\equiv\xi(r, t, s)$$

as (7.9) and put $\check{w}\equiv U_1(w\otimes 1_G)U_1^*$. For z in $K(G, B(\mathscr{K}))$ we put

$$\tilde{z}\equiv\int z(a)\otimes\lambda(a)da .$$

Since

$$\begin{aligned}&\mathrm{Ad}\,((w\otimes 1_G)\check{w})(\tilde{z}\otimes 1_G\otimes 1_G)\\&=\int z(a)\otimes\lambda(a)\otimes\lambda(a)\otimes\lambda(a)da\end{aligned}$$

and

$$\mathrm{Ad}\,w(\tilde{z}\otimes 1_G)=\int z(a)\otimes\lambda(a)\otimes\lambda(a)da$$

by direct culculation, we have

$$\mathrm{Ad}\,(w\otimes 1)\check{w}\circ\iota\otimes\gamma=\iota\otimes\gamma\circ\mathrm{Ad}\,w \tag{8.11}$$

on $\tilde{z}\otimes 1_G$. Since the set of all $\tilde{z}$ is weakly dense in $B(\mathscr{K}\otimes L^2(G))$, (8.11) holds on $M_\alpha\otimes 1_G$ and hence β satisfies (2.6), which shows that β is a dual action of G on M_α.

Finally we shall show that M is isomorphic to $N\otimes_\beta^d G$, where $N\equiv M_\alpha$. Since

$$w^*\beta(x)w=x\otimes 1_G=\alpha(x) \tag{8.12}$$

for x in N and

$$\begin{aligned}(w^*(1_N\otimes f)w\xi)(s, t)&=\Delta(t)^{1/2}((1_N\otimes f)w\xi)(s, t^{-1}s)\\&=f(t^{-1}s)\Delta(t)^{1/2}(w\xi)(s, t^{-1}s)=f(t^{-1}s)\xi(s, t) \qquad (8.13)\\&=\alpha_t(1_{\mathscr{K}}\otimes f)\xi(s, t)=(\alpha(1_{\mathscr{K}}\otimes f)\xi)(s, t)\end{aligned}$$

for f in $L^\infty(G)$, we have $N\otimes_\beta^d G=\mathrm{Ad}\,w\circ\alpha(M)$ and hence M is isomorphic to $N\otimes_\beta^d G$. Q.E.D.

Theorem 8.3 gives a sufficient condition under which Haga's factorization holds for a crossed product, [7], in the following.

Theorem 8.4. *Let* α *be an action of* G *on* M. *If* $1_{\mathscr{X}}\otimes L^\infty(G)$ *is a von Neumann subalgebra of* M *satisfying* $\alpha_t(1_{\mathscr{X}}\otimes f)=1_{\mathscr{X}}\otimes f_{t^{-1}}$ *for all* $t\in G$, *then* $M\otimes_\alpha G$ *is isomorphic to* $M^\alpha\otimes B(L^2(G))$.

Proof. By virtue of Theorem 8.3 we have a von Neumann algebra N and a dual action β of G on N such that M is isomorphic to $N\otimes_\beta^d G$. We set $\rho\equiv\mathrm{Ad}\, w\circ\alpha$. It follows from (8.12) and (8.13) that $\rho(x)=\beta(x)$ for $x\in N\equiv M_\alpha$ and $\rho(1_{\mathscr{X}}\otimes f)=1_N\otimes f$ for $f\in L^\infty(G)$. Since

$$(\rho\otimes\iota)(\alpha(x))=\rho(x)\otimes 1_G=\beta(x)\otimes 1_G$$

$$=\hat{\beta}(\beta(x))=\hat{\beta}(\rho(x)) \qquad \text{(By (2.12))}$$

for $x\in N\equiv M_\alpha$ and

$$(\rho\otimes\iota)\alpha(1_{\mathscr{X}}\otimes f)=(\rho\otimes\iota)(1_{\mathscr{X}}\otimes\varepsilon f)=1_N\otimes\varepsilon f$$

$$=\hat{\beta}(1_N\otimes f)=\hat{\beta}(\rho(1_{\mathscr{X}}\otimes f)),$$

we know that $(\rho\otimes\iota)\circ\alpha=\hat{\beta}\circ\rho$ and hence α is an action dual to β through the isomorphism ρ. Therefore, by Corollary 7.4, we have a desired result. Q. E. D.

9. Appendix

In this section we shall give a few comments on our results considered when we use the left regular representation of G on $L^2(G)$. Let J be a unitary involution on $L^2(G)$ defined by $(J\xi)(t)\equiv\Delta(t)^{1/2}\xi(t^{-1})$. We set

$$\alpha'\equiv(\mathrm{Ad}\,1_M\otimes J)\circ\alpha, \qquad \delta'\equiv(\mathrm{Ad}\,J\otimes J)\circ\delta\circ(\mathrm{Ad}\,J)^{-1}$$

$$\beta'\equiv(\mathrm{Ad}\,1_N\otimes J)\circ\beta, \qquad \gamma'\equiv(\mathrm{Ad}\,J\otimes J)\circ\gamma\circ(\mathrm{Ad}\,J)^{-1}$$

$$M\otimes_{\alpha'}G\equiv\{\alpha'(M),\ 1_M\otimes R(G)'\}'', \qquad N\otimes_{\beta'}^d G\equiv\{\beta'(N),\ 1_N\otimes L^\infty(G)\}''.$$

Then we have

(9.1) $$(\alpha'\otimes\iota)\circ\alpha'=(\iota\otimes\delta')\circ\alpha', \qquad (\alpha'(x)\xi)(t)=\alpha_t^{-1}(x)\xi(t)$$

(9.2) $$(\beta'\otimes\iota)\circ\beta'=(\iota\otimes\gamma')\circ\beta'$$

$$(\delta' f)(s, t) = f(ts), \qquad \gamma'\lambda'(t) = \lambda'(t)\otimes\lambda'(t)$$

$$M\otimes_{\alpha'}G \sim M\otimes_{\alpha}G, \qquad N\otimes^{d}_{\beta'}G \sim N\otimes^{d}_{\beta}G.$$

If we define $\hat{\alpha}'$ and $\hat{\beta}'$ by

$$\hat{\alpha}'(y) \equiv \mathrm{Ad}\, 1_M\otimes W'(y\otimes 1_G), \qquad \hat{\beta}'(z) \equiv \mathrm{Ad}\, 1_N\otimes V^*(z\otimes 1_G)$$

for $y\in M\otimes_{\alpha'}G$ and $z\in N\otimes^{d}_{\beta'}G$, then

$$\hat{\alpha}'(\alpha'(x)) = \alpha'(x)\otimes 1_G, \qquad \hat{\alpha}'(1_M\otimes\lambda'(r)) = 1_M\otimes\lambda'(r)\otimes\lambda'(r)$$

$$\hat{\beta}'(\beta'(x)) = \beta'(x)\otimes 1_G, \qquad \hat{\beta}'(1_N\otimes f) = 1_N\otimes\varepsilon' f,$$

where $(\varepsilon' f)(s, t) \equiv f(st^{-1})$. Besides, $\hat{\alpha}'$ and $\hat{\beta}'$ satisfying (9.2) and (9.1), respectively.

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Correction. In [13] the unitary V in Theorem 2 should be read as U defined below, and Theorem 3 should be replaced by Corollary without assuming the unimodularness. α in Theorem 4 shoud be read $\hat{\beta}$.