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Limit Theorems of Occupation Times for Markov Processes

By

Yuji KASAHARA*)

§0. Introduction

Let X_t be a temporally homogeneous Markov process with values in an abstract space and $f(x)$ be a measurable function on this space. The limiting distribution of random variable

$$\frac{1}{u(t)}\int_0^t f(X_s)\,ds \qquad \text{as } t\longrightarrow\infty,$$

where $u(t)$ is some normalizing function, has been investigated by many authors. A most general limit theorem was obtained by Darling and Kac [1], who also showed that, under suitable condition, the limit distribution must be Mittag-Leffler distribution. However, they confined themselves to the case where $f(x)$ is nonnegative. C. Stone [5] derived a limit theorem for processes including 1-dimensional diffusion processes with the infinitesimal generator $\frac{d}{dm}\frac{d}{dx}$. It is not assumed that $f(x)$ is nonnegative, but it is essential that $f(x)$ is not null-charged; $\int f(x)m(dx)\neq 0$.

In this paper we study the case where $f(x)$ is null-charged. If X_t is positively recurrent, the problem above can be reduced to the central limit theorem (see Tanaka [6]). A similar problem was treated by Dobrusin [2], who studied limit theorems for the 1-dimensional simple random walk.

The aim of this paper is to give a limit theorem for most general processes. Contrary to the case of [1], the limiting distribution is

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*) Department of Mathematics, Kyoto University, Kyoto.

bilateral Mittag-Leffler distribution (for the definition, see Appendix). We also prove that, under suitable conditions, the limiting distribution must be bilateral Mittag-Leffler distribution.

To prove these theorems, we use the method of Darling and Kac; we calculate the Laplace transform of the moments of $\int_0^t f(X_s)ds$ and appeal to Tauberian theorem. However, matters are more complicated since $f(x)$ may take negative values in our case.

In section 1, we will state our theorems in a general form and we give, as an example, a limit theorem for 1-dimensional Brownian motion. Sections 2 and 3 are devoted to the proof of the theorems in section 1. In the last two sections, we apply the main theorems to symmetric stable processes and 1-dimensional diffusion processes.

§1. Main Theorems

Let $X=(X_t, P_x)$ be a temporally homogeneous Markov process with state space $(E, \mathscr{B})$, where E is a locally compact Hausdorff space and $\mathscr{B}$ the Borel σ-field of E. We assume there is a Radon measure $\nu(dx)$ such that the transition probability $p(t, x, dy)$ is absolutely continuous with respect to $\nu(dy)$;

$$p(t, x, dy)=p(t, x, y)\nu(dy).$$

Let us denote by G_s $(s>0)$ the Green operator of X;

$$G_s f(x)=\int_E G_s(x, y)f(y)\nu(dy)$$

for any bounded measurable function f, where

$$G_s(x, y)=\int_0^\infty e^{-st}p(t, x, y)dt.$$

In the sequel, we assume that $G_s(x, y)$ has following representation;

$$G_s(x, y)=h(s)+u(x, y)+\varepsilon(x, y; s).$$

[Assumptions]

(A) $f(x)$ is a bounded measurable function on E such that $\sup_{x\in E}|G_s f(x)|$ is bounded as $s\to 0$.

(B) $\lim_{s\to 0} h(s)=\infty$.

(C) $C=\iint u(x, y)f(x)f(y)\nu(dx)\nu(dy)\neq 0$.

(D) There exists a measurable function $1\leqq\rho(x)<\infty$ satisfying the following;

(D.1) $$\int|f(y)|\rho(y)\nu(dy)<\infty.$$

(D.2) $$\frac{1}{\rho(x)}\int|u(x, y)f(y)|\rho(y)\nu(dy)$$

is bounded in x.

(D.3) $$\frac{1}{\rho(x)}\int|\varepsilon(x, y; s)f(y)|\rho(y)\nu(dy)$$

converges to 0 uniformly in x as $s\to 0$.

Suppose (A)~(D) are satisfied. Then we easily see that $f(x)$ satisfies the following condition;

(N) $$\int f(x)\nu(dx)=0.$$

This condition plays an essential part in the sequel. The condition (B) is, roughly speaking, equivalent to the recurrence of X_t, and in many cases, (C) is satisfied if $f(x)\not\equiv 0$ ν-a.e.. (D) is a rather technical assumption, and in a special case such as $f(x)$ has compact support, (D) can be replaced by a more natural assumption (D$'$);

(D$'$) (D.1$'$) $\int|f(x)|\nu(dx)<\infty$.

(D.2$'$) $u(x, y)$ is locally integrable and $\int|u(x, y)f(y)|\nu(dy)$ is bounded on $\{\xi; f(\xi)\neq 0\}$.

(D.3$'$) $\lim_{s\to 0}\int|\varepsilon(x, y; s)f(y)|\nu(dy)=0$, and the convergence is uniform on $\{\xi; f(\xi)\neq 0\}$.

Theorem 1. *If X and $f(x)$ satisfy (A)~(D) (or (D′)), and if $h(s)=s^{-\alpha}L(1/s)$ with $L(1/s)$ slowly varying as $s\to 0$, then, $0\leqq\alpha\leqq 1$, $C>0$ and for each $x\in E$,*

$$\lim_{t\to\infty} P_x\left\{\frac{1}{\sqrt{Ch(1/t)}}\int_0^t f(X_s)ds<u\right\}=\tilde{g}_{\alpha/2}(u),\ u\in\boldsymbol{R}^1$$

where $\tilde{g}_{\alpha/2}(u)=\frac{1}{\pi\alpha}\int_{-\infty}^{u}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j!}\sin\frac{\pi\alpha}{2}j\Gamma\left(\frac{\alpha}{2}j+1\right)|y|^{j-1}dy.$

Theorem 1 has the following converse;

Theorem 2. *If X and $f(x)$ satisfy (A)~(D) (or (D′)), and if there is a nondegenerate distribution function $G(x)$ such that*

$$\lim_{t\to\infty} P_x\left\{\frac{1}{u(t)}\int_0^t f(X_s)ds<u\right\}=G(u),\quad \text{a.e.}\,u$$

holds for some $x\in E$ and for some appropriate nondecreasing function $u(t)\nearrow\infty$, then $h(s)=s^{-\alpha}L(1/s)$ for some α $(0\leqq\alpha\leqq 1)$ and slowly varying $L(1/s)$. Hence $G(u)=\tilde{g}_{\alpha/2}(bu)$ with suitable constant b.

Example. Let X_t be a 1-dimensional standard Brownian motion and $f(x)$ be a bounded Borel function such that

$$f\not\equiv 0 \text{ a.e.},\quad \int|x^3f(x)|dx<\infty,\quad\text{and}\quad \int f(x)dx=0.$$

Since $G_s(x, y)=\frac{1}{\sqrt{2s}}e^{-\sqrt{2s}|x-y|}$ with $\nu(dx)=dx$, (B)~(D) are satisfied if we set $h(s)=1/\sqrt{2s}$, $u(x, y)=-|x-y|$, $\varepsilon(x, y; s)=\frac{1}{\sqrt{2s}}(e^{-\sqrt{2s}|x-y|}-1+\sqrt{2s}|x-y|)$ and $\rho(x)=|x|+1$. To show that (A) is satisfied, we have only to notice that

$$|G_s(x, y)-G_s(x, 0)|\leqq|y|.$$

In fact,

$$|G_sf(x)|=\left|\int G_s(x, y)f(y)dy\right|=\left|\int(G_s(x, y)-G_s(x, 0))f(y)dy\right|$$
$$\leqq\int|yf(y)|dy\qquad(<\infty).$$

Hence we obtain the following;

$$\lim_{t\to\infty} P_x\left\{\frac{1}{C_1 t^{1/4}}\int_0^t f(X_s)\,ds < u\right\} = \tilde{g}_{1/4}(u),\ u\in \boldsymbol{R}^1,$$

where $(C_1)^2 = -\frac{1}{\sqrt{2}}\iint |x-y|\,f(x)f(y)\,dxdy.$

Remark. The result in the example above can be obtained by another method. Let $f(x)$ ($\neq 0$ a.e.) be a bounded measurable function such that $xf(x)$ is summable and such that $\int f(x)dx = 0$. Then

$$G(x) = 2\int_{-\infty}^{x}(x-y)f(y)dy \text{ and } F(x) = 2\int_{-\infty}^{x} f(y)\,dy$$

are bounded functions. By Itô's formula, we obtain

$$G(X_t) - G(X_0) = \int_0^t F(X_s)dX_s + \int_0^t f(X_s)ds.$$

Since $G(x)$ is bounded, we have only to show

$$\lim_{t\to\infty} E_x\left\{\exp\frac{\lambda}{C_1 t^{1/4}}\int_0^t F(X_s)dX_s\right\} = E_{1/2}(\lambda^2),$$

where $E_{1/2}(\lambda^2) = \sum_{n=1}^{\infty}\lambda^{2n}/\Gamma(n/2+1) = \int_{-\infty}^{\infty} e^{\lambda x}d\tilde{g}_{1/4}(x)$

$$= \int_0^\infty e^{\lambda^2 x}dg_{1/2}(x), \quad \text{(see Appendix)}.$$

To prove this we use the Cameron-Martin formula;

$$E_x\left\{\exp\frac{\lambda}{C_1 t^{1/4}}\int_0^t F(X_s)dX_s\right\}$$

$$= E_x\left\{\exp\frac{\lambda^2}{2C_1^2 t^{1/2}}\int_0^t F(X_s)^2 ds\cdot\exp\left(\frac{\lambda}{C_1 t^{1/4}}\int_0^t F(X_s)dX_s - \frac{\lambda^2}{2C_1^2 t^{1/2}}\int_0^t F(X_s)^2 ds\right)\right\}$$

$$= E_x\left\{\exp\frac{\lambda^2}{2C_1^2 t^{1/2}}\int_0^t F(X_s^t)^2 ds\right\}$$

where X^t is the solution of the following stochastic differential equation;

$$dX_s^t = dX_s + \frac{\lambda F(X_s^t)}{C_1 t^{1/4}} ds.$$

Then using the method of C. Stone [5], we can prove

$$\lim_{t\to\infty} E_x\left\{\exp \frac{\lambda^2}{2C_1^2 t^{1/2}} \int_0^t F(X_s^t)^2 ds\right\} = E_{1/2}(\lambda^2).$$

This proves our assertion.

§2. Auxialiary Results

In order to prove the theorems in section 1, we need some auxialiary results. Throughout this section we assume (A)~(D). To simplify the notations, for any measurable function $u(x)$ defined on E, we denote $\sup_{x\in E} |u(x)|/\rho(x)$ by $\|u\|$.

Notice that $\lim_{n\to\infty} \|u_n\| = 0$ is followed by $\lim_{n\to\infty} u_n(x) = 0$ for each x.

Lemma 2.1. *Let* $g(x) = \int_E u(x, y) f(y) \nu(dy)$. *Then,*

(i) $\overline{\lim_{s\to 0}} \|G_s f(x) - g(x)\| = 0.$

(ii) $g(x)$ *is bounded on* E.

(iii) $\overline{\lim_{s\to 0}} \|G_s(fg) - Ch(s)\| < \infty.$

Proof. (i) follows immediately from (D.3), and (ii) from (A) and (i). By the definition of $g(x)$, we have,

$$G_s(fg)(x) - Ch(s)$$

$$= \int u(x, y) f(y) g(y) \nu(dy) + \int \varepsilon(x, y; s) f(y) g(y) \nu(dy).$$

Since $g(x)$ is bounded, (D.2) and (D.3) imply (iii). Q.E.D.

Lemma 2.2. *For any* $v_s(x)$, $s > 0$ $(\overline{\lim_{s\to 0}} \|v_s\| < \infty)$,

(i) $\overline{\lim_{s\to 0}} \left\| \frac{G_s(f v_s)}{h(s)} \right\| \leq K \overline{\lim_{s\to 0}} \|v_s\|.$

(ii) $\overline{\lim_{s\to 0}} \left\| \frac{G_s f(G_s(fv_s))}{h(s)} \right\| \leqq K \overline{\lim_{s\to 0}} \|v_s\|$

where K is a positive constant which is independent of v_s.

Proof. Set $a(s)=\int f(y)v_s(y)\nu(dy)$, then we easily see that

$$\overline{\lim_{s\to 0}} \|G_s(fv_s)(x)-a(s)h(s)\| \leqq K_1 \overline{\lim_{s\to 0}} \|v_s\| \tag{2.1}$$

where $K_1=\left\|\int |u(x, y)f(y)|\rho(y)\nu(dy)\right\|$ $(<\infty)$ and that

$$\overline{\lim_{s\to 0}} |a(s)| \leqq \int |f(y)|\rho(y)\nu(dy)\cdot \overline{\lim_{s\to 0}} \|v_s\| . \tag{2.2}$$

Since $\|a(s)\| \leqq |a(s)|$, we obtain by (2.1) and (2.2) that

$$\overline{\lim_{s\to 0}} \left\| \frac{G_s(fv_s)}{h(s)} \right\| \leqq K_2 \overline{\lim_{s\to 0}} \|v_s\|$$

where $K_2=\int |f(y)|\rho(y)\nu(dy)$ $(<\infty)$.
Thus (i) is proved, and furthermore, using (i) and (2.1), (2.2), we have,

$$\begin{aligned}\overline{\lim_{s\to 0}} \left\| \frac{G_s(f(G_s(fv_s)))}{h(s)} \right\| &\leqq \overline{\lim_{s\to 0}} \left\| \frac{G_s f(G_s(fv_s)-a(s)h(s))}{h(s)} \right\| + \overline{\lim_{s\to 0}} |a(s)| \|G_s f\| \\ &\leqq K_2 \overline{\lim_{s\to 0}} \|G_s(fv_s)-a(s)h(s)\| + \overline{\lim_{s\to 0}} \sup_x |G_s f(x)| \overline{\lim_{s\to 0}} |a(s)| \\ &\leqq K_1K_2 \overline{\lim_{s\to 0}} \|v_s\| + K_2K_3 \overline{\lim_{s\to 0}} \|v_s\| ,\end{aligned}$$

where $K_3=\overline{\lim_{s\to 0}} \sup_x |G_s f(x)|$. Q.E.D.

Lemma 2.3. *Let* $v_1(s, x)=G_s f(x)$ *and* $u_1(s, x)=\frac{G_s(fv_1)}{h(s)}$,

and $$u_{n+1}(s, x)=\frac{1}{h(s)}(G_s f(G_s(fu_n)))(x) ,$$

$$v_{n+1}(s, x)=\frac{1}{h(s)}(G_s f(G_s(fv_n)))(x) , \quad n=1, 2,\dots .$$

Then,

(i) $\overline{\lim}_{s\to 0} \|u_n(x)-C^n\|=0, \qquad n=1, 2,\dots.$

(ii) $\overline{\lim}_{s\to 0} \|v_n(x)\|<\infty, \qquad n=1, 2,\dots.$

Proof. We prove the assertion by induction. By Lemma 2.1 we have

$$\overline{\lim_{s\to 0}} \left\|\frac{G_s(fg)}{h(s)}-C\right\|=0,$$

$$\overline{\lim_{s\to 0}} \|G_sf-g\|=0.$$

Hence, using Lemma 2.2 (i),

$$\begin{aligned}\overline{\lim_{s\to 0}} \|u_1-C\| &\leqq \overline{\lim_{s\to 0}} \left\|\frac{G_s(fg)}{h(s)}-C\right\| + \overline{\lim_{s\to 0}} \left\|\frac{G_s(f(G_sf-g))}{h(s)}\right\| \\ &= \overline{\lim_{s\to 0}} \left\|\frac{G_s(f(G_sf-g))}{h(s)}\right\| \\ &\leqq K \overline{\lim_{s\to 0}} \|G_sf-g\|=0.\end{aligned}$$

$\overline{\lim}_{s\to 0} \|v_1\|<\infty$ follows immediately from (A).

Next we assume (i) and (ii) are valid for n. Then using Lemma 2.2 (ii),

$$\begin{aligned}\overline{\lim_{s\to 0}} \|u_{n+1}-C^{n+1}\| &\leqq \overline{\lim_{s\to 0}} \left\|\frac{G_sf(G_sf(u_n-C^n))}{h(s)}\right\| \\ &\quad + \overline{\lim_{s\to 0}} \left\|C^n\left(\frac{G_sfG_sf}{h(s)}-C\right)\right\| \\ &\leqq K \overline{\lim_{s\to 0}} \|u_n-C^n\| + |C|^n \overline{\lim_{s\to 0}} \|u_1-C\|=0,\end{aligned}$$

and

$$\overline{\lim_{s\to 0}} \|v_{n+1}\| \leqq K \overline{\lim_{s\to 0}} \|v_n\|.$$

Now the induction is completed. Q.E.D.

As an easy corollary of Lemma 2.3, we obtain the following;

Lemma 2.4. *For each* $x\in E$,

$$\lim_{s\to 0}\frac{1}{h(s)^{n/2}}\overbrace{G_s f(G_s f(\cdots(G_s f)\cdots))}^{n}(x)=\frac{1+(-1)^n}{2}\cdot C^n,$$

$$n=1,\ 2,\dots .$$

We assumed in this section (A)~(D). However we remark that Lemma 2.4 is of course valid if we assume (D′) instead of (D). The proof turns out to be easier, so the details are omitted.

§3. Proof of the Main Theorems

Throughout this section we fix $x_0\in E$ and $E_{x_0}\{\cdot\}$, $P_{x_0}\{\cdot\}$ are denoted simply $E\{\cdot\}$, $P\{\cdot\}$ respectively. Now changing the variable we have

$$\text{(3.1)}\qquad s\int_0^\infty e^{-st}E\left\{\left(\int_0^t f(X_\tau)d\tau\right)^2\right\}dt$$

$$=2!\int G_s(x_0,\ x_1)f(x_1)\nu(dx_1)\int G_s(x_1,\ x_2)f(x_2)\nu(dx_2).$$

Hence Lemma 2.4 provides us with

$$\text{(3.2)}\qquad \lim_{s\to 0}\frac{s}{h(s)}\int_0^\infty e^{-st}E\left\{\left(\int_0^t f(X_\tau)d\tau\right)^2\right\}dt=2C.$$

Notice that the left-hand side is of course nonnegative and consequently C is nonnegative. (3.2) can be generalized easily as follows;

$$\text{(3.3)}\qquad \lim_{s\to 0}\frac{s}{\{Ch(s)\}^{n/2}}\int_0^\infty e^{-st}E\left\{\left(\int_0^t f(X_\tau)d\tau\right)^n\right\}dt=\frac{1+(-1)^n}{2}n!,$$

(see [1]).

Since the integrand is not necessarily increasing, we cannot apply Tauberian theorem even if $h(s)$ varies regularly. So we have to make a detour if we want to evaluate the asymptotic behaviour of the moments of $\int_0^t f(X_\tau)d\tau$.

Let T be a nonnegative random variable which is independent of X such that $P\{T>x\}=e^{-x}$. Then we can rewrite (3.3) as follows;

$$(3.4)\qquad \lim_{s\to 0} E\left\{\left(\frac{1}{\sqrt{Ch(s)}}\int_0^{T/s} f(X_\tau)d\tau\right)^n\right\}=\frac{1+(-1)^n}{2}n!.$$

The right-hand side of (3.4) gives the n-th moment of the bilateral exponential distribution which belongs to the determinate case. Therefore (3.4) implies

$$(3.5)\qquad \lim_{s\to 0} P\left\{\frac{1}{\sqrt{Ch(s)}}\int_0^{T/s} f(X_\tau)d\tau<x\right\}=\frac{1}{2}\int_{-\infty}^{x} e^{-|y|}dy.$$

Consequently,

$$(3.6)\qquad \lim_{s\to 0} P\left\{\left|\frac{1}{\sqrt{Ch(s)}}\int_0^{T/s} f(X_\tau)d\tau\right|<x\right\}=1-e^{-x},\ x>0.$$

We next introduce another process;

$$(3.7)\qquad M_t=g(X_t)+\int_0^t f(X_\tau)d\tau, \qquad t\geqq 0,$$

where $g(x)$ is the bounded function defined in Lemma 2.1. Taking in mind that $G_s f(x)\to g(x)$ (the convergence being dominated by a positive constant), we see that M_t is a martingale. Since $g(x)$ is bounded, (3.5) and (3.6) provide us with

$$(3.8)\qquad \lim_{s\to 0} P\left\{\frac{1}{\sqrt{Ch(s)}}M_{T/s}<x\right\}=\frac{1}{2}\int_{-\infty}^{x} e^{-|y|}dy,$$

and

$$(3.9)\qquad \lim_{s\to 0} P\left\{\frac{1}{\sqrt{Ch(s)}}|M_{T/s}|<x\right\}=1-e^{-x}.$$

It also follows from (3.4) that the moment of arbitrary order of $\frac{1}{\sqrt{Ch(s)}}M_{T/s}$ is bounded. Hence we obtain,

$$\lim_{s\to 0} E\left\{\left(\frac{1}{\sqrt{Ch(s)}}M_{T/s}\right)^n\right\}=\frac{1+(-1)^n}{2}n!,$$

$$\lim_{s\to 0} E\left\{\left|\frac{1}{\sqrt{Ch(s)}}M_{T/s}\right|^n\right\}=n!,$$

or equivalently,

$$\lim_{s\to 0}\int_0^\infty e^{-t}E\Big\{\Big(\frac{1}{\sqrt{Ch(s)}}M_{t/s}\Big)^n\Big\}dt=\frac{1+(-1)^n}{2}n!,$$

$$\lim_{s\to 0}\int_0^\infty e^{-t}E\Big\{\Big|\frac{1}{\sqrt{Ch(s)}}M_{t/s}\Big|^n\Big\}dt=n!.$$

Changing the variables, we have,

$$\lim_{s\to 0}\frac{s}{\{Ch(s)\}^{n/2}}\int_0^\infty e^{-st}E(M_t^n)dt=\frac{1+(-1)^n}{2}n!, \tag{3.10}$$

$$\lim_{s\to 0}\frac{s}{\{Ch(s)\}^{n/2}}\int_0^\infty e^{-st}E(|M_t|^n)dt=n!,\ n=1,2,\dots. \tag{3.11}$$

Consequently we also have,

$$\lim_{s\to 0}\frac{s}{\{Ch(s)\}^{n/2}}\int_0^\infty e^{-st}E\{|M_t|^n+M_t^n\}dt=\frac{3+(-1)^n}{2}n!. \tag{3.12}$$

Since M_t is a martingale, both $E\{|M_t|^n\}$ and $E\{|M_t|^n+M_t^n\}$ are non-decreasing in t $(n=1,2,\dots)$. Hence, if $h(s)=s^{-\alpha}L(1/s)$, we can apply the Karamata's Tauberian theorem. By (3.11) and (3.12) we obtain,

$$\lim_{t\to\infty}\frac{1}{\Big\{Ch\Big(\frac{1}{t}\Big)\Big\}^{n/2}}E\{|M_t|^n\}=n!/\Gamma\Big(\frac{\alpha n}{2}+1\Big), \tag{3.13}$$

$$\lim_{t\to\infty}\frac{1}{\Big\{Ch\Big(\frac{1}{t}\Big)\Big\}^{n/2}}E\{|M_t|^n+M_t^n\}=\frac{3+(-1)^n}{2}n!/\Gamma\Big(\frac{\alpha n}{2}+1\Big), \tag{3.14}$$

and consequently we have,

$$\lim_{t\to\infty}\frac{1}{\Big\{Ch\Big(\frac{1}{t}\Big)\Big\}^{n/2}}E\{M_t^n\}=\frac{1+(-1)^n}{2}n!/\Gamma\Big(\frac{\alpha n}{2}+1\Big),\quad n=1,2\dots. \tag{3.15}$$

The right-hand side of (3.15) is the n-th moment of the bilateral Mittag-Leffler distribution, which belongs to the determinate case. $0\leqq\alpha\leqq 1$ is rather trivial by Lemma 2.1 (iii). Thus the proof of Theorem 1 is completed.

We next prove Theorem 2. We need little modification to the proof of Theorem 2 in [1]. Since $u(t)$ is nondecreasing, we can choose a nondecreasing function $\phi(t)$ with values in $[0, \infty]$ and a sequence $s_n \to 0$, so that

$$\frac{1}{\sqrt{Ch(s_n)}} u(t/s_n) \longrightarrow \phi(t), \quad n \longrightarrow \infty,$$

at each continuity point of $\phi(t)$. Then,

$$\begin{aligned}(3.16) \quad & \lim_{n\to\infty} \int_0^\infty e^{-t} P\left\{\frac{1}{\sqrt{Ch(s_n)}} \left|\int_0^{t/s_n} f(X_\tau) d\tau\right| < x\right\} dt \\ &= \lim_{n\to\infty} \int_0^\infty e^{-t} P\left\{\frac{u(t/s_n)}{\sqrt{Ch(s_n)}} \frac{1}{u(t/s_n)} \left|\int_0^{t/s_n} f(X_\tau) d\tau\right| < x\right\} dt \\ &= \int_0^\infty e^{-t} \tilde{G}(x/\phi(t)) dt, \qquad \text{a.e. } x > 0,\end{aligned}$$

with trivial conventions; $\tilde{G}\left(\frac{x}{0}\right) = 1$ and $\tilde{G}\left(\frac{x}{\infty}\right) = \tilde{G}(0)$ where $\tilde{G}(x) = G(x+0) - G(-x-0)$, $x \geqq 0$.

Now (3.6) and (3.16) provides us with

$$(3.17) \qquad \int_0^\infty e^{-t} \tilde{G}(x/\phi(t)) dt = 1 - e^{-x}.$$

letting $x \to \infty$, we have $\tilde{G}(x/\phi(t)) \to 1$, a.e.$t$. Since $\tilde{G}(0) < 1$ by the assumption, we obtain $\phi(t) < \infty$, $t > 0$. Similarly we also have $\phi(t) > 0$, $t > 0$. Darling and Kac [1] proved that (3.17) determines $\phi(t)$ uniquely, which implies

$$\lim_{s\to 0} \frac{u(t/s)}{\sqrt{Ch(s)}} = \phi(t).$$

This proves that $h(s)$ varies regularly for some exponent α. Since $0 \leqq \alpha \leqq 1$ is rather trivial as in the proof of Theorem 1, our assertion is now proved.

§4. Limit Theorems for Symmetric Stable Processes

Let X_t be an additive process on $\boldsymbol{R}^n$ $(n=1, 2)$ such that $E_0\{e^{i<\xi, X_t>}\} = e^{-t|\xi|^\alpha}$. We assume that X_t is recurrent; *i.e.* $1 \leqq \alpha \leqq 2$ if $n=1$, and $\alpha = 2$ if $n=2$. Green kernel $G(x, y)$ (with respect to Lebesgue measure) has the following representation;

$$G_s(x, y)=\begin{cases}\dfrac{1}{\pi}\displaystyle\int_0^\infty \frac{\cos(x-y)\xi}{s+\xi^\alpha}d\xi & \text{if } n=1,\ 1\leqq\alpha\leqq 2\\[2ex] \dfrac{1}{2\pi}\displaystyle\int_0^\infty \frac{\cos|x-y|\xi}{\sqrt{s+\xi^2}}d\xi & \text{if } n=2,\ \alpha=2\end{cases}$$

Therefore we have,

$$G_s(x, y)=h(s)+u(x, y)+\varepsilon(x, y; s)$$

where

$$h(s)=\begin{cases}\dfrac{1}{\alpha\sin(\pi/\alpha)}s^{1/\alpha-1} & \text{if } n=1<\alpha\leqq 2\\[2ex] \dfrac{1}{\pi}\log(1/s) & \text{if } n=\alpha=1\\[2ex] \dfrac{1}{4\pi}\log(4/s)+\gamma/2 & \text{if } n=\alpha=2\end{cases}$$

$$u(x, y)=\begin{cases}\dfrac{1}{2\cos(\pi\alpha/2)\Gamma(\alpha)}\dfrac{1}{|x-y|^{1-\alpha}} & \text{if } n=1<\alpha\leqq 2\\[2ex] \dfrac{1}{\pi}\log\dfrac{1}{|x-y|} & \text{if } n=\alpha=1\\[2ex] \dfrac{1}{2\pi}\log\dfrac{1}{|x-y|} & \text{if } n=\alpha=2\end{cases}$$

and $\varepsilon(x, y; s)$ converges to 0 uniformly on each compact set in $\boldsymbol{R}^1\times\boldsymbol{R}^1$ or $\boldsymbol{R}^2\times\boldsymbol{R}^2$. Let $f(x)$ be a bounded Borel function with compact support such that $\int f(x)dx=0$. Using a similar argument which we used in the example in section 1, we see that the assumptions (A)$\sim$(D′) are satisfied. Hence we obtain

Theorem 3.

$$\lim_{t\to\infty} P\left\{\frac{1}{\sqrt{Ch(1/t)}}\int_0^t f(X_s)ds<u\right\}=\tilde{g}_{\beta/2}(u)$$

where $\beta=\begin{cases}1-1/\alpha & \textit{if}\ \ n=1\leqq\alpha\leqq 2\\ 0 & \textit{if}\ \ n=\alpha=2\end{cases}$

and $C=\iint u(x, y)f(x)f(y)dxdy.$

§5. Limit Theorems for 1-Dimensional Diffusion Processes

Let $m(dx)$ be a nonnegative Radon measure on $\boldsymbol{R}^1$. Then we can obtain a generalized diffusion process X_t with the infinitesimal generator $\frac{d}{dm}\frac{d}{dx}$ from 1-dimensional Brownian motion by means of time change (see [5]). If the support of $m(dx)$ is an interval Q, then X_t becomes a diffusion process on Q with reflecting barrier when the boundary is finite.

Remark that X_t is a recurrent, conservative Markov process on $E=\operatorname{supp} m(dx)$.[1)]

Now let $\{\phi(x, \lambda), \psi(x, \lambda)\}$ be the system of the solutions of the following equations.

$$\phi(x, \lambda)=1-\lambda\int_{x_1}^{x}(x-y)\phi(y, \lambda)m(dy)$$

$$\psi(x, \lambda)=x-\lambda\int_{x_1}^{x}(x-y)\psi(y, \lambda)m(dy)-x_1, \qquad -\infty<x<\infty.$$

where $\int_y^x=\int_{[y,x)}$ if $y<x$, and $=-\int_{[x,y)}$ if $x<y$.

Then it is well known that the following hold for each $s>0$.

$$1\leqq\phi(x, -s)\leqq e^{s\sigma(x)}$$

$$|\psi(x, -s)|\leqq|x-x_1|e^{s\sigma(x)}$$

where $\sigma(x)=\int_{x_1}^{x}(x-\xi)m(d\xi)$.

We next define $h_i(s)$, $i=1, 2$.

$$h_i(s)=\lim_{x\to(-1)^i\infty}(-1)^i\frac{\psi(x, -s)}{\phi(x, -s)}, \ s>0, i=1, 2.$$

Then $u_i(x, s)=\phi(x, -s)-(-1)^i\psi(x, -s)/h_i(s)$, $i=1, 2$ are positive solutions of $\left(\frac{d}{dm}\frac{d}{dx}-s\right)u=0$; $u_1(\cdot, s)$ is nondecreasing and $u_2(\cdot, s)$ nonincreasing.

1) We assume that E contains at least two points, say x_1, x_2, $(x_1>x_2)$.

Green kernel of X_t with respect to $m(dx)$ is given by the following;

$$G_s(x, y)=\begin{cases} h(s)u_1(x, s)u_2(y, s) & \text{if } x \leqq y \\ h(s)u_2(x, s)u_1(y, s) & \text{if } y<x \end{cases}$$

where $h(s)=\left(\frac{1}{h_1(s)}+\frac{1}{h_2(s)}\right)^{-1}$.

Notice that $\lim_{s\to 0} sh(s)=m(-\infty, \infty)^{-1}$.

Lemma 5.1.

$$|G_s(x, y)-G_s(x, x_1)| \leqq 2\{sh(s)\sigma(y)+|y|\}e^{2s\sigma(y)} \qquad s>0,\ x,\ y \in \boldsymbol{R}^1$$

Proof. Since we have the explicit representation of $G_s(x, y)$, it is not difficult to prove the assertion. Q. E. D.

By Lemma 5.1, it is easy to see that the assumption (A) is satisfied if $f(x)$ fulfils the following;

(A′) $f(x)$ is a bounded Borel function with compact support such that $\int f(x)m(dx)=0$ but $f(x)\not\equiv 0$ m-a.e.

In order to prove (D′) is satisfied, we need further assumption.

(D″) The limit $\theta=\lim_{s\to 0} h(s)/h_1(s)$ exists.

Remark that (D″) is valid whenever $m(-\infty, x_1)<\infty$. In fact, $\theta= m(-\infty, x_1)/m(-\infty, \infty)$.

Lemma 5.2. *Let*

$$u(x, y)=-(x\vee y)-\theta\cdot(x+y)+(\sigma(x)+\sigma(y))/m(-\infty, \infty).$$

Then $\varepsilon(x, y; s)=G_s(x, y)-h(s)-u(x, y)\to 0$, $s\to 0$, *the convergence being uniform on each compact set of* $\boldsymbol{R}^1\times\boldsymbol{R}^1$.

This result is obtained with H. Watanabe and will be published elsewhere.

With $h(s)$ and $u(x, y)$ given above, we can see that (B)~(D′) are satisfied.

Finally we need conditions for the regular variation of $h(s)$. But using the results in [4], we easily obtain that if $m(dx)$ satisfies the following condition (R), then $h(s)$ varies regularly at 0 with exponent $-\alpha$.

(R) $m(-x, x)$ varies regularly at ∞ with exponent $1/\alpha-1$ $(0\leqq\alpha\leqq 1)$ and satisfies one of the following conditions.

(R.1) $m[0, x)\sim cm(-x, 0)$ as $x\to\infty$, with some positive constant c.

(R.2) $\lim_{x\to\infty}\frac{m(-\lambda x, 0)}{m[0, x)}=0[\text{or }\infty]$ for each $\lambda>0$.

We remark that if (R) is satisfied then (D″) is also satisfied. Therefore we obtain the following;

Theorem 4. *If* (A′) *and* (R) *are satisfied, then for each* $x\in E$

$$\lim_{t\to\infty} P_x\left\{\frac{1}{\sqrt{Ch(1/t)}}\int_0^t f(X_s)\,ds<u\right\}=\tilde{g}_{\alpha/2}(u)$$

where $C=-\frac{1}{2}\iint|x-y|f(x)f(y)m(dx)m(dy)$.

Example.

If $m[0, x)\sim x^{\beta}$ as $x\to\infty$ and $m(-x, 0)\sim x^{\gamma}c^{x}$ as $x\to\infty$ for some nonnegative constants β and γ, then

$$h_1(s)\sim\log(1/s)\qquad(s\downarrow 0)$$

$$h_2(s)\sim\text{const}\cdot s^{-1/(1+\beta)}\qquad(s\downarrow 0)$$

and consequently,

$$h(s)\sim\log(1/s)\qquad(s\downarrow 0).$$

Therefore we obtain

$$\lim_{t\to\infty} P_x\left\{\frac{1}{\sqrt{C\log t}}\int_0^t f(X_s)\,ds<u\right\}=\frac{1}{2}\int_{-\infty}^{u}e^{-|y|}dy$$

where $C=-\frac{1}{2}\iint|x-y|f(x)f(y)m(dx)m(dy)$.

In case $m(-\infty, 0)=0$, matters become clearer. As an easy corollary of Theorem 4 in [4], we have the following;

Theorem 5. *Suppose* (A′) *is satisfied. If* $m[0, x)$ *varies regularly at* ∞ *with exponent* β $(0\leqq\beta\leqq\infty)$ *then for each* $x\in E$,

$$\lim_{t\to\infty} P_x\left\{\frac{1}{\sqrt{Ch(1/t)}}\int_0^t f(x_s)\,ds<u\right\}=\tilde{g}_{1/(2(\beta+1))}(u).$$

Theorem 6. *Suppose* (A′) *is satisfied. If there exists a non-degenerate distribution function* $G(u)$ *such that*

$$\lim_{t\to\infty} P_x\left\{\frac{1}{u(t)}\int_0^t f(X_s)ds<u\right\}=G(u) \quad \text{a.e.}\, u$$

holds for some nondecreasing function $u(t)\uparrow\infty$, *then* $m[0, x)$ *varies regularly at* ∞ *with some exponent* β $(0\leqq\beta\leqq\infty)$. *Hence* $G(u)=\tilde{g}_{\frac{1}{2}(\beta+1)^{-1}}(bu)$ *with appropriate constant* b.

Appendix

The distribution function of Mittag-Leffler distribution of order α $(0\leqq\alpha<1)$ is given by

$$g_\alpha(x)=\frac{1}{\pi\alpha}\int_0^x \sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j!}\sin\pi\alpha j\Gamma(\alpha j+1)y^{j-1}dy,\ x>0$$

and the moments of this distribution are given by $k!/\Gamma(\alpha k+1)$, $k=0, 1, 2,\ldots$, which belongs to the determinate case.

Bilateral Mittag-Leffler distribution of order α $(0\leqq\alpha<1)$ is the distribution the moment of which is $\frac{1+(-1)^k}{2}\,\frac{k!}{\Gamma(\alpha k+1)}$, which also belongs to the determinate case. Hence it is easy to see that the distribution function of bilateral Mittag-Leffler distribution is given by

$$\tilde{g}_\alpha(x)=\frac{1}{2\pi\alpha}\int_{-\infty}^x \sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j!}\sin\pi\alpha j\Gamma(\alpha j+1)\,|y|^{j-1}dy.$$

We remark that for the special case of $\alpha=1/2$ we obtain

$$\tilde{g}_{1/2}(x)=\frac{1}{2\sqrt{\pi}}\int_{-\infty}^x e^{-y^2/4}dy.$$

and of $\alpha=0$,

$$\tilde{g}_0(x)=\frac{1}{2}\int_{-\infty}^{x} e^{-|y|}dy.$$

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