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Wave Front Sets and Singular Supports of Convolutions

by

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§ 1. Introduction

In the theory of convolution equations one uses the equation

$$(1\cdot 1)\qquad \text{co sing supp } (T*S) = \text{co sing supp } T * \text{co sing supp } S$$

where T and S are distributions with compact support, sing supp means the singular support and co means the convex hull. However this is only true under additional conditions, e.g. if supp T or supp S consists of a finite number of points [4] or if T is the characteristic function of a compact convex polyhedron [1], but if T is the characteristic function of a ball there is a distribution $S\in\mathcal{E}'(\boldsymbol{R}^n)$ such that (1.1) does not hold [1]. This occurs also if T is the characteristic function of a convex compact set with smooth boundary [2]. We will give a necessary condition on T such that (1.1) holds for any $S\in\mathcal{E}'(\boldsymbol{R}^n)$, from which follows that (1.1) does not hold for some S if T is the characteristic function of a compact set with smooth boundary (not necessarily convex). A counterexample shows that our condition is not sufficient. Details will appear elsewhere.

§ 2. Convolutions and wave front sets

We need the notion of the wave front set of a distribution introduced by Hörmander [5]; for details we refer to this paper.

Definition 2.1. $WF(T)=\bigcap\{\gamma(A); AT\in\mathscr{C}^\infty(\boldsymbol{R}^n)\}$ *where A runs over all properly supported pseudodifferential operators of order O*

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and $\gamma(A)$ denotes the characteristic set of A.

$WF(T)$ is a closed conical set in $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ whose projection on the first coordinate is the singular support of T. It can be calculated by Fourier transform in the following way.

Proposition 2.2. *$(x, \xi) \notin WF(T)$ iff there is a function $\varphi \in \mathcal{D}(\mathbb{R}^n)$ with $\varphi(x) \neq 0$ such that for any integer N there exists a constant C_N with*

$$\widehat{\varphi T}(\eta) \leqq C_N(1+|\eta|)^{-N}$$

for any η in a conic neighborhood of ξ.

Examples: 1) Let Ω be an open set with smooth boundary, T the characteristic function of Ω. The singular support of T is $\partial\Omega$, and one sees easily that the fibre of $WF(T)$ over x consists of the vectors tn, n the normal to $\partial\Omega$ at x and $t \neq 0$ real.

2) Let T be the distribution defined by

$$T(\varphi) = \lim_{y_1 \to 0+} \int \frac{\varphi(x_1, \cdots, x_n)}{x_1 + iy_1} dx_1 \cdots dx_n \quad \varphi \in \mathcal{D}(\mathbb{R}^n)$$

We have $WF(T) = \{(x, \xi); x_1 = 0, \ \xi = t(1, 0 \cdots 0), t > 0\}$.

The first example can be calculated by means of deriving T in directions tangential to $\partial\Omega$ at x and applying the definition, the second follows from Proposition 2.2. It follows also from [3]. With these preparations it is now easy to prove

Proposition 2.3. *Let T,S be distributions with compact support then we have for the convolution $T*S$*

$$(2\cdot1) \qquad WF(T*S) \subset \{(x+y, \xi) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}); \\ (x, \xi) \in WF(T), (y, \xi) \in WF(S)\}.$$

This is a consequence of Theorem 2.5.14 [5], but can also be proved directly in a simple way by using Proposition 2.2 and a suitable partition of unity.

From this we deduce now our necessary condition

Proposition 2.4. *Let T be a distribution with compact support such that the equation* $(1\cdot 1)$ *holds for any distribution S with compact support, then the fibre of $WF(T)$ over every extreme point x of cosing supp T has to be all of $\boldsymbol{R}^n\setminus\{0\}$.*

If there is a $\xi\in\boldsymbol{R}^n\setminus\{0\}$ with $(x,\xi)\notin WF(T)$, we can suppose $\xi=(1,0\cdots 0)$. Take S to be equal to ψS_1 where S_1 is the distribution from Example 2 and $\psi\in\mathcal{D}(\boldsymbol{R}^n)$, $\psi(0)\neq 0$. The inclusion (2.1) shows that $(x,\eta)\notin WF(T*S)$ for all $\eta\in\boldsymbol{R}^n\setminus\{0\}$ so $x\notin$ sing supp $T*S$.

Corollary 2.5. *If T is the characteristic function of an open bounded set with smooth boundary then there is a distribution $S\in\mathcal{E}'(\boldsymbol{R}^n)$ such that* $(1\cdot 1)$ *does not hold.*

Corollary 2.6. *Let T, S be distributions with compact support for which* (1.1) *holds, then for every extreme point $z\in co$ sing supp $(T*S)$, there are extreme points $x\in co$ sing supp T, $y\in co$ sing supp S and $\xi\in\boldsymbol{R}^n\setminus\{0\}$ such that $(x,\xi)\in WF(T)$, $(y,\xi)\in WF(S)$.*

§ 3. A counterexample

Let $\{\eta_k\}$ be a sequence of points which is dense in the unit sphere of $\boldsymbol{R}^n$, $t_j=\exp 2j$, $E=\{\eta\in\boldsymbol{R}^n; |\eta|=\exp(2j+1), j=1,2\cdots\}$ and $\xi_j^k=t_j\eta_k$. Theorem 5.2 of [4] shows that there is a continuous function f_k with support in the unit ball such that sing supp $f_k=\{0\}$ and $|\xi_j^k|\hat{f}_k(\xi_j^k)$ does not converge to 0 when $j\to\infty$ but

$$\sup\{|\hat{f}_k(\zeta)||\xi|^N; \xi\in E, \zeta\in\boldsymbol{C}^n, |\zeta-\xi|\leqq m\ \log|\xi|\}<\infty$$

so that f_k is not slowly decreasing. By a Baire category argument one shows that there is a continuous function f with compact support, sing supp $f=\{0\}$ such that $|\xi_j^k|\hat{f}(\xi_j^k)$ does not converge to 0 for $j\to\infty$, which means that $(0,\eta_k)\in WF(f)$ for all k, and such that $\hat{f}$ is not slowly decreasing. Since wave front sets are closed $WF(f)=\{(0,\xi);\xi\in\boldsymbol{R}^n\setminus\{0\}\}$ and since $\hat{f}$ is not slowly decreasing, there is a distribution S with compact support, $S\notin\mathcal{D}(\boldsymbol{R}^n)$ such that $f*S\in\mathcal{D}(\boldsymbol{R}^n)$.

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