

Reflection of C^∞ Singularities for a Class of Operators With Multiple Characteristics

by

Jacques CHAZARAIN*

Abstract

Let P , a partial differential operator with real principal part and constant multiplicity characteristics. We have shown in some previous works [1], [2] that several results about operators with simple characteristics can be generalized to these operators with multiple characteristics if we add an hypothesis, named Levi's condition, on the lower order terms. For instance, there is still propagation along bicharacteristics for the singularities of distributions u such that $Pu \in C^\infty(\Omega)$; if moreover the principal part of P is hyperbolic, the Cauchy problem is well posed in the C^∞ setting.

In this paper, we shall give a generalization to such operators of the theorem of Lax-Nirenberg [6] concerning the reflection at the boundary $\partial\Omega$ of the singularities of distributions u satisfying $Pu \in C^\infty(\bar{\Omega})$. In order to get a more precise result of regularity up to the boundary, we shall differ from Nirenberg's proof in the details, but the principle will be the same. That is, we factorise micro locally P in an elliptic and two hyperbolic factors satisfying the Levi condition; then the proof of the theorem will be reduce to the study of micro local regularity results for elliptic and hyperbolic boundary problems.

§ 1. Statement of the Theorem of Reflection

First, we have to introduce some notations and definitions. Let $P(y, D_y)$ a classical pseudo-differential operator of degree m , we denote its principal symbol $\sigma_m(p)$ by the corresponding small letter $p(y, \eta)$. Let, (y^0, η^0) a point of the characteristic variety $p^{-1}(0)$ of p . The hypothesis (\mathcal{L}_1) of real constant multiplicity is given by the

Definition 1. p has constant multiplicity, say s , near (y^0, η^0) if there exist a real symbol $\tilde{p}(y, \eta)$ with simple characteristics such that

$$p(y, \eta) = [\tilde{p}(y, \eta)]^s \text{ near } (y^0, \eta^0).$$

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* Nice University. Department of Mathematics, Parc Valrose, Nice.

We call \tilde{p} the reduce part of p near (y^0, η^0) ; the bicharacteristic of p through (y^0, η^0) is defined as those of \tilde{p} . Now, we can formulate the Levi condition (\mathcal{L}_2)

Definition 2. p satisfies the Levi condition at (y^0, η^0) if given a solution φ near y^0 of

$$\tilde{p}(y, d\varphi(y)) = 0, d\varphi(y^0) = \eta^0,$$

and a C_0^∞ function α such that $d\varphi(y) \neq 0$ on the support of α , we have for large λ

$$e^{-i\lambda\varphi} P(y, D_y) (\alpha e^{i\lambda\varphi}) = O(\lambda^{m-s})$$

This is, in fact, a condition on the terms of order $> m-s$ of P (see Chazarain [1], [2] for more details). We denote by (\mathcal{L}) the union of the conditions (\mathcal{L}_1) and (\mathcal{L}_2) . The theorem of reflection is local, even micro-local, so it will be sufficient to state it in the case where Ω is the half space $\mathbf{R}_+^{n+1} = \{(t, x) | t > 0\}$, we call (τ, ξ) the dual variables.

We shall often consider pseudo-differential operators $P(t, x, D_t, D_x)$ which are differential in t , that means precisely that

$$P = \sum_{j=0}^m P_j(t, x, D_x) D_t^{m-j}$$

with P_j a classical pseudo-differential operator of degree $\leq j$ varying smoothly with $t \geq 0$. The hyperplane $t=0$ is said non characteristic for P if the symbol $p_0(0, x, \xi)$ is a function of x alone and is never zero. Consider an operator of this type and assume it satisfies (\mathcal{L}) in $T^*\bar{\mathbf{R}}_+^{n+1} = (\bar{\mathbf{R}}_+ \times \mathbf{R}^n) \times \mathbf{R}^{n+1}$. Given $(x^0, \xi^0) \in T^*\mathbf{R}^n \setminus 0$, we call τ_1, \dots, τ_k , the real roots in τ of

$$p(0, x^0, \tau, \xi^0) = 0$$

and s_1, \dots, s_k their multiplicities. The bicharacteristics C_1, \dots, C_k of P through the points $(0, x^0; \tau_j, \xi^0)$ $j=1, \dots, k$ are called the reflected bicharacteristics from (x^0, ξ^0) . We assume they hint transversally the hyperplane $t=0$, that means, using (\mathcal{L}_1) , that the τ_j are simple roots of the reduce part \tilde{p}_j of p near $(0, x^0; \tau_j, \xi^0)$. Therefore, there exists a conical neighbourhood $U \times \Gamma$ of (x^0, ξ^0) and a $T > 0$, such that the roots τ_j are

smooth functions $\tau_j(t, x, \xi)$ in $[0, T] \times U \times \Gamma$.

We can now state the main theorem, we use Hörmander [4] notation WF for the C^∞ singular spectrum.

Theorem 1. (Reflection of the regularity). *Let $P(t, x, D_t, D_x)$ a pseudodifferential operator of degree m , differential in t , which satisfies the condition (\mathcal{L}) in $T^*\bar{\mathbf{R}}_+^{n+1}$ and such that $t=0$ is non characteristic. Let $\lambda_0 = (x^0, \xi^0) \in T^*\mathbf{R}^n \setminus 0$, we assume that the reflected bicharacteristics C_1, \dots, C_k from λ_0 hit transversally the hyperplane $t=0$. Given an integer $k_0 \in [0, k]$ we define the integers $m_1 = s_1 + \dots + s_{k_0}$, $m_2 = s_{k_0+1} + \dots + s_k$, $m' = m - m_1 - m_2$. Let a distribution $u \in C^\infty(\bar{\mathbf{R}}_+; \mathcal{D}'(\mathbf{R}^n))$ which satisfies the following conditions:*

- (1) $Pu = f \in C^\infty(\bar{\mathbf{R}}_+^{n+1})$
- (2) $WF(u)$ does not meet the bicharacteristics C_1, \dots, C_{k_0}
- (3) $\lambda_0 \notin WF(\gamma_0^j u) \quad j=0, \dots, m_2 + \frac{m'}{2} - 1$
(with $\gamma_{t_0}^j u = D_{t_0}^j u|_{t=t_0}$).

Then, it follows that

- (4) all the bicharacteristics C_1, \dots, C_k do not meet $WF(u)$
- (5) for every $j \geq 0$ $\lambda_0 \notin WF(\gamma_0^j u)$.

In fact, we have also a much more precise result: there exists a pseudodifferential operator $a(x, D_x)$ elliptic at λ_0 and a $T > 0$ such that

$$(6) \quad a(x, D_x)u(t, x) \in C^\infty([0, T] \times \mathbf{R}^n).$$

The last affirmation of the theorem is a result of regularity up to the boundary. In fact, information on $u|_{\mathbf{R}_+^{n+1}}$ and on the traces $\gamma_0^j u$ give nothing concerning the regularity up to the boundary. For example, it is easy to verify, that the function

$$u(t, x) = \exp i \left(x - \frac{1}{t} \right)^2, \quad t > 0$$

is in $C^\infty(\bar{\mathbf{R}}_+; \mathcal{D}')$ and $u|_{\mathbf{R}_+^{n+1}} \in C^\infty(\mathbf{R}_+^{n+1})$, $\gamma_0^j u = 0$ $j \geq 0$, but nevertheless

$$u \in C^\infty(\bar{\mathbf{R}}_+^{n+1}).$$

In order to describe easily the regularity up to the boundary, we introduce a closed set in $T^*\mathbf{R}^n \setminus 0$ that we call the “boundary singular spectrum” ∂WF .

Definition 3. *Let a distribution $u \in C^\infty(\bar{\mathbf{R}}_+; \mathcal{D}'(\bar{\mathbf{R}}^n))$, we say that $\lambda_0 \in \partial WF(u)$ if there exists a pseudo-differential operator $a(x, D_x)$ elliptic at λ_0 and a $T > 0$ such that*

$$a(x, D_x)u(t, x) \in C^\infty([0, T] \times \mathbf{R}^n).$$

The example above shows that $\partial WF(u)$ has nothing to do with the closure of $WF(u)$ in $T^*\bar{\mathbf{R}}_+^{n+1}$.

It is clear from the definition that $\lambda_0 \in \partial WF(u)$ implies

$$(7) \quad \lambda_0 \in \partial WF(D_t^j u), \quad \lambda_0 \in \partial WF\left(\int_0^t u(s, \cdot) ds\right);$$

in that case, we remark also that we have $\lambda_0 \in \partial WF_{t_0}(u)$ for $t_0 < T$ if ∂WF_{t_0} denote the boundary singular spectrum relative to the half space

$$\{(t, x) \mid t > t_0\}.$$

We can give another definition using the Fourier transform in the x variable:

Proposition 1. *Let $u \in C^\infty(\bar{\mathbf{R}}_+; \mathcal{D}')$, we have $(x^0, \xi^0) \in \partial WF(u)$ if there exists a conical neighbourhood $U \times \Gamma$ of (x^0, ξ^0) and a $T > 0$ such for every $\alpha \in C_0^\infty(U)$, $k \in \mathbf{N}$, $N \in \mathbf{R}$, there exists C such that*

$$(8) \quad |\mathcal{F}_x(\alpha \cdot D_t^k u)(t, \xi)| \leq C(1 + |\xi|)^{-N}$$

for every $\xi \in \Gamma$ and $t \in [0, T]$.

The proof is a simple but a little tedious exercise.

Using this new characterisation, we see immediately that $(x^0, \xi^0) \in \partial WF(u)$ implies the existence of “conic box” $\Gamma(T, U)$ such that

$$(9) \quad WF(u) \cap \Gamma(T, U) = \emptyset,$$

where $U \times \Gamma$ is conical neighbourhood of (x^0, ξ^0) , $T > 0$, and the conic

box is by definition:

$$\Gamma(T, U) = \left\{ (t, x; \tau, \xi) \left| \begin{array}{l} t \in [0, T], \quad x \in U \\ \xi \in \Gamma, \quad \tau \in \mathbf{R} \end{array} \right. \right\}$$

In particular, a conic box is a neighbourhood of every point of the form $(0, x^0; \tau, \xi^0)$ with $\tau \in \mathbf{R}$.

We remark also that the boundary singular spectrum describe the regularity up to the boundary; for instance, it is easy to show that $u(t, x)$ is C^∞ up to the boundary near x^0 iff $(x^0, \xi) \notin \partial WF(u)$ for every ξ .

§ 2. Proof of Theorem 1

Coming back to the theorem 1, we see that the conclusion (6) means exactly $\lambda_0 \notin \partial WF(u)$. The conclusion (4) is a trivial consequence of (6) and, using the theorem of propagation of singularities for such operators (Chazarain [2]), the conclusion (5) follows also from (6) and (9). So it remains to prove (6); to do so we shall see during the proof that it is sufficient to assume $\lambda_0 \notin \partial WF(f)$ at the place of (1).

The first ingredient of the proof is a generalization to our case of a result of micro-local factorization of Nirenberg [6].

Proposition 2. *Let P , satisfies the hypothesis of Theorem 1. Then there exists a conical box $\Gamma(T, U)$ and pseudo-differential operators H_1, H_2, Q, R of degree m_1, m_2, m', m and differential in t , such that*

$$(10) \quad P = H_1 \cdot Q \cdot H_2 + R$$

(11) *the condition (\mathcal{L}) is satisfied by H_1 and H_2 in the box $\Gamma(T, U)$ and their principal symbol are given by*

$$h_1 = \prod_{j=1}^{k_0} (\tau - \tau_j(t, x, \xi))^{s_j}, \quad h_2 = \prod_{j=k_0+1}^k (\tau - \tau_j)^{s_j}$$

(12) *the remainder R has the form*

$$R = \sum_{j=0}^{m-1} R_j(t, x, D_x) D_t^j, \text{ and the complete symbol of } R_j \text{ is in } S^{-\infty}([0, T] \times U; \Gamma)$$

the operator Q is elliptic in the box $\Gamma(T, U)$.

Proof. It is sufficient to give the proof in the case where p has only one real root $\tau_1(t, x, \xi)$ for $(t, x, \xi) \in [0, T] \times U \times \Gamma$, let s its multiplicity. We can write

$$p = (\tau - \tau_1(t, x, \xi))^s \cdot q(t, x, \tau, \xi)$$

with q a polynomial of degree $m - s$ in τ such that

$$(13) \quad q(t, x, \tau_1(t, x, \xi), \xi) \neq 0 \text{ is } [0, T] \times U \times \Gamma.$$

By a transposition argument it will be also sufficient to prove a factorization of the form

$$P = H_1 \cdot Q + R.$$

We write the complete symbols of P , H_1 and Q in the form

$$P(t, x, \tau, \xi) = p(t, x, \tau, \xi) + \sum_0^{m-1} (\tau - \tau_1)^j a_j(t, x, \xi)$$

$$H_1(t, x, \tau, \xi) = (\tau - \tau_1)^s + \sum_0^{s-1} (\tau - \tau_1)^j b_j(t, x, \xi)$$

$$Q(t, x, \tau, \xi) = q(t, x, \tau, \xi) + \sum_0^{m-s-1} (\tau - \tau_1)^j c_j(t, x, \xi)$$

where a_j , b_j , c_j are classical symbols of degree $m - 1 - j$, $s - 1 - j$, $m - s - j - 1$, with an asymptotic expansion

$$a_j \sim \sum_{k \leq 0} a_{j,k}, \quad b_j \sim \sum b_{j,k}, \quad c_j \sim \sum c_{j,k}$$

in homogeneous components. To satisfy the condition (12), we have to calculate the symbols $b_{j,k}$ and $c_{j,k}$.

Equating the components of degree $m - 1$ for H_1 , Q and P in the box $\Gamma(T, U)$, we get the equation

$$\begin{aligned} & \left[\sum_{j=0}^{s-1} (\tau - \tau_1)^j b_{j,0} \right] \cdot q + D_t q - \sum_j \frac{\partial \tau_1}{\partial \xi_j} D_j q \\ & + (\tau - \tau_1)^s \left[\sum_{j=0}^{m-s-1} (\tau - \tau_1)^j c_{j,0} \right] = \sum_{j=0}^{m-1} (\tau - \tau_1)^j a_{j,0}. \end{aligned}$$

We make $\tau = \tau_1$, so we obtain $b_{0,0}(t, x, \zeta)$ using (13); after having derivate in τ , we make again $\tau = \tau_1$ so we get $b_{1,0}$ and so on up to $b_{s-1,0}$. Then, we keep alone, in the left hand side, the term

$$(\tau - \tau_1)^s \left[\sum_0^{m-s-1} (\tau - \tau_1)^j c_{j,0} \right].$$

By the construction of the $b_{j,0}$, we know that the right hand side is a polynomial in τ , divisible by $(\tau - \tau_1)^s$. After division, we obtain $c_{j,0}$ by identification. The same kind of procedure can be repeated to obtain the other symbols $b_{j,k}$ and $c_{j,k}$ in the box $\Gamma(T, U)$. Then, we extend them outside $\Gamma(T, U)$ as symbols of the same degree.

We define R by the equality

$$P = H_1 \cdot Q + R$$

and by construction it satisfies (12). It remains to show that H_1 satisfies the Levi condition (\mathcal{L}_2) at points of $\Gamma(T, U)$.

For this purpose, we consider a convenient solution $\varphi(t, x)$ of

$$\partial_t \varphi - \tau_1(t, x, \partial_x \varphi) = 0$$

and a smooth solution $\alpha(t, x)$ with small support. From the equality

$$e^{-i\lambda\varphi} P(\alpha e^{i\lambda\varphi}) = e^{-i\lambda\varphi} (H_1 \cdot Q)(\alpha e^{i\lambda\varphi}) + O(\lambda^{-\infty})$$

we deduce

$$e^{-i\lambda\varphi} H_1(\alpha_1 e^{i\lambda\varphi}) = O(\lambda^{m-s})$$

with

$$\alpha_1 = e^{-i\lambda\varphi} Q(\alpha e^{i\lambda\varphi}).$$

From (13), we have $\alpha_1 \sim \lambda^{m-s} q(t, x, \partial_t \varphi, \partial_x \varphi)$;

so we deduce

$$e^{-i\lambda\varphi} H_1(\alpha_1 e^{i\lambda\varphi}) = O(\lambda^0)$$

from which it follows easily that (\mathcal{L}_2) is satisfied by H_1 .

The next ingredients in the proof of theorem 1 are results of micro-local regularity for non-strict hyperbolic Cauchy problems and Dirichlet boundary problems.

Theorem 2. (Hyperbolic case). *Let $H(t, x, D_t, D_x)$ an operator which satisfies the hypothesis of theorem 1 in a box $\Gamma(T, U)$. We assume moreover that h is hyperbolic in that box, that is to say*

$h(t, x, \tau, \xi) = 0$ has only real roots in τ for $(t, x, \xi) \in [0, T] \times U \times \Gamma$.

Consider $u \in C^\infty(\bar{\mathbf{R}}_+; \mathcal{D}')$, we pose

$$(14) \quad Hu = f, \quad \gamma_0^j = v_j \quad j=0, \dots, m-1.$$

and we assume

$$(15) \quad (x^0, \xi^0) \notin \partial WF(f), \quad (x^0, \xi^0) \notin WF(v_j) \quad j=0, \dots, m-1.$$

Then

$$(16) \quad (x^0, \xi^0) \notin \partial WF(u).$$

The proof will be given in the next paragraph.

Theorem 3. (Elliptic case). *Let $Q(t, x, D_t, D_x)$ a pseudo-differential operator of degree $m' = 2d$, differential in t , and elliptic in a conic box $\Gamma(T, U)$. Consider $u \in C^\infty(\bar{\mathbf{R}}_+; \mathcal{D}')$, we pose*

$$(17) \quad Qu = f \quad \gamma' u = w \quad \text{with} \quad \gamma' u = (\gamma_0 u, \dots, \gamma_{d-1} u)$$

and we assume

$$(18) \quad (x^0, \xi^0) \notin \partial WF(f), \quad (x^0, \xi^0) \notin WF(w),$$

Then

$$(19) \quad (x^0, \xi^0) \notin \partial WF(u).$$

the proof will be given also in the next paragraph.

Now we shall finish the proof of theorem 1, it will be a simple calculation of WF and ∂WF assuming theorem 2 and theorem 3. We shall proceed in three steps, each step corresponds to factor of the factorization (10) of P .

First step. We define

$$(20) \quad v = QH_2 u \quad \text{and} \quad f_1 = f - Ru,$$

using (1) we can write

$$(21) \quad H_1 v = f_1.$$

With these notations, we have the

Lemma 1. *There exists a conical box $\Gamma(T, U)$ such that for every $k \geq 0$ and $t \in [0, T]$*

$$(22) \quad WF(\gamma_t^k v) \cap (U \times \Gamma) = \phi.$$

Proof of the Lemma. The property of the singular spectrum (cf. Hörmander [4]) and (20), (21) implies the inclusions

$$(23) \quad WF(v) \cap WF(u) \subset h_1^{-1}(0) \cup WF(f_1).$$

On the other hand, if the roots $\tau_j(t, x, \xi)$ are defined for $(t, x, \xi) \in [0, T] \times U \times \Gamma$, the theorem of traces [4] shows that, for $t \in [0, T]$,

$$(24) \quad (U \times \Gamma) \cap WF(\gamma_t^k v) \subset \{(x, \xi) \mid (t, x, \tau_j(t, x, \xi), \xi) \in WF(u) \\ \text{for some } j \in [1, k]\}.$$

Taking $\Gamma(T, U)$ small enough, the property (12) and the hypothesis $(x^0, \xi^0) \notin \partial WF(f)$ imply

$$(25) \quad (U \times \Gamma) \cap WF(\gamma_t^k f_1) = \phi \quad \text{for } t \in [0, T], \quad k \geq 0.$$

Combining (23), (24), (25) and (11) we obtain

$$(26) \quad (U \times \Gamma) \cap WF(\gamma_t^k v) \subset \{(x, \xi) \mid (t, x, \tau_j(t, x, \xi), \xi) \in WF(u) \\ \text{for some } j \in [1, k_0]\}.$$

But the hypothesis (2) allows us to find a box $\Gamma(T, U)$ such that, for $j \in [1, k_0]$,

$$(27) \quad \{(t, x; \tau_j(t, x, \xi), \xi) \mid (x, \xi) \in U \times \Gamma, t \in [0, T]\} \cap WF(u) = \phi.$$

So, the Lemma follows from (26) and (27).

It is easy to see that $(x^0, \xi^0) \notin \partial WF(f_1)$, so using (22), we can apply the theorem 2 to the backward Cauchy problem for time $t \leq T$

$$H_1 v = f_1 \quad (x^0, \xi^0) \notin \gamma_x^j v \quad j = 0, \dots, m_1 - 1$$

and we deduce, with the remark (7), that

$$(28) \quad (x^0, \xi^0) \notin \partial WF(v).$$

Second step. We define

$$(29) \quad w = H_2 u, \quad h_j = \gamma_0^j w$$

so w is a solution of the following Dirichlet problem

$$(30) \quad \begin{cases} Qw = v \\ \gamma_0^j w = h_j \quad j=0, \dots, \frac{m'}{2}-1. \end{cases}$$

As the degree of H_2 is m_2 , we deduce from (3) that $(x^0, \xi^0) \notin WF(h_j)$ for $j=0, \dots, \frac{m'}{2}-1$.

On the other hand, the proposition 2 shows that Q is elliptic in a box $\Gamma(T, U)$, so using also (28) the theorem 3 implies that

$$(31) \quad (x^0, \xi^0) \notin \partial WF(w).$$

Third step. The distribution u satisfies the Cauchy problem

$$(32) \quad \begin{cases} H_2 u = w \\ \gamma_0^j u = k_j \quad j=0, \dots, m_2-1. \end{cases}$$

With (3) and (31), the theorem 2 implies finally that

$$(x^0, \xi^0) \notin \partial WF(u),$$

the theorem 1 is proved.

Remark. There is no new difficulties to generalize to our case, the theorem of Majda and Osher [5] concerning more general boundary problems. In return, the generalization of the result of Taylor [7] to systems with constant multiplicities seems to be more difficult, due to the lack of a nice Levi condition for systems.

§ 3. Proofs of Theorems 2 and 3

Proof of theorem 2. We shall first reduce to the case where H satisfies the hypothesis of theorem 1 in all $T^*\bar{\mathbf{R}}_+^{n+1}$. To do so, we extend smoothly the roots $\tau_j(t, x, \xi)$ to $\bar{\mathbf{R}}_+^{n+1} \times \mathbf{R}^n \setminus 0$ keeping them distincts and homogeneous in ξ . We define an operator M by

$$M(t, x, D_t, D_x) = (D_t - \tau_1(t, x, D_x))^{s_1} \dots (\dots)^{s_k}$$

it satisfies by construction the hypothesis of theorem 1 in $T^*\bar{\mathbf{R}}_+^{n+1}$. Then, we obtain a prolongation \tilde{H} of H by taking

$$\tilde{H} = \alpha(x, D_x) \cdot H + (I - \alpha) \cdot M,$$

where $\alpha(x, \xi)$ is a non-negative symbol identical to one in a neighbourhood $U' \times I'$ of (x^0, ξ^0) and with "compact" conic support in $U \times I$. So \tilde{H} and H coincide in a box $I'(T, U')$ and \tilde{H} satisfies the hypothesis in all $T^* \bar{\mathbf{R}}^{n+1}$. We define $\tilde{f} = f + (\tilde{H} - H)u$, then we are reduced to the same problem for \tilde{H}

$$(33) \quad \tilde{H}u = \tilde{f} \quad \gamma_0^j u = v_j \quad j=0, \dots, m-1$$

It is clear, from the construction of \tilde{H} , that

$$(34) \quad (x^0, \xi^0) \notin \partial WF(\tilde{f}).$$

To simplify the notations, we skip the tilda in the rest of the proof. From the theory of Cauchy problem for such operator (Chazarain [1]), we know that that (33) has an unique solution u . We decompose it as

$$u = u_1 + u_2$$

with u_1 and u_2 defined as solutions of

$$\begin{cases} Hu_1 = 0 \\ \gamma_0^j u_1 = v_j \quad j=0, \dots, m-1 \end{cases} \quad \begin{cases} Hu_2 = f \\ \gamma_0^j u_2 = 0 \quad j=0, \dots, m-1 \end{cases}$$

We begin by u_1 . It is sufficient to consider the case where

$$v_j = 0, \quad j=0, \dots, m-2 \quad \text{and} \quad v_{m-1} = v.$$

We have proved in the reference above, that u_1 can be expressed in term of v with Fourier integral operator; in particular for $t \geq 0$ small enough, we have

$$(35) \quad u_1(t, x) = \sum_{j=1}^m \int e^{i\varphi_j(t, x, \xi)} d_j(t, x, \xi) \hat{v}(\xi) d\xi.$$

The phase φ_j is defined as the solution of

$$\partial_t \varphi_j - \tau_j(t, x, \partial_x \varphi_j) = 0 \quad \varphi_j(0, x, \xi) = x \cdot \xi,$$

and the amplitude d_j is a symbol of degree $s_j - m$ which satisfies some transport equation.

To prove that $(x^0, \xi^0) \notin \partial WF(u_1)$, we have to consider a pseudo-differential operator $a(x, D_x)$; whose symbol is supported in a conic neighbourhood $U_1 \times I_1$ of (x^0, ξ^0) to be chosen latter and elliptic at that point.

From (35), we obtain

$$(36) \quad a(x, D)u_1(t, x) = \sum_{j=1}^m \int_{\mathbf{R}^n} e^{i\varphi_j(t, x, \xi)} e_j(t, x, \xi) \widehat{v}(\xi) d\xi$$

with

$$e_j(t, x, \xi) = e^{-i\varphi_j} a(x, D) (d_j e^{i\varphi_j}).$$

It is well known that e_j has the following expansion

$$(37) \quad e_j(t, x, \xi) \sim \sum_{\alpha \geq 0} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} a(x, \partial_x \varphi) D_v^{\alpha} (d_j e^{i\rho_j}) |_{v=x}$$

with

$$\rho_j(t, x, y, \xi) = \varphi_j(t, x, \xi) - \varphi_j(t, y, \xi) - \langle \partial_x \varphi(t, x, \xi), x - y \rangle.$$

By hypothesis $(x^0, \xi^0) \notin WF(v)$, so we can find $U \times \Gamma$ small such that $(U \times \Gamma) \cap WF(v) = \emptyset$. Now we can choose $U_1 \times \Gamma_1$ and $T > 0$ in order that $(x, \xi) \in U \times \Gamma$ implies $(x, \partial_x \varphi_j(t, x, \xi)) \in U_1 \times \Gamma_1$ for every $t \in [0, T]$, this possibility follows from the fact that $\partial_x \varphi_j(0, x, \xi) = \xi$. With this choice it is easy to verify, using (37), that we have

$$(38) \quad a(x, D)u_1(t, x) \in C^{\infty}([0, T] \times \mathbf{R}^n).$$

Using the Duhamel principle, the solution u_2 is given by

$$(39) \quad u_2(t, x) = \sum_1^m \int_0^t \int_{\mathbf{R}^n} e^{i\varphi_j(t, s, x, \xi)} d_j(t, s, x, \xi) \widehat{f}(s, \xi) d\xi ds$$

where the variable s in φ_j and d_j means that they are relative to the Cauchy problem with initial data at time $t=s$, and $\widehat{f}(s, \xi) = (\mathcal{F}_x f(s, \cdot))(\xi)$. We pose

$$(40) \quad w_j(t, s, x) = \int_{\mathbf{R}^n} e^{i\varphi_j(t, s, x, \xi)} d_j(t, s, x, \xi) \widehat{f}(s, \xi) d\xi.$$

Using the same kind of arguments as for u_1 , we can find a $T > 0$ and pseudodifferential operator $a(x, D_x)$ elliptic at (x^0, ξ^0) such that

$$(41) \quad a(x, D_x)w_j(t, s, x) \in C^{\infty}([0, T] \times [0, T] \times \mathbf{R}^n).$$

After an integration of (41) with respect to $s \in [0, T]$ we obtain from (39), (40), (41)

$$(42) \quad a(x, D_x)u_2(t, x) \in C^{\infty}([0, T] \times \mathbf{R}^n).$$

Finally, $(x^0, \xi^0) \in \partial WF(u)$ follows from (38), (42) and the theorem 2 is proved.

Proof of the theorem 3. The hypothesis (18) on the data implies the existence of a pseudo-differential operator $a(x, D_x)$ elliptic at (x^0, ξ^0) , with support in $U \times \Gamma$, such that

$$(43) \quad \begin{cases} a(x, D_x)f = g \in C^\infty([0, T] \times \mathbf{R}^n) \\ a(x, D_x)w_j = h_j \in C^\infty(\mathbf{R}^n) \quad j=0, \dots, d-1. \end{cases}$$

After composition of (17) on the left by $a(x, D)$ we obtain

$$(44) \quad \begin{cases} Q_1 u = g \\ \gamma_1' u = h \end{cases}$$

with $Q_1 = a.Q$ and $\gamma_1' = a.\gamma'$. As $a(x, D)$ is elliptic at (x^0, ξ^0) , the problem (44) is still elliptic in a small box $\Gamma'(T', U')$. From the theory of the Dirichlet problem, we know that the application

$$Q: u \longrightarrow (Q_1 u, \gamma_1' u)$$

has a micro-local left parametrix

$$\mathcal{I}: (g, h) \longrightarrow u.$$

More precisely, we have

$$\mathcal{I}.Q = I + \mathcal{R}$$

where \mathcal{R} is a pseudo-differential operator whose complete symbol is rapidly decreasing in a smaller box $\Gamma''(T'', U'')$. We do not recall here the construction of this parametrix, it is done with the help of the Calderon projector, as in Hörmander [3], but the calculations of symbols are performed only in the box $\Gamma'(T', U')$.

We compose (44) on the left with \mathcal{I} , we obtain

$$(45) \quad u = \mathcal{I}(g, h) + \mathcal{R}u.$$

As g and h are smooth, we have

$$(46) \quad \mathcal{I}(g, h) \in C^\infty([0, T'] \times \mathbf{R}^n),$$

and the smoothness property of \mathcal{R} implies that

$$(47) \quad (x^0, \xi^0) \notin \partial WF(\mathcal{R}u).$$

Combining (45), (46), (47) we obtain finally

$$(x^0, \xi^0) \notin \partial WF(u).$$

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