*Publ. RIMS, Kyoto Univ.*
**12** Suppl. (1977), 69-88.

# Sojourn Times and Asymptotic Properties of the Scattering Matrix

by

V. GUILLEMIN

## § 1. Introduction

In scattering theory one usually has the following set-up or some simple variant of it. One has a Hilbert space $H$ and two one-parameter groups of unitary operators $\mathcal{U}$ and $\mathcal{U}_0$ on $H$. One supposes that the wave operators

$$W^{\pm}=\lim_{t\to\pm\infty}\mathcal{U}_0(-t)\mathcal{U}(t)$$

and their inverses exist and are, therefore, unitary operators from $H$ to $H$ intertwing $\mathcal{U}$ and $\mathcal{U}_0$. the scattering operator is $S=W^+(W^-)^{-1}$, and it intertwines $\mathcal{U}_0$ with itself.

Now in the cases of interest for scattering theory $\mathcal{U}_0$ has a uniform continuous spectrum. This means there exists a Hilbert space $K$ and an isomorphism of Hilbert spaces $\rho: H\to L^2(\boldsymbol{R}, K)$ such that $\rho\mathcal{U}_0(t)\rho^{-1}$ is the operator "multiplication by $e^{it\sigma}$" for $\sigma\in\boldsymbol{R}$. Since $S$ commutes with $\mathcal{U}_0$, $\rho S\rho^{-1}$ commutes with multiplication by $e^{it\sigma}$ for all $t$, and must then necessarily be of the form "multiplication by $S(\sigma)$" where $S(\sigma): K\to K$ is for each $\sigma\in\boldsymbol{R}$ a unitary operator. The subject of this talk will be the asymptotic behavior of $S(\sigma)$ for large values of $\sigma$. Our purpose will be to examine this asymptotic behavior in special cases and attempt to discern some general pattern.

## § 2. A simple example

Let $X$ be a compact manifold and $\vec{v}$ a vector field on $X\times\boldsymbol{R}$ which has positive $\boldsymbol{R}$ component and is equal to $\partial/\partial t$ outside $\{(x,t), a<t<b\}$.

Received April 22, 1976.
* Department of Mathematics, M. I. T., Cambridge, Mass. 02139, U.S.A.

Let $f_s: X\times \boldsymbol{R}\to X\times \boldsymbol{R}$ be the flow associated with $\vec{v}$. Let $H$ be the Hilbert space of $L^2$ half-densities on $X\times\boldsymbol{R}$. Then $f_s$ induces on $H$ a one-parameter group of unitary operators $\mathcal{U}(s): H\to H$. Let $\mathcal{U}_0$ be the corresponding group for $\partial/\partial t$. Then $\mathcal{U}_0(-t)\mathcal{U}(t)$ is independent of $t$ for large $t$, so the wave operators exist. Moreover, they are of the form $(f^\pm)^*$ where $f^\pm: X\times\boldsymbol{R}\to X\times\boldsymbol{R}$ are diffeomorphisms. It follows that the scattering operator is of the form $g^*$ where $g=(f^-)^{-1}f^+$. $g$ commutes with the group of translations $(x, t)\to(x, t+q)$; therefore, it must be of the form

$$(x, t)\to(h(x), t+T(x))$$

where $h:X\to X$ is a diffeomorphism and $T$ a smooth function on $X.h$ and $T$ can be computed as follows. For $x\in X$ and $t_0<a$ follow the trajectory of the point $(x, t_0)$ with respect to the vector field $\vec{v}$. For $t_1>b$ this trajectory will be a line parallel to the $t$-axis with $x$-coordinate $h(x)$. (See figure 1).

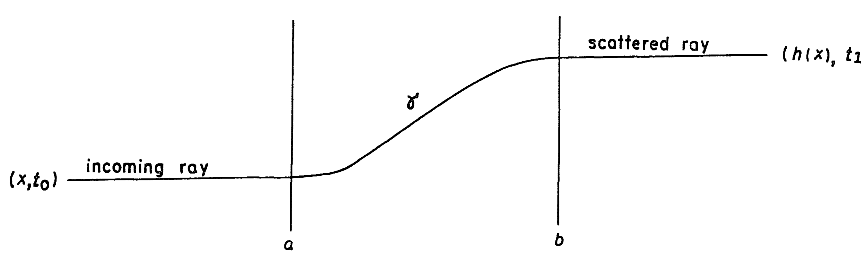


Figure 1.

Moreover $T(x)+t_1-t_0$ will be the time taken to go from $(x, t_0)$ to $(h(x), t_1)$ along the curve $\gamma$. We will call $T(x)$ the "sojourn time" of $\gamma$.

Let $K=L^2(X)$. A spectral representation of $\mathcal{U}_0$ is given by the Fourier transform

$$H=L^2(X\times\boldsymbol{R})=L^2(\boldsymbol{R}, K)\xrightarrow{A_t}L^2(\boldsymbol{R}, K)$$

which converts $\mathcal{U}_0$ into multiplication by $e^{it\sigma}$, and the scattering operator into

$$S(\sigma)=e^{i\sigma T}h^*. \tag{2·1}$$

Note that *the oscillatory part of* (2·1) *determines the sojourn times* *of the scattered rays.*

## § 3. Acoustical scattering: opaque obstacles

Let $G$ be a compact convex subregion of $\boldsymbol{R}^n$. Let $H_G$ be the Hilbert space completion of the space of all pairs $(f, g)$, $f,g\in C_0^\infty(\boldsymbol{R}^n-G)$ with respect to the energy norm

$$\int(|\nabla f|^2+|g|^2)\,dx\,.$$

The mixed boundary problem

$$\frac{\partial^2}{\partial t^2}u-\Delta u=0,\ u\in L^2(\boldsymbol{R}^n-G),\ u\equiv 0 \text{ ou } \partial G;$$

has a unique solution with Cauchy data $u(x,0)=f$, $\partial u/\partial t(x,0)=g$, $(f,g)\in H$. Denote the Cauchy data at time $t$ of this solution by $\mathcal{U}_G(t)(f,g)$. One easily verifies that $\mathcal{U}_G(t): H\rightarrow H$ is unitary. If $G$ is the empty set, we will denote the corresponding Hilbert space and unitary group by $H_0$ and $\mathcal{U}_0$. Since $H_0\supset H_G$ the operators

$$\mathcal{U}_0(-t)\mathcal{U}_G(t) \tag{3·1}$$

make sense as linear transformations from $H_G$ to $H_0$. It turns out that if the dimension $n$ is odd, the limits $W^\pm$ of (3·1) as $t$ tend to $\pm\infty$ and their inverses exist, and hence, so does the scattering operator $S=W^+(W^-)^{-1}$. The infinitesimal generator of $\mathcal{U}_0$ is the operator

$$\begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}. \tag{3·2}$$

Let $K=L^2(S^{n-1})$. Given any function $f\in K$ we can associate with it a generalized eigenfunction of $\Delta$ of eigenvalue $\lambda$, namely

$$\int f(\omega)e^{i(x\cdot\omega)\sqrt{\lambda}}d\omega\,. \tag{3·3}$$

It is not hard, using (3·2), (3·3) and the Fourier inversion formula, to construct a spectral resolution

$$\rho\colon H\rightarrow L^2(\boldsymbol{R}, K)$$

of $\mathcal{U}_0$ (See Lax-Phillips [5]). Therefore the scattering operator is a unitary operator

$$S(\sigma): L^2(S^{n-1}) \to L^2(S^{n-1})$$

for each $\sigma \in \boldsymbol{R}$. Given $(\omega, \theta) \in S^{n-1} \times S^{n-1}$ we will denote by $S(\sigma, \omega, \theta)$ the Schwartz kernel of this operator. Our main result is

**Theorem 1.** *For $\omega \neq \theta$ $S(\sigma, \omega, \theta)$ is a smooth function of all three variables. Moreover for fixed $\omega \neq \theta$*

$$S(\sigma, \omega, \theta)\left(\frac{\sigma}{2\pi i}\right)^{(1-n)/2} = J^{-1/2} e^{iT\sigma} + O(\sigma^{-1}) \tag{3·4}$$

*where $T$ is the sojourn time of the unique scattered ray with direction of incidence $\omega$ and direction of reflection $\theta$ and $J$ is the scattering differential cross-section at $(\omega, \theta)$.*

We must explain the last two terms. Let $a$ be a large positive number so that the ball of radius $a$ contains the region $G$. Since $G$ is convex there exists one point $x \in G$ such that $\omega$ and $-\theta$ make equal angles, both less than 90°, with $\vec{n}(x)$. (See figure 2.)

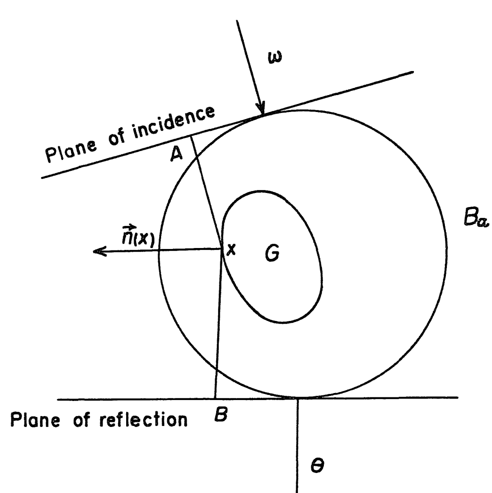


Figure 2.

Let $\gamma$ be the union of the line segments joining $A$ to $x$ and $x$ to $B$. We will call $\gamma$ the *scattered ray* with angle of incidence $\omega$ and angle of scattering $\theta$. The length of $\gamma$ minus the normalizing factor $2a$ will be called the *sojourn time of* $\gamma$. Finally consider the map of the plane of incidence in figure

2 into $S^{n-1}$ which maps $A'$ to $\omega'$ as follows. For $A'$ near $A$ on the plane of incidence go along the incident ray from $A'$ with direction $\omega$ until it hits $G$ at a point $x'$ near $x$. Let $\omega'$ be the direction of the reflected ray at $x'$. The map $A'\to\omega'$ is well-defined (for $A'$ near $A$) and is differentiable. Its Jacobian at $A$ is by definition the *differential cross-section* at $\gamma$.

## § 4. Acoustical Scattering: refracting media

Theorem 1 is due to Andrew Majda. (See [8].) It is stated in a rather different form in [8], but can be converted to our form without too much effort. (See appendix A below.) The proof is rather difficult, because of the problems posed by "glancing rays". Rather than attempt to outline it here, we will sketch the proof of a similar result for refracting media, for which one doesn't encounter glancing rays.

Let $G=\sum G_{ij}dx_i dx_j$ be a Riemannian metric on $\boldsymbol{R}^n$ which is identical with the ordinary Euclidean metric except on some compact set. Let $\Delta_G$ be its Laplace-Beltrami operator. Let $H_G$ be the Hilbert space completion of the set of pairs $(f, g), f, g\in C_0^\infty(\boldsymbol{R}^n)$ with respect to the norm

$$\int_{\boldsymbol{R}^n}\left(\Sigma G_{ij}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}+|g|^2\right)dx$$

and let $\mathcal{U}_G$ be the one-parameter group of unitary operators on $H$ associated with the wave equation: $\partial^2/\partial^2 t-\Delta_G$. If $G$ is the standard Euclidean metric then $\mathcal{U}_G$ and $H_G$ are the $\mathcal{U}_0$ and $H_0$ defined in § 3. (Note that as topological vector spaces $H$ and $H_0$ are the same.) It turns out that, for odd dimensions, the wave operators

$$W^{\pm}=\lim_{t\to\pm\infty}\mathcal{U}_0(-t)\mathcal{U}_G(t)$$

and their inverses exist, just as for the Dirichlet problem. Therefore, the scattering operator also exists; and, just as in § 3, it induces a unitary transformation on the $\sigma$-th generalized eigenspace of $\mathcal{U}_0$

$$S(\sigma): L^2(S^{n-1})\to L^2(S^{n-1}).$$

Choose a number, $a$, so large that the metric $G$ is Euclidean outside of the ball of radius $a$. Let $\gamma$ be a geodesic which for large negative and positive times lies outside of $B_a$. (See figure 3.)

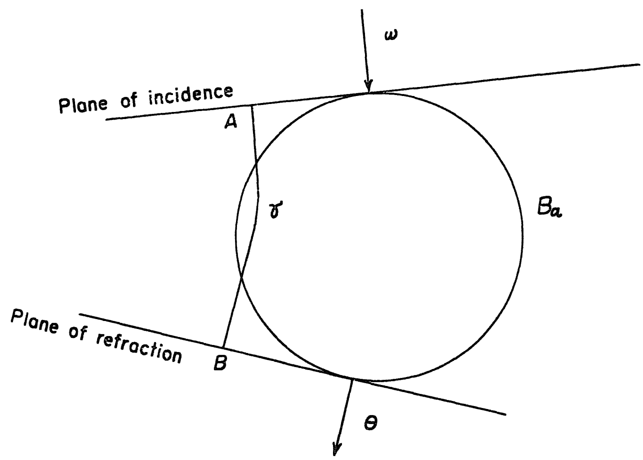


Figure 3.

Then $\gamma$ consists of two line segments with directions $\omega$ and $\theta$ plus a curved arc lying in $B_a$. We will call $\omega$ the *direction of incidence* of $\gamma$ and $\theta$ the *direction of scattering*. We will define the *sojourn time* of $\gamma$ to be the length of the arc joining $B$ to $A$ minus the normalizing factor $2a$. Geodesic flow induces a map from the plane of incidence into the $n-1$ sphere, and the Jacobian of this map at $A$ will be defined to be the *scattering cross-section* at $\gamma$. We will call $\gamma$ *non-degenerate* if the scattering cross-section is non-zero.

**Theorem 2.** *Let $\omega$ and $\theta \in S^{n-1}$ be fixed with $\theta \neq \omega$. Suppose there are only a finite number of rays $\gamma_1, \cdots, \gamma_N$ with direction of incidence $\omega$ and direction of scattering $\theta$, each $\gamma_i$ being non-degenerate. Then*

$$(4\cdot 1) \qquad S(\sigma, \omega, \theta)\left(\frac{\sigma}{2\pi i}\right)^{(i-n)/2} = \sum_{i=1}^{N} i^{n_i}|J_i|^{-1/2}e^{i\sigma T_i} + O(\sigma^{-1})$$

*where $T_i$ is the sojourn time of $\gamma_i$, $J_i$ the differential cross-section and $n_i$ the number of conjugate points, counted with multiplicity, along $\gamma_i$.*

The idea of the proof is to exhibit the wave operators $W_G^{\pm}$ as solutions of a hyperbolic partial differential equation. To do so we start with the spectral resolution $\rho: H_0 \to L^2(\boldsymbol{R}, K)$ of the unitary group $\mathcal{U}_0$. (See the previous section.) Denote by $\widehat{W}_G{}^+$ the composition of the following sequence of transformations

$$H \xrightarrow{W_G{}^+} H_0 \xrightarrow{\rho} L^2(\boldsymbol{R}, K) \xrightarrow{A_t} L^2(\boldsymbol{R}, K)$$

the last arrow being Fourier transform in the $\boldsymbol{R}$ variable. $\widehat{W}_G{}^+$ intertwines the one-parameter group $\mathcal{U}_G$ and the one-parameter group of translations on the real line. The infinitesimal generator of $\mathcal{U}_G$ is the operator

$$A_G = \begin{pmatrix} 0 & 1 \\ \Delta_G & 0 \end{pmatrix}$$

and the generator of the translation group is $\partial/\partial t$, so the Schwartz kernel of $\widehat{W}_G{}^+$ satisfies the equation

$$A_G \widehat{W}_G{}^+(x, t, \omega) - \frac{\partial}{\partial t} \widehat{W}_G{}^+(x, t, \omega) = 0 \tag{4·2}$$

for $x \in \boldsymbol{R}^n$, $\omega \in S^{n-1}$. Now when $G$ is the standard Euclidean metric, $W^+$ is the identity and $\widehat{W}_G{}^+ = \hat{\rho}$. It follows from Huygens' principle, that no matter what $G$ is, the Schwartz kernels of $\widehat{W}_G{}^+$ and $\hat{\rho}$ are equal when $t$ is large. However (4·2) is a hyperbolic equation, so its solution is uniquely determined by its values for large positive $t$. Pursuing this line of reasoning a little further one can show that $\widehat{W}_G{}^+$ is a Fourier integral operator, and write down quite explicitly its associated canonical relation and its leading symbol. Similar results hold for $\widehat{W}_G{}^-$; and the composition formula for Fourier integral operators given in Hörmander [4], § 4 shows that $\hat{S}$ is itself a sum of Fourier integral operators. Finally, having computed the top symbol of $\hat{S}$, which, as we've just intimated, is not too formidable a job, one easily obtains the result (4·1) on the asymptotic behavior of $S$ since $S$ is just the Fourier transform in $t$ of $\hat{S}$. The details of this computation can be found in [3].

## § 5. The automorphic wave equation

Our list of examples is still too small, and the example in § 2 too simple-minded, for us to draw any general conclusions from them. The general conclusion we would like to draw is that the periods of oscillation of the scattering matrix for $\sigma$ large are intimately related to the sojourn times of the scattered rays. In this section we will discuss an example which is quite different from the others; so the fact that we will be able to derive a formula for the scattering matrix which is formally identical with (3·4), makes our general conclusion a litte bit more plausible. This example is due to Faddeev, and for lack of time we will describe it in its barest outlines. A detailed and very readable account of it can be found in Lax-Phillips [6], and we will frequently refer to this paper below.

Let $H$ be the upper half-plane provided with its Poincaré metric, $ds^2=(dx^2+dy^2)/y^2$. Let $\Gamma$ be a discrete group of fractional linear transformations such that $X=H/\Gamma$ has finite area. For simplicity we'll assume $\Gamma$ contains no elliptic transformations. Then $X=H/\Gamma$ is a Rieman nian manifold which looks geometrically like a compact surface with a finite number of tentacles (cusps) attached. (See figure 4.)

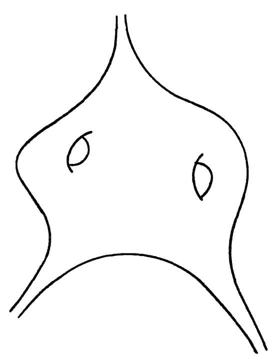

Figure 4.

By a theorem of Siegel ([9], Ch. 1), $X$ is the disjoint union of a compact subset $X_0=X_0^a$ and a finite number of open sets $X_i=X_i^a$, $i=1, \cdots, 1$, called *cusp neighborhoods*, such that each $X_i$ is isometric to the set $-1/2\leqq \mathrm{Re}\, z \leqq 1/2$, $\mathrm{Im}\, z>a$, in the upper half-plane (figure 5 below.)

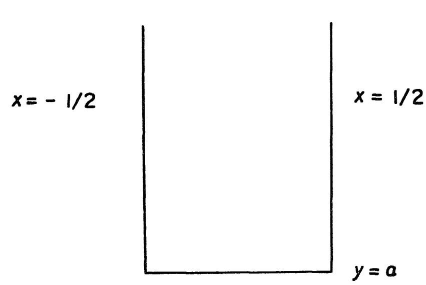


Figure 5.

(In this figure the lines $x=1/2$ and $x=-1/2$ are identified.) There is such a Siegel decomposition for every sufficiently large $a$, and, for fixed $a$, it is unique. We will assume a choice of suitable $a$ has been made from now on.

Consider the intrinsic Laplace-Beltrami operator, $\Delta$, on $X$ and its associated wave equation

$$(\partial^2/\partial t^2)u-\Delta u=0 \tag{5·1}$$

It is well-known that $\Delta$ has a large discrete spectrum, so the scattering techniques described in this paper won't apply per se to (5·1). However, if $H$ is the Hilbert space of Cauchy data on $X$ (with the same energy norm as in § 4) and $H_1$ the subspace of Cauchy data spanned by the proper eigenfunctions of $\Delta$, then in $H_1{}^\perp$ an appropriate scattering theory can be set up. Its vague outlines are as follows: Assume for simplicity that there is only one cusp. Then there is a fundamental domain $F$ for $\Gamma$ which looks like the region in figure 5, but with a broken curve consisting of circular arcs in place of the line $y=a$. (In figure 6 we've drawn the fundamental region for $\Gamma=SL(2, \mathbb{Z})$. The wave equation associated with figure 5 should be regarded as the "free" system and that associated with figure 6 as the "perturbed" system.)

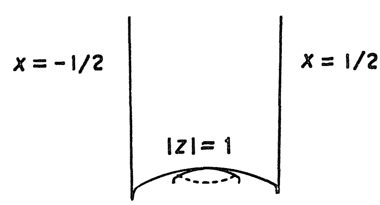


Figure 6.

The wave equation for $H/\Gamma$ is somewhat like the wave equation for an obstacle except that instead of having zero boundary data on the boundary of the obstacle as in § 3, one has periodic boundary data on the components of the boundary of $F$ which get identified by $\Gamma$. We won't go into details here, but refer the reader to [6], Ch. VI.

By a scattered geodesic or scattered ray we will mean a geodesic $\gamma=\gamma(t)$ which lies outside of $X_0$ both for all sufficiently negative $t$ and for all sufficiently positive $t$. If it lies in the exterior region $X_i$ for large negative $t$ and in the exterior region $X_j$ for large positive $t$, we will say that $\gamma$ is *scattered from the i-th cusp neighborhood into the j-th cusp neighborhood.* The sojourn time of $\gamma$ will be the total time elapsed from the time it first enters $X_0$ to the time it leaves $X_0$ forever. We will show in appendix B that there are only a countable number of scattered geodesics, $\gamma_j$, $j=1, 2, \cdots$, and that their sojourn times tend to infinity with $j$.

Lax-Phillips prove in [6] that the scattering matrix $S(\sigma)$ associated with the problem just described is an $m\times m$ unitary matrix for each $\sigma$, $m$ being the number of cusps. Our main result is the following explicit formula for the *i-jth* component of this matrix.

**Theorem 3.** *Let $T_1 \leq T_2 \leq \cdots$ be the sojourn times of the rays scattered from the ith cusp neighborhood into the j-th cusp neighborhood, and let $c(\sigma)$ be the function*

$$c(\sigma)=\int_{-\infty}^{\infty}\frac{d_q}{(1+q^2)^{1/2+i\sigma}}\,.$$

*Then*

$$S_{ij}(\sigma)=ac(\sigma)\sum e^{-T_k(\sqrt{-1}\sigma+1/2)}\,. \tag{5·2}$$

A couple remarks are in order.

1) The series (5·2) only converges for $\operatorname{Im}\sigma \leqq -3/2$. Therefore, the right hand side of (5·2) has to be understood as the meromorphic continuation of this series to the whole complex plane. One of the deep consequences of the scattering theory is that (5·2) can be so extended.

2) We can rewrite (5·2) in the form

$$S_{ij}(\sigma)=ac(\sigma)\sum J_k{}^{-1/2}e^{-\sqrt{-1}T_k}, \qquad J_k=e^{-T_k}.$$

The geodesic flow, in the neighborhood of $\gamma_k$, gives us a map from the boundary of the $i$-th cusp neighborhood to the boundary of the $j$-th cusp neighborhood. One can show that $J_k$ is the Jacobian, at $\gamma_k$ of this mapping (computed, of course, using the Siegel coordinates). This shows that the amplitude of the $k$-th oscillation in (5·2) is a "differential cross-section" just like the amplitude of the kth oscillation in formula (4·1).

We will give the proof of theorem 3 in the appendix B.

A last remark: We conjecture that if $X$ is an arbitrary Riemann surface of finite area (not necessarily of the form $H/\Gamma$) then there is a formula analogous to (5·2); however it is an *asymptotic* formula. We have been able to prove such a formula for the case when the cusp neighborhoods are isometric to those given by figure 5, the proof being a variant of the proof of theorem 2, sketched in § 4.

## Appendix A

Let $G$ be a smooth convex subset of $\boldsymbol{R}^n$ and let $\gamma_0$ be a ray reflected off $G$ with angle of incidence $\omega_0$ and angle of reflection $\theta_0$. Suppose $\gamma_0$ encounters the obstacle at the point $y_0$. Our purpose is to obtain a formula for the scattering differential cross-section at $(\omega_0, \theta_0)$ in terms of the Gaussian curvature $K(y_0)$ of the surface $\partial G$ at $y_0$:

**Theorem.** *Let $c_0/2$ be the cosine of the angle that $\gamma_0$ makes with the normal direction at $y_0$. Then the scattering differential cross-section at $(\omega_0, \theta_0)$ is equal to $4c_0^{n-3}K(y_0)$.*

*Proof.* Let us choose coordinates $(x_1, \cdots, x_n)$ in $\boldsymbol{R}^n$ such that $\omega_0$ is the unit vector pointing in the direction of the positive $x_n$-axis. Then the plane of incidence in figure 2 is just the plane $x_n = -1$. Let $x_0$ be the point where $\gamma_0$ intersects the plane of incidence. Consider, for each point $x$ on the plane of incidence near $x_0$, the ray with initial direction $\omega_0$ and initial position $x$. Let $y$ be the point where it is reflected off the obstacle and $\theta$ the direction in which it is reflected. By definition the differential cross-section is the Jacobian determinant at $x_0$ of the map $x \to \theta$. $\theta$ can be determined from $x$ by the set of equations

$$\theta = cn - \omega \tag{A·1}$$

$$\theta \cdot n = \omega \cdot n = c/2$$

$n$ being the unit outward normal at $y$. Here $\omega = \omega_0$ is fixed and $\theta$, $c$ and $n$ are functions of $x = (x_1, x_2, \cdots, x_{n-1}, -1)$. We will compute the Jacobian determinant, $J$, from the formula

(A·2) $$\frac{\partial\theta}{\partial x_1}\wedge\cdots\wedge\frac{\partial\theta}{\partial x_{n-1}}\wedge\theta=Jdx_1\wedge\cdots\wedge dx_n\,.$$

Making the substitution (A·1) we get for the left hand side of (A·2):

(A·3) $$c^n\left(\frac{\partial n}{\partial x_1}\wedge\cdots\wedge\frac{\partial n}{\partial x_{n-1}}\wedge n\right)-c^{n-1}\left(\frac{\partial n}{\partial x_1}\wedge\cdots\wedge\frac{\partial n}{\partial x_{n-1}}\wedge\omega\right)$$
$$-\sum_{i=1}^{n-1}\frac{\partial c}{\partial x_i}c^{n-2}\left(\frac{\partial n}{\partial x_1}\wedge\cdots\wedge n\wedge\cdots\wedge\frac{\partial n}{\partial x_{n-1}}\right)\wedge\omega$$

the "$n$" in the bottom line occuring in the i-th place. Let

(A·4) $$\omega=c/2n+a_1\frac{\partial n}{\partial x_1}+\cdots+a_{n-1}\frac{\partial n}{\partial x_{n-1}}\,.$$

Substituting (A·4) for $\omega$ in the second term in the top line of (A·3) we get for the whole top line of (A·3)

(A·5) $$c^n/2\left(\frac{\partial n}{\partial x_1}\wedge\cdots\wedge\frac{\partial n}{\partial x_{n-1}}\wedge n\right).$$

In the bottom line of (A·3) we can interchange $\omega$ and $n$ and write, for $\frac{\partial c}{\partial x_i}$, $2\omega\cdot\frac{\partial n}{\partial x_i}$, getting

(A·6) $$2\sum c^{n-2}\left(\frac{\partial n}{\partial x_1}\wedge\cdots\wedge\left(\omega\cdot\frac{\partial n}{\partial x_i}\right)\omega\wedge\cdots\wedge\frac{\partial n}{\partial x_{n-1}}\right)\wedge n\,.$$

Finally, making use of (A·4), (A·6) can be written as:

$$2c^{n-2}\sum a_i\left(\omega\cdot\frac{\partial n}{\partial x_i}\right)\left(\frac{\partial n}{\partial x_1}\wedge\cdots\wedge\frac{\partial n}{\partial x_{n-1}}\right)\wedge n$$

or

$$2c^{n-2}\omega\cdot(\omega-c/2n)\left(\frac{\partial n}{\partial x_1}\wedge\cdots\wedge\frac{\partial n}{\partial x_{n-1}}\wedge n\right)$$

or

$$2c^{n-2}\left(\frac{\partial n}{\partial x_1}\wedge\cdots\wedge\frac{\partial n}{\partial x_{n-1}}\wedge n\right)-c^n/2\left(\frac{\partial n}{\partial x_1}\wedge\cdots\wedge\frac{\partial n}{\partial x_{n-1}}\wedge n\right).$$

The second term cancels (A·5); so the upshot of our computation is that (A·3) is equal to

(A·7) $$2c^{n-2}\left(\frac{\partial n}{\partial x_1}\wedge\cdots\wedge\frac{\partial n}{\partial x_{n-1}}\wedge n\right).$$

Let $Hdx_1\wedge\cdots\wedge dx_{n-1}$ be the curvature form of the surface of the obstacle at the point on the surface corresponding to $(x_1, \cdots, x_{n-1})$. Then

$$Hdx_1\wedge\cdots\wedge dx_{n-1}\wedge dx_n=\frac{\partial n}{\partial x_1}\wedge\cdots\wedge\frac{\partial n}{\partial x_{n-1}}\wedge n \tag{A·8}$$

(by the formula for the curvature as the Jacobian of the Gauss map.) Now

$$H\,dx_1\wedge\cdots\wedge dx_{n-1}=K\,dA$$

where $K$ is the scalar curvature and $dA$ the area form on the surface. Moreover

$$dx_1\wedge\cdots\wedge dx_{n-1}=(n\cdot\omega)\,dA=(c/2)\,dA$$

since $\omega$ is the unit vector pointing along the positive $x_n$-axis. Thus $H=(2/c)K$. Substituting this into (A·7) and (A·2) we get $J=4c^{n-3}K$ as asserted. Q.E.D.

We mention one consequence of this theorem (pointed out by Majda in [8].) Take $\omega_0=-\theta_0$ in the theorem. This means that the ray $\gamma_0$ is reflected off the surface at $y_0$ in the normal direction; so $c=1$ and the scattering cross-section is equal to 4 times the curvature of the surface at $y_0$. Thus (3·4) becomes

$$\lim|S(\sigma, \omega, -\omega)|\left(\frac{\sigma}{2\pi i}\right)^{(1-n)/2}=(4K)^{-1/2}. \tag{A·9}$$

Now $\omega=n(y_0)$ is just the image of $y_0$ under the Gauss map, so we have proved that *the asymptotic behavior of the scattering matrix at* $(\omega, -\omega)$ *determines the Gaussian curvature of the surface* $\partial G$ *at the preimage of* $\omega$ *under the Gauss map.* By an old result of Hermann Weyl a convex surface is determined uniquely up to Euclidean motions by the values of the curvature at the pre-image points of the Gauss map; so this result can be restated as follows: *the asymptotic behavior of the scattering amplitude determines the shape of the scatterer.*

## Appendix B

This appendix is devoted to the proof of theorem 3. We begin by recalling the following familiar fact.

**Lemma.** *Geodesics on the upper half-plane consist either of*

a) *The half-lines* $\mathrm{Re}\, z=c$, $\mathrm{Im}\, z>0$

*or*

b) *the half-circles* $|z-c|=d$ *with* $c$ *and* $d$ *real,* $d$ *positive, and* $\mathrm{Im}\, z>0$.

(*See for example Spivak* [10], *pg* 430.)

**Corollary 1.** *Geodesics on the manifold* $X=H/\Gamma$ *are projections of half-lines of type a) or half-circles of type b).*

**Corollary 2.** *A scattered geodesic has the property that for large negative and positive times it corresponds to a vertical line in figure* 5 (*i.e. after we have mapped the appropriate cusp neighborhoods onto the standard cusp neighborhood exhibited in figure* 5.)

Now choose a fundamental domain for $\Gamma$ in $H$ such that the ith cusp, $v_i$, is at $\infty$ and $X_i$ is the standard cusp neighborhood defined by $y>a$, $-\frac{1}{2}\leqq x\leqq\frac{1}{2}$. The jth cusp $v_j=(x_0, 0)$ will then be one of the vertices of the fundamental domain lying on the real axis, and its cusp neighborhood will be bounded by two geodesics which are perpendicular to the $x$-axis at $x_0$. (See the figure below.)

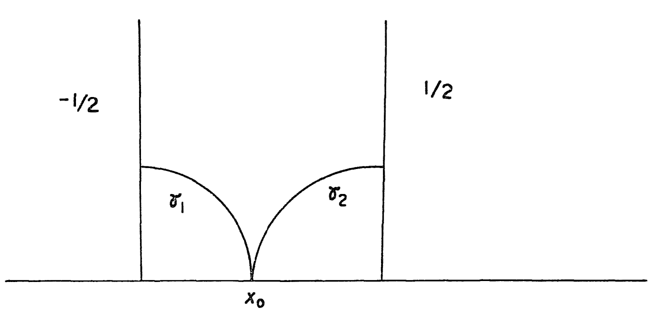


Figure 7.

The lines $x=-\frac{1}{2}$ and $x=\frac{1}{2}$ are, of course, identified by the parabolic transformation $z\to z+1$ which by assumption belongs to $\Gamma$, and the two bounding geodesics $\gamma_1$ and $\gamma_2$ are identified by a parabolic transformation, $T\in\Gamma$, having $v_j$ as a fixed point. We will assume that none of these four boundary curves represents a scattered ray.*

We will now describe a way of constructing scattered rays. The construction we're about to describe in fact gives all of them in a more or less unambigous way. Let $A \in \Gamma$. Let $\gamma$ be a geodesic on $H$ which is perpendicular to the real axis at $x_0$ and lies between $\gamma_1$ and $\gamma_2$ as in figure 8.

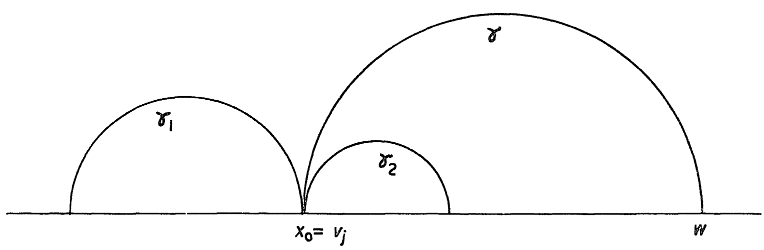


Figure 8.

Since $\gamma$ is a geodesic it is a half-circle with center on the $x$-axis. Theorefore, $A\gamma$ will also be a half-circle with center on the $x$-axis. Of course the center of $A\gamma$ can be at infinity; i.e. $A\gamma$ can be a straight line perpendicular to the $x$-axis. We claim that *if this is the case then projection of $\gamma$ onto the manifold $X=H/\Gamma$ is a scattered ray joining the ith to the jth cusp neighborhood.*

*Proof*: $\gamma$ and $A\gamma$ get identified on $X$. $Av_j$ is a point on the real axis which can't be $\infty$, since $\infty=v_i$ and $v_i \neq v_j$ mod $\Gamma$. Let $\gamma$ be parametrized so that $\gamma(-\infty)=v_j$ and $\gamma(+\infty)=w$. (See figure 8). Then $A\gamma(t)$ will tend to $\infty$ as $t$ tend to $+\infty$; and, by applying an appropriate iterate of the map $z \rightarrow z+1$ to it, $A\gamma(t)$ will tend to infinity along a straight line $l_c$: $x=c$, $-\frac{1}{2} \leqq c < \frac{1}{2}$. That is its image in $X=H/\Gamma$ will scatter into the ith cusp neighborhood as $t \rightarrow +\infty$. This proves the above claim.

Arguing backwards one can show that all scattered rays associated with the *j-i*th cusp neighborhoods can be constructed this way. Note, however, that not every $A \in \Gamma$ *will give rise to a scattered ray,* because

* It causes no essential problems if one of the bounding curves is a scattered ray but does slightly complicate the argument below.

it can happen that none of the geodesics, $A\gamma$, for $\gamma$ lying between $\gamma_1$ and $\gamma_2$ as in figure 4, are straight lines. Note also that *for a given $A$ there is at most one $\gamma$ which works.* For if $\gamma$ is a half-circle centered on the $x$-axis with vertices at $v_j$ and $\omega$ as in figure 8, then $w=A^{-1}(\infty)$; so $\gamma$ is uniquely determined by $A$. From the forgoing we can already conclude

**Theorem B1.** *There are only a countable number of scattered rays.*

*Proof*: $\Gamma$ is a discrete subgroup of $SL(2, \boldsymbol{R})$; so it is countable. (Exhaust $SL(2, \boldsymbol{R})$ by a countable number of compacts.) Hence the set of $A\in\Gamma$ which give rise to scattered rays is countable. Q.E.D.

It can also happen that $A$ and $B\in\Gamma$ give rise to the same scattered ray. We claim that this can happen *if and only if $A=PB$ where $P$ is an iterate of the translation $z\to z+1$.* In fact suppose $A$ and $B$ map $\gamma$ as in figure 8 onto straight lines representing the same scattered geodesic in the ith cusp neighborhood. Then there exists a $P$ such that $PA$ and $B$ map $\gamma$ onto the same straight line. Replacing $A$ by $PA$ we can assume that $A$ and $B$ map $\gamma$ onto the same straight line, and that this straight line lies in the fundamental strip $-\frac{1}{2}<x<\frac{1}{2}$. $BA^{-1}$ leaves this straight line fixed and maps infinity to infinity. Since points sufficiently far out on this line lie in the fundamental region and can't be conjugated one into the other by elements of $\Gamma$ other than the identity, $BA^{-1}$ must be the identity. Q.E.D.

Suppose now we are given an $A\in\Gamma$ which defines a scattered ray according to the prescription outlined above. Let $T_A$ be the sojourn time of this ray. It turns out that there is a very simple procedure for computing $T_A$. Let $\varphi$ be an isometry of the strip $-\frac{1}{2}\leqq x\leqq\frac{1}{2}$ onto the ith cusp neighborhood mapping $\infty$ onto $v_j$. Then $A\circ\varphi$ is a linear fractional transformation of the form

$$z\to(\alpha z+\beta)/(cz+d),\ \begin{pmatrix}\alpha & \beta\\ c & d\end{pmatrix}\in SL(2,\boldsymbol{R}). \tag{B·1}$$

We will prove

$$T_A=2\log ca \tag{B·2}$$

the "$a$" here being the "$a$" in figure 2.

*Proof*: Let $\gamma$ be the geodesic through the cusp $v_j$ representing the given scattered ray as in figure 8, and let $\gamma' = \varphi^{-1}(\gamma)$. By construction $\gamma'$ is a straight line perpendicular to the $x$-axis; in other words, $\gamma'$ is an infinite half-circle with one vertex at $\infty$ and the other vertex at $(A\circ\varphi)^{-1}(\infty) = -d/c$. In terms of the coordinate system associated with $\varphi$ this ray enters the compact region when it intersects the line $y=a$, i.e. it enters at the point $z_0 = -\dfrac{d}{c} + \sqrt{-1}a$. It leaves the compact region when $A\gamma$ intersects the line $y=a$; i.e. when $\mathrm{Im}((\alpha z+\beta)/(cz+d)) = \mathrm{a}$, the $z$ here being a point on the line $\varphi^{-1}(\gamma)$, $z = -d/c + s\sqrt{-1}$. Substituting $-d/c + s\sqrt{-1}$ for $z$ in this equation we get

$$a = (-\alpha d/c + \beta)/sc = \frac{\alpha d - \beta c}{sc^2} = \frac{1}{sc^2},$$

so $s = 1/ac^2$. This proves that in the coordinate system of the $jth$ cusp neighborhood the portion of the scattered ray which lies in the compact region is just the line segment:

$$\{-d/c + s\sqrt{-1}\,, 1/ac^2 < s < a\}.$$

Since this line segment is vert:cal, the Poincaré metric restricted to it is $dy/y$, and the sojourn time is easily computed to be $\mathrm{Log}\, c^2a + \mathrm{Log}\, a = 2\mathrm{Log}\, ca$, proving (B·2).

We will now compute the $i$-$i$th entry of the scattering matrix following the prescription of Lax-Phillips [6], Chapter 8. Here Lax-Phillips prove that the $i$-$j$th entry of the scattering matrix is $a^{-2i\sigma}$ times the zero Fourier coefficient in the $j$-th cusp neighborhood of the Eisenstein series $e^i(\sigma, z)$ associated with the $i$-th cusp neighborhood. If we use the coordinates introduced in the discussion above (with $v_i$ at $\infty$) and as above let $\varphi$ be an isometry of the fundamental strip $-\frac{1}{2} \leqq x \leqq \frac{1}{2}$ onto the $j$-th cusp neighborhood then

(B·3) $$e^i(z, \sigma) = \sum A^* y^{1/2+i\sigma}$$

the sum taken over the left cosets $\Gamma_0 \backslash \Gamma$ where $\Gamma_0$ is the cyclic subgroup of $\Gamma$ generated by $z \to z+1$. The $A$'s are representative elements of the cosets. It is clear that (B·3) is independent of the choice of these representatives. The zero Fourier coefficient of $e^i(z, \sigma)$ in the $j$-th cusp neighborhood is

$$\int_{-1/2}^{1/2} \varphi^* e^i(z, \sigma)\, dx$$

and the theorem of Lax-Phillips alluded to above says that

$$e^{-2i \log a} \int_{-1/2}^{1/2} \varphi^* e^i(z, \sigma)\, dx = s_{ij}(\sigma) y^{1/2-i\sigma} \tag{B·4}$$

$s_{ij}(\sigma)$ being the $i$-$j$th entry of the scattering matrix. The contribution of the $A$-th term of (B·3) to (B·4) is

$$e^{-2i \log a} \int_{-1/2}^{1/2} \varphi^* A^* y^{1/2+\sigma} dx\,. \tag{B·5}$$

Let $A \circ \varphi$ be the linear fractional transformation, $z \to (\alpha z + \beta)/(cz+d)$. Then

$$\varphi^* A^* y = \mathrm{Im}\, \frac{\alpha z + \beta}{cz+d} = \mathrm{Im}\, \frac{(\alpha z + \beta)(c\bar{z}+d)}{|cz+d|^2}$$

$$= \frac{(\alpha d - \beta c) y}{|cz+d|^2} = \frac{y}{|cz+d|^2}$$

$$= \frac{y}{(cx+d)^2 + c^2 y^2}\,.$$

If we let $r = -d/c$ and $q = (x-r)/y$ we get

$$\varphi^* A^* y = \frac{1}{c^2 y} \frac{1}{1+q^2}\,. \tag{B·6}$$

Therefore, the contribution of the term involving $A$ to (B·4) is

$$e^{-2j\log a} y \left(\frac{1}{c^2 y}\right)^{1/2+i\sigma} \left[ \int_{\lambda(-1/2-r)}^{\lambda(1/2+r)} \frac{dq}{(1+q^2)^{1/2+i\sigma}} \right],$$

or

$$a y^{1/2-i\sigma} e^{-2 \log ca (1/2+i\sigma)} I(\lambda),$$

where $\lambda = \frac{1}{y}$ and $I(\lambda)$ is the integral in brackets. Dividing this expression by $y^{1/2-i\sigma}$ we get for the contribution of the $A$-th term to the scattering matrix

$$a e^{-2 \log ca (1/2+i\sigma)} I(\lambda)\,. \tag{B·7}$$

Each of these terms individually depends on $\lambda$, but the sum is independent of $\lambda$ so it will be unaffected if we let $\lambda$ tend to $\infty$. What happens if we interchange the summation and limit operations ignoring for the moment questions of convergence? For $\lim_{\lambda\to\infty} I(\lambda)$ we get

$$(\mathrm{B}\cdot 8)\qquad c(\sigma)=\begin{cases}\displaystyle\int_{-\infty}^{\infty}\frac{dq}{(1+q^2)^{1/2+i\sigma}} & \text{if } r=-\dfrac{d}{c} \text{ lies in } \left(-\dfrac{1}{2},\ \dfrac{1}{2}\right)\\ 0 & \text{if } r \text{ doesn't lie in } \left(-\dfrac{1}{2},\dfrac{1}{2}\right).\end{cases}$$

Let us now interpret this result geometrically. First of all given $A\in\Gamma$ when does $A$ determine a scattered ray? As we've seen the answer is if and only if some geodesic $\gamma$ lying between $\gamma_1$ and $\gamma_2$ as in figure 8 gets mapped onto a vertical line by $A$. This happens if and only if $A\circ\varphi$ maps one of the lines $l_\rho: x=\rho,\ -\frac{1}{2}<\rho<\frac{1}{2}$ onto a vertical line, and this happens if and only if the point $(A\circ\varphi)^{-1}(\infty)=-d/c$ lies on the interval $(-\frac{1}{2},\frac{1}{2})$. Thus the $A$'s making non-trivial contributions to the scattering matrix are precisely the $A$'s that account for the scattered rays. We showed above that two $A$'s define the same scattered ray if and only if they belong to the same left coset with respect to $\Gamma_0$. Since the Eisenstein series (B·3) is defined by taking precisely one representative from each coset the non-trivial summands correspond in a 1-1 fashion to the scattered rays. Finally the exponent occuring in (B·7) is, by (B·2), just the sojourn time, $T_A$, of the corresponding scattered ray. To justify the interchange of summation and limits in the argument above we note that by Lehner [7] (p. 159) and Lax-Phillips [6] (p. 8.12, formula 8.26) the sum of the terms $|c|^{-r}$ converges absolutely for $r\geqq 4$, so for $\operatorname{Im}\sigma\leqq -3/2$ the sum of the terms (B·7) converges absolutely. Thus our final result is that, for $\operatorname{Im}\sigma\leqq -3/2$

$$S_{ij}(\sigma)=ac(\sigma)\sum e^{-T_k(1/2+\sqrt{-1}\sigma)}$$

summed over all the sojourn times of rays which are scattered from the $i$-th to the $j$-th cusp neighborhood. This is what we set out to prove.

## References

[1] Faddeev, L., "Expansion in eigenfunctions of the Laplace operator in the fundamental domain of a discrete group in the Lobacevski plane". *Trudy Moscov. Mat. Obsc.*, vol. **17** (1967), 323-350.

[2] Faddeev, L. and Pavlov, B., "Scattering theory and automorphic functions" *Seminar*

*of Steklov Math. Institute of Leningrad*, vol. **27** (1972), 161–193,
[3] Guillemin, V., "Notes on scattering theory", (zeroxed notes) M. I. T. Math. Dept.
[4] Hörmander, "Fourier integral operators I", *Acta Math.*, vol. **127** (1971), 79–183.
[5] Lax, P. and Phillips, R., *Scattering Theory*, Academic Press, New York (1967).
[6] Lax, P. and Phillips, R., "Scattering theory for automorphic functions" preprint, NYU.
[7] Lehner, J., *Discontinuous groups and automorphic functions*, AMS Math. Surveys, N° 8, 1964.
[8] Majda, A., "High frequency asymptotics for the scattering matrix and the inverse problem of acoustical scattering", *Comm. Pure Appl. Math.* (to appear).
[9] Siegel, C. L., *Topics in Complex Function Theory*, Vol. 2, Wiley (Interscience), New York, (1973).
[10] Spivak, M., *A comprehensive introduction of differential geometry*, Publish or Perish (Boston) (1975).