

Introduction to Microlocal Analysis

by

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§ 1. Hyperfunctions, Microfunctions and the Exact Sequence of Sato

The most important example of a hyperfunction is the δ -function of Dirac. It can be characterized up to constant multiple by the differential equation

$$(1.1) \quad (xD_x + 1)f(x) = 0,$$

and the support property

$$(1.2) \quad f(x) = 0 \quad \text{if } x > 0 \text{ or } x < 0.$$

Even if we omit the condition (1.2), we cannot find any real analytic function other than zero which satisfies (1.1) in a neighborhood of zero. Therefore it is necessary to generalize the concept of functions.

If we consider (1.1) in the complex plane, its solution is

$$(1.3) \quad F(z) = \text{const.} \frac{1}{z}.$$

It is holomorphic in the upper half plane and also in the lower half plane. If we can attach some meaning to the phrase "the boundary value of $F(z)$ from upper (lower) half plane", we can find non trivial solutions of (1.1). These are hyperfunctions

$$(1.4) \quad \frac{1}{x + i0}$$

and

$$(1.5) \quad \frac{1}{x - i0}.$$

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These two solutions form a basis of the solution space of (1.1). Now we can find a solution which satisfies (1.2) taking the difference of (1.4) and (1.5). Adjusting the constant, we have the well-known formula

$$(1.6) \quad \delta(x) = -\frac{1}{2\pi i} \left(\frac{1}{x+i0} - \frac{1}{x-i0} \right).$$

Thus $\delta(x)$ can be expressed as a sum of two boundary values.

Definition 1.7. A hyperfunction is a sum of real boundary values of holomorphic functions.

To give rigorous and mathematical meaning to the above definition we use cohomology theory, the theory of residue class. (It is only a sky-scraper made of the same material which we use to construct rational numbers from integers.) In the case of one independent variable, the space of hyperfunctions in an open set U in \mathbf{R} is defined by

$$(1.8) \quad \mathcal{B}(U) = (\mathcal{O}(V_+) \oplus \mathcal{O}(V_-)) / \mathcal{O}(V).$$

Here V is a complex neighborhood of U which contains U as a closed subset, and $V_1(V_2)$ is the intersection of V and the upper (the lower) half plane. (Fig. 1.9)

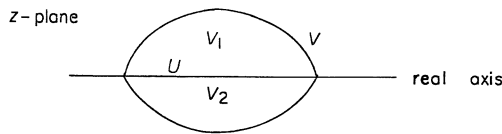


Fig. 1.9

If $F_1(z)$ is holomorphic in V_1 we write the hyperfunction defined as the boundary value of $F_1(z)$ like

$$(1.10) \quad F_1(x+i0).$$

Similarly we define a hyperfunction

$$(1.11) \quad F_2(x-i0).$$

A hyperfunction $u(x)$ has an expression

$$(1.12) \quad f(x) = F_1(x+i0) + F_2(x-i0).$$

If $\tilde{f}(x) = \tilde{F}_1(x+i0) + \tilde{F}_2(x-i0)$ is another hyperfunction, $f(x) = \tilde{f}(x)$ if and only if

$$(1.13) \quad F_1(x+i0) - \tilde{F}_1(x+i0) \in \mathcal{A},$$

$$(1.14) \quad F_2(x-i0) - \tilde{F}_2(x-i0) \in \mathcal{A},$$

and

$$(1.15) \quad (F_1(x+i0) - \tilde{F}_1(x+i0)) + (F_2(x-i0) - \tilde{F}_2(x-i0)) = 0.$$

We can summarize (1.12)~(1.15) in the following exact sequence—the exact sequence of Sato.

$$(1.16) \quad 0 \rightarrow \mathcal{A} \xrightarrow{\iota} \mathcal{B} \xrightarrow{sp_1 \oplus sp_2} \mathcal{E}_1 \oplus \mathcal{E}_2 \rightarrow 0.$$

In (1.16) $\mathcal{E}_1(\mathcal{E}_2)$ is the space of the element (1.10) ((1.11)) where two elements like (1.13) ((1.14)) are identified.

Exercise. 1.17 Examine that differential operators operate on \mathcal{B} , \mathcal{E}_1 and \mathcal{E}_2 as local operators.

We have characterized $\delta(x)$ by (1.1) and (1.2). To characterize (1.4) and (1.5) we can use (1.16). (1.4) ((1.5)) is an element in \mathcal{B} whose image under $sp_2(sp_1)$ vanishes. We can illustrate these three cases of “support property” by the following diagram. (Fig. 1.18).

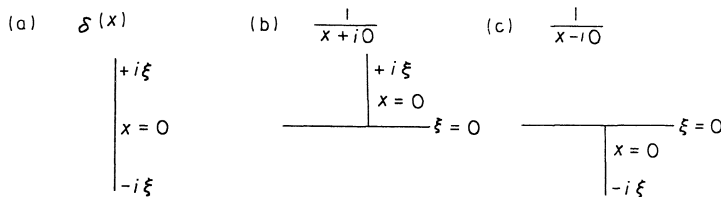


Fig. 1.18 The notation $\pm i\xi$ will be understood when we explain the case of several independent variables.

Remark. 1.19 The diagram for a general hyperfunction is not so simple. Nevertheless it is important to note that a kind of hyperfunction as $F(x+i0)(F(x-i0))$ has no support below (above) the axis $\{\xi=0\}$.

Before proceeding to the case of several independent variables, we give several examples of hyperfunctions with one independent variable. We consider a differential equation with one complex parameter λ .

$$(1.20) \quad (xD_x - \lambda)u = 0.$$

Two independent solutions of (1.20) are

$$(1.21) \quad (x+i0)^\lambda$$

and

$$(1.22) \quad (x-i0)^\lambda$$

for $\lambda \neq 0, 1, 2, \dots$. Here we choose the branch such that $z^\lambda = 1$ for $z=1$. The diagrams for $(x+i0)^\lambda$ and $(x-i0)^\lambda$ are (b) and (c) of (1.18). For $\lambda = -1, -2, \dots$, we can construct a solution of (1.20) whose diagram is like (a) of (1.18). The solutions are the $(-\lambda+1)$ -th derivative of $\delta(x)$

$$(1.23) \quad \delta^{(-\lambda+1)}(x), \quad \lambda = -1, -2, \dots.$$

In contrast, for $\lambda \notin \mathbf{Z}$ we cannot construct such solutions. Instead we can construct two more solutions with characteristic support property. (Fig. 1.24).

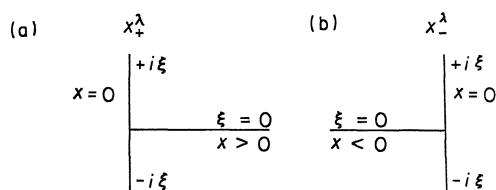


Fig. 1.24

Exercise. 1.25 Express x_+^λ and x_-^λ as linear combinations of (1.21) and (1.22). Show that they are meromorphic with respect to λ and have simple poles at $\lambda = -1, -2, \dots$. (Hint. Compare the values at $x = \pm 1$, and calculate their Laurent expansions at $\lambda \in \mathbf{Z}$.)

$$(1.26) \quad x_+^\lambda = \frac{1}{2i \sin \pi \lambda} \{(-x+i0)^\lambda - (-x-i0)^\lambda\},$$

$$x_-^\lambda = \frac{1}{2i \sin \pi \lambda} \{(x+i0)^\lambda - (x-i0)^\lambda\}.$$

For $\lambda = 0, 1, 2, \dots$, (1.21) and (1.22) are the same and there is one more independent solution of (1.20). Thus we have the following three

solutions with characteristic support property any two of which form a basis of the solution space of (1.20). (Fig. 1.27)

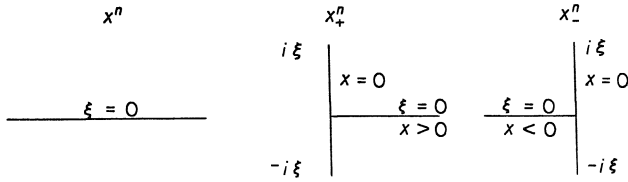


Fig. 1.27

$$(1.28) \quad x_+^n = -\frac{x^n}{2\pi i} \{ \log(-x+i0) - \log(-x-i0) \},$$

$$(1.29) \quad x_-^n = \frac{x^n}{2\pi i} \{ \log(x+i0) - \log(x-i0) \}.$$

Summary. 1.30 The dimension of the solution space of (1.20) is 2. The structure of support of solutions are classified into three classes.

- I. $\lambda \notin \mathbb{Z}$
- II. $\lambda = 0, 1, 2, \dots$
- III. $\lambda = -1, -2, \dots$

Later we explain that this difference is due to the difference of algebraic structure of the equations.

Exercise. 1.31 Show that

$$(1.32) \quad |x|^\lambda = x_+^\lambda + x_-^\lambda$$

is a meromorphic function of λ with simple poles at $\lambda = -1, -3, \dots$. Calculate its Laurent expansion.

To illustrate how the structure of solution space become simple, we state some theorems about more general ordinary differential equations. (The proof is easy, try yourself.)

Theorem 1.33. $P(x, D) = \sum_{\alpha=0}^m a_\alpha(x) D^\alpha$ is an ordinary differential operator defined in a neighborhood U of the origin, $ord_0 a_m$ denotes the

degree of zero of $a_m(x)$ at $x=0$.

(a) $\mathcal{B}(U)^P \rightarrow \mathcal{B}(U) \rightarrow 0$ is exact.

(b) $\dim \mathcal{B}(U)^P = m + ord_0 a_m$

where $\mathcal{B}(U)^P$ represents the space of hyperfunction solutions in U of the equation $P(x, D)u=0$.

Now we go back to the formula (1.6). It is easy to see that the well-known formula

$$(1.34) \quad \int f(x) \delta(x-y) dx = f(y)$$

is equivalent to the Cauchy's integral formula (Fig. 1.35)

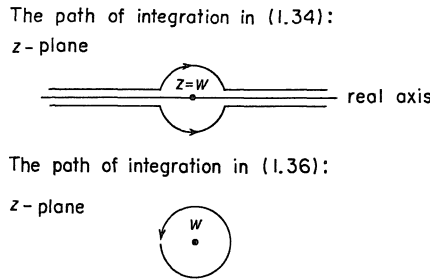


Fig. 1.35

$$(1.36) \quad \frac{1}{2\pi i} \oint \frac{F(z)}{z-w} dz = F(w).$$

(1.36) tells us every function imaginable can be expressed as a hyperfunction, that is, as a sum of boundary values. In fact, take any "function of one real variable" $f(x)$. We assume its support is contained in the interval $I=[-1, 1]$. Then we can define a holomorphic function outside I by

$$(1.37) \quad F(z) = \frac{1}{2\pi i} \int \frac{f(x)}{x-z} dx.$$

Then we have

$$(1.38) \quad f(x) = F(x+i0) - F(x-i0).$$

Exercise. 1.39 Show that for any real analytic function $g(x)$

$$\int f(x)g(x)dx = \int \{F(x+i0) - F(x-i0)\}g(x) dx .$$

This argument can be easily extended to the case of several independent variables using the formula

$$(1.40) \quad F(w_1, \dots, w_n) = \frac{1}{(2\pi i)^n} \oint \dots \oint \frac{F(z_1, \dots, z_n)}{(z_1 - w_1) \dots (z_n - w_n)} dz_1 \dots dz_n .$$

Exercise. 1.41 Find the holomorphic region of the function defined by

$$(1.42) \quad F(z_1, \dots, z_n) = \frac{1}{(2\pi i)^n} \int \dots \int \frac{f(x_1, \dots, x_n)}{(x_1 - z_1) \dots (x_n - z_n)} dx_1 \dots dx_n$$

where $f(x_1, \dots, x_n)$ is a function with bounded support.

Thus any hyperfunction can be expressed as a sum of 2^n boundary values of a holomorphic function which is holomorphic in 2^n quadrants

$$(1.43) \quad V_{\varepsilon_1 \dots \varepsilon_n} = \{(z_1, \dots, z_n) \in \mathbf{C}^n \mid (\text{Im } z_i) \varepsilon_i > 0\} \quad \text{for } i=1, \dots, n$$

where $\varepsilon_i = \pm 1$. (Fig. 1.44)

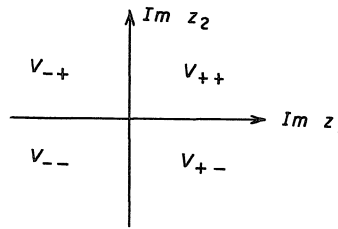


Fig.1.44

More generally we can define the boundary value from a tube domain.

Let U be an open set in \mathbf{R}^n and V be a complex neighborhood of U in $\mathbf{C}^n = \mathbf{R}^n + \sqrt{-1}\mathbf{R}^n$ which contains U as a closed subset. V is real $2n$ dimensional and U is embedded in V as a submanifold of real codimensions n . Let us consider n -dimensional cone Γ . Γ is an open convex subset of \mathbf{R}^n which is closed under the multiplication by positive real numbers. Then $V \cap (U + \sqrt{-1}\Gamma)$ is a union of n -dimensional cones attached to real points in U and is called a tube domain. When we consider

this tube domain only near x we also denote it by $x+i0\Gamma$. We shall use the notation $z=(z_1, \dots, z_n)$. Let $F(z)$ be a holomorphic function in $V \cap (U+\sqrt{-1}\Gamma)$, then we denote by

$$(1.45) \quad F(x+i0\Gamma)$$

the boundary value to U of $F(z)$. Here the symbol 0 in front of Γ means a residue class of tangent vectors of X at x modulo tangent vectors of the real manifold U .

Now we can define a hyperfunction as a sum of boundary values.

$$(1.46) \quad f(x) = \sum_j F_j(x+i0\Gamma_j).$$

To complete the definition we must define when such expression represents zero. It requires the cohomology theory and we omit it.

Before we proceed on, let us explain some geometrical concepts.

Let M be an n dimensional real analytic manifold and X be its complexification. You may imagine $M=\mathbf{R}^n$ and $X=\mathbf{R}^n+\sqrt{-1}\mathbf{R}^n$. There is an exact sequence which defines a normal vector at $x \in M$.

$$(1.47) \quad 0 \leftarrow (T_M X)_x \leftarrow (TX)_x \leftarrow (TM)_x \leftarrow 0.$$

The dual exact sequence

$$(1.48) \quad 0 \rightarrow (T_M^* X)_x \rightarrow (T^* X)_x \rightarrow (T^* M)_x \rightarrow 0$$

defines a conormal vector — a cotangent vector of X at x which is zero at tangent vectors of M at x . There is a canonical isomorphism

$$(1.49) \quad \sqrt{-1} T^* M \cong T_M^* X.$$

$x \in M$ is identified with a zero vector in $\sqrt{-1}(T^* M)_{x,*}$. Then we define

$$(1.50) \quad \sqrt{-1} S^* M \cong S_M^* X$$

as $(\sqrt{-1} T^* M - M)/\mathbf{R}_+$ and $(T_M^* X - M)/\mathbf{R}_+$ (residue class modulo real positive multiple). We denote an element of (1.49) and (1.50) by $(x, i\xi)$ and $(x, i\infty\xi)$. There is a natural projection $\sqrt{-1} T^* M \rightarrow M$ and $\sqrt{-1} S^* M \rightarrow M$. We denote them by π . $(x, i\infty\xi)$ represents an infinitesimal half space of X attached to M at x . (Fig. 1.51)

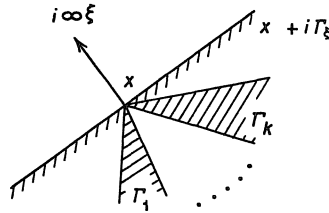


Fig. 1.51

We denote this half space by $x + i\Gamma_\xi$.

$$(1.52) \quad x + i\Gamma_\xi = \{x' + i0y \mid x' \approx x, \langle i0y, i\infty\xi \rangle = -\langle y, \xi \rangle > 0\}.$$

Definition (1.53) (Microanalyticity). A hyperfunction is said to be microanalytic at $(x, i\infty\xi)$ if it can be expressed as a sum of boundary values of holomorphic functions which are holomorphic in tube domains contained in $x + i\Gamma_\xi$. (Fig. 1.54)

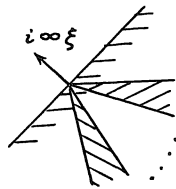


Fig. 1.54

Definition 1.55 ($\mathcal{E}_{(x, i\infty\xi)}$). A residue class of hyperfunctions at x modulo those which are microanalytic at $(x, i\infty\xi)$ defines a microfunction at $(x, i\infty\xi)$.

We collect $\mathcal{E}_{(x, i\infty\xi)}$ to define the sheaf \mathcal{E} of microfunctions. This is a sheaf on $\sqrt{-1}S^*M$.

Theorem 1.56. (The exact sequence of Sato)

$$0 \rightarrow \mathcal{A} \xrightarrow{\iota} \mathcal{B} \xrightarrow{sp} \pi_* \mathcal{E} \rightarrow 0.$$

Here $\pi_* \mathcal{E}$ is a sheaf on M whose stalk at x is the set of microfunctions defined on $\pi^{-1}(x)$.

For a hyperfunction $f(x)$, support of $spf(x)$ is called the singular

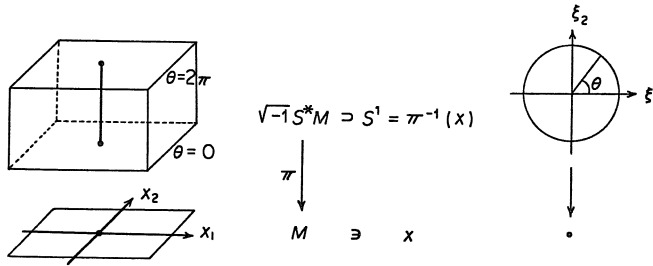
* In this sense M is called the zero section of $\sqrt{-1}T^*M$.

spectrum of $f(x)$. We denote it by $S.S.f(x)$.*) It is a closed subset in $\sqrt{-1}S^*M$ where $f(x)$ is not microanalytic.

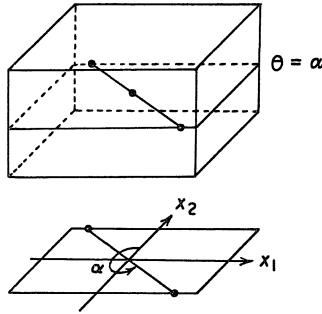
Exercise. 1.57 Show that

$$S.S.F(x + i0\Gamma) \subset \{(x, i\infty\xi) \mid \langle i0\Gamma, i\infty\xi \rangle \leq 0\}.$$

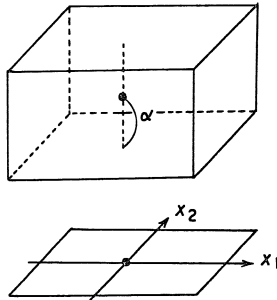
Example 1.58 $S.S.\delta(x_1)\delta(x_2)$



Example 1.59 $S.S.(x_1 \cos \alpha + x_2 \sin \alpha + i0)^\lambda$



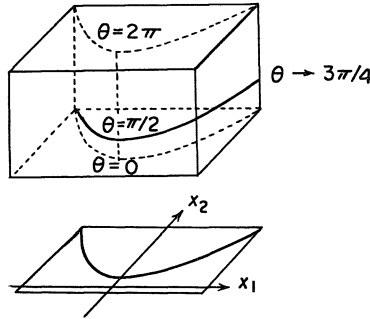
Example 1.60 $S.S.\{(x_1 \cos \alpha + x_2 \sin \alpha) + i(-x_1 \sin \alpha + x_2 \cos \alpha)^2 + i0\}^\lambda$



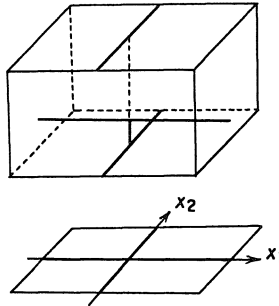
*) We also denote by $S.S. f(x)$ the support of a microfunction $f(x)$. A hyperfunction $f(x)$ is real analytic if and only if $S.S. f(x) = \emptyset$.

Exercise. 1.61 Show that $\{(x_1 \cos \alpha + x_2 \sin \alpha) + i(-x_1 \sin \alpha + x_2 \cos \alpha)^2 - i0\}^\lambda$ cannot be defined.

Example 1.62 $S.S.(x_2^2 - x_1^2 - m^2 + i0)^\lambda \ (x_2 > 0)$



Example 1.63 $S.S. \frac{1}{x_1 + i0} \frac{1}{x_2 + i0}$



Exercise 1.64 Compare $S.S. \left(\frac{1}{x_1 + i0} + \frac{1}{x_2 + i0} \right)$ with (1.63).

In examples (1.59) ~ (1.62) each hyperfunction can be expressed as a boundary value from a half space. Therefore its singular spectrum consists of at most one point in $\pi^{-1}(x)$. In contrast, singular spectrum of $\delta(x_1) \cdot \delta(x_2)$ spreads all over $\pi^{-1}(0)$. This is concordant with the well-known plane wave expansion of $\delta(x)$.

$$(1.65) \quad \delta(x) = \frac{(n-1)!}{(-2\pi i)^n} \int_{S^{n-1}} \frac{\omega_{n-1}(\xi)}{\langle x, \xi \rangle + i0} dx$$

where $\langle x, \xi \rangle = x_1 \xi_1 + \dots + x_n \xi_n$ and $\omega_{n-1}(\xi)$ is the surface element on S^{n-1} .

Let Γ_1 and Γ_2 be cones such that $\Gamma_1 \cup \Gamma_2$ is convex. Then Cousin's

lemma in the theory of functions with several complex variables asserts that there is a decomposition

$$(1.66) \quad F(x + i0(\Gamma_1 \cap \Gamma_2)) = F_1(x + i0\Gamma_1) + F_2(x + i0\Gamma_2).$$

We can proceed with this decomposition until Γ reaches at a half space. (Fig. 1.67). From this point of view Exercise (1.57) and Definition (1.53) may be well understood.

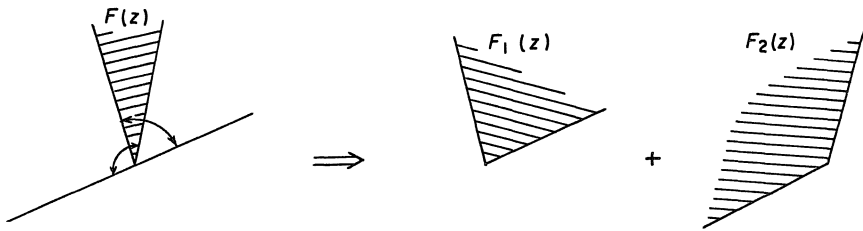


Fig. 1. 67

Another way to understand the “singular spectrum” is the Fourier transform, and micro-analyticity and macro-causality in physics are connected by the Fourier transform. Here we do not enter into this subject.

§ 2. Micro-differential Operators and Holonomic Systems

In § 1 the sheaf \mathcal{B} of hyperfunctions was introduced naturally through the study of differential equations and the sheaf \mathcal{E} of microfunctions was introduced through the study of the structure of manifolds. The third step is to generalize the concept of differential operators so as to fit with \mathcal{E} and $\sqrt{-1}S^*M$.

Let us consider a differential operator $D_1 = \partial/\partial x_1$. A solution of the equation

$$(2.1) \quad D_1 f(x) = g(x)$$

always exists but is not unique. Its ambiguity is a hyperfunction $h(x_2, \dots, x_n)$ independent of x_1 . It is easy to see that

$$(2.2) \quad S.S.h \subset \{(x, i\infty\xi) \in \sqrt{-1}S^*M \mid \xi_1 = 0\}.$$

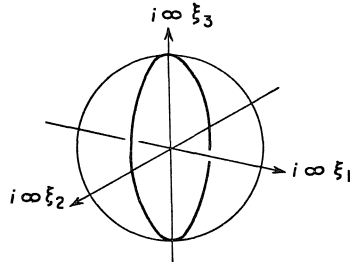


Fig. 2.3 S.S. $h(x_2, x_3)$

Therefore if we consider (2.1) microlocally in the neighborhood of a point $(x, i\infty\xi)$ in $\sqrt{-1}S^*M$ with $\xi_1 \neq 0$, the microfunction solution is unique. In other words, D_1^{-1} is well defined micro-locally on $\sqrt{-1}S^*M$ if $\xi_1 \neq 0$.

Definition 2.4. Let $(x_0, \infty\xi_0)$ be a point of P^*X^* , and $P_j(x, \xi)$ $(-\infty < j \leq m)$ be a holomorphic function defined at (x_0, ξ_0) and homogeneous of degree j with respect to ξ . We require that

$$(2.5) \quad -^j \sqrt{|P_j(x, \xi)|} / (-j)! \quad (-\infty < j \leq m)$$

are locally uniformly bounded. Then the formal expression

$$(2.6) \quad P(x, D) = \sum_{j=-\infty}^m P_j(x, D)$$

represents a microdifferential operator defined at $(x_0, \infty\xi_0)$. Microdifferential operators defined at $(x_0, \infty\xi_0)$ form a non commutative ring $\mathcal{E}_{(x_0, \infty\xi_0)}$.

Through the two-fold embedding

$$(2.7) \quad \sqrt{-1}S^*M \rightarrow P^*X$$

we consider P^*X as a complexification of $\sqrt{-1}S^*M$. Then $\mathcal{E}_{(x, i\infty\xi)}$ operates on $\mathcal{E}_{(x, i\infty\xi)}$.

Example 2.8. Let $P(x, D) = \sum_{j=-\infty}^0 a_j(x) (-j)! D^j$.

$$P(x, D)\theta(x) = \sum_{j=-\infty}^0 a_j(x) x_+^{-j}.$$

* $P^*X = (T^*X - X) / (\mathbb{C} - \{0\})$ (See (1.50)).

Then at $(0, i\infty dx) = (\sum_{j=-\infty}^0 a_j(x)x^{-j})\theta(x)$ where $\sum_{j=-\infty}^0 a_j(x)x^{-j}$ is convergent because of (2.5). Thus a microdifferential operator can be replaced by a multiplication of a holomorphic function in this case.

Definition 2.9. We call $P_m(x, \xi)$ in the expression (2.6) the principal symbol of $P(x, D)$. We denote it by $\sigma(P)$. It has an intrinsic meaning as a homogeneous holomorphic function on T^*X independent of coordinate transformations.

$$(2.10) \quad \sigma(P \cdot Q) = \sigma(P)\sigma(Q)$$

$$(2.11) \quad \sigma([P, Q]) = \{\sigma(P), \sigma(Q)\}$$

where $\{ \ , \ }$ means the Poisson bracket

$$(2.12) \quad \{f, g\} = \sum_{j=1}^n \left(\frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial \xi_j} \right).$$

Through out this section and § 4, we shall see how “commutative” principal symbols control “non-commutative” micro-differential operators.

Theorem 2.13.

If $\sigma(P)(x_0, \infty \xi_0) \neq 0$, $P(x, D)$ has a unique inverse in $\mathcal{E}_{(x_0, \infty \xi_0)}$. We call $P(x, D)$ elliptic.

Corollary 2.14. (Sato’s fundamental theorem) If a microfunction $f(x)$ satisfies a micro-differential equation

$$(2.15) \quad P(x, D)f(x) = 0,$$

then we have

$$(2.16) \quad S.S.f(x) \subset \{(x, i\infty \xi) \in \sqrt{-1}S^*M \mid \sigma(P)(x, i\xi) = 0\}.$$

Thus a system of microdifferential equations enables us to estimate the singular spectrum of a microfunction which satisfies the system.

Exercise 2.17. Prove Weyl’s lemma from (2.16) and (1.56). Weyl’s lemma: a solution of an elliptic linear differential equation $Pu=0$ is real analytic.

Exercise 2.18. Determine $S.S.\delta(x_1)\cdots\delta(x_r)$ ($1\leq r\leq n$) by the above method.

Let us consider a system of micro-differential equations

$$(2.19) \quad P_1u=0, \dots, P_Nu=0.$$

A solution of (2.15) satisfies

$$(2.20) \quad Qu=0 \quad \text{for } Q=R_1P_1+\dots+R_NP_N.$$

But $\sigma(P_1)(x, i\xi)=0, \dots, \sigma(P_N)(x, i\xi)=0$ do not always imply $\sigma(Q)(x, i\xi)=0$. Thus we require a more intrinsic definition of what is a system of micro-differential equations.

Definition 2.21. A system of micro-differential equations is a coherent left \mathcal{E} -Module. This means that a system \mathcal{M} has a representation (not unique)

$$(2.22) \quad \begin{array}{ccccccc} 0 & \longleftarrow & \mathcal{M} & \longleftarrow & \mathcal{E}^s & \longleftarrow & \mathcal{E}^t \\ & & \cup & & \cup & & \\ & & P_1u_1+\dots+P_su_s & \longleftarrow & (P_1, \dots, P_s) & & \\ & & & & \cup & & \\ & & & & (\sum_j R_jP_{j1}, \dots, \sum_j R_jP_{js}) & \longleftarrow & (R_1, \dots, R_t) \end{array}$$

where $u_1, \dots, u_s \in \mathcal{M}$ and $(P_{jk})_{\substack{j=1, \dots, t \\ k=1, \dots, s}}$ is a matrix of micro-differential operators. This representation corresponds to a usual expression

$$(2.23) \quad \sum_{k=1}^s P_{jk}u_k=0 \quad (j=1, \dots, t).^{*)}$$

Exercise 2.24. Show that differential equations

$$(2.25) \quad (xD-\lambda)u=0$$

and

$$(2.26) \quad (xD-\lambda+1)v=0$$

are different representations of the same *differential* equation if and only

* $Hom_{\mathcal{C}}(\mathcal{M}, \mathcal{C})$ represents the sheaf of microfunction solutions of \mathcal{M} . The image of a set of generators (u_1, \dots, u_s) by an \mathcal{E} -homomorphism from \mathcal{M} to \mathcal{C} represents a solution of (2.23).

if $\lambda \neq 0$. If we consider (2.25) and (2.26) as a micro-differential equation at $(0, i\infty\xi)$ ($\xi \neq 0$) they represent the same equation.

Definition 2.27. The characteristic variety of \mathcal{M} is the support of \mathcal{M} as a sheaf on P^*X . We denote it by $S.S.\mathcal{M}$.

From now on we assume that $s=1$ and we fix a generator u and a coherent left ideal \mathcal{J} such that

$$(2.28) \quad \mathcal{M} = \mathcal{E}u = \mathcal{E}/\mathcal{J}.$$

This means

$$(2.29) \quad Pu = 0 \quad \text{for } P \in \mathcal{J}.$$

Then we have

$$(2.30) \quad S.S.\mathcal{M} = \{(x, \infty\xi) \in P^*X \mid \sigma(P)(x, \xi) = 0 \text{ for } \forall P \in \mathcal{J}\}.$$

Exercise 2.31. Prove (2.30) using Theorem (2.13).

Let V be a subvariety in T^*X . V is called involutory when

$$(2.32) \quad f \equiv g \equiv 0 \text{ on } V \text{ implies } \{f, g\} \equiv 0 \text{ on } V,$$

and isotropic when

$$(2.33) \quad \sum_{j=1}^n d\xi_j \wedge dx_j|_V \equiv 0.$$

If V is conic, this is equivalent to

$$(2.34) \quad \sum_{j=1}^n \xi_j dx_j|_V \equiv 0.$$

A hypersurface in T^*X is involutory. A submanifold defined by $x_1=0$ and $\xi_1=0$ is not involutory. A submanifold defined by $x_1=\dots=x_r=\xi_{r+1}=\dots=\xi_n=0$ is involutory and isotropic.

The dimension of an involutory subvariety is equal to or more than n , and the dimension of an isotropic subvariety is equal to or less than n .

We identify a conic (in ξ) subvariety (other than the zero section X) in T^*X with a subvariety in P^*X .

Theorem 2.35. (Fundamental theorem in the theory of linear differential equations). *The characteristic variety of a system is involutory.*

Corollary 2.36. $\dim S.S.\mathcal{M} \geq n$.

Exercise 2.37. Prove (2.35) under the assumption that

$$(2.38) \quad J = \text{the ideal in } \mathcal{O}_{T^*X} \text{ generated by } \{\sigma(P) \mid P \in \mathcal{G}\}$$

is reduced, that is, $f(x, \xi) \in J$ if and only if $f(x, \xi) \equiv 0$ on $S.S.\mathcal{M}$. (Hint. Use (2.11).)

Definition 2.39. An involutory subvariety with n dimensions or equivalent to say an isotropic subvariety with n dimensions is called holonomic. A system \mathcal{M} is called holonomic if $S.S.\mathcal{M}$ is holonomic.

Theorem 2.40. *The solution space of a linear differential holonomic system is finite dimensional.*

A function with a natural origin maybe satisfies a holonomic system. "Study functions microlocally through holonomic systems." This is the principle of MICROLOCAL CALCULUS and Theorem (2.40) gives a basis to our principle. We call solutions of a holonomic system holonomic hyper-(micro-)functions.

Let Y be an irreducible subvariety in X . If Y is non-singular the conormal bundle T_Y^*X can be defined as (1.48). In general we define T_Y^*X to be the Zariski closure in T^*X of the conormal bundle of the non singular part of Y .

Theorem 2.41. T_Y^*X is an irreducible conic holonomic subvariety. Conversely if A is an irreducible conic holonomic subvariety, we have

$$(2.42) \quad A = T_Y^*X$$

where $Y = \pi(A)$.

Exercise 2.43. Prove theorem (2.41). It is sufficient to prove it when $Y = \pi(A)$ is a nonsingular submanifold.

Let Ω_X^n be the sheaf of n forms on X . We can define a sheaf of (micro =) differential operators which operate on $\sqrt{\Omega_X^n}$.

$$(2.44) \quad \tilde{\mathcal{D}}_X = \sqrt{\Omega_X^n} \otimes \mathcal{D}_X \otimes \sqrt{\Omega_X^{n-1}}$$

and

$$(2.45) \quad \tilde{\mathcal{E}}_X = \sqrt{\Omega_X^n} \otimes \mathcal{E}_X \otimes \sqrt{\Omega_X^{n-1}}.$$

This means that $\sqrt{dx}P(x, D_x)\sqrt{dx}^{-1}$ and $\sqrt{dy}Q(y, D_y)\sqrt{dy}^{-1}$ are equal if and only if

$$(2.46) \quad \sqrt{\frac{dx}{dy}} P(x, D_x) \sqrt{\frac{dx}{dy}}^{-1} = Q(y, D_y)$$

where x and y denote two local coordinates. In this sheaf not only principal symbol but also the following next symbol is invariant.

$$(2.47) \quad P_{m-1}(x, \xi) - \frac{1}{2} \sum_{j=1}^n \frac{\partial^2 P_m}{\partial x_j \partial \xi_j}(x, \xi).$$

Exercise 2.48. Check in some simple cases (2.47) is invariant under the coordinate transformation.

Then it is easy to see that for $P(x, D) \in \mathcal{D}_X$ or \mathcal{E}_X , $\tilde{L}_P = \sqrt{dx}^{-1} L_P \sqrt{dx}$ is a first order linear differential operator defined in T^*X independent of local coordinates, where

$$(2.49) \quad L_P = H_{P_m} + P_{m-1}(x, \xi) - \frac{1}{2} \sum_{j=1}^n \frac{\partial^2 P_m}{\partial x_j \partial \xi_j}.$$

Let $\mathcal{M} = \mathcal{E}u = \mathcal{E}/\mathcal{I}$ be a holonomic system whose support is a nonsingular holonomic manifold A . Then for $P(x, D) \in \mathcal{I}$, L_P is a differential operator on A and the system of differential equations on A

$$(2.50) \quad \tilde{L}_P s = 0, \quad P(x, D) \in \mathcal{I}$$

* $H_{P_m}(f) = \{P_m, f\}$.

where $s \in \sqrt{\Omega_A^n} \otimes \sqrt{\Omega_X^{n-1}}$ has finite dimensional solutions. Here H_{P_m} operates on $\sqrt{\Omega_A^n}$ as the Lie derivative $L_{H_{P_m}}$ ***).

From now on we assume that \mathcal{M} is simple on A , that is, J in (2.38) is reduced. Then (2.50) has a unique (up to constant multiple) analytic solution on A . We denote it by $\sigma_A(u)$ and call it the principal symbol of the holonomic system $\mathcal{E}u = \mathcal{E}/\mathcal{G}$.

Definition 2.51.

$\sigma_A(u)$ is homogeneous in ξ and we call its homogeneous degree the order and denote it by $\text{ord}_A(u)$.***)

Example 2.52.

$$\begin{aligned} \mathcal{G} &= \mathcal{D}x_1 + \dots + \mathcal{D}x_r + \mathcal{D}D_{r+1} + \dots + \mathcal{D}D_n, \\ A &= \{(x, \xi) \mid x_1 = \dots = x_r = \xi_{r+1} = \dots = \xi_n = 0\}, \\ \sigma_A(u) &= \sqrt{\frac{d\xi_1 \dots d\xi_r dx_{r+1} \dots dx_n}{dx_1 \dots dx_r dx_{r+1} \dots dx_n}}, \\ \text{ord}_A(u) &= \frac{r}{2}. \end{aligned}$$

Example 2.53.

$$\begin{aligned} \mathcal{G} &= \mathcal{D}(x_1 D_1 - \alpha) + \mathcal{D}(x_2 D_2 - \beta) + \mathcal{D}D_3 + \dots + \mathcal{D}D_n, \\ A_1 &= \{x_1 = x_2 = \xi_3 = \dots = \xi_n = 0\}, \\ A_2 &= \{x_1 = \xi_2 = \xi_3 = \dots = \xi_n = 0\}, \\ \sigma_{A_1}(u) &= \xi_1^{-\alpha-1} \xi_2^{-\beta-1} \sqrt{\frac{d\xi_1 d\xi_2 dx_3 \dots dx_n}{dx_1 dx_2 dx_3 \dots dx_n}}, \end{aligned}$$

* s is well-defined up to constant multiple. If A and M are real, s is defined without ambiguity as a section of $\sqrt{\mathcal{C}V_A} \otimes \sqrt{\mathcal{C}V_M^{-1}}$ where $\mathcal{C}V$ denotes the sheaf of volume elements.

** $L_{\sum_{i=1}^n a_i(x) \frac{\partial}{\partial x_i}} f(x) \sqrt{dx_1 \wedge \dots \wedge dx_n} = \left\{ \sum_{i=1}^n a_i(x) \frac{\partial f}{\partial x_i}(x) + \frac{1}{2} \sum_{i=1}^n (-)^{i-1} \frac{\partial a_i}{\partial x_i} f(x) \right\} \sqrt{dx_1 \wedge \dots \wedge dx_n}$.

*** Note that ‘principal symbol’ and ‘order’ are defined only if we fix a generator of \mathcal{M} . An exchange of generators causes a difference of an integer in orders. See Exercise (2.24).

$$\text{ord}_{A_1}(u) = -\alpha - \beta - 1,$$

$$\sigma_{A_2}(u) = \xi_1^{-\alpha-1} x_2^\beta \sqrt{\frac{d\xi_1 dx_2 dx_3 \cdots dx_n}{dx_1 dx_2 dx_3 \cdots dx_n}},$$

$$\text{ord}_{A_2}(u) = -\alpha - \frac{1}{2}.$$

Example 2.54.

$$\mathcal{J} = \mathcal{D} \left(\frac{1}{2} x_1 D_1 + x_2 D_2 + \alpha \right) + \mathcal{D} \left\{ \frac{1}{2} \left(\frac{1}{2} D_1 + x_1 D_2 \right) D_1 + \beta D_2 \right\},$$

$$A_1 = \{\xi_1 = x_2 = 0\},$$

$$A_2 = \{x_2 = x_1^2, \xi_1 + 2x_1 \xi_2 = 0\},$$

$$\sigma_{A_1}(u) = x_1^{-2\beta} \xi_2^{\alpha-\beta-1} \sqrt{\frac{dx_1 d\xi_2}{dx_1 dx_2}},$$

$$\text{ord}_{A_1}(u) = \alpha - \beta - \frac{1}{2},$$

$$\sigma_{A_2}(u) = x_1^{2\beta-1} \xi_2^{\alpha+\beta-3/2} \sqrt{\frac{dx_1 d\xi_2}{dx_1 dx_2}},$$

$$\text{ord}_{A_2}(u) = \alpha + \beta - 1.$$

To study singular points of a characteristic variety only a singularity of codimension 1 (that is of dimensions $n-1$) plays an essential role. In this sense it is important to determine if two (or more) irreducible components of the characteristic variety have a codimension 1 intersection or not. The structure of a holonomic system is visualized by ‘holonomy diagram’ which represents the configuration of irreducible components of the characteristic variety.

§ 3. Several Operations on Functions and Systems

In this section we show by an example how the Feynman amplitude can be defined as a hyperfunction. In the course we explain several operations (substitution, integration, product and compactification).

Consider the following Feynman graph (Fig. 3.1).

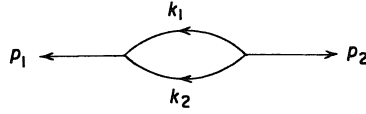


Fig. 3. 1

The Feynman amplitude with parameter λ_i of this graph is an integral of the form

$$(3.2) \quad \iint \delta(p_1 - k_1 - k_2) \delta(p_2 + k_1 + k_2) \frac{1}{(k_1^2 - m_1^2 + i0)^{\lambda_1}} \times \frac{1}{(k_2^2 - m_2^2 + i0)^{\lambda_2}} dk_1 dk_2 .^{*)}$$

To define (3.2) as a hyperfunction we prepare several operations.

Let $f: N \rightarrow M, y \mapsto x = f(y)$ be a real analytic map. We have the following diagram

$$(3.3) \quad \begin{array}{ccc} N \times \sqrt{-1}T^*M & & (y, \sqrt{-1}\xi) \\ \rho \swarrow M & \searrow \varpi & \rho \swarrow \searrow \varpi \\ \sqrt{-1}T^*N & \sqrt{-1}T^*M \left(y, \sqrt{-1}\xi \cdot \frac{\partial f}{\partial y} \right) & (f(y), \sqrt{-1}\xi) \end{array}$$

We define T_N^*M by an exact sequence

$$(3.4) \quad 0 \rightarrow T_N^*M \rightarrow N \times_M T^*M \rightarrow T^*N \rightarrow 0 .$$

In a local coordinate

$$(3.5) \quad \sum_{i=1}^n \xi_i dx_i \in T_N^*M$$

if and only if $\sum_{i=1}^n \xi_i \frac{\partial f_i}{\partial y_j} = 0$ for any j .

We shall not only identify the zero section of $\sqrt{-1}T^*M$ with M but also identify a conic set of $\sqrt{-1}T^*M - M$ with the corresponding set in $\sqrt{-1}S^*M$. We consider a hyperfunction $u(x)$ as a microfunction defined *all over* $\sqrt{-1}T^*M$. Therefore it is real analytic if and only if $S.S.u(x) \subset M$.^{**)}

(i) Substitution.

Let $u(x)$ be a microfunction on $\sqrt{-1}T^*M$. $u(f(y))$ is defined as

* p_j and k_i are ν dimensional Minkowski vectors.
 ** We consider the relations between T^*X and P^*X , and \mathcal{D}_X and \mathcal{E}_X likewise.

a microfunction at $(y, \sqrt{-1}\eta) \in \sqrt{-1}T^*N$ if

$$(3.6) \quad \rho: \varpi^{-1}(S.S.u(x)) \rightarrow \sqrt{-1}T^*N$$

is proper in a neighborhood of $(y, \sqrt{-1}\eta)$. For a hyperfunction this is equivalent to

$$(3.7) \quad \varpi^{-1}(S.S.u(x)) \cap \sqrt{-1}T_N^*M = N^*.$$

We can estimate $S.S.u(f(y))$.

$$(3.8) \quad S.S.u(f(y)) \subset \rho(\varpi^{-1}(S.S.u(x))).$$

(ii) Integration.

Let $v(y)dy$ be a microfunction density on $\sqrt{-1}T^*N$. $\int_{f^{-1}(x)} v(y)dy$ is defined as a microfunction at $(x, \sqrt{-1}\xi) \in \sqrt{-1}T^*M$ if

$$(3.9) \quad \varpi: \rho^{-1}(S.S.v(y)dy) \rightarrow \sqrt{-1}T^*M$$

is proper. We can estimate $S.S. \int_{f^{-1}(x)} v(y)dy$.

$$(3.10) \quad S.S. \left(\int_{f^{-1}(x)} v(y)dy \right) \subset \varpi(\rho^{-1}(S.S.v(y)dy)).$$

(iii) Product.

Let $u_1(x)$ and $u_2(x)$ be hyperfunctions on M . $u_1(x) \cdot u_2(x)$ is defined as a hyperfunction on M if

$$(3.11) \quad S.S.u_1(x) \cap S.S.^a u_2(x) \subset M$$

where $S.S.^a u_2(x) = \{(x, \sqrt{-1}\xi) \in \sqrt{-1}T^*M \mid (x, -\sqrt{-1}\xi) \in S.S.u_2(x)\}$.

We can estimate $S.S.(u_1(x)u_2(x))$.

$$(3.12) \quad S.S.(u_1(x)u_2(x)) \subset \{(x, i\xi) \in \sqrt{-1}T^*M \mid \exists (x, i\xi_l) \in S.S.u_l(x) \\ (l=1, 2) \text{ such that } \xi_1 + \xi_2 = \xi\}.$$

We show how to apply these formulas (i) ~ (iii) to define (3.2) and estimate its singular spectrum.

First we define the integrand $\vartheta_G(p, k)$ as a hyperfunction on $N = \{(p_1, p_2, k_1, k_2)\}$. We denote by x_j and v_i the dual coordinates of p_j and k_i

* N means zero vectors.

in $\sqrt{-1}T^*N$. (3·6) is always satisfied if f is a smooth map.*) Therefore $\delta(p_1 - k_1 - k_2)$, $\delta(p_2 + k_1 + k_2)$, $(k_1^2 - m_1^2 + i0)^{-\lambda_1}$ and $(k_2^2 - m_2^2 + i0)^{-\lambda_2}$ are well-defined. Their singular spectra are given by

$$\begin{aligned}
 (3\cdot 13) \quad S.S.\delta(p_1 - k_1 - k_2) &= \{(p, k, \sqrt{-1}x_1 d(p_1 - k_1 - k_2)) \in \sqrt{-1}T^*N | \\
 &\quad p_1 - k_1 - k_2 = 0\}, \\
 S.S.\delta(p_2 + k_1 + k_2) &= \{(p, k, \sqrt{-1}x_2(p_2 + k_1 + k_2)) \in \sqrt{-1}T^*N | \\
 &\quad p_2 + k_1 + k_2 = 0\}, \\
 S.S.(k_1^2 - m_1^2 + i0)^{-\lambda_1} &= \left\{ \left(p, k, \sqrt{-1} \frac{\alpha_1}{2} d(k_1^2 - m_1^2) \right) \in \sqrt{-1}T^*N | \right. \\
 &\quad \left. \alpha_1 \geq 0, \quad \alpha_1(k_1^2 - m_1^2) = 0 \right\}, \\
 S.S.(k_2^2 - m_2^2 + i0)^{-\lambda_2} &= \left\{ \left(p, k, \sqrt{-1} \frac{\alpha_2}{2} d(k_2^2 - m_2^2) \right) \in \sqrt{-1}T^*N | \right. \\
 &\quad \left. \alpha_2 \geq 0, \quad \alpha_2(k_2^2 - m_2^2) = 0 \right\}.
 \end{aligned}$$

Then using (3·12) we have

$$\begin{aligned}
 (3\cdot 14) \quad S.S.\mathcal{O}_G(p, k) &= \{(p, k, \sqrt{-1}(x, v)) \in \sqrt{-1}T^*N | \\
 &\quad \exists \alpha_l \geq 0 \text{ such that } p_1 - k_1 - k_2 = 0, p_2 + k_1 + k_2 = 0, \\
 &\quad \alpha_1(k_1^2 - m_1^2) = 0, \alpha_2(k_2^2 - m_2^2) = 0, v_1 + x_1 - x_2 - \alpha_1 k_1 = 0, \\
 &\quad v_2 + x_1 - x_2 - \alpha_2 k_2 = 0\}.
 \end{aligned}$$

If we apply (3·10)**) we have

$$\begin{aligned}
 (3\cdot 15) \quad S.S.F_G(p) &\subset \{(p, \sqrt{-1}x) \in \sqrt{-1}T^*M | \exists k_l, \alpha_l \geq 0, \\
 &\quad \alpha_l \geq 0, p_1 - k_1 - k_2 = 0, p_2 + k_1 + k_2 = 0, \alpha_l(k_l^2 - m_l^2) = 0, \\
 &\quad x_1 - x_2 = \alpha_l k_l \quad (l=1, 2)\}.
 \end{aligned}$$

These are the well-known Landau equations. There are two irreducible components. One on which $\alpha_l \neq 0$ ($l=1, 2$) is called the leading Landau singularity and is denoted by A_G .

The other is the conormal bundle of the submanifold $p_1 + p_2 = 0$, on which $\alpha_l = 0$ ($l=1, 2$). We denote it by $A_{\text{point}, \tau(1, 2)}$ or A_{pt} in short.

The condition (3·9) fails on A_{pt} . To overcome this difficulty we

* f is smooth by definition if $T_N^*M=N$. See also the footnote on page 28.

** Note that $\rho(N \times T^*M) = \{(p, k, \sqrt{-1}(x, 0)) \in \sqrt{-1}T^*N\}$

construct a compactification of (k_1, k_2) -space. We projectify each ν vector k_1, k_2 .

$$(3.16) \quad \{(p_1, p_2, (k_1, s_1), (k_2, s_2))\} \cong \mathbf{R}^\nu \times \mathbf{R}^\nu \times \mathbf{P}^\nu \times \mathbf{P}^\nu.$$

Carrying out the integration of δ functions we can rewrite (3.2) as an integral over the space of loop momentum.

$$(3.17) \quad N_{\text{loop}} = \{(p, k) \in N \mid p_1 - k_1 - k_2 = p_2 + k_1 + k_2 = 0\}.$$

The closure of N_{loop} in (3.16) is defined by

$$(3.18) \quad p_1 - \frac{k_1}{s_1} - \frac{k_2}{s_2} = 0.$$

Therefore it is non singular and as a local coordinate we may take p_1 and (k_1, s_1) . Abbreviating suffix 1, we define the integrand at infinity as the form

$$(3.19) \quad |s|^{2\alpha_1 + \lambda_2 - \nu - 1} (k^2 - s^2 m_1^2 + i0)^{-\lambda_1} ((k - sp)^2 - s^2 m_2^2 + i0)^{-\lambda_2} d(s, k).$$

This is a hyperfunction density along the fiber with parameters λ_1, λ_2 . It is meromorphic in λ_1, λ_2 with simple poles at

$$(3.20) \quad \lambda_1 + \lambda_2 - \frac{\nu}{2} = 0, -1, \dots$$

Exercise 3.21. Examine the well-definedness of (3.19). (Hint. $+i0$ plays a role.)

Exercise 3.22. Show that the singular spectrum of the integral of (3.19) is contained in $A_G \cup A_{pt}$.

We have explained how to define a hyperfunction and estimate its singular spectrum. Further we shall explain how to construct the system which it satisfies and how to estimate its characteristic variety.

We define the formal adjoint $P^*(x', D_{x'})$ of a microdifferential operator $P(x, D_x)$ by

$$(3.23) \quad P(x, D) \delta(x - x') = P^*(x', D_{x'}) \delta(x - x').$$

Then \mathcal{E}_x operates on $\mathcal{E}_x \otimes \mathcal{Q}_x^n$ from the right by

$$(3.24) \quad (Q(x, D) dx) P(x, D) = P^*(x, D) Q(x, D) dx.$$

We define $\mathcal{E}_{Y \dashrightarrow X}$ to be the following holonomic system tensored by \mathcal{Q}_X^n from the right.

(3.25)

$$\begin{aligned} & \mathcal{E}_{Y \dashrightarrow X} \delta(x-f(y)) \otimes \mathcal{Q}_X^n \\ &= \mathcal{E}_{Y \times X} / \left(\sum_{i=1}^n \mathcal{E}_{Y \times X}(x_i - f_i(y)) + \sum_{j=1}^m \mathcal{E}_{Y \times X} \left(D_{y_j} - \sum_{i=1}^n \frac{\partial f_i}{\partial y_j} D_{x_i} \right) \otimes \mathcal{Q}_X^n \right). \end{aligned}$$

We denote the generator $\delta(x-f(y)) \otimes dx$ by $1_{Y \rightarrow X}$.

The characteristic variety of $\mathcal{E}_{Y \rightarrow X}$ is the conormal bundle of the graph Γ of $f: Y \rightarrow X$. We identify it with $Y \times T^*X$.

(3.26)
$$T_r^*(Y \times X) \cong Y \times T^*X$$

$\mathcal{E}_{Y \rightarrow X}$ is a (left \mathcal{E}_Y , right \mathcal{E}_X)-Module.

Several operations on systems.

(i) Substitution.

Let \mathcal{M}_X be a system on T^*X . The induced system

(3.27)
$$\mathcal{M}_Y = f^* \mathcal{M}_X = \rho_* (\mathcal{E}_{Y \rightarrow X} \otimes_X \mathcal{M}_X)$$

is defined as a system at $(y, \eta) \in T^*Y$ if

(3.28)
$$\rho: \varpi^{-1}(S.S.\mathcal{M}_X) \rightarrow T^*Y$$

is proper (hence with finite fibers) in a neighborhood of (y, η) . If (3.28) holds we say \mathcal{M}_X is non-characteristic with respect to f . We can estimate $S.S.\mathcal{M}_Y$.

(3.29)
$$S.S.\mathcal{M}_Y \subset \rho(\varpi^{-1}(S.S.\mathcal{M}_X)).$$

If \mathcal{M}_X is holonomic, \mathcal{M}_Y is holonomic. If \mathcal{M}_X is simple holonomic with a generator u and ϖ is transversal to $A = S.S.\mathcal{M}_X$ and $\rho|_{\varpi^{-1}(A)}$ is an embedding then $f^*u = 1_{Y \rightarrow X} \otimes u \in \mathcal{M}_Y$ generates a simple holonomic system and

(3.30)
$$\text{ord}_{\rho(\varpi^{-1}(A))}(f^*u) = \text{ord}_A(u).$$

Exercise 3.31. If Y is a submanifold of codimension 1 in X and $\mathcal{D}_X = \mathcal{D}_X / \mathcal{D}_X P(x, D)$, what is the condition (3.32) and what is \mathcal{M}_Y ?

Exercise 3.32. If $Y = \{x_1 = \dots = x_d = 0\} \subset X = \{(x_1, \dots, x_d, y_1, \dots, y_{n-d})\}$ and $\mathcal{M}_X = \mathcal{E}_X u = \mathcal{E}_X / \mathcal{I}_X$, then the ideal which annihilates $1_{Y \rightarrow X} \otimes u$ consists of $P(y, D_y)$ such that $P(y, D_y) + \sum_{i=1}^d x_i R_i(x, y, D_x, D_y) \in \mathcal{I}_X$.

(ii) Integration

Let \mathcal{M}_Y be a right coherent Module on T^*Y . The induced system

$$(3.31) \quad f_* \mathcal{M}_Y = \varpi_* (\mathcal{M}_Y \otimes_{\mathcal{E}_Y} \mathcal{E}_{Y \rightarrow X})$$

is defined as a right coherent Module at $(x, \xi) \in T^*X$ if

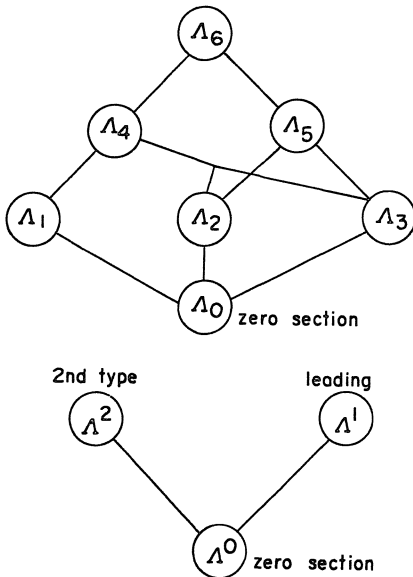
$$(3.32) \quad \varpi: \rho^{-1}(S.S. \mathcal{M}_Y) \rightarrow T^*X$$

is proper and with finite fibers. We can estimate $S.S.f_* \mathcal{M}_Y$.

$$(3.33) \quad S.S.f_* \mathcal{M}_Y \subset \varpi(\rho^{-1}(S.S. \mathcal{M}_Y)).$$

If \mathcal{M}_Y is holonomic, $f_* \mathcal{M}_Y$ is holonomic. If \mathcal{M}_Y is simple holonomic with a generator u and ρ is transversal to $A = S.S. \mathcal{M}_Y$ and $\varpi|_{\rho^{-1}(A)}$ is an embedding, then $f_* u = u \otimes 1_{Y \rightarrow X} \in f_* \mathcal{M}_Y$ generates a simple holonomic right Module and

$$(3.34) \quad \begin{aligned} \text{ord}_A(dy^{-1} \otimes u) - \frac{1}{2} \dim Y \\ = \text{ord}_{\rho^{-1}(A)}(dx^{-1} \otimes f_* u) - \frac{1}{2} \dim X. \end{aligned}$$



$$\begin{aligned} \pi(A_i) \quad (i=1, \dots, 6) \\ i=1 \quad t=0 \\ 2 \quad k^2 - t^2 m_2^2 = 0 \\ 3 \quad (tp - k)^2 - t^2 m_1^2 = 0 \\ 4 \quad t = k^2 = 0 \\ 5 \quad k^2 - t^2 m_2^2 = 0, \\ \quad tp^2 - 2p \cdot k + t(m_2^2 - m_1^2) = 0 \\ 6 \quad t = k^2 = p \cdot k = 0 \end{aligned}$$

$$\begin{aligned} \pi(A^i) \quad (i=1, 2) \\ i=1 \quad p^2 - (m_1 + m_2)^2 = 0 \\ 2 \quad p^2 = 0 \end{aligned}$$

Each segment represents an intersection of codimension 1.

Fig. 3.36 The holonomy diagram of a self energy graph.

Exercise 3.35. If f is a projection $Y = \{(x, y)\} \rightarrow X = \{x\}$ and $\mathcal{M}_Y \mathcal{E}_Y u = \mathcal{E}_Y / \mathcal{J}_Y$, then the ideal which annihilates $u \otimes 1_{Y \rightarrow X}$ consists of $P(x, D_x)$ such that $P(x, D_x) + \sum_{i=1}^m D_{y_i} R_i(x, y, D_x, D_y) \in \mathcal{J}_Y$. The condition (3.23) is slightly restrictive. It can be weakened for holonomic systems under some natural conditions so that the finiteness of the fiber is unnecessary. We give the holonomy diagram of the integrand and the integral of (3.2) after carrying out an integration of δ function. (Fig. 3.36)

§ 4. Quantized Contact Transformation

A contact transformation

$$(4.1) \quad F: T^*X \rightarrow T^*Y$$

is an isomorphism defined by

$$(4.2) \quad \begin{aligned} \eta_i &= f_i(x, \xi) \\ y_i &= g_i(x, \xi) \quad (i=1, \dots, n), \end{aligned}$$

such that $f_i(x, \xi)$ is homogeneous of degree 1 with respect to ξ and $g_i(x, \xi)$ of degree 0 and the canonical 1 form $\sum_{i=1}^n \eta_i dy_i$ is pulled back to $\sum_{i=1}^n \xi_i dx_i$.

$$(4.3) \quad \begin{aligned} F^* \left(\sum_{i=1}^n \eta_i dy_i \right) &= \sum_{i=1}^n f_i(x, \xi) dg_i(x, \xi) \\ &= \sum_{i=1}^n \xi_i dx_i. \end{aligned}$$

There are two projections $\pi_X: T^*(X \times Y) \rightarrow T^*X$ and $\pi_Y: T^*(X \times Y) \rightarrow T^*Y$ defined by $\pi_X(x, y, \xi, -\eta) = (x, \xi)$ and $\pi_Y(x, y, \xi, -\eta) = (y, \eta)$.

Let us consider the graph A of F .

$$(4.4) \quad \begin{aligned} A &= \{(x, y, \xi, -\eta) \in T^*(X \times Y) \mid \eta_i = f_i(x, \xi), \\ &\quad y_i = g_i(x, \xi), \quad (i=1, \dots, n)\}. \end{aligned}$$

A is holonomic, and conversely if the graph of an isomorphism is holonomic it is a contact transformation.

Exercise 4.5. Check the above equivalence.

A holonomic manifold is given by a generating function. We explain this in the case $A \subset T^*X = \{(x_1, \dots, x_n; \xi_1, \dots, \xi_n)\}$.

We write $x' = (x_1, \dots, x_d)$, $x'' = (x_{d+1}, \dots, x_n)$, $\xi' = (\xi_1, \dots, \xi_d)$ and $\xi'' = (\xi_{d+1}, \dots, \xi_n)$. A function of the form $F(x', \xi'')$ and homogeneous of degree 1 with respect to ξ (we assume $d \neq n$) is called a generating function. A defined by

$$(4.6) \quad A = \left\{ (x, \xi) \in T^*X \mid \xi' = -\frac{\partial F}{\partial x'}(x', \xi''), \quad x'' = \frac{\partial F}{\partial \xi''}(x', \xi'') \right\}$$

is holonomic and any holonomic manifold is given likewise.

Exercise 4.7. Check the above equivalence.

Example 4.8.

Consider the icecream cone diagram (Fig. 4.8).

Icecream cone diagram G .

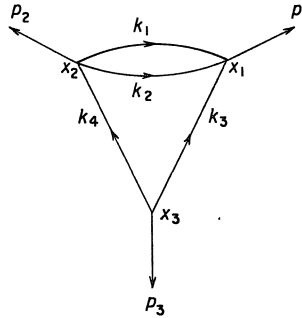


Fig. 4.8

The leading Landau singularity A_G is given by Landau equations.

$$(4.9) \quad \begin{aligned} p_1 &= k_1 + k_2 + k_3, & p_2 &= -k_1 - k_2 + k_4, & p_3 &= -k_3 - k_4, \\ k_l^2 &= m_l^2 & (l=1, 2, 3), \\ x_1 - x_2 &= \alpha_1 k_1 = \alpha_2 k_2, & x_1 - x_3 &= \alpha_3 k_3, & x_2 - x_3 &= \alpha_4 k_4. \end{aligned}$$

We can take a generating function

$$(4.10) \quad \begin{aligned} H(x) &= m_1 \sqrt{(x_1 - x_2)^2} + m_2 \sqrt{(x_1 - x_2)^2} + m_3 \sqrt{(x_1 - x_3)^2} \\ &\quad + m_4 \sqrt{(x_2 - x_3)^2}. \end{aligned}$$

Exercise 4.11. Check $p = \frac{\partial H}{\partial x}$ is equivalent to (4.9)

(Hint. $k_1 = \frac{m_1(x_1 - x_2)}{\sqrt{(x_1 - x_2)^2}}$, $\alpha_1 = \frac{\sqrt{(x_1 - x_2)^2}}{m_1}$ etc.)

We call $H(x)$ a potential function. Consider a generating function

$$(4.11) \quad \sum_{j=1}^n p_j \tilde{x}_j - (m_3 \sqrt{(x_1 - x_3)^2} + m_4 \sqrt{(x_2 - x_3)^2}).$$

It defines the following contact transformation. (Here we let $k_3 = \frac{m_3(x_1 - x_3)}{\sqrt{(x_1 - x_3)^2}}$

and $k_4 = \frac{m_4(x_2 - x_3)}{\sqrt{(x_2 - x_3)^2}}.$)

$$(4.12) \quad \begin{aligned} \tilde{p}_1 &= p_1 - k_3, & \tilde{p}_2 &= p_2 - k_4, & \tilde{p}_3 &= p_3 + k_3 + k_4, \\ \tilde{x}_j &= x_j & (j &= 1, 2, 3). \end{aligned}$$

(4.12) reduces (4.9) to

$$(4.13) \quad \begin{aligned} \tilde{p}_1 &= k_1 + k_2, & \tilde{p}_2 &= -k_1 - k_2, & \tilde{p}_3 &= 0, \\ k_l^2 &= m_l^2 & (l &= 1, 2), \\ \tilde{x}_1 - \tilde{x}_2 &= \alpha_1 k_1 = \alpha_2 k_2. \end{aligned}$$

This is the leading Landau singularity $A_{G_{I_0 - \{3,4\}}}$ of the subgraph $G_{I_0 - \{3,4\}}$ obtained from (4.7) by deleting two lines k_3 and k_4 . (Fig. 4.14)

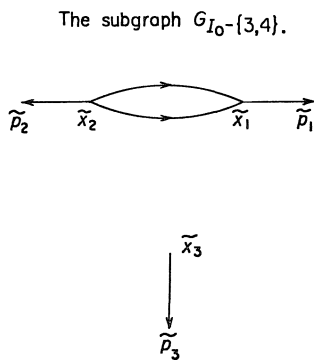


Fig. 4.14

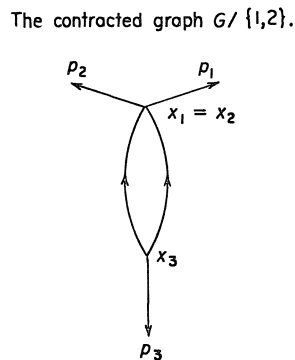


Fig. 4.15

Exercise 4.15. Into which the non-leading component $A_{G/\{1,2\},\tau}$ (corresponding to the contracted graph $G/\{1,2\}$, Fig. (4.15)) is transformed

by the above contact transformation? Show that A_G and $A_{G',(1,2),r}$ have a codimension 1 intersection.

Let $A \subset T^*(X \times Y)$ be the graph of a contact transformation (4.2) and \mathcal{M} be a simple holonomic system with generator $K(x, y)$ with the characteristic variety A . On $\mathcal{M} \otimes_{\Omega_X^n} \mathcal{E}_X$ operates from the left and \mathcal{E}_Y from the right.

The equation

$$(4.16) \quad \begin{aligned} P(x, D_x)K(x, y)dy \\ = K(x, y)dyQ(y, D_y) \\ = Q^*(y, D_y)K(x, y)dy \end{aligned}$$

gives an isomorphism between \mathcal{E}_X and \mathcal{E}_Y . Because \mathcal{M} is simple, there exist microdifferential equations of the forms

$$(4.17) \quad \begin{aligned} \{D_{y_i} - f_i(x, D_x) + \text{lower order}\}K(x, y) = 0 \\ \{y_i - g_i(x, D_x) + \text{lower order}\}K(x, y) = 0 \quad (i=1, \dots, n). \end{aligned}$$

We can eliminate D_{y_i} 's and y_i 's from the lower order terms of (4.17) and get the isomorphism (4.16).*) Therefore principal symbols are invariant under a quantized contact transformation. But we cannot specify a unique quantized contact transformation for a given contact transformation.

If M and N are real, and $F: T^*M \rightarrow T^*N$ is a real contact transformation, the integral transformation defined by the kernel function $K(x, y)dy$ is an isomorphism from \mathcal{E}_N to \mathcal{E}_M .

$$(4.18) \quad \begin{aligned} \mathcal{E}_N \rightarrow \mathcal{E}_M \\ u(y) \mapsto \int K(x, y)u(y)dy. \end{aligned}$$

Exercise 4.18. Apply the formulas given in § 3 to study (4.16).

Exercise 4.19. Show that these two isomorphisms of \mathcal{E} and \mathcal{E} are compatible.

* We call this isomorphism a quantized contact transformation.

Example 4.20. Quantization of (4.12).

We take as a kernel function

$$(4.19) \quad F_{G_{I_0-\{1,2\}}}(p-\tilde{p})d\tilde{p}$$

where $F_{G_{I_0-\{1,2\}}}(p)$ is the Feynman integral of the subgraph obtained from (4.8) by deleting two lines k_1 and k_2 .

Then it transforms $F_{G_{I_0-\{1,2\}}} |_{A_{G_{I_0-\{3,4\}} \cup A_{p^i, \tau}}$ into $F_G |_{A_G \cup A_{G/\{1,2\}, \tau}}$.

Structure Theorem 4.21. *Let \mathcal{M} be a system of microdifferential equations with one generator. Assume that*

$$(4.22) \quad \text{the characteristic variety } V \text{ is non-singular of codimensions } d \text{ (} 1 \leq d \leq n-1 \text{) and } \sum_{i=0}^n \xi_i dx_i |_V \neq 0, \text{ and that}$$

$$(4.23) \quad \text{the symbol ideal } J \text{ is reduced.}$$

Then \mathcal{M} is transformed into

$$(4.24) \quad D_1 u = 0, \dots, D_d u = 0.$$

To prove this theorem we first straighten the characteristic variety by a suitable contact transformation. Then by a corresponding quantized contact transformation the system is transformed into

$$(4.25) \quad (D_1 + Q_1(x, D))v = 0, \dots, (D_d + Q_d(x, D))v = 0.$$

Then using the following two theorems (4.25) is reduced to (4.24).

Theorem 4.26. *Let $P(x, D)$ and $Q(x, D)$ be microdifferential operators with the same principal symbol $P_m(x, \xi)$. Consider a point (x_0, ξ_0) where $P_m(x, \xi)$ vanishes and assume that*

$$(4.27) \quad d_{(x, \xi)} P_m(x, \xi) \not\propto \sum_{i=1}^n \xi_i dx_i.$$

Then there exists an elliptic microdifferential operator $R(x, D)$ such that

$$(4.28) \quad RPR^{-1} = Q.$$

Theorem 4.29. *Let $P(x, D)$ be a microdifferential operator de-*

fixed at $(0, \dots, 0, \infty(1, 0, \dots, 0)) \in P^*X$. Assume that $\sigma(P)(0, \dots, 0, 1, 0, \dots, \xi_n)/\xi_n^p$ is holomorphic and never vanishes at $\xi_n=0$. Then any microdifferential operator $Q(x, D)$ can be divided by $P(x, D)$ with residue $R(x, D) = \sum_{j=1}^{p-1} R^{(j)}(x, D') D_n^j$ where $D' = (D_1, \dots, D_{n-1})$.

$$(4.30) \quad Q = SP + R.$$

We shall explain structure theorems for holonomic systems and holonomic hyperfunctions.

(i) Let \mathcal{M} be a holonomic system with support A . We assume that A is non-singular and \mathcal{M} is simple.

By a suitable contact transformation, A is transformed into

$$(4.21) \quad x_1 = \xi_2 = \dots = \xi_n.$$

But (4.27) fails to hold and the structure admits a parameter λ and takes the form

$$(4.22) \quad \mathcal{E}_x/\mathcal{E}_x(x_1 D_1 - \lambda) + \mathcal{E}_x D_2 + \dots + \mathcal{E}_x D_n^*).$$

(ii) Let $u(x)$ be a hyperfunction such that $\text{sp } u$ satisfies a holonomic system $\mathcal{M} = \mathcal{E}u$ with support A , a conormal bundle of a hypersurface $\{f(x) = 0\}$ where $\text{grad } f(x) \neq 0$. We assume that \mathcal{M} is simple and $\text{ord}_A(u) = -\left(\lambda + \frac{1}{2}\right)$. We take $x_1 = f(x)$, then by a suitable elliptic microdifferential operator $Q(x, D)$ of order 0 defined at $(0, dx_1) \in P^*X$ we can represent $\mathcal{M} = \mathcal{E}v$ as follows.

$$(4.23) \quad (x_1 D_1 - \lambda)v = 0, \quad D_2 v = 0, \dots, D_n v = 0,$$

where $v = Q(x, D)u$.

Then using (2.8) and (1.56) we can determine the form of $u(x)$.

$$(4.24) \quad \begin{aligned} \text{I. } (\lambda \notin \mathbf{Z}) \quad & F_1(x) (f(x) + i0)^\lambda + F_2(x) (f(x) - i0)^\lambda + F_3(x) \\ \text{II. } (\lambda = 0, 1, 2, \dots) \quad & F_1(x) f(x)^\lambda \log(f(x) + i0) \\ & + F_2(x) f(x)^\lambda \log(f(x) - i0) + F_3(x) \\ \text{III. } (\lambda = -1, -2, \dots) \quad & F_1(x) (f(x) + i0)^\lambda + F_2(x) (f(x) - i0)^\lambda \\ & + F_3(x) \log(f(x) + i0) + F_4(x) \log(f(x) - i0) \end{aligned}$$

* Different λ 's give isomorphic systems if and only if the difference is an integer. If we use micro-differential operators of fractional order any λ gives the same system.

where $F_1(x)$ and $F_2(x)$ are real analytic and either $\equiv 0$ or $\neq 0$, $F_3(x)$ and $F_4(x)$ are real analytic.

Example 4.25.

We have transformed

$$F_G|_{A_G \cup A_{G/(1,2),\tau}} \text{ to } F_{G_{I_0-\{3,4\}}}|_{A_{G_{I_0-\{3,4\}}} \cup A_{\text{pt},\tau}}$$

in (4.20). Now we can determine the structure of $F_{G_{I_0-\{3,4\}}}$.

In general $F_G(p)$ satisfies a simple holonomic system of order

$$(4.26) \quad \text{ord}_{A_G} F_G = \frac{1}{2} \{N + \nu(n - N)\}$$

where $N = \# \{\text{internal lines}\}$ and $n = \# \{\text{vertices}\}$. We can write

$$(4.27) \quad F_{G_{I_0-\{3,4\}}}(\tilde{p}_1, \tilde{p}_2, \tilde{p}_3) = f(\tilde{p}_1) \delta(\tilde{p}_1 + \tilde{p}_2) \delta(\tilde{p}_3).$$

Then $f(\tilde{p}_1)$ is a holonomic hyperfunction with singular spectrum in

$$(4.28) \quad \{(\tilde{p}_1, \sqrt{-1} \infty \tilde{p}_{11} \cdot d\tilde{p}_{11}) | \tilde{p}_1^2 = (m_1 + m_2)^2\}$$

and of order $1 - \frac{\nu}{2}$ or $\lambda = \frac{\nu - 3}{2}$.

Thus we have one of the canonical forms of (4.24), where $F_2(x) \equiv 0$.*)

Similarly the structure of a Feynman amplitude at a generic point of any intersection $A_{G/I_1,\tau} \cup A_{G/I_1 \cup I_2,\tau}$ of codimension 1 is one of the forms I~III with $F_2(x) \equiv 0$.*)

(iii) Let \mathcal{M} be a holonomic system with support $A = A_1 \cup A_2$. We assume that A_1 , A_2 and $A_1 \cap A_2$ are non-singular and $A_1 \cap A_2$ has dimensions $n - 1$. Moreover we assume that $T_p(A_1 \cap A_2) = (T_p A_1) \cap (T_p A_2)$ for any $p \in A_1 \cap A_2$, and the symbol ideal J is reduced. Then \mathcal{M} is transformed to Example (2.53). In (2.53) $\beta = \text{ord}_{A_2}(u) - \text{ord}_{A_1}(u) - \frac{1}{2}$ has an essential meaning. According to the value of β , \mathcal{M} is classified into three classes. This we have already explained in Summarize (1.30) by examining the solution space, where we use λ in place of β .

Now as a final remark we explain the difference of their algebraic properties. (The proof is straightforward.)

* In case III, $F_4(x) \equiv 0$.

Theorem 4.29.

- I. If $\beta \notin \mathbf{Z}$, \mathcal{M} is an irreducible \mathcal{E}_X -Module.
II. If $\beta = 0, 1, 2, \dots$ there is an exact sequence of \mathcal{E}_X -Modules

$$0 \rightarrow \mathcal{M}_1 \rightarrow \mathcal{M} \rightarrow \mathcal{M}_2 \rightarrow 0$$

where $\text{supp } \mathcal{M}_i \subset A_i$ ($i=1, 2$).

- III. If $\beta = -1, -2, \dots$ there is an exact sequence of \mathcal{E}_X -Modules

$$0 \rightarrow \mathcal{M}_2 \rightarrow \mathcal{M} \rightarrow \mathcal{M}_1 \rightarrow 0$$

where $\text{supp } \mathcal{M}_i \subset A_i$ ($i=1, 2$).