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Existence and Continuation of Holomorphic Solutions of Partial Differential Equations

by

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Abstract

Necessary conditions and sufficient conditions are given for the existence and the continuation of holomorphic solutions of partial differential equations near the characteristic boundary of an open subset of C^n.**

§ 1. Definitions and Results

Let Ω be an open subset of $M=C^{n+1}$ defined by the equation:

$$\Omega=\{z\in C^{n+1}|\varphi(z)>0\}$$

where φ is a real analytic function with real values and $d\varphi(z)\neq 0$ when $\varphi(z)=0$.

If $\mathcal{O}$ designates the sheaf of holomorphic functions on M, we denote by $\mathcal{O}^+$ the sheaf on $N=\partial\Omega$ defined by this stalk at $z\in N$:

$$\mathcal{O}_z{}^+=\varinjlim_{z\in\omega}\mathcal{O}(\omega^+),\ \omega^+=\omega\cap\Omega$$

where ω runs over a fundamental system of neighbourhoods of z in M.

Let P be a differential operator with holomorphic coefficients defined in a neighbourhood of N:

$$P(z, D_z)=\sum_{|\alpha|\leqq m}a_\alpha D_z{}^\alpha.$$

We give necessary conditions and sufficient conditions to have one of the following properties:

$(PR)_z$ Continuation: $f\in\mathcal{O}_z{}^+$ and $Pf\in\mathcal{O}_z\Rightarrow f\in\mathcal{O}_z$

$(EX)_z$ Existence: $\forall g\in\mathcal{O}_z{}^+$, $\exists f\in\mathcal{O}_z{}^+$ so that $Pf=g$

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** An integral version of this paper has appeared in [3].

If N is non-characteristic with respect to P at z, it is known that we have the properties $(PR)_z$ and $(EX)_z$ ([5], [1]).

If N is simply characteristic with respect to P at z, Y. Tsuno ([4]) has given a geometric sufficient condition and a necessary condition to have the property $(PR)_z$.

We will compare later his conditions with ours.

Later, we shall need the $\bar{\partial}_b$-system of tangential Cauchy-Riemann equations on N. We recall its definition briefly.

The problem being local, we can suppose that $\frac{\partial\varphi}{\partial\bar{z}_0}\neq 0$ near $z=(z_0, \cdots, z_n)$. We can define $\bar{\partial}_b$ on N by:

$$\bar{\partial}_b u=0 \Leftrightarrow X_j u=0\,, \quad j=1, \cdots, n\,,$$

where $X_j=\frac{\partial}{\partial\bar{z}_j}-\frac{\partial\varphi}{\partial\bar{z}_j}\left(\frac{\partial\varphi}{\partial\bar{z}_0}\right)^{-1}\frac{\partial}{\partial\bar{z}_0}$.

If we denote by Σ the real characteristic variety of the system $\bar{\partial}_b$ in S^*N, Σ is a reunion of two cotangent vector fields Σ^+ and Σ^- where

$$\Sigma^+(z)\,(z, \xi(z))\in S^*N,\ \xi_j(z)=-i\frac{\partial\varphi}{\partial\bar{z}_j}(z), \quad j=0, \cdots, n,$$

Σ^- is the antipodal of Σ^+ in S^*N.

Now we define an operator on N associated with P. Let (Y_j), $j=0, \cdots, n$, be a family of $n+1$ complex vector fields on N defined by

$$Y_j=\frac{\partial}{\partial z_j}-\frac{\partial\varphi}{\partial z_j}\left(\frac{\partial\varphi}{\partial\bar{z}_0}\right)^{-1}\frac{\partial}{\partial\bar{z}_0}\,.$$

We have $[Y_j, Y_k]=0$, $\forall j, k=0, \cdots, n$ and the family Y_j, $j=0, \cdots, n$ is linearly independent, so we can define the operator P_b on N by:

$$P_b=\sum_{|\alpha|\leq m}\tilde{a}_\alpha Y^\alpha \quad \text{where} \quad \tilde{a}_\alpha=a_\alpha|_N\,.$$

Note that:

N is characteristic with respect to P at $z\Leftrightarrow\sigma(P_b)(\Sigma^+(z))=0$ (where $\sigma(Q)$ designates the principal symbol of a differential operator Q).

Let $p_b=0$ be a reduced equation of the complex characteristic variety of the operator P_b.

Definition. The generalized Levi-form of (Ω, P) at z, denoted by

L_z, is the hermitian form on $\boldsymbol{C}^{n+1}$ defined by

$$\tau \in \boldsymbol{C}^{n+1},\ L_z(\tau) = \sum_{1\leq j,k\leq n} \{\sigma(X_k), \overline{\sigma(X_j)}\}(\Sigma^+(z))\tau_k\bar{\tau}_j$$

$$+2\operatorname{Re}\sum_{1\leq j\leq n}\{\sigma(X_j), \bar{p}_b\}(\Sigma^+(z))\tau_j\tau_{n+1}$$

$$+\{p_b, \bar{p}_b\}(\Sigma^+(z))|\tau_{n+1}|^2,$$

where $\{f, g\}$ designates the Poisson bracket of two homogeneous functions on S^*M.

Suppose that N is characteristic for P at z, that is:

$$\sigma(P)(z, d\varphi(z)) = 0$$

then we have the following:

Theorem I. *If there is* $\tau \in \boldsymbol{C}^{n+1}$ *so that* $L_z(\tau) < 0$, *we have the property* $(PR)_z$.

Theorem II. *If* Ω *is strictly pseudo-convex at* z *we have*

$$\det L_z > 0 \Rightarrow (EX)_z \text{ and no } (PR)_z,$$

$$\det L_z < 0 \Rightarrow (PR)_z \text{ and no } (EX)_z.$$

Remark I. After a change of coordinate near $z^0 \in N$ which transforms the function φ in Ψ defined by

$$\Psi(z) = \operatorname{Im} z_0 + \sum_{1\leq j,k\leq n} \frac{\partial^2\varphi}{\partial z_j \partial \bar{z}_k}(z^0) z_j \bar{z}_k$$

the generalized Levi-form has the same signature as that of the hermitian form

$$-L_{z^0}(\tau) = \sum_{1\leq j,k\leq n} \frac{\partial^2\varphi}{\partial z_j \partial \bar{z}_k}(z^0)\tau_j\bar{\tau}_k$$

$$-2\operatorname{Im}\sum_{1\leq j\leq n} p_{(j)}(z^0, d\varphi(z^0))\tau_j\bar{\tau}_{n+1}$$

$$+\sum_{1\leq j,k\leq n} p^{(j)}(z^0, d\varphi(z^0))\cdot\overline{p^{(k)}(z^0, d\varphi(z^0))}\frac{\partial^2\varphi}{\partial z_j \partial \bar{z}_k}(z^0)|\tau_{n+1}|^2.$$

Here we have used the usual notations

$$p_{(j)}=\frac{\partial p}{\partial z_j}, \quad p^{(j)}=\frac{\partial p}{\partial \xi_j}.$$

Remark II. (Y. Tsuno's result) Suppose that the normal of N at z is simply characteristic and denote by $(z(\tau), \xi(\tau))$ the complex bicharacteristic curve of P issued from $(z, d\varphi(z))$. The result of [4] which is comparable with ours is:

If there is $\tau_0 \in \boldsymbol{C}$ so that $\frac{d^2}{dt^2}\varphi(z(t\tau_0))|_{t=0}<0$ $(t\in \boldsymbol{R})$ then we have the property $(PR)_z$.

In fact that is exactly:

If there is $\tau=(\tau_0, \cdots, \tau_{n+1}) \in \boldsymbol{C}^{n+1}$ so that:

$$\forall j=1, \cdots, n, \ \tau_j=\tau_{n+1}\cdot p^{(j)}(z^0, d\varphi(z^0)) \quad \text{and} \quad L_z(\tau)<0$$

we have $(PR)_z$.

Remark III. By geometrical arguments, we can show that the condition to have the property $(PR)_z$ can be applied if only φ is in the class of C^3.

§ 2. Sketch of the Proof

We prove these results in two steps. To begin with we show that $(PR)_z$ and $(EX)_z$ are equivalent to properties of the sheaf of microfunction solutions of an induced system of differential equations on N. It is a consequence of a more general result of Kashiwara-Kawai [2], but we give a direct elementary proof and we calculate explicitly the induced system. More precisely the first result is the following:

If $\tilde{\mathscr{D}}$ designates the sheaf on N of the differential operators and $\tilde{\mathscr{C}}$ the sheaf on S^*N of microfunctions, if $\mathscr{M}_{(\bar{\partial}_b, P_b)}$ designates the $\tilde{\mathscr{D}}$-module associated to the system of differential equations $(\bar{\partial}_b, P_b)$, we have

Lemma I. (i) $(PR)_z \Leftrightarrow \mathscr{H}om_{\tilde{\mathscr{D}}}(\mathscr{M}_{(\bar{\partial}_b, p_b)}, \tilde{\mathscr{C}})_{\Sigma^+(z)}=0$.

(ii) *If Ω is strictly pseudo-convex at z, we have* $(EX)_z \Leftrightarrow \mathscr{E}xt^1_{\tilde{\mathscr{D}}}(\mathscr{M}_{(\bar{\partial}_b, p_b)}, \tilde{\mathscr{C}})_{\Sigma^+(z)}=0$.

The second step is to study the vanishing of the group $\mathscr{E}xt^k_{\tilde{\mathscr{D}}}(\mathscr{M}_{(\bar{\partial}_b, P_b)},$

$\widetilde{\mathscr{C}})_{\Sigma^+(z)}$. For this we make use of the structure theorem of [S.K.K] for a system of microdifferential equations which has a non involutive real characteristic variety.

We show that the generalized Levi-form of the system $(\bar{\partial}_b, P_b)$ in the sense of [S.K.K] is L_z. The only difficulty is to prove the following:

Lemma II. *The compex characteristic variety* $SS(\mathscr{M}_{(\bar{\partial}_b, P_b)})$ *of the system* $(\bar{\partial}_b, P_b)$ *is defined by the equation*

(1) $SS(\mathscr{M}_{(\bar{\partial}_b, P_b)}) = \{z^* \in P^*Y | \sigma(X_1)(z^*) = \cdots = \sigma(X_n)(z^*) = p_b(z^*) = 0\}$

where P^*Y *is the projective bundle of a complexification* Y *of* N.

In fact we have in general:

$$SS(\mathscr{M}_{(\bar{\partial}_b, P_b)}) \subset V$$

where V denotes the right hand side of (1).

To prove $V \subset SS(\mathscr{M}_{(\bar{\partial}_b, P_b)})$, we must prove that:
if $z^* \in V$, $\forall Q_0, \cdots, Q_n \in \widetilde{\mathscr{E}}_{z^*}$, we have

$$\sum Q_i \hat{X}_i + Q_0 \hat{P}_b = 1_{\widetilde{\mathscr{E}}}$$

where $\hat{R}$ denotes the complexification of $R \in \widetilde{\mathscr{D}}$ and $\widetilde{\mathscr{E}}$ the sheaf on P^*Y of the microdifferential operators ([S.K.K]).

To prove this, using the Frobenius theorem, we find a local coordinate of Y, $Z_1, \cdots, Z_{2n+1}$ so that $\hat{X}_i$ is transformed to $\dfrac{\partial}{\partial Z_i}$ $\forall i = 1, \cdots, n$, and in this situation we prove the lemma 2 using the symbolic calculus on micro-differential operators of [S.K.K].

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