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Complex-dimensional Integral and Light-cone Singularities

by

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Abstract

The notion of a complex-dimensional integral is introduced in the complex n-dimensional Minkowski space. Its basic properties, such as Lorentz invariance, are investigated. Complex-dimensional invariant delta functions $\Delta_n(x; m^2)$, $\Delta^{(1)}{}_n(x; m^2)$, etc. are explicitly calculated in position space. It is proposed to define products of singular functions in the ordinary Minkowski space by analytically continuing the corresponding n-dimensional ones to $n=4$. The light-cone singularities of $[\Delta(x; m^2)]^2$, $\Delta(x; m^2) \times \Delta^{(1)}(x; m^2)$ and $[\Delta^{(1)}(x; m^2)]^2$ are shown to be unambiguously determined in this way.

Recently, in quantum field theory, much attention has been paid to complex-dimensional regularization [1]. The momentum-space Feynman integral is regularized by considering it in the complex n-dimensional space formally. The extension of the dimension 4 to the complex dimension n is easily done in the Feynman-parametric representation of the Feynman integral. The purpose of my talk is to formulate the theory of complex-dimensional integrals in the general framework and apply it to regularizing singular products in position space. Detailed accounts are presented in my papers [2, 3].

The complex n-dimensional Minkowski space M^n is a product of a one-dimensional Euclidean space $\boldsymbol{R}$ and a complex $(n-1)$-dimensional space E^{n-1} such that the scalar product in M^n is defined by the difference between the product in $\boldsymbol{R}$ and the scalar product in E^{n-1}. Here E^{n-1} is an abstract vector space equipped with a real-valued, symmetric scalar product. Except for the case in which n is a positive integer, however, E^{n-1} is *not* a topological space and therefore the number of linearly independent vectors in it is *indefinite* because it has no complete basis. It is assumed that any finite-dimensional subspace of E^{n-1} is a Euclidean

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space. The notion of components $p_1, p_2, \cdots$ of a vector $\boldsymbol{p} \in E^{n-1}$ is meaningful only with reference to such a subspace. The index μ of a vector $p_\mu \in M^n$ takes discrete values only when one works in a finite-dimensional subspace of M^n.

Let $F(p_\mu)$ be a tempered distribution (or a Fourier hyperfunction) of scalar products $p^2, px^{(1)}, \cdots, px^{(k)}$, where p_μ is an integration vector and $x_\mu{}^{(1)}, \cdots, x_\mu{}^{(k)}$ are constant vectors of M^n. Then I define the complex-dimensional integral of $F(p_\mu)$ by

$$(1)\qquad \int d^n p F(p_\mu) \equiv \frac{2\pi^{(n-k-1)/2}}{\Gamma((n-k-1)/2)} \int_{-\infty}^{+\infty} dp_0 \int_{-\infty}^{+\infty} dp_1 \cdots \int_{-\infty}^{+\infty} dp_k \int_0^\infty dp_\perp p_\perp^{n-k-2} F(p_0; p_1, \cdots, p_k; p_\perp).$$

Here $p_1, \cdots, p_k$ are orthogonal coordinates in a generically k-dimensional subspace spanned by the spatial parts $\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(k)}$ of $x_\mu{}^{(1)}, \cdots, x_\mu{}^{(k)}$, and

$$(2)\qquad p_\perp{}^2 \equiv \boldsymbol{p}^2 - \sum_{j=1}^{k} p_j{}^2 .$$

If $\boldsymbol{x}^{(1)}, \cdots, \boldsymbol{x}^{(k)}$ happen to be linearly dependent, that is, for example, F is independent of p_k, then setting $p_\perp'^2 = p_\perp^2 + p_k{}^2$, one can easily see that (1) reduces to the expression which is the same as (1) except that k is replaced by $k-1$. Thus the definition (1) does not intrinsically depend on k. From this fact it follows that (1) is invariant under a translation of the integration vector p_μ, as it should be. Of course, (1) reduces to the ordinary n-dimensional multiple integral when n is a positive integer.

The complex-dimensional integral defined by (1) is not manifestly Lorentz invariant, but its Lorentz invariance can be proved. More precisely, (1) can be shown to be a quantity depending only on scalar products formed from $x_\mu{}^{(1)}, \cdots, x_\mu{}^{(k)}$. The proof is carried out by reducing the problem to that for the complex-dimensional Fourier transform[1)]

$$(3)\qquad \int d^n p e^{-ipx} \varphi(p^2) = (2\pi)^{(n-1)/2} \int_{-\infty}^{+\infty} dp_0 \int_0^\infty d|\boldsymbol{p}|\, |\boldsymbol{p}|^{(n-1)/2} |\boldsymbol{x}|^{-(n-3)/2} J_{(n-3)/2}(|\boldsymbol{p}||\boldsymbol{x}|)\, e^{-ip_0 x_0} \varphi(p_0{}^2 - |\boldsymbol{p}|^2),$$

where J_ν denotes a Bessel function.

[1] The right-hand side follows from the polar-coodinate form of (1) with $k=1$.

The complex-dimensional invariant delta functions are defined by

$$\Delta_n(x; m^2) \equiv -i(2\pi)^{-n+1} \int d^n p \epsilon(p_0)\delta(p^2 - m^2) e^{-ipx}, \tag{4}$$

$$\Delta^{(1)}{}_n(x; m^2) \equiv (2\pi)^{-n+1} \int d^n p \delta(p^2 - m^2) e^{-ipx}, \tag{5}$$

etc. Their explicit expressions can be calculated by using (3). For example,

$$\Delta_n(x; m^2) = -\epsilon(x_0) \frac{(\sqrt{x^2}/m)^{(2-n)/2}}{2^{n/2}\pi^{(n-2)/2}} J_{(2-n)/2}(m\sqrt{x^2})\,\theta(x^2). \tag{6}$$

It is easy to extend the definition of the complex-dimensional integral to the case in which the integrand is a Lorentz-covariant quantity $G_{\mu\cdots\nu}$, which is defined by

$$H \equiv y^\mu \cdots z^\nu G_{\mu\cdots\nu}, \tag{7}$$

where H is a Lorentz-invariant quantity and $y_\mu, \cdots, z_\nu$ are artificially introduced constant vectors in M^n. For example, consider $G_{\mu\nu} = p_\mu p_\nu F(p^2, px)$. The complex-dimensional integral of $y^\mu z^\nu G_{\mu\nu}$ is given by (1). Because of the Lorentz invariance of (1) and the proportionality in y_μ and z_ν, I can write

$$\int d^n p (y^\mu p_\mu)(z^\nu p_\nu) F(p^2, px) = (y^\mu x_\mu)(z^\nu x_\nu)\Phi_1(x^2) + (y^\mu z_\mu)\Phi_2(x^2), \tag{8}$$

where Φ_1 and Φ_2 depend only on x^2. On introducing an abstract metric tensor $g_{\mu\nu}$ of M^n, I rewrite (8) as

$$\int d^n p\, p_\mu p_\nu F(p^2, px) = x_\mu x_\nu \Phi_1(x^2) + g_{\mu\nu}\Phi_2(x^2). \tag{9}$$

Then it can be proved that the formula

$$g^{\mu\nu} \int d^n p\, p_\mu p_\nu F(p_\sigma) = \int d^n p\, p^2 F(p_\sigma) \tag{10}$$

always holds if and only if one sets[2)]

$$g_\mu{}^\mu = n. \tag{11}$$

The proof is carried out by showing that to prove (10) is equivalent

[2] Necessity of (11) is well known and is shown easily.

to proving

$$\left(g^{\mu\nu}\frac{\partial}{\partial x^{\mu}}\frac{\partial}{\partial x^{\nu}}+m^2\right)\varDelta^{(1)}{}_n(x;m^2)=0. \tag{12}$$

Finally, I mention the complex-dimensional regularization of singular products in position space. As is well known, the invariant delta functions in the ordinary Minkowski space exhibit light-cone singularities:

$$\varDelta(x;m^2)=-\frac{\epsilon(x_0)}{2\pi}\left[\delta(x^2)-\frac{m^2}{4}\theta(x^2)+\cdots\right], \tag{13}$$

$$\varDelta^{(1)}(x;m^2)=-\frac{1}{2\pi^2}\left[\mathrm{P}\frac{1}{x^2}-\frac{m^2}{4}\left(\log\frac{m^2|x^2|}{4}+2\gamma-1\right)+\cdots\right], \tag{14}$$

where P and γ denote Cauchy's principal value and Euler's constant, respectively. Therefore their products are not well defined. The complex-dimensional extensions $\varDelta_n$ and $\varDelta^{(1)}{}_n$ are, however, continuous on the light cone $x^2=0$ if Re $n<2$. In that region, therefore, any product of $\varDelta_n$ and $\varDelta^{(1)}{}_n$ is always well defined. What I propose is to define singular products in the ordinary Minkowski space by analytically continuing in n the corresponding complex n-dimensional products to $n=4$. After lengthy calculations, I have found that the products $(\varDelta_n)^2$, $\varDelta_n\varDelta^{(1)}{}_n$, and $(\varDelta^{(1)}{}_n)^2$ have no pole at $n=4$. Accordingly, I obtain the regularized expressions for $\varDelta^2$, $\varDelta\varDelta^{(1)}$, and $(\varDelta^{(1)})^2$ unambiguously [2]. They are consistent with another way of definitions

$$\varDelta(x;m^2)\varDelta^{(1)}(x;m^2)=2\epsilon(x_0)\ \mathrm{Im}[\varDelta_F(x;m^2)]^2, \tag{15}$$

$$[\varDelta^{(1)}(x;m^2)]^2-[\varDelta(x;m^2)]^2=4\ \mathrm{Re}[\varDelta_F(x;m^2)]^2 \tag{16}$$

where $2\varDelta_F\equiv i\epsilon(x_0)\varDelta+\varDelta^{(1)}$ is a boundary value of an analytic function.

References

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