Publ. RIMS, Kyoto Univ.
12Suppl. (1977), 357-366.

Caustics and Microfunctions

by

Frédéric PHAM*

It has been known from the 19^{th} Century that the intensity of light on a caustic cannot be understood by mere geometrical optics, but only by the *geometrical limit of wave optics.* After recalling (in § 1) what the physical problem is, I shall relate it to a problem in microfunction theory.

§ 1. The Geometrical Limit of Wave Optics

A stationary wave of frequency ω can be written

$$\tilde{u}(x, t) = \tilde{u}(x) e^{i\omega t} \quad (x, t) \in \boldsymbol{R}^3 \times \boldsymbol{R} .$$

The following are two special cases:

1°/ *The plane wave with wave vector* $\vec{k}$:

$$\tilde{u}(x) = ae^{-i\vec{k}.\vec{x}}$$

($\vec{k}^2 = \omega^2 c^2$, with c the speed of light, $c=1$ if the units are suitably chosen);

2°/ *the spherical wave emitted by a point source* y:

$$\tilde{u}(x) = a \frac{e^{-i\omega\varphi(x, y)}}{\varphi(x, y)}$$

(with $\varphi(x, y)$ = distance from x to the point y).

Consider now the general case. Given a wave surface $Y \subset \boldsymbol{R}^3$ (surface where $\tilde{u}(x)$ has constant phase) we may consider that $\tilde{u}(x)$ is a superposition of spherical waves emitted by every point of the wave surface: this is the Huyghens-Fresnel principle, the basis of diffraction theory (see for instance [5], § 59); this principle was introduced heuristically by Fresnel in 1816, and although it gave the first strong evidence for the

Received August 3, 1976.

* Département de Mathématiques, Université de Nice, Parc Valrose, 06 Nice (France)

wave theory of light, it is not at all easily deduced from the wave equation (nor is it easily formulated in precise fashion: see [3], Chap. VIII). Here we shall admit the following vague formulation:in the high frequency limit the wave function is "approximately" given by a surface integral

$$\tilde{u}(x) \simeq \frac{\omega}{2\pi} \int_Y \frac{e^{-i\omega\varphi(x,y)}}{\varphi(x,y)} a(y)\, dy \tag{1}$$

where $a(y)$ is interpreted as the amplitude of the "secundary wavelet" emitted by the point y of Y (the normalization factor $\frac{\omega}{2\pi}$ will be accounted for below).

When ω is large the phase $\omega\varphi(x, y)$ of the integrand oscillates rapidly along Y, except at points where $d_y\varphi(x, y) = 0$; such "critical" points give the main contribution to the integral (1) (*"principle of stationary phase"*).

We shall denote by $\sum \subset X \times Y$ the set of all critical points

$$\sum = \{(x_0, y_0) \in X \times Y;\ d_y\varphi(x_0, y_0) = 0\}$$

i.e. the set of couples (x_0, y_0) such that the straight line $\overline{x_0 y_0}$ is perpendicular to the wave surface Y at y_0 (Fig. 1).

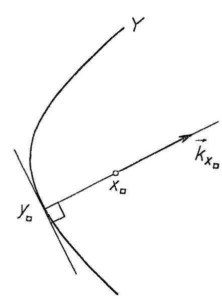

Fig. 1. A wave surface and wave vector·

At a "general" critical point (x_0, y_0) we may suppose that the Hessian $\mathrm{Hess}_y\,\varphi\left(= \det\left(\frac{\partial^2\varphi}{\partial y_i \partial y_j}\right)\right)$ does not vanish (i.e. (x_0, y_0) is a non degenerate quadratic critical point). It then follows from the usual formula of stationary phase in classical analysis that in the high ω limit the contribution of the point (x_0, y_0) to the integral (1) is given by

$$\tilde{u}(x_0) \simeq a(x_0, y_0)\, e^{-i\omega\varphi(x_0, y_0)} \tag{2}$$

where $a(x_0, y_0) = a(y_0)\, e^{i(\pi/4)\sigma} \dfrac{|\mathrm{Hess}_y \varphi(x_0, y_0)|^{-1/2}}{\varphi(x_0, y_0)}$

(σ is the signature of the quadratic form $\partial_y{}^2\varphi(x_0, y_0)$). One may notice that $a(x_0, y_0)$ is a slowly varying function of x_0, which goes to $a(y_0)$ as x_0 approaches y_0 (the normalizing factor $\dfrac{\omega}{2\pi}$ in (1) has been introduced precisely for that purpose). Therefore, the wave function $\tilde{u}(x_0)$ behaves locally like a plane wave with wave vector $\vec{k}_{x_0} = \omega\, \mathrm{grad}_{x_0}\varphi(x_0, y_0)$ (see Fig. 1).

The integral curves of the vector field $\vec{k}_{x_0}$ are straight lines and we thus recover geometrical optics, where the light rays are the straight lines orthogonal to the wave surface.

But the above analysis rested on the hypothesis that $\mathrm{Hess}_y\varphi \neq 0$, *not valid above the caustic* (envelope of the family of light rays). Before pushing it further, let us make a

Parenthesis : Geometrical Optics "A la Thom"

A caustic point is the intersection of two "infinitely near" light rays, i.e. a point x_0 such that $\varphi(x_0, \cdot)$ is stationary for two "infinitely near" points y_0. In other words the caustic K is the "bifurcation locus "of the family of functions φ (as functions of $y \in Y$, with parameter $x \in X$). Given such a family of functions, a topologist will like to consider its "*discriminant locus*" Δ, defined as the image of the critical set of the mapping

$$\Phi : X \times Y \longrightarrow X \times \boldsymbol{R}$$

$$x, y \longrightarrow x, \varphi(x, y)$$

here the critical set is the already introduced set $\sum$, whereas $\Delta = \Phi(\sum)$

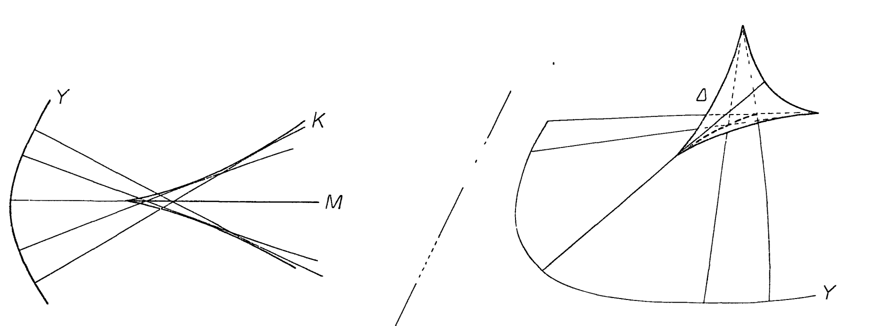

Fig. 2. A caustic K, its discriminant locus Δ and the Maxwell set M.

$=\{(x_0, t_0)\in X\times \boldsymbol{R} | x_0$ is at distance t_0 from Y along some light ray$\}$

Fig. 2 shows a typical caustic and its discriminant locus (the celebrated "swallow's tail" of Thom's classification).

From Thom's theory of deformations of functions, one knows that in the "generic" situation, the critical set $\sum$ is smooth and n-dimensional (where $n=\dim X$), and finite over its image Δ (a point of Δ comes from a finite number of points of $\sum$), whereas Δ itself is finite over X. Furthermore the "apparent contour" of Δ on X is the union of the caustic K and of the "Maxwell set" M (here M is the set of points x_0 which lie on two different light rays at the same distance from Y:see Fig. 2)

What I want to stress here is that the discriminant locus Δ is a nice geometrical object which carries all the information on geometrical optics in any generic situation. I shall now show that it also "carries" (in the precise mathematical sense of being the support of a sheaf) a nice analytic object which describes the asymptotic ($\omega\rightarrow+\infty$) limit of wave optics.

§ 2. Asymptotic Integrals and Microfunctions

Considering (1) as a function of ω, (and writing $\tilde{u}(x,\omega)$, instead of $\tilde{u}(x)$), let us perform a Fourier transformation. Calling t the variable conjugate to ω, we get

$$\tilde{u}(x,\omega)=\int e^{-i\omega t}u(x,t)\,dt \tag{3}$$

where

$$u(x,t)=\frac{1}{2\pi i}\frac{\partial}{\partial t}\int\delta(t-\varphi(x,y))\frac{a(y)}{\varphi(x,y)}\,dy\,. \tag{4}$$

From formula (3) we see that if we add to $u(x,t)$ a function analytic on the $\operatorname{Im} t<0$ side we only change $\tilde{u}(x,\omega)$ by exponentially decreasing terms (as $\omega\rightarrow+\infty$). This means that if we are interested in the asymptotic behaviour of $\tilde{u}(x,\omega)$ up to exponentially decreasing terms, we must consider $u(x,t)$ in equations (3) (4) as a *microfunction in the variable* t. I now proceed to explain how an integral such as (4) defines a microfunction.

2.1 Let us consider the following general geometric situation.

$$\varphi : X \times Y \longrightarrow \boldsymbol{R}$$

is a real analytic function on $X\times Y$ where X[resp. Y] is an n-dimensional [resp. a k-dimensional] real analytic manifold. All the considerations below are local and can be put in sheaf language but for simplicity let us fix one point (x_0, y_0) and consider the corresponding germs (*pointwise situation*). Let us assume

Assumption 1. *φ is of "finite singular type", i.e. the critical set $\sum$ is finite over X in the algebraic sense*: this means that the *complexified* critical set $\sum$ is finite over the complexified manifold X, a condition which can be expressed in algebraic fashion by asking the quotient module.

$$\mathcal{A}_{\Sigma}=\mathcal{A}_{X\times Y}/(\varphi'_{y_1}, \cdots, \varphi'_{y_k})$$

to be a (free) finitely generated $\mathcal{A}_X$-module, where $\mathcal{A}$ denotes the ring of complex valued analytic functions on the subscript manifold.

We shall sometimes need also

Assumption 2. *φ is "non degenerate", i.e. the analytic set $\sum$ is reduced and smooth* (algebraic translation: $\mathcal{A}_{\Sigma}$ is a regular ring).

Both assumptions are satisfied for generic choice of φ. Furthermore, given any function $\varphi_0(y)$ satisfying assumption 1, we can get a function $\varphi(x, y)$ satisfying assumptions 1 and 2 by taking a versal deformation of φ_0.

Proposition. *Under assumption* 1, *the integral*

$$u(x, t) = \int \delta(t-\varphi(x, y))\, a(x, y)\, dy_1 \cdots dy_k\,, \tag{5}$$

where $a \in \mathcal{A}_{X\times Y}$, defines a family, analytic with respect to the parameter x, of microfunctions in one variable t (on the $\operatorname{Im} t>0$ *side) with support on the discriminant locus Δ.*

Sketch of the proof. We replace the δ-(micro) function by its defining analytic function $\frac{-1}{2\pi i(t-\varphi(x, y))}$, and integrate over a small ball B in Y space, surrounding y_0. For $\operatorname{Im} t>0$ (x real) there is no trouble, the denominator does not vanish and the integral is analytic in x and t. For t real, $(x, t) \notin \Delta$ there is no trouble either because $d_y\varphi$ vanishes nowhere along the zero locus of the denominator, so that one can deform the real ball of integration

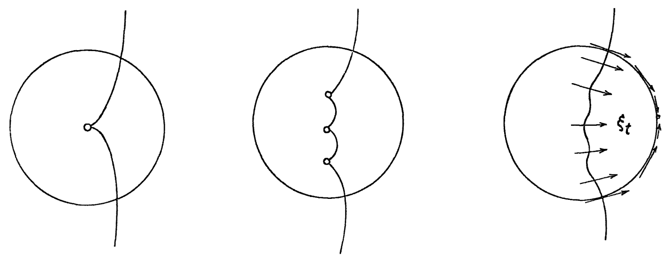

Fig. 3. Singularities of the integrand of (5) for various choices of (x, t).

by pushing it in the imaginary direction $i\xi_t$, where ξ_t is a vector field tangent to the sphere, and such that $d_y\varphi(\xi_t)>0$ on the zero locus of the denominator (see Fig. 3). For $(x, t) \in \Delta$ this procedure fails because the zero locus of the denominator has singularities. But if (x, t) is close enough to (x_0, t_0) none of those singularities will escape the ball, so that the integral on a bigger ball B' will yield the same microfunction (by the above reasoning the integral on the spherical shell $B'-B$ will be analytic). Therefore, the resulting microfunction $u(x, t)$ does not depend on the choice of the ball B (provided B has been chosen small enough to start with).

Remark. Seen as a microfunction in the $n+1$ variables (x, t) (instead of just one variable t, with parameter x), $u(x, t)$ is supported by the conormal bundle of Δ, i.e. the set of codirections

$$\tilde{\Delta} = \{(x_0 + i\infty d_x\varphi(x_0, y_0),\quad t_0 + i\infty 1) \mid d_y\varphi(x_0, y_0) = 0\}.$$

2.2 Given φ, we now study the set of all integrals (5) when a varies in $\mathcal{A}_{X\times Y}$. Let $\mathscr{C}_\varphi^0$ be that set of microfunctions, and

$$\mathscr{C}_\varphi^\lambda = \partial_t^{-\lambda} \mathscr{C}_\varphi^0 \quad (\lambda \in \boldsymbol{C}).$$

We shall be especially interested in the case when $\lambda \in \boldsymbol{N}$: recall that derivating a microfunction u by a negative power ∂_t^{-r} corresponds through Fourier transform to multiplying $\tilde{u}$ by ω^{-r}, an operation which increases the regularity of the microfunction u [resp. of the asymptotic expansion $\tilde{u}$].

Let

$$\mathscr{A}_X\{\{\partial_t^{-1}\}\} = \left\{ \sum_{r=0}^{\infty} c_r(x) \partial_t^{-r} \in \boldsymbol{C}[[x, \partial_t^{-1}]]; \right.$$
$$\left. \sum_{r=0}^{\infty} c_r(x) \frac{T^r}{r!} \in \boldsymbol{C}\{x_1, \cdots, x_n, T\} \right\}$$

be the ring of microdifferential operators of zero order in ∂_t^{-1}, with coefficients depending only on x. Our main results are summarized in the following

Theorem 1. *Under assumption* 1, *one has a filtration*

(0) $\mathscr{C}_\varphi^0 \supset \mathscr{C}_\varphi^1 \supset \mathscr{C}_\varphi^2 \supset \cdots$; *furthermore,*

(i) $\mathscr{C}_\varphi^0$ *is a finitely generated* $\mathscr{A}_X\{\{\partial_t^{-1}\}\}$*-module;*

(ii) *every* $u \in \mathscr{C}_\varphi^0$ *is a microfunction with regular singularity* (*i.e. a solution of a differential equation with regular singularity*).

Theorem 2. *Under assumptions* 1 *and* 2, *the module* $\mathscr{C}_\varphi^{k/2} = \partial_t^{k/2} \mathscr{C}_\varphi^0$ *depends on* φ *only through* Δ (*one must understand that different functions* φ *with different values of* $k = \dim Y$ *may give rise to the same discriminant locus* Δ). *This module will be written* $\mathscr{C}^0_{\Delta|X\times\boldsymbol{R}}$ *and will be called the module of microfunctions holomorphic on the discriminant* Δ.

Remark. The operation $\partial_t^{k/2}$ may be understood as a "normalization factor": for instance in the $k=2$ case (i.e. when Y is a surface) the formula $\mathscr{C}^0_{\Delta|X\times\boldsymbol{R}} = \partial_t \mathscr{C}_\varphi^0$ accounts for the ∂_t factor in equation (4), corresponding to the $\frac{\omega}{2\pi}$ factor in equation (1).

Let me give a rough idea of the proof of theorem 1. Integrating by parts, it is easy to see that the integral $\int \delta(t-\varphi(x,y))\omega$ (where

$\omega=a(x, y)\, dy_1\wedge\cdots\wedge dy_k)$ depends only on the class of ω modulo $d_y\varphi\wedge Z^{k-1}(\Omega^{\cdot}_{\mathrm{rel}})$, where $(\Omega^{\cdot}_{\mathrm{rel}}, d_y)$ is the complex of germs of relative differential forms of the space $X\times Y$ over X, and Z means "cocycles" (according to the Poincaré Lemma $Z^{k-1}(\Omega^{\cdot}_{\mathrm{rel}})=d_y\Omega^{k-2}_{\mathrm{rel}}$ for $k>1$).

Set $G=\begin{cases}\Omega^k_{\mathrm{rel}}/d_y\varphi\wedge d_y\Omega^{k-2}_{\mathrm{rel}} & \text{if } k>1\\ \Omega^1_{\mathrm{rel}}/d_y\varphi\cdot\mathcal{O}_x\{\varphi\} & \text{if } k=1.\end{cases}$ One verifies at once that the operation ∂_t^{-1} on the integral corresponds to the following operation on G:

$$\begin{aligned} D^{-1}: G &\longrightarrow G\\ [\omega=d\chi] &\longmapsto [d_y\varphi\wedge\chi]\end{aligned}$$

i.e. the inverse of the Gauss-Manin connection on G (see [6]). Part (i) of theorem 1 then follows from the following

Theorem 3.*) *G is a free $\mathcal{A}_X\{\{D^{-1}\}\}$-module of rank μ, where μ is the Milnor number of the germ φ at (x_0, y_0) (μ=rank of the free $\mathcal{A}_X$-module $\mathcal{A}_\Sigma$).*

The well known regularity property of the Gauss-Manin connection then yields part (ii) of Theorem 1.

Remark. Under assumptions 1 and 2 one proves that the integration epimorphism $G\to\mathscr{C}_\varphi^0$ is an *isomorphism,* so that in that case $\mathscr{C}_\varphi^0$ is free of rank μ over $\mathcal{A}_X\{\{\partial_t^{-1}\}\}$.

2.3 *Exponents measuring the singular behaviour*

It follows from part ii) of theorem 1 that every microfunction u in $\mathscr{C}_\varphi^0$ (or in $\mathscr{C}^0_{\Delta|X\times\boldsymbol{R}}$) can be given in the neighbourhood of any point $(x_0, t_0)\in\Delta$ by a convergent expansion

$$u(x_0, t)=\Big[\sum_{\substack{\alpha\in A+\boldsymbol{N}\subset\boldsymbol{C}\\ p\in B\subset\boldsymbol{N}}} c_{p,\alpha}(x_0).(t-t_0)^\alpha\log^p(t-t_0)\Big] \tag{6}$$

where $p\in B$, a finite set of natural integers, whilst $\alpha\in A+\boldsymbol{N}$, with A a finite set of complex numbers (in the formula the parameter x is

* The "formal" analogue of this theorem was proved by Malgrange in [6]. The difficult part in theorem 3 is the proof that one can make series converge in $\mathcal{A}_X\{\{D^{-1}\}\}$. This proof also relies on an idea of Malgrange (see [7] for further details).

fixed: $x=x_0$, so that u is a microfunction in one variable).

The smallest real part Re α of all α's appearing in expansion (6) measures, in some sense, the singular behaviour of the microfunction u (with regular singularity) at point (x_0, t_0): smaller that exponent is, the more singular u will be. Now it is obvious that multiplication by ∂_t^{-1}, or by a microdifferential operator in $\mathscr{A}_X\{\{\partial_t^{-1}\}\}$, can but increase that exponent. Part (i) of Theorem 1 thus warrants the existence of a highest lower bound $\beta(x_0, t_0)$ for the exponents of all microfunctions in $\mathscr{C}^0_{\Delta|x\times\boldsymbol{R}}$, at a given point (x_0, t_0). Going back to the physical problem of § 1, easy considerations on Fourier transforms show that the wave amplitude $\tilde{u}(x)$ behaves, as the frequency ω goes to infinity, not worse than $\omega^{\gamma(x)}$, where $\gamma(x)=-\inf\{\beta(x,t)+1 \mid (x,t)\in\Delta\}$ (not unexpectedly, $\gamma(0)=0$ outside the caustic).

Notice that this is just a pointwise estimate (for fixed x), and that the problem of finding *uniform estimates* is up to now unsolved, except in simpler cases.

Historical Comments and Open Problems

Exponents measuring the asymptotic behaviour of oscillatory integrals have been defined and computed for the first time by V. I. Arnold in [1] (for simple singularities) and in [2] (for more general singularities. including all generic singularities if $\dim X\leq 10$). Then J. J. Duistermaat [4] proved that all quasihomogeneous singularities satisfy the uniform estimates conjectured by Arnold. Weaker than the conjecture on uniform estimates is the conjecture on the *semi-continuity of Arnold's exponent*: this conjecture was recently disproved by a counterexample of Varchenko (about which I first heard from J. M. Kantor in this conference).

One of my aims when starting this work (a full exposition of which will be published in [7]) was to prove Arnold's conjecture: I hoped that the microfunction techniques were best suited (through "coherent sheaf" arguments) for proving such semi-continuity properties. In the meanwhile, I realized that my results gave much better control on the sheaf G (see Theorem 3) than on the sheaf $\mathscr{C}_\varphi^0$, so that I had better hope to prove semicontinuity properties of "generalized" β exponents defined directly from G. A module defined in purely algebraic fashion, G may be considered as kind of a "complexification" of $\mathscr{C}_{\varphi}^{0}$, describing the singular behaviour of integrals like (5) over arbitrary *complex* integration cycles: one is thus led to define a "generalized" β exponent, whose semi-continuity is *not* disproved by Varchenko's counterexample (V.I. Arnold: private communication).

References

[1] Arnold, V. I., Integrals of rapidly oscillating functions and singularities of the projections of Lagrangean manifolds, *Funct. Anal. and its Appli.*, **6**, **3** (1972), 61–62.

[2] Arnold, V. I., Remarks on the method of stationary phase and Coxeter numbers, *Usp. Mat. Nauk* **28**, **5** (1973), 17–44.

[3] Born, M., Wolf, E., *Principles of optics* (Pergamon Press 1975).

[4] Duistermaat, J. J., Oscillatory integrals, Lagrange immersions and unfolding of singularities, *Comm. Pure and Applied Maths.* **27** (1974), 207–281.

[5] Landau, L., Lifschitz, E., *Field theory* (Mir ed., Moscow).

[6] Malgrange, B., Intégrales asymptotiques et monodromie, *Ann. Sc. Ec. Norm. Sup.* **7** (1974), 405–430.

[7] Pham, F., Caustiques, phase stationnaire et microfonctions, *Acta Math. Vietnamica* (to be published).