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Zonal Spherical Functions on Some Symmetric Spaces

by

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§ 0. Introduction

Let G be a real semisimple Lie group with finite center, and K a maximal compact subgroup of G. A zonal spherical function on the symmetric space $X=G/K$ is an simulatneous eigenfunction $\varphi(x)$ of all the invariant differential operators on X satisfying $\varphi(kx)=\varphi(x)$ for any $x\in X$, $k\in K$, and $\varphi(eK)=1$, where e is the identity element in G. By the Cartan decomposition $G=KAK$, $\varphi(x)$ is considered as a function on A. And by the separation of variables, we obtain differential operators on A from the invariant differential operators, which are called their radial components. In this paper, we investigate the radial components of the invariant differential operators and the zonal spherical functions when G is a real, complex or quanternion unimodular group. The eigenvalues of the zonal spherical functions is parametrized by the element in $\mathfrak{a}^*$. Therefore, the system of differential equations on A satisfied by the zonal spherical function has as many parameters as dim $\mathfrak{a}$. However, we can construct a new system of differential equations which admits the other parameter ν. It is shown that the zonal spherical function on the real, complex or quaternion unimodular group corresponds to the case in which $\nu=\frac{1}{2}$, 1, 2, respectively.

§ 1. Radial Components of Invariant Differential Operators

Let $\mathfrak{a}$ be a vector space of dimension n, and $\mathfrak{a}^*$ its dual space. $\mathfrak{a}^*$ is generated by e_i, $i=1, 2, \cdots$, where $e_i(H)=t_i$ for $H=(t_1, \cdots, t_n)\in\mathfrak{a}$.

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First we will define n differential operators $\Delta_i^{(\nu)}$, $i=1, 2, \cdots, n$, by the following formula,

$$\Delta(\zeta, \nu) = \frac{1}{\delta(H)} \sum_{s\in\mathfrak{S}_n} (\det s)\, e^{2\rho(sH)} \prod_{i=1}^{n} (\zeta + D_{t_{s(i)}} + (n+1-2i)\nu)$$
$$= \zeta^n + \Delta_1^{(\nu)}\zeta^{n-1} + \Delta_2^{(\nu)}\zeta^{n-2} + \cdots + \Delta_n^{(\nu)}.$$

Here

$$\delta(H) = \prod_{i<j} (e^{t_i - t_j} - e^{t_j - t_i})$$

$$\rho(H) = \frac{1}{2} \sum_{i<j} (t_i - t_j)$$

$$sH = (t_{s(1)}, \cdots, t_{s(n)}),$$

for $H=(t_1, \cdots, t_n) \in \mathfrak{a}$, $s\in\mathfrak{S}_n$, and ζ is an indeterminate. For example

$$\Delta_1^{(\nu)} = D_{t_1} + \cdots + D_{t_n}.$$

$$\Delta_2^{(\nu)} = \sum_{i<j} (D_{t_i}D_{t_j} - \nu \operatorname{cth}(t_i - t_j)(D_{t_i} - D_{t_j})) - 2\langle\rho, \rho\rangle\nu^2.$$

Here $\langle\ ,\ \rangle$ is the inner product on $\mathfrak{a}^*$ defined by $\langle e_i, e_j\rangle = \delta_{ij}$:

Therem 1. The operators $\Delta_i^{(\nu)}$, $i=1, 2, \cdots, n$, are commutative with each other. And under the condition $\sum_{i=1}^{n} t_i = 0$, the radial components of generators of the algebra of the invariant differential operators on symmetric spaces $SL(n, \boldsymbol{R})/SO(n)$ $SL(n, \boldsymbol{C})/SU(n)$, $SL(n, \boldsymbol{H})/Sp(n)$ are given by the above operators, if we substitute ν for $\frac{1}{2}$, 1, 2 respectively.

Proof. In complex unimodular group case, the radial components of invariant differential operators are known (cf. [1]). And in real unimodular group case, it is easy to compute the radial components of invariant differential operators by using a well-known formula called Cappelli's identity. In these cases, the operators $\Delta_i^{(\nu)}$, $i=1, 2, \cdots, n$, are commutative, and by this fact, we can prove the commutativity of $\Delta_i^{(\nu)}$ for any fixed ν. If we know the commutativity, it is easy to check quaternion unimodular group case.

Next, we investigate the system of differential equations

$$\mathcal{M}_\lambda^{(\nu)};\ \Delta(\zeta,\nu)u=\prod_{i=1}^{n}(\zeta+\lambda_i)u \text{ for any } \zeta.$$

Here $\lambda=(\lambda_1,\cdots,\lambda_n)\in\mathfrak{a}^*$ and we assume $\sum_{i=1}^{n}\lambda_i=0$. This means that a solution u of this system is a simultaneous eigenfunction of the differential operators $\Delta_i^{(\nu)}$, $i=1, 2, \cdots, n$. Following Harish-Chandra [2], we can construct $n!$ solutions $\Phi_{s\lambda}^{(\nu)}(H)$ $s\in\mathfrak{S}_n$ of this system. $\Phi_\lambda^{(\nu)}(H)$ is defined as follows.

$$\Phi_\lambda^{(\nu)}(H)=\sum_{\mu\in L}\Gamma_\mu^{(\nu)}(\lambda)e^{(\lambda-2\nu\rho-\mu)(H)},$$

where $L=\{m_1\alpha_1+\cdots+m_{n-1}\alpha_{n-1};\ m_i\in N\ i=1,2,\cdots,n-1\}$, $\alpha_i=e_i-e_{i+1}$, $N=\{0,1,2,\cdots\}$. And the coefficients $\Gamma_\mu^{(\nu)}(\lambda)$ satisfy the recursion formulas

$$\sum_{s\in\mathfrak{S}_n}(\det s)\Big(\prod_{i=1}^{n}(\zeta+\tau_i(\lambda-\mu,s,\nu))-\prod_{i=1}^{n}(\zeta+\lambda_i)\Big)\Gamma_{\mu+2(s\rho-\rho)}^{(\nu)}(\lambda)=0,$$

for any ζ. Here

$$\begin{aligned}\tau(\lambda,s,\nu)&=\lambda+2(\nu-1)(s\rho-\rho)\\&=(\tau_1(\lambda,s,\nu),\cdots,\tau_n(\lambda,s,\nu)).\end{aligned}$$

$\Phi_\lambda^{(\nu)}(H)$ is holomorphic in the positive Weyl chamber $C=\{H\in\mathfrak{a};\ \alpha_i(H)>0\ i=1,\cdots,n-1\}$.

§ 2. An Analogue of Gegenbauer's Function in Two Variables

In case $n=2$, the system $\mathcal{M}_\lambda^{(\nu)}$ is well-known Gegenbauer's differential equation by taking a suitable coordinate system. In this section, we will obtain integral representation and recursion formulas for the functions satisfying the system $\mathcal{M}_\lambda^{(\nu)}$ in case $n=3$.

We set $\sigma_i=2e_i-\frac{2}{3}\sum_{j=1}^{3}e_j$ $i=1, 2, 3$, $x_1=\frac{1}{3}\sum_{i=1}^{3}e^{\sigma_i(H)}$, $x_2=\frac{1}{3}\sum_{i=1}^{3}e^{\sigma_i(-H)}$, and assume that $\sum_{i=1}^{3}e_i(H)=0$ for $H\in\mathfrak{a}$ in this section. we represent the operators $\Delta_2^{(\nu)}$, $\Delta_3^{(\nu)}$ by x_1, x_2, then

$$\begin{aligned}\Delta_2^{(\nu)}&=(x_2-x_1^2)D_1^2+(1-x_1x_2)D_1D_2\\&\quad+(x_1-x_2)D_2^2-(3\nu+1)(x_1D_1+x_2D_2)-\nu^2\end{aligned}$$

$$\Delta_3^{(\nu)}=(1-3x_1x_2+2x_1^3)D_1^3+3(x_1-2x_2^2+x_1^2x_2)D_1^2D_2$$

$$-3(x_2-2x_1^2+x_1x_2^2)D_1D_2^2$$
$$-(1-3x_1x_2+2x_2^3)D_2^3-3(3\nu+2)((x_2-x_1^2)D_1^2-(x_1-x_2^2)D_1^2)$$
$$+(3\nu+1)(3\nu+2)(x_1D_1-x_2D_2).$$

First we have the following recursion formulas.

Theorem 2. There are two recursion formuras between the functions $\Phi_\lambda^{(\nu)}(H)$, $\lambda\in\mathfrak{a}^*$, if we normalize the initial value by $\Gamma_0^{(\nu)}(\lambda)=\dfrac{I(\lambda,\nu)}{I(2\nu\rho,\nu)}$, $I(\lambda,\nu)=\prod_{i<j}B\left(\dfrac{\lambda_i-\lambda_j}{2},\nu\right)$ ($B(x,y)$ is the beta function).

$$(*)\qquad\begin{cases} 3\prod_{i<j}\langle\lambda, e_i-e_j\rangle x_1\Phi_\lambda^{(\nu)}=\sum_{k=1}^{3}\prod_{i<j}\langle\lambda+\nu\sigma_k, e_i-e_j\rangle\Phi_{\lambda+\sigma_k}^{(\nu)} \\ \\ 3\prod_{i<j}\langle\lambda, e_i-e_j\rangle x_2\Phi_\lambda^{(\nu)}=\sum_{k=1}^{3}\prod_{i<j}\langle\lambda-\nu\sigma_k, e_i-e_j\rangle\Phi_{\lambda-\sigma_k}^{(\nu)} \end{cases}$$

Now, we consider an integral representation of a solution of the system $\mathcal{M}_\lambda^{(\nu)}$.

Theorem 3. Set

$$G_\nu(x_1,x_2;u_1,u_2)=\int_0^\infty u_0^{\nu-1}(P_1P_2P_3)^{-\nu}du_0 \text{ for } \mathrm{Re}\,\nu>0,$$

$$\varphi_{p_1p_2}^{(\nu)}(x_1,x_2)=c_\nu(p_1,p_2)\int_0^\infty\int_0^\infty u_1^{p_1-1}u_2^{p_2-1}G_\nu(x_1,x_2;u_1,u_2)\,du_1du_2$$

for $0<\mathrm{Re}\,p_i<\mathrm{Re}\,2\nu\,(i=1,2)$, where

$$P_i=u_0+(1+u_1e^{\sigma_i(H)})(1+u_2e^{\sigma_i(-H)})$$

$$c_\nu(p_1,p_2)=\frac{1}{B(p_1,2\nu-p_1)B(p_2,2\nu-p_2)B(\nu,2\nu)}.$$

Then $\varphi_{p_1p_2}^{(\nu)}(x)$ has following properties.

(1) $\varphi_{p_1p_2}^{(\nu)}$ is a solution of the system $\mathcal{M}_\lambda^{(\nu)}$, where

$$p_1=\frac{\lambda_2-\lambda_1+2\nu}{2},\quad p_2=\frac{\lambda_3-\lambda_2+2\nu}{2}.$$

(2) $\varphi_{p_1p_2}^{(\nu)}(1,1)=1.$

(3) (Generating function.)

$$G_\nu(x, u) = \sum_{m,n=1}^{\infty} \frac{(2\nu, m)(2\nu, n)}{m!\, n!} (-1)^{m+n} \varphi_{mn}^{(\nu)}(x) u_1{}^m u_2{}^n.$$

(4) (Functional equation.)

$$\varphi_{2\nu-p_1,\, p_1+p_2-\nu}^{(\nu)} = \varphi_{p_1+p_2-\nu,\, 2\nu-p_2}^{(\nu)} = \varphi_{p_1p_2}^{(\nu)}.$$

(5) There is a relation between $\varphi_{p_1p_2}^{(\nu)}$ and $\Phi_{s\lambda}^{(\nu)}$, $s \in \mathfrak{S}_3$.

$$\varphi_{p_1p_2}^{(\nu)}(x) = \sum_{s \in \mathfrak{S}_3} \Phi_{s\lambda}^{(\nu)}(H) \quad \text{for } H \in C.$$

(6) The recursion formulas (*) are also valid if we change $\Phi_\lambda{}^{(\nu)}$ by $\varphi_{p_1p_2}^{(\nu)}$.

The function $\varphi_{p_1p_2}^{(\nu)}(x)$ is an analogue of Gegenbauer's function in two variables. $\varphi_{p_1p_2}^{(\nu)}$ satisfies many interesting properties including (1) $\sim$(6) in Theorem 3. And if we substitute ν for $\frac{1}{2}$, 1, 2, $\varphi_{p_1p_2}^{(\nu)}$ is a zonal spherical function on $SL(3, \boldsymbol{R})/SO(3)$, $SL(3, \boldsymbol{C})/SU(3)$, $SL(3, \boldsymbol{H})/Sp(3)$, respectively.

Remark. After the work, I knew that Prof. Koornwinder ([3]) has obtained the differential operators $\Delta_2{}^{(\nu)}$, $\Delta_3{}^{(\nu)}$, and investigated the orthogonal polynomials $\varphi_{mn}^{(\nu)}(x)$ $(m, n \in \boldsymbol{N})$.

References

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