Publ. RIMS, Kyoto Univ.
12 Suppl. (1977), 469-480.

On the Holonomic System of f^s and b-Function

by

By Tamaki YANO*

M. Sato initiated a theory of prehomogenous vector spaces in 1961 and defined a-, b- and c-functions. The general theory of b-function has extensively been developed since 1972, and an amount of results have ever been obtained [8], [3], [6]. Micro-local calculus finds its good application in the area of the b-function theory. This report is concerned with an algorithm to determine (micro-local) b-functions.

As for the general theory of b-function, we refer the reader to [1], [2] and [5]. Concerning an application of micro-local calculus to P.V. and b-functions (i.e. § 3), see [4].

§ 1. b-Functions, Prehomogeneous Vector Spaces

The b-function $b(s)$ associated with a local holomorphic function is defined to be a polynomial of minimum degree satisfying

(1) $$P(s, x, D)f^{s+1}=b(s)f^s,$$

where P denotes the differential operator $P(s, x, D)=\sum s^j P_j(x, D)$. The following is a famous example of this type of equality.

(2) $$\Delta(\sum_{i=1}^{n} x_i^2)^{s+1}=4(s+1)\left(s+\frac{n}{2}\right)(\sum x_i^2)^s.$$

Define the gamma factor $\gamma(s)=\prod\Gamma(s+\alpha_i)$ when $b(s)=\prod(s+\alpha_i)$. then, equation (1) turns out to be

(3) $$P(s, x, D)\frac{1}{\gamma(s+1)}f^{s+1}=\frac{1}{\gamma(s)}f^s.$$

In view of this formula, we can readily see that f^s depends meromorphical-

Received October 12, 1976.
* Graduate School (attached to RIMS), Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606, Japan.

ly on s and its poles occur only at $-\alpha_j-\nu$, where ν runs over the non-negative integers[1].

The b-function of f is related to the local monodromy of $f^{-1}(0)$, and hence to the asymptotic expansion of $I(h)=\int \exp((\sqrt{-1}/h)f(x))\cdot\phi(x)\,dx$, $\phi\in C_0^\infty$. For example, when $f=\sum x_i^2$, the factor $\left(s+\frac{n}{2}\right)$ in (2) has a relation with the fact that the local monodromy map of $f^{-1}(0)$ at 0 is $(-1)^n=\exp(2\pi i(n/2))$, and the asymptotic behavior of $I(h)$ is $O(h^{n/2})$.

In general, the roots of $b(s)=0$ is strictly negative and rational [5].

It should be recalled that $\sum x_i^2$ is relatively invariant under the action of $GO(n)$[2]. Given more generally, a complex vector space V of dimension n and a linear algebraic subgroup G of $GL(V)$, the couple (G, V) is called a prehomogeneous vector space (abbreviated as P.V.), if there exists a proper algebraic subset S of V such that $V\backslash S$ is a single G-orbit. Assume S is an irreducible hypersurface defined by $f=0$. Then, f is a homogeneous polynomial called a fundamental relative invariant (abbreviated as F.R.I.) of (G, V). Under certain conditions on G, there exists a nonzero polynomial $b(s)$ and $f^*(y)$ such that

$$f^*(D)f(x)^{s+1}=b(s)f(x)^s. \tag{4}$$

This $b(s)$ is called the b-function of (G, V), which will be analyzed later.

§ 2. The Holonomic System of f^s

We investigate the structure of $b(s)$ through the differential equations satisfied by f^s. Let

$$\mathcal{M}=\mathcal{D}[s]f^s/\mathcal{D}[s]f^{s+1}=\mathcal{D}[s]/\mathcal{J}(s)+\mathcal{D}[s]f$$

Here, $\mathcal{D}[s]=\mathcal{D}\otimes\boldsymbol{C}[s]$ and $\mathcal{J}(s)=\{P(s)\in\mathcal{D}[s]\,|\,P(s)f^s=0\}$. It should be noted that s acts on $\mathcal{M}$ by defining $P(s)f^s\mapsto sP(s)f^s$. Then, equation (1) reads "$b(s)=0$ in $\mathcal{E}nd_{\mathcal{D}}(\mathcal{M})$". That is, $b(s)$ is nothing but the minimal polynomial of s in $\mathcal{M}$. For $\alpha\in\boldsymbol{C}$, we define $\mathcal{N}_\alpha=\mathcal{D}/\mathcal{J}(\alpha)$, $\mathcal{J}(\alpha)=\{P\in\mathcal{D}\,|\,P=Q(\alpha)$ for $Q(s)\in\mathcal{J}(s)\}$. $\mathcal{M}$ and $\mathcal{N}_\alpha$ are known to be holonomic and $\check{SS}(\mathcal{M})\subset W\cap((f^{-1}(0))$, $\check{SS}(\mathcal{N}_\alpha)\subset W_0$, where $W=\Big\{\Big(x,$

$s\dfrac{df}{f}\Big)\Big| s\in \boldsymbol{C},\ f(x)\neq 0\Big\}^{\text{closure}} \subset T^*X$, and $W_0=(W\cap(f^{-1}(0)))\cup(\xi=0)$.
Moreover, $\mathcal{N}_\alpha \overset{\sim}{\to} \mathcal{D}f^\alpha$ for generic $\alpha\in \boldsymbol{C}$[3].

When (G, V) is a P.V. with finite G-orbits and f is its F.R.I., W_0 proves to be a union of Lagrangians of following type.

$$\Lambda_j = T^*_{Gx_j}V = \{(x, y)\in V\times V^* \mid x\in Gx_j,\ y\in V_x{}^*\}^{\text{closure}}$$

This is called the conormal bundle of the orbit Gx_j. Since G acts contragrediently on V^*, G acts on $V\times V^*$. A Lagrangian Λ is called a good Lagrangian if Λ has a Zariski dense G-orbit and $\Lambda\subset W$.

Theorem 1. *Let Λ be a good Lagrangian. Then it follows in a neighborhood of a generic point of Λ that*

1° *W is non-singular and Λ is definable in W by the equation $s(x, y)=0$, where $s(x, y)=\langle x, y\rangle/r$ $(r=\deg f)$ and*

2° *$\mathcal{N}_\alpha$ is a simple holonomic system and $\{\langle Ax, D_x\rangle-\alpha\delta\chi(A) \mid A\in \mathfrak{g}\}$ forms an involutory basis of $\mathcal{J}(\alpha)$.*

We use later this theorem and determine b-function with the aid of the holonomy diagram. When a holonomic system $\mathcal{L}$ is given, we can depict the structure of $\mathcal{L}$ by the holonomy diagram of $\mathcal{L}$. This diagram consists of vertices and segments: the former correspond to the irreducible components of $\check{SS}(\mathcal{L})$, and the latter are the ones joining the vertices which have one codimensional intersection[4]. Here follows some examples in the case $\mathcal{L}=\mathcal{D}f^s$ for generic $s\in \boldsymbol{C}$. The number at the right-hand side of a vertex denotes the order of the vertex.

Example 1. $f=\sum_{i=1}^{n} x_i^2$

This is the fundamental relative invariant of $(GO(n), \square)$.

$$\Lambda_0=\boldsymbol{C}^n\times\{0\},\quad \Lambda_1=\{(x,\xi)\mid \xi /\!/ x,\ f(x)=0\},$$

$$\Lambda_2=\{0\}\times \boldsymbol{C}^n.\quad \Lambda_0\cap\Lambda_1=\{(x, O)\mid f(x)=0\},$$

$$\Lambda_1\cap\Lambda_2=\{(0,\xi)\mid \textstyle\sum \xi_i^2=0\}.$$

The factor of b-function between Λ_1 and Λ_2 is determined by Theorem 3.

$$(2-1)s+\left(\frac{n}{2}-\frac{1}{2}\right)+\frac{1}{2}=s+\frac{n}{2}\,.\quad b(s)=(s+1)\left(s+\frac{n}{2}\right).$$

Example 2. $f=\det(x_{ij})$, (x_{ij}) being an $n\times n$ matrix.

$$\Lambda_k=(\text{conormal bundle of } \{x|\operatorname{rank}(x)=n-k\}).$$

codim $\pi(\Lambda_k)=k^2$. This is the relative invariant of a prehomogeneous vector space $(SL(n)\times GL(n),\ \square\otimes\square)$.

$$b(s)=\prod_{\nu=1}^{n}(s+\nu).$$

Example 3. $f=x(y^3+xz)$

$\Lambda_0=\boldsymbol{C}^3\times\{0\}$, $\Lambda_1=\{(0,y,z;\xi,0,0)\}$, $\Lambda_4=\{0\}\times\boldsymbol{C}^3$.

$\Lambda_2=$(The conormal bundle of $y^3+xz=0$).

$\Lambda_3=\{(0,0,z;\xi,\eta,0)\}$. $\Lambda_1\cap\Lambda_2\cap\Lambda_3=\{(0,0,z;\xi,0,0)\}$

$\Lambda_2\cap\Lambda_4=\{(0,0,0;\xi,0,\zeta)$. $\Lambda_3\cap\Lambda_4=\{(0,0,0;\xi,\eta,0)\}$.

Note that Λ_4 (the conormal of the origin) is over Λ_3. The index of the intersection (Λ_2)—(Λ_4) is $(m;n)=(2;1)$. Hence, $c=2$, $c_1=4$.

$$b(s)=(s+1)\prod_{\nu=1}^{5}\left(s+\frac{1}{2}+\frac{\nu}{6}\right).$$

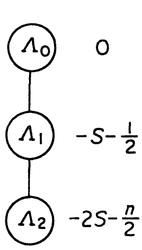

Fig. 1.

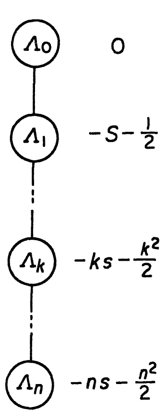

Fig. 2.

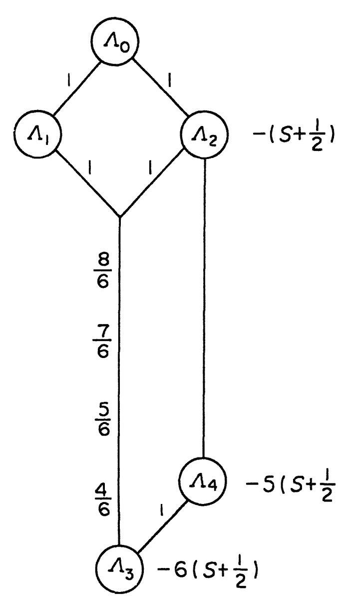

Fig. 3.

Example 4. f=the discriminant of $p(u)$, where $p(u)=u^5+x_1u^4+x_2u^3+x_3u^2+x_4u+x_5$.

$\Lambda_0=\boldsymbol{C}^n\times\{0\}$, Λ_k (resp. $\Lambda_{k,h}$) is the conormal bundle of x when $p(u)$ has exact k-th multiple (resp. k-th and h-th multiple) root. codim $\pi(\Lambda_k)=k-1$, codim $(\Lambda_{k,h})=(k-1)+(h-1)$. This is one of the examples of polynomials associated with Coxeter groups. For the detailed discussion of this type of polynomials and Lie algebra associated with them, we refer the reader to T.Yano-J.Sekiguchi [7].

$$b(s)=\prod_{\substack{1\leqq\nu\leqq n-1\\ 2\leqq n\leqq 5}}\left(s+\frac{1}{2}+\frac{\nu}{n}\right).$$

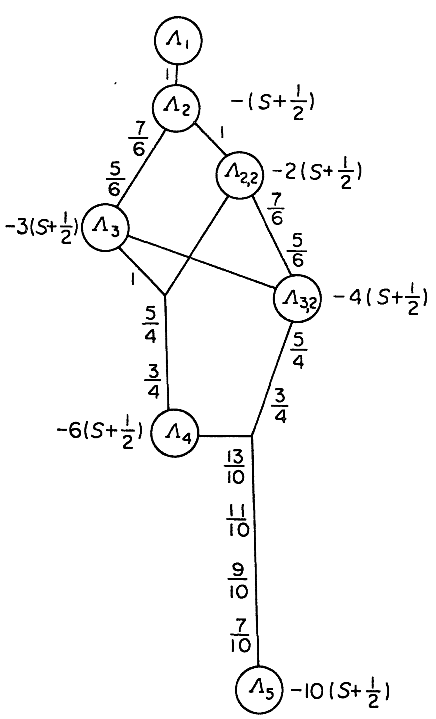

Fig. 4.

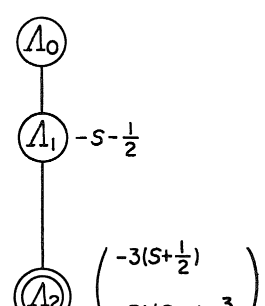

Fig. 6.

Example 5. $f=f_1^{s_1}f_2^{s_2}$, $f_1=\det X$, $f_2=\det\widetilde{X}$.

$$X=\begin{bmatrix} x_1 & x_2 & x_3\\ x_2 & x_4 & x_5\\ x_3 & x_5 & x_6\end{bmatrix},\quad Y=\begin{bmatrix} y_1\\ y_2\\ y_3\end{bmatrix},\quad \widetilde{X}=\begin{bmatrix} X & Y\\ {}^tY & 0\end{bmatrix}.$$

These f_1 and f_2 are fundamental relative invariants of $(GL(3)\times GL(1),\ \square\square\otimes 1+\square\otimes\square)$. The action of $g=(g_1, g_2)$ is $g(X, Y)=(g_1X^tg_1, g_1Y^tg_2)$. The codimension of $\pi(\Lambda)$ is encircled. The triplet of integers under the right-hand side denotes (rank $\widetilde{X}$, rank X, rank Y).

$$b_\nu(s)=[s_1+1]_{\nu_1}\left[(s_2+1)\left(s_2+\frac{3}{2}\right)\right]_{\nu_1}\left[\left(s_1+s_2+\frac{3}{2}\right)(s_1+s_2+2)\right]_{\nu_1+\nu_2}$$

$$P_\nu(s)f_1^{s_1+\nu_1}f_2^{s_2+\nu_2}=b_\nu(s)f_1^{s_1}f_2^{s_2},$$

for an operator $P_\nu(s)$.

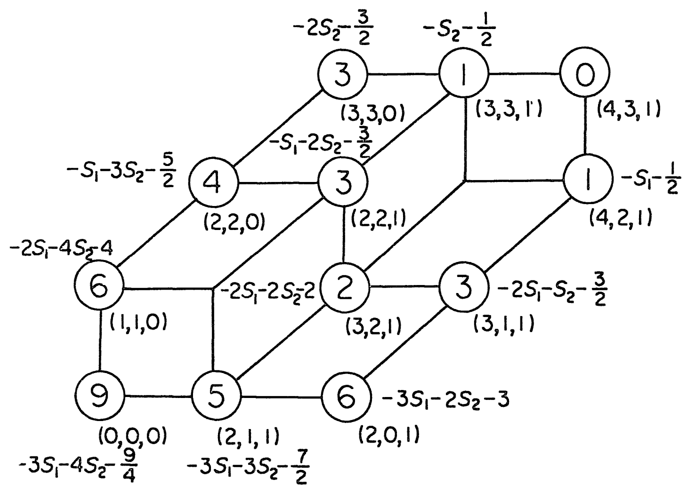

Fig. 5.

Example 6. $f=x^{p_1}+y^{p_2}+z^{p_3}-xyz\ \ 3\leq p_1\leq p_2\leq p_3>3$. $\Lambda_0=\boldsymbol{C}^3\times\{0\}$, $\Lambda_1=$ (The conormal bundle of $f=0$), $\Lambda_2=\{0\}\times\boldsymbol{C}^3$. In this example, $\mathcal{D}f^s$ has multiplicity 4 on Λ_2, and the order consists of four numbers. (See § 4).

$$b(s)=(s+1)(s+1)^2\left(\prod_{\substack{1\leq\nu\leq p_i-1\\ i=1,2,3}}\left(s+1+\frac{\nu}{p_i}\right)\right)_{\text{red}},$$

§ 3. Micro-Local b-Functions

Let Λ be a good Lagrangian of $\overset{\vee}{SS}(\mathcal{N}_s)$. Then, there follows

Theorem 2. *There is an invertible operator $P_\Lambda\in\overset{\vee}{\mathcal{E}}$ in a neighborhood of the generic point of Λ which defines the unique monic poly-*

nomial $b_\Lambda(s)$, by the equation

(5) $$fu_s = b_\Lambda(s) P_\Lambda u_s\,,$$

where u_s is the generator of $\mathfrak{N}_s$. This $b_\Lambda(s)$ is called the micro-local b-function on Λ.

There are two remarks which deserve mentioning.

1° Since $f^*(D) fu_s = b(s) u_s$ holds thanks to (4) and $f^*(D)$ is invertible on $\{0\}\times \boldsymbol{C}^n$, $b(s) = b_\Lambda(s)$ for $\Lambda = \{0\}\times \boldsymbol{C}^n$, when f is an F.R.I. of certain class of P.V.

2° It is crucial that Λ is simple and $P_\Lambda \in \check{\mathcal{E}}$ is invertible. For instance, when $f = x(y^3 + xz)$, $\Lambda = \{0\}\times \boldsymbol{C}^n$ (cf. Example 3), one can take

$$P_\Lambda^{-1} = 2^{-4}3^{-5} D_y^4 D_z^{-1}(D_y^2 + 3yD_xD_z) + 2^{-3}3^{-1} D_xD_y(D_y^2 + 2yD_xD_z) + 2^{-2}D_x^2D_z$$

and $b_\Lambda(s) = (s+1)\left(s+\frac{2}{3}\right)\left(s+\frac{5}{6}\right)\left(s+\frac{7}{6}\right)\left(s+\frac{4}{3}\right)$. In this case, $b(s) = (s+1)\, b_\Lambda(s)$.

Let Λ_1 and Λ_2 be good Lagrangians, satisfying $\dim(\Lambda_1 \cap \Lambda_2) = n-1$, with their holonomy diagram as depicted in Fig. 7. Then,

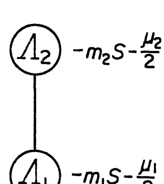

Fig. 7.

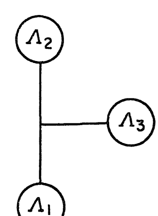

Fig. 8.

Theorem 3. *Suppose Λ_1, Λ_2 and $\Lambda_1 \cap \Lambda_2$ are nonsingular and $T_p(\Lambda_1 \cap \Lambda_2) = T_p(\Lambda_1) \cap T_p(\Lambda_2)$, $p \in \Lambda_1 \cap \Lambda_2$. Then*[5],

$$b_{\Lambda_1}(s) = [(m_1 - m_2)s + (\mu_1 - \mu_2 + 1)/2]_{m_1 - m_2} b_{\Lambda_2}(s).$$

The key to the proof of this theorem lies in the following two lemmata.

Lemma 1. *For any* α, *there is no module* $\mathcal{L}$ *satisfying the conditions a)* $\check{SS}(\mathcal{L})=\Lambda_1$, *and b)* $\mathcal{L}$ *is a submodule of* $\check{\mathcal{E}}u_\alpha$ *and a quotient module of* $\check{\mathcal{E}}u_{\alpha+1}$.

This can be easily verified if based on the structure theorem of holonomic system. By use of this lemma, we can prove

$$b_{\Lambda_2}(s)\,|\,b_{\Lambda_1}(s).$$

Lemma 2. *Let* α *be such that there is a quotient of* $\check{\mathcal{E}}u_\alpha$ *of which singular support is* Λ_1, *but there is no such quotient of* $\check{\mathcal{E}}u_{\alpha+1}$. *Then,* $(s-\alpha)\,|\,(b_{\Lambda_1}/b_{\Lambda_2})$.

Once they are established, we have only to seek for α's which satisfy the condition of lemma 2. Using the structure theorem again we eventually reach the formula,

$$\prod_{\lambda=0}^{m_1-m_2-1}(s+((\mu_1-\mu_2+1)/2+\lambda)/(m_1-m_2))\,b_{\Lambda_2}|\,b_{\Lambda_1}$$

Note that the degrees of both sides coinside. Hence this is an equality.

As an example, we determine b-function of Example 2. Since $f(x)$ is invertible at a generic point of Λ_1, $b_{\Lambda_0}=1$ by $fu_s=1\cdot f\cdot u_s$. $b_{\Lambda_k}/b_{\Lambda_{k-1}}=s+\beta$, $\beta=(k^2-(k-1)^2+1)/2=k$. Therefore, $b_{\Lambda_n}=\prod_{\nu=1}^{n}(s+\nu)$.

Even when Λ_2 is singular, there is a similar theorem, which holds on the basis of the canonical form of the simple holonomic system near $\Lambda_1\cup\Lambda_2$.

Lemmata 1 and 2 are also valid in that case, and we can prove

Theorem 4. *Assume* Λ_1 *is non-singular. Then, with a pair of mutually prime natural numbers* $(m;n)$ *associated to* $\Lambda_1\cap\Lambda_2$, *it follows*

$$b_{\Lambda_1}(s)=\prod_{k=0}^{n}[cs+(c_1+k)/(n+m)]_c b_{\Lambda_2}(s),$$

where $c=(m_1-m_2)/(n+1)$ *and* $c_1=\frac{1}{2}((n+m)(\mu_1-\mu_2)/(n+1)+m)$.

Theorem 3 can be considered as a special case $m=1$, $n=0$.

As an application, we explain Example 3. In this case the intersection of Λ_2 and Λ_4 has the index $(m;n)=(2;1)$. Thus $c=2$, $c_1=4$.

$$b_{\Lambda_4}(s)=\left(2s+\frac{4}{3}\right)\left(2s+\frac{4}{3}+1\right)\left(2s+\frac{5}{3}\right)\left(2s+\frac{5}{3}+1\right)b_{\Lambda_2}.$$

Therefore, $b_{\Lambda_4}=\prod_{\nu=1}^{5}(s+\frac{1}{2}+\frac{\nu}{2})$. (Cf. Remark 2° after Theorem 2.)

There follows also a theorem when the holonomy diagram of three good Lagrangians are as depicted in Fig. 8, where Λ_1 and Λ_2 impliy to intersect transversally and Λ_3 does contact to $\Lambda_1\cap\Lambda_2$. We omit the details of this case.

Even when (G, V) has several fundamental invariants, almost the same procedure can be applied in such cases, b-function is defined by

$$P_\nu(s, x, D)f^{s+\nu}=b_\nu(s)f^s,$$

where $s=(s_1, \cdots, s_l)$ and $\nu=(\nu_1, \cdots, \nu_l)\in N_0^l$. Example 5 is one of such cases.

§ 4. The Case of Isolated Singularity

When f has an isolated singularity at $0\in C^n$, the holonomy diagram of $\mathcal{N}_s(s\in C)$ consists of only three Lagrangians. But the multiplicity of $T^*_{\{0\}}X$ is greater than one in general (cf. Examples 1 and 6). In this case, we use the module $\tilde{\mathcal{M}}=(s+1)\mathcal{M}=\mathcal{D}[s]/\mathcal{J}(s)+\mathcal{D}[s](\mathfrak{a}+\mathcal{O}f)$ where $\mathfrak{a}=\sum\mathcal{O}\frac{\partial f}{\partial x_i}$. Then $\check{SS}(\tilde{\mathcal{M}})=T^*_{\{0\}}X$, and hence $\tilde{\mathcal{M}}$ is a finite direct sum of $\mathcal{B}_{pt}=\mathcal{D}\delta(x)$. Therefore, if we write $\tilde{b}(s)$ for the minimal polynomial of s in $\mathcal{H}om_{\mathcal{D}}(\tilde{\mathcal{M}}, \mathcal{B}_{pt})$, we readily have $b(s)=(s+1)\tilde{b}(s)$. We determine $\tilde{b}(s)$ by constructing eigenfunctions of s.

It the case when f is a quasi-homogeneous, the action of s is semi-simple and furthermore, we can easily determine $\tilde{b}(s)$. For instance, when $f=\sum x_i^2$, $\tilde{\mathcal{M}}=\mathcal{D}/\sum\mathcal{D}x_i=\mathcal{B}_{pt}$. In view of equations $s\cdot f^s=X_0f^s$ with $X_0=\sum x_iD_i/2$, and $X_0\delta=-(n/2)\delta$, $\delta(x)$ is an eigenfunction of s in $\mathcal{H}om(\tilde{\mathcal{M}}, \mathcal{B}_{pt})\cong C\delta(x)$ belonging to the eigenvalue $-n/2$. Thus $\tilde{b}(s)=\left(s+\frac{n}{2}\right)$ and we again arrive at $b(s)=(s+1)\left(s+\frac{n}{2}\right)$.

On the other hand, when f is non-quasi-homogeneous, s may not be semi-simple. Indeed, in Example 6, -1 proves to be a non-semi-simple double root. In this example, $\mathcal{J}(s)$ is generated by $f_iD_j-f_jD_i$, $x_is-A_i(x, D)$ and $s^2+A(x, D)s+B(x, D)$ where A_i and A are of first order and B is of second order operator. Thus, $\tilde{\mathcal{M}}$ is generated by the residue classes of 1 and s, and has a presentation

$$0\leftarrow\tilde{\mathscr{M}}\leftarrow\mathscr{D}^{2}\leftarrow\mathscr{D}^{11}.$$

Therefore, $\mathscr{H}om_{\mathscr{D}}(\tilde{\mathscr{M}}, \mathscr{B}_{pt})$ is a subspace of $\mathscr{B}_{pt}^{2}$, and the action of s can be realized as a matrix $\begin{pmatrix} & 1 \\ -B & -A\end{pmatrix}$. Then, $e_1=\begin{pmatrix}0 \\ \delta(x)\end{pmatrix}$ and $e_2=\begin{pmatrix}\delta(x) \\ -\delta(x)\end{pmatrix}$ form a root subspace belonging to the eigenvalue -1. Other eigenvalues are $-\left(1+\frac{\nu}{p_i}\right)$ $1\leqq\nu\leqq p_i-1$, and hence

$$\tilde{b}(s)=(s+1)^{2}\left(\prod_{i=1}^{3}\prod_{1\leq\nu\leq p_i-1}\left(s+1+\frac{\nu}{p_i}\right)\right)_{\mathrm{red}}$$

Here, we use the notation that if $a(s)=\prod(s+\alpha_i)^{\varepsilon_i}$, $\varepsilon_i\geqq 1$, $\alpha_i\neq\alpha_j$ $(i\neq j)$, then $(a(s))_{\mathrm{red}}=\prod(s+\alpha_i)$.

It is known that $\exp(2\pi is)$ is equivalent to the local monodromy map of $f^{-1}(0)$ at 0 when f is of isolated singularity [6].

When $\{-\alpha_j\}$ is the set of characteristic roots of s in $\mathscr{H}om_{\mathscr{D}}(\tilde{\mathscr{M}}, \mathscr{B}_{pt})$ (counted repeatedly according to the multiplicity), we sometimes use the notation that

$$\tilde{P}(t)=\sum t^{\alpha_j}.$$

For instance, in Example 6,

$$\tilde{P}(t)=-t+t(1-t)\sum_{i=1}^{3}\frac{1}{1-t^{1/p_i}}.$$

When f is of isolated singularity and weighted homogeneous with weights $(a_1, \cdots, a_n)$, the following explicit formula holds.

$$\tilde{P}(t)=\prod_{i=1}^{n}\frac{t^{a_i}-t}{1-t^{a_i}}.$$

Example 7. $f=x^{n_1}+y^{n_2}+cx^{m_1}y^{m_2}$. $0\neq c\in\mathbf{C}$.

Case 1. $\frac{n_i}{2}\leq m_i\leq n_i-2$ $(i=1,2)$, or $m_i=\frac{n_i-1}{2}$, $\frac{2}{3}n_j\leq m_j\leq n_j-2$ $(i,j=1,2)$. Then, s is semi-simple and

$$\tilde{P}(t)=\frac{(t^{\alpha}-t)(t^{\beta}-t)}{(1-t^{\alpha})(1-t^{\beta})}+t^{1-a\alpha-b\beta}(1-t)\frac{(1-t^{a\alpha})(1-t^{b\beta})}{(1-t^{\alpha})(1-t_{\beta})},$$

where $\alpha=1/n_1$, $\beta=1/n_2$, and $a=n_1-m_1-1$, $b=n_2-m_2-1$. In this case, $cx^{m_1}y^{m_2}$ in f can be considered as a term of higher order deformation with respect to $x^{n_1}+y^{n_2}$. Therefore, the local monodromy of $f^{-1}(0)$ at 0 is the same as that of $x^{n_1}+y^{n_2}$. b-function, however, varies as indicated above.

Case 2. $2\leq m_i\leq\frac{n_i}{2}$ $(i=1,2)$, or $m_i=\frac{n_i+1}{2}$, $2\leq m_j\leq\frac{n_j}{3}$ $(i,j=1,2)$. Then, if $d=\text{g.c.d.}(m_1,m_2)\geq 2$, eigenvalues $-\frac{r}{d}$ $(1\leq r\leq d-1)$ are non-semi-simple double roots.

$$\tilde{P}(t)=t+\frac{t^{a_{11}+a_{12}}(1-t)(1-t^{(m_1-1)a_{11}})}{(1-t^{a_{11}})(1-t^{a_{12}})}+\frac{t^{a_{21}+a_{22}}(1-t)(1-t^{(m_2-1)a_{22}})}{(1-t^{a_{21}})(1-t^{a_{22}})},$$

where

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}=\begin{pmatrix} \frac{n_2-m_2}{m_1n_2} & \frac{1}{n_2} \\ \frac{1}{n_1} & \frac{n_1-m_1}{m_2n_1} \end{pmatrix}.$$

The characteristic polynomial $\chi(\tau)$ of local monodromy of $f^{-1}(0)$ at 0 is given by, when $\text{g.c.d.}(n_1-m_1,m_2)=\text{g.c.d.}(m_1,n_2-m_2)=1$,

$$\chi(\tau)=\frac{(\tau^{m_2n_1}-1)(\tau^{m_1n_2}-1)}{(\tau^{n_1}-1)(\tau^{n_2}-1)}(\tau-1).$$

In this way, we can determine all b-functions of isolated singularity with modality less than 3, [1] [2].

Notes

(1) More exactly, the order of pole can be improved by the use of reduced b-function, $\bar{b}(s)$. Let $b_\nu(s)$ be a polynomial of minimum degree among $b'(s)$ satisfying

$$P(s,x,D)f^{s+\nu}=b'(s)f^s.$$

Then, $b_\nu(s)$ has the following structure for $\nu\gg 0$.

$$b_\nu(s)=[\bar{b}(s)]_\nu c(s+\nu).$$

Here, $\bar{b}(s)$ and $c(s)$ are polynomials. By the aid of $\bar{b}(s)$ which is called the reduced b-function, we define the gamma factor, $\bar{\gamma}(s)=\prod\Gamma(s+\alpha_j)$, when $\bar{b}(s)=\prod(s+\alpha_j)$. Then, $\frac{1}{\bar{\gamma}(s)}f^s$ depends holomorphically on s.

(2) $GO(n)=\{A\in GL(n)\,|\,{}^tAA=c\cdot I_n,\ c\neq 0\}$

(3) This isomorphism holds if and only if $b(\alpha-n)\neq 0$, $n\in N$.

(4) When $\mathcal{L}$ has the structure of direct sum near the intersection, we do not write a segment.

(5) $[p(s)]_k=p(s)p(s+1)\cdots p(s+k-1)$ for a rational function $p(s)$.

References

[1] Yano, T., On the theory of b-functions, to appear in *Publ. of RIMS, Kyoto Univ.*

[2] ———, On the theory of b-functions 2, in preparation

[3] ———, On the theory of b-functions —Examples— Master thesis presented to *Kyoto Univ.*, 1974, in Japanese.

[4] Kashiwara, M., Kimura, T., Muro, M., Microlocal calculus of simple holonomic systems and its applications, to appear.

[5] Kashiwara, M., B-functions and holonomic systems, Inventiones Math. **38** (1976), 33-53.

[6] Malgrange, B., Sur les polynomes de I. N. Bernstein, to appear.

[7] Yano, T. and Sekiguchi, J., On the weighted homogeneous polynomials associated with Coxeter groups, to appear.

[8] Kashiwara, M., b-functions and the singularity of hypersurface, *RIMS Kokyuroku* **225** (1975), 16-53, in Japanese.