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On Lorentz Invariant Distributions

by

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The study of Lorentz invariant distributions has a long history. Lorentz invariant one-point distributions were extensively investigated by P.-D. Methée [2]. It was shown by K. Hepp [1] that in the case of n-point Lorentz invariant tempered distributions with supports for one-point only in in $\overline{V}_+{}^n$ the problem of the description of Lorentz invariant distributions is reduced to that of the description of the rotation invariant tempered distributions of $n-1$ three-vectors.

The purpose of the present report is to give an effective description of Lorentz invariant (and covariant) distributions in the above case. We study the rotation invariant distributions by means of $SO(3)-$ harmonic analysis on the space $S'(\boldsymbol{R}^{3n})$. The reader is referred to [5] for further details.

The following notations will remain fixed throughout. $Y_{lm}(\theta,\varphi)$ denote the spherical harmonics, i.e., the eigenvectors of the spherical part of the three-dimensional Laplace operator, $Y_{lm}(x)$ denote corresponding harmonic polynomial $Y_{lm}(x)=r^l Y_{lm}(\theta,\varphi)$, $x\in\boldsymbol{R}^3$, where $f(r,\Omega)$ is the function $f(x)$ in spherical coordinates.

Let $S(\overline{\boldsymbol{R}}_+\times\widehat{SO}((3))$ be the space of the sequences $\{\varphi_{lm}(t)\}$ ($l=0, 1,\cdots$; $m=-l,\cdots,l$) of infinitely differentiable functions $\varphi_{lm}(t)$ on $\boldsymbol{R}_+$, whose derivatives all have continuous extensions to $\overline{\boldsymbol{R}}_+$ and are of fast decrease at infinity ($t, l\to\infty$). The topology on $S(\overline{\boldsymbol{R}}_+\times\widehat{SO}(3))$ is defined by the seminorms

$$\sup_{\substack{l,m,\overline{\boldsymbol{R}}_+\\ \alpha\leq q}}(2l+1)^p t^{1/2(l-k)_+}(1+t)^n|\varphi^{(\alpha)}(t)|$$

where $(l-k)_+=\max(l-k,0)$ and p, n, q, k are integers.

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With the distribution $f(x)\in S'(\boldsymbol{R}^3)$ and the harmonic polynomial $Y_{lm}(x)$ we relate the distribution $f_{lm}(t)\in S'(\overline{\boldsymbol{R}}_+)$

$$(f_{lm}(t),\varphi(t))=(f(x),Y_{lm}(x)\varphi(|x|^2)).$$

$f_{lm}(t)$ is said to be the spherical harmonic of the distribution $f(x)$. Using the estimates for derivatives of the polynomial $Y_{lm}(x)$ one can prove that the sequence $f_{lm}(t)$ belongs to $S'(\overline{\boldsymbol{R}}_+\times\widehat{SO}(3))$.

The following theorem shows that the spherical harmonics f_{lm} determine the distribution f completely.

Theorem 1. *The relation*

$$(f(x),\varphi(x))=\sum_{l,m}(f_{lm}(t),t^{-l/2}\int_{S^2}d\Omega\overline{Y}_{lm}(\Omega)\varphi(t^{1/2},\Omega))$$

implies the isomorphism between two topological spaces: $S'(\boldsymbol{R}^3)$ and $S'(\overline{\boldsymbol{R}}_+\times\widehat{SO}(3))$.

In our applications we shall use a multivariable version of Theorem 1, which is easy to prove.

For the description of the rotation invariant distributions it is necessary to consider generalized Wigner's symbol, which may be expressed in terms of the Clebsh-Gordan coefficients $(l_1m_1l_2m_2|l_1l_2jm)$ and the 3-j symbol of Wigner

$$\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}.$$

More precisely we have

$$\begin{aligned}\begin{pmatrix} l_1\cdots l_n \\ m_1\cdots m_n \end{pmatrix}_{j_1,\cdots,j_{n-3}} &=\sum_{\underline{p}}(l_1m_1l_2m_2|l_1l_2j_1p_1)\cdots(j_{n-2}p_{n-2}l_{n-2}m_{n-2}|j_{n-2}l_{n-2}j_{n-3}p_{n-3}). \\ &\qquad\times\begin{pmatrix} j_{n-3} & l_{n-1} & l_n \\ p_{n-3} & m_{n-1} & m_n \end{pmatrix}.\end{aligned}$$

The properties of the generalized Wigner's symbols are analogous to those of the 3-j symlols of Wigner. This symbol isn't zero only for $1_1,\cdots,l_n$; $j_1,\cdots,j_{n-3}$ satisfying the polygonal condition. The polygonal condition for the natural numbers $l_1,\cdots,l_n$; $j_1,\cdots,j_{n-3}$ is as follows: one can construct the polygon such that $l_1, \cdots, l_n$ correspond to the lengths of sides and $j_1 \cdots, j_{n-3}$ correspond to the lengths of the diagonals which get going at the vertex where sides l_1 and l_n intersect. By P_n we denote the set of $l_1, \cdots, l_n$; $j_1, \cdots, j_{n-3}$ satisfing the polygonal condition.

We relate $p=(l_1, \cdots, l_n;\ j_1, \cdots, j_{n-3}) \in P_n$ to the invariant spherical harmonic

$$Y_p=\sum_{\underline{m}} \begin{pmatrix} l_i & \cdots & l_n \\ m_1 & \cdots & m_n \end{pmatrix}_{\underline{j}} Y_{l_1 m_1}(\Omega_1) \cdots Y_{l_n m_n}(\Omega_n).$$

It corresponds to the invariant polynomial

$$Y_p(x_1, \cdots, x_n)=r_1^{l_1} \cdots r_n^{l_n} Y_p(\Omega_1, \cdots, \Omega_n).$$

We define the invariant harmonic f_p of the distribution $f(x_1, \cdots, x_n) \in S'(\boldsymbol{R}^{3n})$ by

$$(f_p(\underline{t}), \Psi(\underline{t}))=(f(\underline{x}), Y_p(\underline{x}) \Psi(|x_1|^2, \cdots, |x_n|^2))$$

where $\Psi \in S(\overline{\boldsymbol{R}}_+^n)$.

Let $S(\overline{\boldsymbol{R}}_+^n P_n)$ be the space of the sequences $\{\Psi_p\}$ ($p=(l_1, \cdots, l_n;\ j_1, \cdots, j_{n-3}) \in P_n$) of infinitely differentiable functions $\Psi_p(t_1, \cdots, t_n)$ on $\boldsymbol{R}_+^n$, whose derivatives all have continuous extensions to $\overline{\boldsymbol{R}}_+^n$ and are of fast decrease at infinity ($\underline{t}, \underline{l} \to \infty$). The topology on $S(\overline{\boldsymbol{R}}_+^n P_n)$ is defined by the seminorms

$$\max_{p \in p_n}(2 \sum_{i=1}^n l_i+1)^K \|\Psi_p(\underline{t})\|_{m,q,s}^{(p)}$$

where the seminorm $\|\Psi(t)\|_{m,q,s}^{(p)}$ equals

$$\sup_{\overline{\boldsymbol{R}}_+^n} t_i^{(l_1-s)_+/2} \cdots t_n^{(l_n-s_n)_+/2}(1+\sum_{i=1}^n t_i)^m |\mathscr{D}^q \Psi(\underline{t})|.$$

The multivariable version of Theorem 1 may be used to obtain the multispherical decomposition of the rotation invariant distribution f. Let us consider the transformation of this decomposition under the rotation from the group $SO(3)$. By integration over $SO(3)$ with respect to normalized Haar measure we have

Theorem 2. *The sequence* $\{f_p\}$ *belongs to* $S'(\overline{\boldsymbol{R}}_+^n P_n)$ *and*

$$(f, \varphi)=\sum_{p \in P_n}(f_p, t_1^{-l_1/2} \cdots t_n^{-l_n/2} \int_{S^{2 \times n}} d^n \Omega \overline{Y}_n(\Omega) \varphi(\underline{t}^{1/2}, \underline{\Omega})).$$

This relation implies the topological isomorphism between the space of $SO(3)$-invariant tempered distributions from $S'(\boldsymbol{R}^{3n})$ and the space $S'(\overline{\boldsymbol{R}}_{+}^{n}P_{n})$.

We restricted our attention to rotation invariant distributions; the generalization to rotation covariant distributions requires only minor and mostly obvious modifications.

Recently A. I. Oksak [4] proved that for any compact group G provided the condition on the system of the standard invariants there exists an isomorphism between the space of the G-invariant distributions and the space of the distributions with the supports in the subset of the G-invariants. The equivalence result may be generalized to G-covariant distributions. However this result seems to be purely esthetical interest for $SO(3)$ group, because there exists no effective description either of above isomorphism or of such object as the distribution with support in algebraic variety of the $SO(3)$-invariants.

We shall now consider the Lorentz invariant distributions with the supports in $\overline{V}_{+}{}^{\mu}\times\boldsymbol{R}^{4n}$. For any $y\in V_{+}$, let $L(y)$ be the Lorentz transformation corresponding to the $A(y)\in SL(2,C)$

$$A(y)=[2(y,y)^{1/2}((y,y)^{1/2}+y_{0})]^{-1/2}\{((y,y)^{1/2}+y_{0})\sigma_{0}+\underline{y}\underline{\sigma}\}$$

Let $g(x)$ be a function on $\boldsymbol{R}^{4}$. By $g(t,r,\theta,\varphi)$ we denote the function $g(t,r\sin\theta\cos\varphi,r\sin\theta\sin\varphi,r\cos\theta)$. For any $p=(l_{1},\cdots,l_{n};\ j_{1},\cdots,j_{n-3})\in P_{n}$ we define the map M_{p} by

$$\begin{aligned}M_{p}\varphi(t_{0},\cdots,t_{2n})&\\ &=t_{n+1}^{-l_{1}/2}\cdots t_{2n}^{-l_{n}/2}\int dy\int_{S^{2\times n}}d^{n}\Omega\overline{Y}_{p}(\underline{\Omega})\delta((y,y)-t_{0})\\ &\quad\times\varphi(y,L(y)(t_{1},t_{n+1}^{1/2},\Omega_{1}),\cdots,L(y)(t_{n},t_{2n}^{1/2},\Omega_{n}))\end{aligned}$$

where $\varphi\in S(\boldsymbol{R}^{4(n+1)})$.

Let $\overline{\boldsymbol{R}}_{+}^{(\mu^{2})}$ denote the interval $\mu^{2}\leqq t<\infty$. The space $S(\overline{\boldsymbol{R}}_{+}^{(\mu^{2})}\times\boldsymbol{R}^{n}\times\overline{\boldsymbol{R}}_{+}^{n}P_{n})$ may be defined in the similar way to the space $S(\overline{\boldsymbol{R}}_{+}^{n}P_{n})$; for a sequence $\{\varphi_{p}(t_{0},\cdots,t_{n},t_{n+1},\cdots,t_{2n})$ an index $p\in P_{n}$ is related to the variables $t_{n+1},\cdots,t_{2n}$ only.

Hepp's result [1] and Theorem 2 lead immediately to

Proposition 1 *For any Lorentz invariant $F\in S'(\overline{V}_{+}^{\mu}\times\boldsymbol{R}^{4n})$ there ex-*

ists a sequence $\{f_p\} \in S'(\overline{\boldsymbol{R}}_+^{(\mu^2)} \times \boldsymbol{R}^n \times R_+^n P_n)$ *such that*

$$(F, \varphi) = \sum_{p \in P_n} (f_p, M_p \varphi).$$

This relation implies the topological isomorphism between the space of $L_+^\uparrow$*-invariant tempered distributions from* $S'(\overline{V}_+^\mu \times \boldsymbol{R}^{4n})$ *and the space* $S'(\overline{\boldsymbol{R}}_+^{(\mu^2)} \times \boldsymbol{R}^n \times \boldsymbol{R}_+^n P_n)$.

A.I.Oksak and I.T.Todorov [3] generalized Hepp's result [1] to Lorentz covariant distributions. This generalization and the covariant version of Theorem 2 may be used to obtain the description of the Lorentz covariant distribution from $S'(\overline{V}_+^\mu \times \boldsymbol{R}^{4n})$.

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