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Isomorphism of a Piecewise Linear Transformation to a Markov Automorphism

By

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D. Ornstein and others proved important isomorphism theorems for a large class of automorphisms, i.e. measure preserving transformations. However, it is also interesting to explicitely determine an isomorphic mapping in each concrete case. R. Adler and B. Weiss [1] explicitely constructed an isomorphism between an ergodic group automorphism of a 2-dimensional torus and a Markov automorphism. Y. Takahashi [8] gave an isomorphism between a β-automorphism and a Markov automorphism. The crucial point of their argument lies in the fact that the metrical entropy coincides with the topological entropy for these automorphisms. Using this fact and Parry's result [3], they showed that the representation mapping of the automorphism in consideration to a Markov subshift (of a symbolic dynamics) is actually an isomorphism in each case.

In this paper, we will explicitely construct an isomorphism of a piecewise linear transformation (a generalization of β-automorphism) to a Markov automorphism. Since the metrical entropy of such a transformation does not always attain its topological entropy, we cannot use the method mentioned above, so instead of topological entropy we use free energy as our main tool.

In § 1 we define the free energy of a Markov subshift and show under certain conditions that a shift invariant measure with the minimal free energy is unique. This is a generalization of Parry's result about topological entropy [3]. In § 2 we define a piecewise linear transformation and investigate its properties. In § 3 we construct an isomorphic mapping from a piecewise linear transformation to a Markov automor-

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§ 1. Markov Potential Function and Its Free Energy

Let $I^{\boldsymbol{z}}$ be an infinite product of state space I, where $I=\{0, 1, \cdots, s\}$ or $\{0, 1, 2, \cdots\}$ and $\boldsymbol{z}=\{\cdots, -1, 0, 1, \cdots\}$. Let $\omega(n)$ be the n-th coordinate of ω in $I^{\boldsymbol{z}}$ and let σ be a shift operator on $I^{\boldsymbol{z}}$ such that

$$(\sigma\omega)(n)=\omega(n+1).$$

Definition 1-1. A function $U: I^{\boldsymbol{z}} \to [0, \infty]$ is called a (simple) Markov potential function, if $U(\omega)=U(\omega(0), \omega(1))$ for all $\omega \in I^{\boldsymbol{z}}$.

Definition 1-2. A Markov potential function U is called irreducible, if, for any $i, j \in I$, there exists a finite sequence $(i_0, i_1, \cdots, i_n)$ such that $i_0=i$, $i_n=j$, and $U(i_k, i_{k+1})<\infty$ for all $0 \leqq k \leqq n-1$.

Definition 1-3. A Markov potential function U is called of Perron-Frobenius type if

(1) U is irreducible

(2) $Q(i,j)$ defined by $Q(i,j)=e^{-U(i,j)}$

has positive right and left eigenvectors, $r=(r_i)$ and $l=(l_i)$ respectively, with a common positive eigenvalue λ, such that

$$\sum_{j \in I} Q(i,j) r_j = \lambda r_i \quad \text{for all} \quad i \in I,$$

$$\sum_{i \in I} Q(i,j) l_i = \lambda l_j \quad \text{for all} \quad j \in I,$$

$$\sum_{i \in I} l_i r_i < \infty .$$

Definition 1-4. Let λ, $r=(r_i)$ and $l=(l_i)$ be the ones defined by Perron-Frobenius type potential U as above. The Gibbs measure $\nu=\nu(U)$ is an ergodic σ-invariant Markov measure with initial distribution

$$\pi_i = l_i r_i / (\sum_{j \in I} l_j r_j) \quad \text{for} \quad i \in I,$$

and transition probabilities

$$P_{ij}=\frac{r_jQ(i,j)}{\lambda r_i} \quad \text{for} \quad i,j\in I.$$

When $I=\{0,1,\cdots,s\}$ $(s<\infty)$, free energy $f_U(\mu)$ is defined by (cf. Spitzer [7]), $f_U(\mu)=\int U d\mu-h_\mu(\sigma)$, where μ is a σ-invariant probability measure on (I^z,σ). Now we extend this definition.

Definition 1-5. Let $I=\{0,1,2,\cdots\}$ and let μ be a σ-invariant probability measure on (I^z,σ) with finite metrical entropy $h_\mu(\sigma)$. Then free energy $f_U(\mu)$ is defined by,

$$f_U(\mu)=\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}}\sum_{x\in I_m{}^n}\mu([x])U_n{}^m(x)-h_\mu(\sigma),$$

where $I_m=\{0,1,\cdots,m,m^*\}$, $m^*=\{m+1,m+2,\cdots\}$, $[x]$ is the cylinder set generated by $x\in I_m{}^n$ and $U_n{}^m$, a function on $I_m{}^n$, is defined in the following way. For $x,y\in I_m'=\{0,1,\cdots,m\}$, $k\geqq 1$, we put

$$U_2{}^m(x,y)=U(x,y),$$

$$U_{k+1}^m(x,\underbrace{m^*,\cdots,m^*}_{k})$$

$$=-\frac{1}{k}\log\sum_{\substack{i_1\geqq m+1\\ \cdots\cdots\\ i_k\geqq m+1}} r_{i_k}\exp[-U(x,i_1)-\sum_{j=1}^{k-1}U(i_j,i_{j+1})]$$

$$U_{k+2}^m(x,\underbrace{m^*,\cdots,m^*}_{k},y)$$

$$=-\frac{1}{k+1}\log\sum_{\substack{i_1\geqq m+1\\ \cdots\cdots\\ i_k\geqq m+1}}\exp[-U(x,i_1)-\sum_{j=1}^{k-1}U(i_j,i_{j+1})-U(i_k,y)]$$

$$U_{k+1}^m(\underbrace{m^*,\cdots,m^*}_{k+1})$$

$$=-\frac{1}{k}\log\sum_{\substack{i_1\geqq m+1\\ \cdots\cdots\\ i_{k+1}\geqq m+1}}\frac{l_{i_1}r_{i_{k+1}}}{\sum_j l_jr_j}\exp[-\sum_{j=1}^{k}U(i_j,i_{j+1})],$$

and

$$U_{k+1}^m(\underbrace{m^*,\cdots,m^*}_{k},y)$$

$$=-\frac{1}{k}\log\sum_{\substack{i_1\geqq m+1\\ \cdots\cdots\\ i_k\geqq m+1}}\frac{l_{i_1}}{\sum_j l_jr_j}\exp[-\sum_{j=1}^{k-1}U(i_j,i_{j+1})-U(i_k,y)]$$

(we define $\log 0=-\infty$). Moreover we define $U_n{}^m$ inductively by

$$(n-1)\,U_n{}^m(x_1,\cdots,x_n)$$
$$=(i-1)\,U_i{}^m(x_1,\cdots,x_i)+(n-i)\,U^m_{n-i+1}(x_i,\cdots,x_n)$$

if $x_j\in I_m$ $(1\leqq j\leqq n)$ and $x_i\in I_m{}'$.

Remark 1. For $x_i\in I_m{}'$ $(1\leqq i\leqq n)$,

$$(n-1)\,U_n{}^m(x_1,\cdots,x_n)=\sum_{i=1}^{n-1}U(x_i,x_{i+1})\,.$$

We get, for $x\in I_m{}^n$,

$$U_n{}^m(x)=-\log\lambda-\frac{1}{n-1}\log P_n{}^m(x)-\frac{1}{n-1}\log r^m_{x_1}+\frac{1}{n-1}\log r^m_{x_n}$$

(including $\infty=\infty$)

where $r_i{}^m=r_i$, $i\in I_m{}'$, $r^m_{m^*}=1$, and $P_n{}^m(x)$, a function on $I_m{}^n$, is defined by the following; for $x,y\in I_m{}'$, $k\geqq 1$

$$P_2{}^m(x,y)=P_{x,y}$$

$$P^m_{k+1}(x,\underbrace{m^*,\cdots,m^*}_{k})=\sum_{\substack{i_1\geqq m+1\\ \cdots\cdots\\ i_k\geqq m+1}}P_{x,i_1}P_{i_1,i_2}\cdots P_{i_{k-1},i_k}$$

$$P^m_{k+2}(x,\underbrace{m^*,\cdots,m^*}_{k},y)=\sum_{\substack{i_1\geqq m+1\\ \cdots\cdots\\ i_k\geqq m+1}}P_{x,i_1}P_{i_1,i_2}\cdots P_{i_{k-1},i_k}P_{i_k,y}$$

$$P^m_{k+1}(\underbrace{m^*,\cdots,m^*}_{k+1})=\sum_{\substack{i_1\geqq m+1\\ \cdots\cdots\\ i_{k+1}\geqq m+1}}\pi_{i_1}P_{i_1,i_2}\cdots P_{i_k,i_{k+1}}$$

$$P^m_{k+1}(\underbrace{m^*,\cdots,m^*}_{k},y)=\sum_{\substack{i_1\geqq m+1\\ \cdots\cdots\\ i_k\geqq m+1}}\pi_{i_1}P_{i_1,i_2}\cdots P_{i_{k-1},i_k}P_{i_k,y}$$

and

$$P_n{}^m(x_1,\cdots,x_n)=P_i{}^m(x_1,\cdots,x_i)\,P^m_{n-i+1}(x_i,\cdots,x_n)$$

where $x_j\in I_m$ $(1\leqq j\leqq n)$, $x_i\in I_m{}'$.

Remark. If $x_i\in I_m{}'$ $(1\leqq j\leqq n)$,

$$P_n{}^m(x_1,\cdots,x_n)=\prod_{i=1}^{n-1}P_{x_i,x_{i+1}}\,.$$

Theorem 1. *Let U be a Markov potential of Perron-Frobenius type on $(I^{\mathbf{z}}, \sigma)$, and let λ be an eigenvalue of $Q=\exp(-U)$ as in Definition* 1-3. *Then, for any σ-invariant probability measure μ with finite entropy, we have*

$$f_U(\mu) \geqq -\log \lambda \tag{1.1}$$

In particular, (1.1) *holds with equality if and only if $\mu=\nu$ and $h_\nu(\sigma)<\infty$, where $\nu=\nu(U)$ is the Gibbs measure defined by U.*

Proof. 1st stage: We will show (1.1) holds with equality if $\mu=\nu$ and $h_\nu(\sigma)<\infty$. We denote by φ_m the σ-algebra generated by the cylinder sets of $I_m^{\mathbf{z}}$, and by $h_\mu(\sigma|\varphi_m)$ we mean the entropy of the factor of σ on φ_m. Evidently,

$$\begin{aligned} h_\mu(\sigma) &= \lim_{m\to\infty} h_\mu(\sigma|\varphi_m) \\ &= \lim_{m\to\infty}\lim_{n\to\infty} \frac{1}{n} \sum_{x\in I_m{}^n} \mu([x]) \log \mu([x]). \end{aligned}$$

Therefore,

$$\begin{aligned} f_U(\nu) &= \overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}} \Big[-\log\lambda - \frac{1}{n-1} \sum_{x\in I_m{}^n} \nu([x]) \log \nu([x]) \\ &\quad + \frac{1}{n-1} \sum_{i\in I_m'} \pi_i \log \pi_i \Big] + \lim_{m\to\infty}\lim_{n\to\infty} \frac{1}{n} \sum_{x\in I_m{}^n} \nu([x]) \log \nu([x]) \\ &= -\log\lambda . \end{aligned}$$

2nd stage: Let μ be a σ-invariant probability measure with finite entropy which does not equal to ν. Let $\{\tilde{\pi}_i\}$ be a distribution such that

(1) $\sum_{i\in I} \tilde{\pi}_i = 1$

(2) $-\sum_{i\in I} \mu([i]) \log \tilde{\pi}_i < \infty$

(3) $\tilde{\pi}_i > 0$ if and only if $\pi_i > 0$.

We define $\tilde{\nu}$ the Markov measure with initial distribution $\tilde{\pi}_i$ and transition probability P_{ij}. (Notice that $\tilde{\nu}$ is not always σ-invariant.) Then it is easy to see that $\tilde{\nu}\sim\nu$. From the σ-invariance of μ, we get

$$\frac{1}{n-1} \sum_{x\in I_m{}^n} \mu([x]) \log P_n{}^m(x) \leqq \sum_{x,y\in I_m'} \mu([x,y]) \log P_{x,y} .$$

Therefore,

$$\overline{\lim_{n\to\infty}}\frac{1}{n-1}\sum_{x\in I_m{}^n}\mu([x])\log P_n{}^m(x)\leqq\sum_{x,y\in I'_m}\mu([x,y])\log P_{x,y}\,.$$

If $\lim_{m\to\infty}\sum_{x,y\in I'_m}\mu([x,y])\log P_{x,y}=-\infty$, then

$$f_U(\mu)=\overline{\lim_{m\to\infty}}\,\overline{\lim_{n\to\infty}}\sum_{x\in I_m{}^n}\mu([x])U_n{}^m(x)-h_\mu(\sigma)=\infty>-\log\lambda\,.$$

If $\lim_{m\to\infty}\sum_{x,y\in I_m'}\mu([x,y])\log P_{x,y}>-\infty$, then

$$\overline{\lim_{m\to\infty}}\,\overline{\lim_{n\to\infty}}\frac{1}{n-1}\sum_{x\in I_m{}^n}\mu([x])\log P_n{}^m(x)$$

$$\leqq\sum_{x,y\in I}\mu([x,y])\log P_{x,y}$$

$$=\frac{1}{n-1}\sum_{x=(x_1,\cdots,x_n)\in I^n}\mu([x])\log P_{x_1,x_2}P_{x_2,x_3}\cdots P_{x_{n-1},x_n}$$

$$=\frac{1}{n-1}\sum_{x\in I^n}\mu([x])\log\tilde{\nu}([x])-\frac{1}{n-1}\sum_{i\in I}\mu([i])\log\tilde{\pi}_i$$

$$\leqq\overline{\lim_{m\to\infty}}\frac{1}{n-1}\sum_{x\in I_m{}^n}\mu([x])\log\tilde{\nu}([x])-\frac{1}{n-1}\sum_{i\in I}\mu([i])\log\tilde{\pi}_i\,.$$

Thus,

$$f_U(\mu)=\overline{\lim_{m\to\infty}}\,\overline{\lim_{n\to\infty}}\sum_{x\in I_m{}^n}\mu([x])U_n{}^m(x)-\lim_{m\to\infty}h_\mu(\sigma|\varphi_m)$$

$$\geqq-\log\lambda-\overline{\lim_{m\to\infty}}\frac{1}{n-1}\sum_{x\in I_m{}^n}\mu([x])\log\tilde{\nu}([x])$$

$$+\frac{1}{n-1}\sum_{i\in I}\mu([i])\log\tilde{\pi}_i+\lim_{m\to\infty}\frac{1}{n}\sum_{x\in I_m{}^n}\mu([x])\log\mu([x])$$

$$\geqq\frac{1}{n-1}\left[\lim_{m\to\infty}\sum_{x\in I_m{}^n}\mu([x])\log\left(\frac{\mu([x])}{\tilde{\nu}([x])}\right)+\sum_{i\in I}\mu([i])\log\tilde{\pi}_i\right]$$

$$-\log\lambda\,.$$

On the other hand, since ν is ergodic and μ is σ-invariant, it follows $\mu\not\perp\nu$. We cite the followings.

Lemma 1 ([5]).

$$\lim_{m\to\infty}\sum_{x\in I_m{}^n}\mu([x])\log\frac{\mu([x])}{\tilde{\nu}([x])}\geqq\sum_{x\in I^n}\mu([x])\log\frac{\mu([x])}{\tilde{\nu}([x])}\,.$$

Lemma 2. (Dobrushin, Gelfand, Yaglom and Perez [5] (p. 20)). *If* μ_1 *and* μ_2 *are probability measures on* I^{z} *and* $\mu_1 \not\prec \mu_2$, *then*

$$\sum_{x \in I^n} \mu_1([x]) \log\left(\frac{\mu_1([x])}{\mu_2([x])}\right) \to \infty \quad \text{as} \quad n \to \infty$$

By the above lemmas, we can take $n \geqq 1$ such that

$$\sum_{i \in I} \mu([i]) \log \tilde{\pi}_i + \sum_{x \in I^n} \mu([x]) \log\left(\frac{\mu([x])}{\tilde{\nu}([x])}\right) > 0\,,$$

and thus we can conclude that

$$f_U(\mu) > -\log \lambda\,.$$

Remark 1-6. If a Markov potential U satisfies the condition in Theorem 1 and takes values only 0 and ∞, then $Q = e^{-U}$ becomes a structure matrix of a Markov subshift [2]. Hence $\inf -f_U(\mu) = -f_U(\nu)$ and we see that ν is the maximal Markov measure [3].

Let U be a Markov potential on (I^{z}, σ) which satisfies the conditions in Theorem 1 and let ν be the Gibbs measure defined by U.

Main Lemma. *Let* (X, T, μ) *be an automorphism with finite entropy. Then* (X, T, μ) *is isomorphic to* (I^{z}, σ, ν), *if there exist a* 1–1 (μ-a.e.) *bi-Borel mapping* $\varphi: X \to I^{z}$, μ-*integrable functions* $V(\omega')$ *and* $h(\omega')$ ($\omega' \in X$) *such that*

(1) $\sigma \cdot \varphi = \varphi \cdot T\,,\quad \mu\text{-a.e.}$

(2) $U(\varphi(\omega')) = V(\omega') + h(\omega')\,,\quad \mu\text{-a.e.}$

$$\int h(\omega')\, d\mu(\omega') = 0\,,$$

(3) $f_U(\nu) = \int d\mu(\omega')\, V(\omega') - h_\mu(T)\,,$

(4) $\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}} \sum_{x \in I_m{}^n} \mu \cdot \varphi^{-1}([x])\, U_n{}^m(x) = \int d\mu \cdot \varphi^{-1} U\,.$

Proof. Let $\mu \cdot \varphi^{-1} = \mu'$, then from the condition (1) μ' is a σ-invariant measure with finite entropy $h_\mu(T)$. Therefore to show that they are isomorphic it is sufficient to prove that $f_U(\mu') = f_U(\nu)$.

$$
\begin{aligned}
f_U(\mu') &= \overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}} \sum \mu'([x])\, U_n{}^m(x) - h_{\mu'}(\sigma) \\
&= \int d\mu(\omega')\, U(\varphi(\omega')) - h_\mu(T) \\
&= \int d\mu(\omega')\, V(\omega') - h_\mu(T) \\
&= f_U(\nu)\,.
\end{aligned}
$$

§ 2. A Definition and Properties of a Piecewise Linear Transformation

We define a piecewise linear transformation according to [6].

Let $\beta=(\beta_0, \beta_1, \cdots, \beta_N)$ is an $N+1$-tuple vector such that $\beta_k>1$ for $0\leqq k\leqq N$ and $\sum_{k=0}^{N}\beta_k{}^{-1}\geqq 1>\sum_{k=0}^{N-1}\beta_k{}^{-1}$. A partition $\{A_i\}_{i=0,\cdots,N}$ of interval $[0, 1)$ is defined by

$$
\begin{aligned}
A_0 &= [0, \beta_0{}^{-1}) \\
A_i &= (\sum_{k=0}^{i-1}\beta_k{}^{-1}, \sum_{k=0}^{i}\beta_k{}^{-1}) \quad i=1, \cdots, N-1\,. \\
A_N &= (\sum_{k=0}^{N-1}\beta_k{}^{-1}, 1)\,.
\end{aligned}
$$

We define mappings $T:[0, 1)\to[0, 1)$ and $\pi;[0, 1)\to S^{\mathbf{N}}$ by

$$
Tx=\beta_i(x-\sum_{k=0}^{i-1}\beta_k{}^{-1}) \quad \text{if} \quad x\in A_i,\ i=0, 1, \cdots, N\,,
$$

and

$$
\pi(x)=(a_0(x), a_1(x), \cdots, a_n(x), \cdots),
$$

$a_n(x)=i$ if $T^n x\in A_i$, respectively, where $S=\{0, 1, \cdots, N\}$ and $S^{\mathbf{N}}$ is its one-sided product. Then the mapping T can be realized as the shift σ on $S^{\mathbf{N}}$ by the mapping π, namely

$$
\pi\cdot T=\sigma\cdot\pi\,.
$$

For convenience we define a sequence corresponding to 1 by

$$
\begin{aligned}
\pi(1) &= (a_0(1), a_1(1), \cdots, a_n(1), \cdots) \\
&= \sup_{x\in[0,1)} (a_0(x), a_1(x), \cdots, a_n(x), \cdots)
\end{aligned}
$$

where supremum is taken over with respect to the lexicographical order. Then it is easy to see that for $x\in[0,1)$

$$x=\sum_{k=0}^{a_0(x)-1}\beta_k{}^{-1}+\beta_{a_0(x)}^{-1}\sum_{k=0}^{a_1(x)-1}\beta_k{}^{-1})+\cdots$$

$$+\beta_{a_0(x)}^{-1}\cdots\beta_{a_n(x)}^{-1}(\sum_{k=0}^{a_{n+1}(x)-1}\beta_k{}^{-1})+\cdots.$$

We put

$$T^j1=\sum_{k=0}^{a_j(1)-1}\beta_k{}^{-1}+\beta_{a_j(1)}^{-1}(\sum_{k=0}^{a_{j+1}(1)-1}\beta_k{}^{-1})+\cdots.$$

We already know the results that the endomorphism T has an invariant probability measure $\nu_\beta{}^+$ equivalent to Lebesgue measure, and its unnormalized density $f(x)$ is given by

$$f(x)=\sum_{n=0}^{\infty}\frac{X_{[0,T^n1)}}{\beta_{a_0(1)}\beta_{a_1(1)}\cdots\beta_{a_n(1)}}(x),$$

where X_A is the characteristic function of A[6].

Let $\pi[0,1)=X_\beta{}^+\subset S^{\mathbf{N}}$ and let $\mu_\beta{}^+$ be an induced measure of $\nu_\beta{}^+$ by π. Then $[X_\beta{}^+,\sigma,\mu_\beta{}^+]$ becomes an endomorphism which is homeomorphic to $([0,1),T,\nu_\beta{}^+)$. We call this endomorphism a piecewise linear endomorphism.

Proposition 2. 1.

(1) *For any* $\omega\in X_\beta{}^-$

$$\sigma^n\omega<(a_0(1),a_1(1),\cdots,a_n(1),\cdots),\ for\ n\geqq 0.$$

(2) *Let* $X_\beta{}^n=\{[a_0(x),\cdots,a_n(x)]:x\in[0,1)\}$, *then*

$$\max\{[b_0,\cdots,b_n]\in X_\beta{}^n\}=[a_0(1),\cdots,a_n(1)].$$

The proofs of this proposition are very easy and we omitt it (c.f. [6]).

The piecewise linear transformation $(X_\beta,\sigma,\mu_\beta)$ which is the natural extension of $(X_\beta{}^+,\sigma,\mu_\beta{}^+)$ is defined by

$$X_\beta=\{\omega\in S^{\mathbf{Z}};(\omega(n),\omega(n+1),\cdots)\in X_\beta{}^+,\ \text{for all}\ n\},$$

$$\mu_\beta([b_0,\cdots,b_n])=\mu_\beta{}^+([b_0,\cdots,b_n]),$$

where $[b_0, \cdots, b_n] \in X_\beta{}^n$. Note that μ_β is well-defined because of σ-invariance of $\mu_\beta{}^+$.

§ 3. Isomorphism of a Piecewise Linear Transformation to a Markov Automorphism

In this section we will prove the following Theorem 2 by appealing to Main Lemma in Section 1 and a mapping φ which is similar to the mapping introduced by Y. Takahashi for β-transformation ([8]).

Theorem 2. *A piecewise linear transformation* $(X_\beta, \sigma, \mu_\beta)$ *is isomorphic to a Markov automorphism.*

To prove this Theorem we begin with the construction of a mapping $\varphi: X_\beta \to I^{\boldsymbol{z}}$ where $I = \{-N+1, \cdots, -1, 0, 1, \cdots\}$.

For $\omega \in X_\beta$, $\tau(\omega)$ is defined by

$$\tau(\omega) = \begin{cases} \sup\{i : i \geqq 1,\ \omega \in B_i\}, \\ 0, \quad \text{if} \quad \omega \in X_\beta \setminus \bigcup_{i \geqq 1} B_i, \end{cases}$$

where $B_i = \{\omega \in X_\beta : (\omega(-i), \cdots, \omega(-1)) = (a_0(1), \cdots, a_{i-1}(1))\}$.

Let $C_i = \{\omega \in X_\beta : \tau(\omega) = i\}$, for $i = 1, 2, \cdots$,

$= \{\omega \in X_\beta : \tau(\omega) = 0, \omega(-1) = -i\}$, for $-N+1 \leqq i \leqq 0$,

$C_\infty = \omega \in X_\beta : \tau(\omega) = \infty\}$.

It is easy to see that $\mu_\beta(C_\infty) = 0$ and $\sigma C_\infty = C_\infty$. Define a mapping $\varphi: X_\beta \setminus C_\infty \to I^{\boldsymbol{z}}$ by

$$(\varphi(\omega))(n) = i \quad \text{if} \quad \omega \in \sigma^n C_i,\ n \in \boldsymbol{Z},\ i \in I.$$

Then, it is easy to see that the mapping φ is a Borel injection and satisfies

$$\varphi \cdot \sigma = \sigma^{-1} \cdot \varphi .$$

Lemma 3-1. (1) *If* $\omega \in C_i$, $1 \leqq i \leqq \infty$, *then*

$$\sigma^{-1} \omega \in C_{i-1} .$$

(2) *If* $\omega \in C_i$, $i \leqq 0$, *and* $\sigma^{-1}\omega \in C_j$, $j \geqq 1$, *then*

$$a_j(1) > -i = |i|.$$

Proof. (1) Suppose $\sigma^{-1}\omega \in C_j$ $j \geqq i$, then

$$(\omega(-j-1), \cdots, \omega(-1)) < (a_0(1), \cdots, a_j(1))$$

and

$$(\omega(-j-1), \cdots, \omega(-2)) = (a_0(1), \cdots, a_{j-1}(1)).$$

Hence

$$\omega(-1) < a_j(1)$$

and so

$$(\omega(-i), \cdots, \omega(-1)) < (\omega(-i), \cdots, \omega(-2), a_j(1)).$$

This contradicts to the maximality of $(\omega(-i), \cdots, \omega(-1))$.

(2) By the assumption, the j-tuple

$$(\omega(-j-1), \cdots, \omega(-2)) = (a_0(1), \cdots, a_{j-1}(1))$$

is maximum. From $\omega \in C_i$, $(i \leqq 0)$, it follows that $\omega(-1) = -i = |i|$ and

$$(\omega(-j-1), \cdots, \omega(-1)) < (a_0(1), \cdots, a_j(1)).$$

Therefore

$$a_j(1) > -i = |i|.$$

We define a Markov potential U on I^{Z} by

$$U(i,j) = \begin{cases} \log \beta_{-i}, & -N+1 \leqq i \leqq 0, \quad -N+1 \leqq j \leqq 0, \\ \log \beta_N, & i=1, \quad -N+1 \leqq j \leqq 0, \\ \log \beta_{a_{i-1}(1)}, & i \geqq 2, \quad j = i-1, \\ \log \beta_{-i}, & -N+1 \leqq i \leqq 0, \quad j > 0 \quad \text{and} \quad a_j(1) > -i, \\ \infty, & \text{otherwise}. \end{cases}$$

It is easy to see that this potential U is irreducible. We show that U is of the Perron-Frobenius type by finding an eigenvalue λ of $Q = e^{-U}$ and corresponding eigenvectors to λ.

Put λ, $l=(l_i)_{i\geq -N+1}$ and $r=(r_i)_{i\geq -N+1}$ as follows,

$$\begin{aligned}
&\lambda=1\,,\\
&l_i=1\,,\quad -N+1\leqq i\leqq 0\,,\\
&\;\;=T^i 1\,,\quad i\geqq 1\,,\\
&r_i=\beta^{-1}_{a_0(1)}\beta^{-1}_{a_1(1)}\cdots\beta^{-1}_{a_{i-1}(1)}\,,\quad i\geqq 1\,,\\
&\;\;=\beta^{-1}_{-i}(1+\sum_{j\geq 1}M_{ij}\beta^{-1}_{a_0(1)}\cdots\beta^{-1}_{a_{j-1}(1)})\,,\quad -N+1\leqq i\leqq 0\,,
\end{aligned}$$

where

$$\begin{aligned}
M_{ij}&=1\,,\quad \text{if}\quad Q(i,j)>0\,,\\
&=0\,,\quad \text{if}\quad Q(i,j)=0\,.
\end{aligned}$$

Then it is easy to see that

$$\begin{aligned}
&\sum_{i\in I} l_i Q(i,j)=l_j=\lambda l_j\,,\quad \text{for all } j\,,\\
&\sum_{j\in I} Q(i,j)r_j=r_i\,,\quad \text{for all } i\,,\\
&\sum_{i\in I} r_i l_i<\infty\,.
\end{aligned}$$

Therefore U is of the Perron-Frobenius type. Let ν be the Gibbs measure corresponding to U. From the fact $-\sum \pi_i \log \pi_i<\infty$,

$$h_\nu(\sigma)<\infty\,.$$

Hence Theorem 1 in Section 1 implies

$$f_U(\nu)=-\log 1=0\,.$$

Now we will define $V(\omega')$ on $(X_\beta,\sigma,\mu_\beta)$ to apply the Main Lemma. Let

$$V(\omega')=\sum_{i\in S} X_i(\omega')\log\beta_i\,,$$

where $X_i(\omega')$ is the characteristic function of $\{\omega':\omega'(-1)=i\}$. Then

$$\begin{aligned}
\int d\mu_\beta(\omega')\,V(\omega')&=\int d\mu_\beta(\omega')\,V(\sigma^{-1}\omega')\\
&=\int\sum_{i\in S} X_{[\omega':\omega'(0)=i]}(\omega')\log\beta_i d\mu_\beta(\omega')\,.
\end{aligned}$$

On the other hand by [4],

$$h_{\mu_\beta}(\sigma) = \int d\mu_\beta \log \phi'(\omega'),$$

where $\phi'(\omega')$ is the derivative of the graph of ϕ-expansion. Hence

$$\int d\mu_\beta(\omega')\,V(\omega') - h_{\mu_\beta}(\sigma) = 0 = f_U(\nu).$$

We have already proved φ satisfies the conditions (1) and (3) in Main Lemma, and hence it suffices to show only the conditions (2) and (4).

1) In case $\omega' \in C_i$, $2 \leqq i < \infty$, it holds $\omega'(-1) = a_{i-1}(1)$ and hence

$$V(\omega') = \log \beta_{a_{i-1}(1)}.$$

On the other hand by Lemma 3-1, it follows

$$(\varphi(\omega'))(0) = i, \quad (\varphi(\omega'))(1) = i-1.$$

Thus

$$U(\varphi(\omega')) = U((\varphi(\omega'))(0), (\varphi(\omega'))(1)).$$
$$= \log \beta_{a_{i-1}(1)}.$$

2) In case $\omega' \in C_i$, $-N+1 \leqq i \leqq 0$, it holds $\omega'(-1) = -i$, and hence

$$V(\omega') = \log \beta_{-i}.$$

From Lemma 3-1,

$$(\varphi(\omega'))(0) = i, \quad (\varphi(\omega'))(1) = j, \quad \text{for some } j,$$
$$U(\varphi(\omega')) = \log \beta_{-i}.$$

3) In case $\omega' \in C_1$, it holds $\omega'(-1) = N$, and hence

$$V(\omega') = \log \beta_N.$$

and

$$(\varphi(\omega'))(0) = i, \quad (\varphi(\omega'))(1) = j, \quad -N+1 \leqq j \leqq 0,$$
$$U(\varphi(\omega')) = \log \beta_N.$$

This implies that

$$U(\varphi(\omega')) = V(\omega'), \quad \mu_\beta\text{-a.e.}$$

Therefore (2) of the Main Lemma is satisfied for $h(\omega') = 0$, $\omega' \in X_\beta$.

To prove (4) of the Main Lemma, put $\mu_\beta' = \mu_\beta \cdot \varphi^{-1}$, then

$$(*)\quad \sum_{x\in I_m{}^n} \mu_\beta'([x])\,U_n{}^m(x)$$

$$= \sum_{x,y\in I_m'} \mu_\beta'([x,y])\,U(x,y)$$

$$+\sum_{i=1}^{n-2}\frac{(i+1)(n-1-i)}{n-1}\sum_{x,y\in I_m'}\mu_\beta'([x,\underbrace{m^*,\cdots,m^*}_{i},y])$$

$$\times U^m_{i+2}(x,\underbrace{m^*,\cdots,m^*}_{i},y)$$

$$+\sum_{i=1}^{n-1}\frac{i}{n-1}\sum_{x\in I_m'}[\mu_\beta'(x,\underbrace{m^*,\cdots,m^*}_{i}])\,U^m_{i+1}(x,\underbrace{m^*,\cdots,m^*}_{i})$$

$$+\mu_\beta'([\underbrace{m^*,\cdots,m^*}_{i},x])\,U^m_{i+1}(\underbrace{m^*,\cdots,m^*}_{i},x)]$$

$$+\mu_\beta'([\underbrace{m^*,\cdots,m^*}_{n}])\,U_n{}^m(\underbrace{m^*,\cdots,m^*}_{n}).$$

On the support of μ_β'

$$U(x,y)\leqq\log\beta_{max}$$

where $\beta_{max}=\max\{\beta_0,\beta_1,\cdots,\beta_N\}$. For $i\geqq1$ we get

$$|U^m_{i+2}(x,\underbrace{m^*,\cdots,m^*}_{i},m)|=\frac{1}{i+1}[U(x,m+i)+\sum_{j=0}^{i-1}U(m+j+1,m+j)]$$

$$\leqq\log\beta_{max},$$

$$|U^m_{i+1}(x,\underbrace{m^*,\cdots,m^*}_{i})|$$

$$=\Bigg|-\frac{1}{i}\log\sum_{j\geqq1}r_{m+j}\exp[-U(x,m+i+j)$$

$$-\sum_{k=1}^{i-1}U(m+k+j+1,m+k+j)]\Bigg|\leqq\log\beta_{max}-\frac{1}{i}\log\frac{\beta^{-m}_{max}}{1-\beta_{max}},$$

$$|U^m_{i+1}(\underbrace{m^*,\cdots,m^*}_{i},m)|$$

$$=\Bigg|-\frac{1}{i}\log\frac{l_{m+i}}{\sum_j l_jr_j}\exp[-\sum_{k=0}^{i-1}U(m+k+1,m+k)]\Bigg|$$

$$\leqq\log\beta_{max}+\left|\frac{1}{i}\log\frac{l_{m+i}}{\sum_j l_jr_j}\right|,$$

$$|U^m_{i+1}(\underbrace{m^*,\cdots,m^*}_{i+1})|$$

$$=\left|-\frac{1}{i}\log\sum_{j\geqq 1}\frac{l_{m+j+i+1}r_{m+j}}{\sum_k l_k r_k}\exp[-\sum_{k=1}^{i}U(m+k+j+1,m+k+j)]\right|$$

$$\leqq\log\beta_{max}+\left|\frac{1}{i}\log\frac{\sum_{j\geqq 1}l_{m+j+i+1}r_{m+j}}{\sum_k l_k r_k}\right|.$$

Moreover for $j\geqq 1$

$$l_j=T^j 1$$

$$=\sum_{k=0}^{a_j(1)-1}\beta_k{}^{-1}+\beta_{a_j(1)}^{-1}(\sum_{k=0}^{a_{j+1}(1)-1}\beta_k{}^{-1})+\cdots$$

$$=\{\beta_{a_0(1)}^{-1}\cdots\beta_{a_{j-1}(1)}^{-1}\sum_{k=0}^{a_j(1)-1}\beta_k{}^{-1}+\beta_{a_0(1)}^{-1}\cdots\beta_{a_j(1)}^{-1}\sum_{k=0}^{a_{j+1}(1)-1}\beta_k{}^{-1}+\cdots\}$$

$$\times\beta_{a_0(1)}\cdots\beta_{a_{j-1}(1)}$$

$$\geqq C\cdot\mu_\beta'([j])\beta_{a_0(1)}\cdots\beta_{a_{j-1}(1)},$$

for some constant C, and

$$\mu_\beta'([k])\leqq K\beta_{min}^{-k}$$

for some constants K, $k\geqq 1$ and $\beta_{min}=\min\{\beta_0,\cdots,\beta_N\}$. Therefore the right-hand terms of (*) become

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}}\ (\text{the first term})=\sum_{x,y\in I}\mu_\beta'([x,y])U(x,y)=\int d\mu_\beta'\cdot U,$$

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}}\ |\text{the second term}|$$

$$\leqq\overline{\lim_{m\to\infty}}\sum_{i=1}^{\infty}(i+1)\mu_\beta'([\underbrace{m^*,\cdots,m^*}_{i},m])\log\beta_{max}$$

$$\leqq\overline{\lim_{m\to\infty}}\sum_{i=1}^{\infty}(i+1)K\beta_{min}^{-m-i}\log\beta_{max}=0,$$

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}}\ |\text{the third term}|$$

$$\leqq\overline{\lim_{m\to\infty}}\max_i\mu_\beta'([\underbrace{m^*,\cdots,m^*}_{i}])\left(i\log\beta_{max}-\log\frac{\beta_{max}^{-m}}{1-\beta_{max}}\right)$$

$$\leqq\overline{\lim_{m\to\infty}}\max_i\sum_{k=m+1}^{\infty}K\beta_{min}^{-k}\left(i\log\beta_{max}-\log\frac{\beta_{max}^{-m}}{1-\beta_{max}}\right)=0,$$

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}}\ |\text{the fourth term}|$$

$$\leqq \overline{\lim_{m\to\infty}} \max_i \mu_\beta'([\underbrace{m^*, \cdots, m^*}_{i}, m]) \left(i \log \beta_{max} + \left| \log \frac{\mu_\beta'([m+i])}{\sum_j l_j r_j} \right| \right)$$

$$= \overline{\lim_{m\to\infty}} |\max_i \mu_\beta'([m+i]) \log \mu_\beta'([m+i])| = 0,$$

$$\overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}}\ |\text{the fifth term}|$$

$$\leqq \overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}} \Big(\sum_{k\geqq m+n+1} \mu_\beta'([k]) \Big(\log \beta_{max} + \Big| \frac{1}{n-1} \log \frac{\sum_{s\geqq 1} l_{m+s+n} r_{m+s}}{\sum_j l_j r_j} \Big| \Big)$$

$$\leqq \overline{\lim_{m\to\infty}}\ \overline{\lim_{n\to\infty}} \frac{1}{n-1} \Big(\sum_{k\geqq m+n+1} \mu_\beta'([k]) \Big) |\log (\sum_{s\geqq 1} \mu_\beta'([m+s+n]))| = 0.$$

Thus (4) of the Main Lemma is proved. Therefore, applying the Main Lemma, we complete the proof of Theorem 2.

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